Research Article

Finite Frequency Vibration Control for Polytopic Active Suspensions via Dynamic Output Feedback

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This paper presents a disturbance attenuation strategy for active suspension systems with frequency band constraints, where dynamic output feedback control is employed in consideration that not all the state variables can be measured on-line. In view of the fact that human are sensitive to the vibration between 4–8 Hz in vertical direction, the H_∞ control based on generalized Kalman-Yakubovich-Popov (KYP) lemma is developed in this specific frequency, in order to achieve the targeted disturbance attenuation. Moreover, practical constraints required in active suspension design are guaranteed in the whole time domain. At the end of the paper, the outstanding performance of the system using finite frequency approach is confirmed by simulation.

1. Introduction

By reason of rough road conditions, passengers in the car are often in a vibration environment which negatively impacts the comfort, mental, and physical health of them, and suspensions are crucial to attenuate the disturbance transferred to passengers [1–3]. Hence, various approaches various approaches have been developed that aim to enhance suspensions' performance such as adaptive control [4], robust control [5, 6], and fuzzy control [7]. Generally speaking, there are three types of suspensions: passive, semiactive, and active suspensions. Compared with the other two kinds of suspensions, active suspensions have a greater potential to improve ride comfort and to guarantee the ride safety due to the existence of active actuator.

There are three performance requirements for active suspension systems. One is the ride comfort, which requires isolation of vibration from road; another one is handling performance mainly described by road holding, which restricts the hop of wheel in order to ensure continuous contact of wheels to road; the last one is the sprung displacement which limits the suspension stroke within an allowable band.

However, these requirements are usually conflicting. For instance, a large suspension displacement may exist if a better rider comfort performance is required . A variety of control strategies have been applied to cope with this conflict [8–12]. In particular, because the H_∞ norm index can measure the vibration attenuation performance of system appropriately [13], many suspension problems are considered by H_∞ control theory [14–22]. In this paper, the handling performance and the suspension stroke are regarded as constraints, and the ride comfort is deemed as the main index to optimize.

Although various control strategies have been applied to promote ride comfort performance of suspension systems, few of them notice the fact that due to the human body structure and other factors people are more sensitive to disturbances in 4–8 Hz than other frequency in vertical direction (ISO2361). Therefore, it is considerable to develop a finite frequency strategy to reduce the negative effect caused by disturbances in 4–8 Hz. The generalized Kalman-Yakubovich-Popov (KYP) lemma, which has been used to solve practical problems [23, 24], is applied to achieve the finite frequency control to active suspensions.

It should be mentioned that the parameters of passengers including model for suspension system could vary due to the mass change of passengers, so how to guarantee the performance of suspension with varying parameters is worth discussing. Meanwhile, considering that the mass of passengers

can be accessed on line, in this paper, the suspension model is described as a polytope, and then a parameter-dependent control law is proposed to assure the above performance requirements, which can also reach a lower conservativeness than control law based on quadratic stability and constant parameter feedback at the same time. In addition, though state feedback may attain a superior performance compared with static output feedback, measuring some states may bring burden to the systems. Thusly, constructing dynamic output feedback is desirable, which can achieve a relative enhanced performance and meanwhile reduce state-measuring sensors.

The paper is organized as follows. In Section 2 the state-space model for quarter car suspension is presented. In Section 3 the theorems which can be used to design the dynamic feedback controller are illustrated. The simulation is presented in Section 4, and the conclusion is in Section 5.

Notation. For a matrix P, P^T, P^*, P^{-1}, P^{-T}, and $P^\perp$ stand for its transpose, conjugate transpose, inverse, transposed inverse, and orthogonal complement, respectively; $\text{sym}(P)$ denotes $P + P^T$. $P > 0$ ($P < 0$) means that matrix P is positive (negative) definite. For a matrix, $\{\cdot\}_i$ stands for the ith line of the matrix. $\|G(j\omega)\|_\infty$ stands for the H_∞ norm of transfer function matrix G. For matrices P and Q, $P \otimes Q$ means the Kronecker product. In symmetric block matrices or complex matrix expressions, we use an asterisk $(*)$ to represent a term that is induced by symmetry.

2. Quarter Car Suspension Model

The model of a quarter car suspension is shown in Figure 1. m_s and m_u stand for sprung and unsprung mass, respectively. z_s, z_u, and z_r denote the sprung, unsprung displacement, and disturbance displacement from the road, respectively. k_s, k_t, c_s, and c_t are the stiffnesses and dampings of the suspension system, respectively. The input of the controller is denoted by u.

Based on the law of Newton, the motion equation of suspension can be denoted as

$$m_s \ddot{z}_s(t) + c_s [\dot{z}_s(t) - \dot{z}_u(t)] + k_s [z_s(t) - z_u(t)] = u(t),$$
$$m_u \ddot{z}_u(t) - c_s [\dot{z}_s(t) - \dot{z}_u(t)] - k_s [z_s(t) - z_u(t)]$$
$$+ k_t [z_u(t) - z_r(t)] - c_t [\dot{z}_r(t) - \dot{z}_u(t)] = -u(t). \tag{1}$$

Define the following state variables and the disturbance input:

$$\xi_1(t) = z_s(t) - z_u(t), \qquad \xi_2(t) = z_u(t) - z_r(t),$$
$$\xi_3(t) = \dot{z}_s(t), \qquad \xi_4(t) = \dot{z}_u(t), \qquad w(t) = \dot{z}_r(t). \tag{2}$$

Then (1) is equivalent to

$$\dot{\xi}(t) = A(m_s)\xi(t) + B(m_s)u(t) + B_1(m_s)w(t), \tag{3}$$

FIGURE 1: The quarter car model.
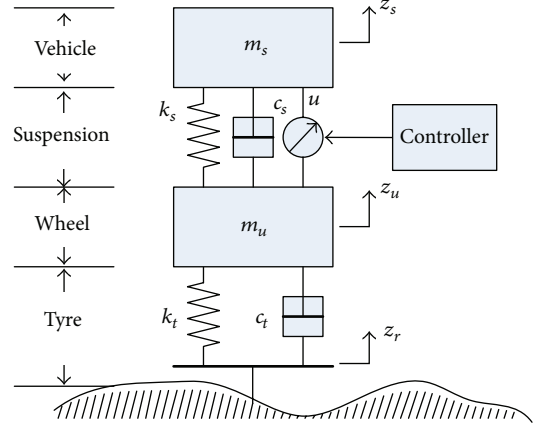
where

$$\xi(t) = \left[\xi_1(t)\,\xi_2(t)\,\xi_3(t)\,\xi_4(t)\right]^T,$$

$$A(m_s) = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_s}{m_s} & 0 & -\dfrac{c_s}{m_s} & \dfrac{c_s}{m_s} \\ \dfrac{k_s}{m_u} & -\dfrac{k_t}{m_u} & \dfrac{c_s}{m_u} & -\dfrac{c_s+c_t}{m_u} \end{bmatrix},$$

$$B(m_s) = \begin{bmatrix} 0 & 0 & \dfrac{1}{m_s} & \dfrac{-1}{m_u} \end{bmatrix}^T,$$

$$B_1(m_s) = \begin{bmatrix} 0 & -1 & 0 & \dfrac{c_t}{m_u} \end{bmatrix}^T. \tag{4}$$

Define

$$z_{o1}(t) = \ddot{z}_s(t), \tag{5}$$

which reflects the acceleration of m_s that contains the body mass of passengers and seat. In the design of control law for suspension system, body acceleration is the main index that needs to be optimized.

The handling performance requires continuous contact of wheel to road, which means that the suspension system needs to guarantee that dynamic tire load is less than static load, namely,

$$k_t(z_u(t) - z_r(t)) < (m_s + m_u)g. \tag{6}$$

The stroke of the suspension could not be so large that it may exceed the maximum, which can be formulated as

$$|z_s(t) - z_u(t)| < z_{\max}. \tag{7}$$

The state space expression is described integrally as

$$\dot{\xi}(t) = A(m_s)\xi(t) + B(m_s)u(t) + B_1w(t),$$
$$z_{o1}(t) = C_1(m_s)\xi(t) + D_1(m_s)u(t),$$
$$z_{o2}(t) = C_2(m_s)\xi(t),$$
$$y(t) = C\xi(t), \tag{8}$$

where A, B, and B_1 are same with the definition in (4), and

$$C_1(m_s) = \begin{bmatrix} -\dfrac{k_s}{m_s} & 0 & -\dfrac{c_s}{m_s} & \dfrac{c_s}{m_s} \end{bmatrix},$$

$$C_2(m_s) = \begin{bmatrix} \dfrac{1}{z_{\max}} & 0 & 0 & 0 \\ 0 & \dfrac{k_t}{(m_s g + m_u g)} & 0 & 0 \end{bmatrix},$$

$$D_1(m_s) = \dfrac{1}{m_s},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

(9)

$z_{o1}(t)$ reflects the acceleration output of m_s; $z_{o2}(t)$ represents for the relative (normalized) constraints output; $y(t)$ stands for the output of measurable states.

The parameter-varying model is depicted as the following polytopic form:

$$(A(m_s), B(m_s), B_1(m_s), C_1(m_s), C_2(m_s), D_1(m_s))$$

$$= \sum_{i=1}^{2} \lambda_i (A_i, B_i, B_{1i}, C_{1i}, C_{2i}, D_{1i}),$$

(10)

where

$$(A_1, B_1, B_{11}, C_{11}, C_{21}, D_{11})$$

$$= (A(m_{s\max}), B(m_{s\max}), B_1(m_{s\max}),$$

$$C_1(m_{s\max}), C_2(m_{s\max}), D_1(m_{s\max})),$$

$$(A_2, B_2, B_{12}, C_{12}, C_{22}, D_{12})$$

$$= (A(m_{s\min}), B(m_{s\min}), B_1(m_{s\min}),$$

$$C_1(m_{s\min}), C_2(m_{s\min}), D_1(m_{s\min})),$$

(11)

$$\lambda_1 = \dfrac{1/m_s - 1/m_{s\min}}{1/m_{s\max} - 1/m_{s\min}},$$

$$\lambda_2 = \dfrac{1/m_{s\max} - 1/m_s}{1/m_{s\max} - 1/m_{s\min}},$$

and $m_{s\max}$, $m_{s\min}$ stand for the maximum and minimum of m_s, respectively.

The form of dynamic output feedback controller is described as

$$\dot{\eta}(t) = A_K(m_s)\eta(t) + B_K(m_s)y(t),$$

$$u(t) = C_K(m_s)\eta(t) + D_K(m_s)y(t).$$

(12)

Substituting (12) into (8), we get

$$\dot{x}(t) = \overline{A}(m_s)x(t) + \overline{B}(m_s)w(t),$$

$$z_{o1}(t) = \overline{C_1}(m_s)x(t),$$

$$z_{o2}(t) = \overline{C_2}(m_s)x(t),$$

(13)

where

$$x(t) = \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix},$$

$$\overline{A}(m_s)$$

$$= \begin{bmatrix} A(m_s) + B(m_s)D_K(m_s)C & B(m_s)C_K(m_s) \\ B_K(m_s)C & A_K(m_s) \end{bmatrix},$$

$$\overline{B}(m_s) = \begin{bmatrix} B_1(m_s) \\ 0 \end{bmatrix},$$

$$\overline{C_1}(m_s)$$

$$= [C_1(m_s) + D_1(m_s)D_K(m_s)C \quad D_1(m_s)C_K(m_s)],$$

$$\overline{C_2}(m_s) = [C_2(m_s) \quad 0].$$

(14)

Denote

$$G(j\omega) = \overline{C_1}(m_s)\left(j\omega I - \overline{A}(m_s)\right)^{-1}\overline{B}(m_s),$$

(15)

as the transfer function from $w(t)$ to $z_{o1}(t)$.

The H_∞ norm of transfer function matrix G is applied to depict the ride comfort performance of suspension system, which is defined as

$$\|G(j\omega)\|_\infty = \sup_{0 \notin w \in L_2[0,+\infty)} \dfrac{\|z_{o2}\|_{L_2}}{\|w\|_{L_2}},$$

(16)

where

$$\|z_{o2}\|_{L_2} = \left(\int_0^{+\infty} \|z_{o2}(t)\|^2 dt\right)^{1/2},$$

$$\|w\|_{L_2} = \left(\int_0^{+\infty} \|w(t)\|^2 dt\right)^{1/2}.$$

(17)

The control target is summarized as follows.

For a certain γ, design a dynamic output feedback in the form of (12) which satisfies

(I) The closed loop system is asymptotically stable.

(II) The finite frequency (from ω_1 to ω_2) H_∞ norm from road disturbance to vehicle acceleration is less than γ. namely,

$$\|G(j\omega)\|_\infty < \gamma. \atop \omega_1 < \omega < \omega_2$$

(18)

(III) The relative constraint responses shown in the following can be satisfied as long as the disturbance energy is less than the maximum of the 2-norm of disturbance input denoted as $w_{\max}$. that is,

$$|\{z_{o2}(t)\}_k| < 1, \quad k = 1, 2.$$

(19)

3. Dynamic Output Feedback Controller Design for Polytopic Suspension System

In this section, we will derive three theorems for the design of output feedback controller that satisfies (19) and one theorem of full frequency controller used for comparison.

3.1. Finite Frequency Design

Theorem 1. *For the given parameters $\gamma, \eta, \rho > 0$, if there exist symmetric matrices $P(m_s)$, positive definite symmetric matrices $P_S(m_s)$, $Q(m_s)$, and general matrix $W(m_s)$ satisfying*

$$\begin{bmatrix} -\operatorname{sym}\,(W(m_s)) & -W^T(m_s)\overline{A}(m_s) + P_S(m_s) & -W^T(m_s) & -W^T(m_s)\overline{B}(m_s) \\ * & -P_S(m_s) & 0 & 0 \\ * & * & -P_S(m_s) & 0 \\ * & * & * & -\eta I \end{bmatrix} < 0, \tag{20}$$

$$\begin{bmatrix} \Omega_1 & \Omega_2 \\ * & \Omega_3 \end{bmatrix} < 0, \tag{21}$$

$$\begin{bmatrix} -I & \sqrt{\rho}\,\{\overline{C}_2(m_s)\}_k \\ * & -P_S(m_s) \end{bmatrix} < 0, \quad k = 1, 2, \tag{22}$$

where

$$\Omega_1 = \begin{bmatrix} -\omega_1\omega_2 Q(m_s) - \operatorname{sym}\,\left(\overline{A}^T(m_s)W(m_s)\right) & P(m_s) - j\omega_c Q(m_s) + W^T(m_s) \\ P(m_s) + j\omega_c Q(m_s) + W(m_s) & -Q(m_s) \end{bmatrix},$$

$$\Omega_2 = \begin{bmatrix} -W^T(m_s)\overline{B}(m_s) & \overline{C}_1^T(m_s) \\ 0 & 0 \end{bmatrix},$$

$$\Omega_3 = \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & -I \end{bmatrix},$$

$$\omega_c = \frac{(\omega_1 + \omega_2)}{2},$$

(23)

then a dynamic output feedback controller exists, which satisfies the requirements of (I), (II) and (III) with $w_{\max} = \rho/\eta$.

Proof. By using Schur complement, inequality (20) is equivalent to

$$\begin{bmatrix} \Gamma & -W^T(m_s)\overline{A}(m_s) + P_S(m_s) \\ * & -P_S(m_s) \end{bmatrix} < 0, \tag{24}$$

where

$$\Gamma = \frac{1}{\eta}W^T(m_s)\overline{B}(m_s)\overline{B}^T(m_s)W(m_s) \\ + W^T(m_s)P_S^{-1}(m_s)W(m_s) + \operatorname{sym}\,(W(m_s)). \tag{25}$$

Multiplying (24) by diag$\{-W^{-T}(m_s), P_S^{-T}(m_s)\}$ from the left side and by diag$\{-W^{-1}(m_s), P_S^{-1}(m_s)\}$ from the right side, respectively, we obtain

$$\begin{bmatrix} \frac{1}{\eta}\overline{B}(m_s)\overline{B}^T(m_s) + P_S^{-1}(m_s) - \operatorname{sym}\,(L(m_s)) & \overline{A}(m_s)P_S^{-1}(m_s) + L^T(m_s) \\ * & -P_S^{-1}(m_s) \end{bmatrix} < 0, \tag{26}$$

where $L = -W^{-1}$.

Applying reciprocal projection theorem (see Appendix A) and choosing $S^T(m_s) = \overline{A}(m_s)P_S^{-1}(m_s)$, $\Psi(m_s) = (1/\eta)\overline{B}(m_s)\overline{B}^T(m_s)$, inequality (26) is equivalent to

$$\overline{A}(m_s)P_S^{-1}(m_s) + P_S^{-1}(m_s)\overline{A}^T(m_s) + \frac{1}{\eta}\overline{B}(m_s)\overline{B}^T(m_s) < 0. \tag{27}$$

Multiplying the above inequality from both the left and the right sides by $P_S(m_s)$, we get

$$P_S(m_s)\overline{A}(m_s) + \overline{A}^T(m_s)P_S(m_s) + \frac{1}{\eta}P_S(m_s)\overline{B}(m_s)\overline{B}^T(m_s)P_S(m_s) < 0, \tag{28}$$

which guarantees

$$P_S(m_s)\overline{A}(m_s) + \overline{A}^T(m_s)P_S(m_s) < 0, \tag{29}$$

obviously. From the standard Lyapunov theory for continuous-time linear system, the closed-loop system (13) is asymptotically stable with $w(t) = 0$.

By substitution, inequality (21) is equivalent to

$$[I \;\; F_B(m_s)]\,\Omega(m_s)\,[I \;\; F_B(m_s)]^T + \operatorname{sym}(F_A(m_s)W(m_s)R) < 0, \tag{30}$$

where

$$F_A(m_s) = \begin{bmatrix} -\overline{A}^T(m_s) \\ I \\ -\overline{B}^T(m_s) \end{bmatrix}, \qquad F_B(m_s) = \begin{bmatrix} \overline{C}_1^T(m_s) \\ 0 \\ 0 \end{bmatrix},$$

$$R = [I \;\; 0 \;\; 0],$$

$$\Omega(m_s) = T\begin{bmatrix} \Phi \otimes P_s(m_s) + \Psi \otimes Q(m_s) & 0 \\ 0 & \Pi \end{bmatrix}T^T, \tag{31}$$

$$\Pi = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}, \qquad \Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\Psi = \begin{bmatrix} -1 & j\omega_c \\ -j\omega_c & -\omega_1\omega_2 \end{bmatrix}, \qquad T = \begin{bmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix}.$$

Based on projection lemma (see Appendix A), inequality (30) is equivalent to

$$F_A^\perp(m_s)[I \;\; F_B(m_s)]\,\Omega(m_s)\,[I \;\; F_B(m_s)]^T(F_A^\perp(m_s))^T < 0, \tag{32}$$

$$(R^T)^\perp[I \;\; F_B(m_s)]\,\Omega(m_s)\,[I \;\; F_B(m_s)]^T(R^{T\perp})^T < 0. \tag{33}$$

Noting that inequality (33) is eternal establishment, we just need to consider inequality (32), which is equivalent to

$$F^T(m_s)\,\Omega(m_s)\,F(m_s) < 0, \tag{34}$$

where

$$F(m_s) = \begin{bmatrix} I & \overline{A}^T(m_s) & 0 & \overline{C}_1^T(m_s) \\ 0 & \overline{B}^T(m_s) & I & 0 \end{bmatrix}^T. \tag{35}$$

Rewrite inequality (34) as

$$\begin{bmatrix} \overline{A}(m_s) & \overline{B}(m_s) \\ I & 0 \\ \overline{C}_1(m_s) & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \Phi \otimes P(m_s) + \Psi \otimes Q(m_s) & 0 \\ 0 & \Pi \end{bmatrix} \times \begin{bmatrix} \overline{A}(m_s) & \overline{B}(m_s) \\ I & 0 \\ \overline{C}_1(m_s) & 0 \\ 0 & I \end{bmatrix} < 0, \tag{36}$$

and then we obtain

$$\begin{bmatrix} \overline{A}(m_s) & \overline{B}(m_s) \\ I & 0 \end{bmatrix}^T (\Phi \otimes P(m_s) + \Psi \otimes Q(m_s)) \times \begin{bmatrix} \overline{A}(m_s) & \overline{B}(m_s) \\ I & 0 \end{bmatrix} + \begin{bmatrix} \overline{C}_1(m_s) & 0 \\ 0 & I \end{bmatrix}^T \times \Pi \begin{bmatrix} \overline{C}_1(m_s) & 0 \\ 0 & I \end{bmatrix} < 0. \tag{37}$$

Applying generalized KYP lemma (see Appendix A), we get

$$\Xi^*\Pi\Xi < 0, \qquad \omega_1 < \omega < \omega_2, \tag{38}$$

where

$$\Xi = \begin{bmatrix} \overline{C}_1(m_s) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} (j\omega I - (\overline{A}(m_s))^{-1}\overline{B}(m_s) \\ I \end{bmatrix}, \tag{39}$$

namely,

$$\sup_{\omega_1 < \omega < \omega_2} \|G(j\omega)\|_\infty < \gamma. \tag{40}$$

Select

$$V(t) = x^T(t)P_S(m_s)x(t) \tag{41}$$

as the Lyapunov function, we obtain

$$\dot{V}(t) = 2x^T(t)P_S(m_s)\overline{A}(m_s)x(t) + 2x^T(t)P_S(m_s)\overline{B}(m_s)w(t). \tag{42}$$

Applying the following inequality

$$2x^T(t)P_S(m_s)\overline{B}(m_s)w(t) \leq \frac{1}{\eta}x^T(t)P_s(m_s)\overline{B}(m_s)\overline{B}^T(m_s)P_S(m_s)(x)t + \eta w^T(t)w(t), \quad \forall \eta > 0, \tag{43}$$

and (28) to (42), we get

$$\dot{V}(t) \le \eta w^T(t) w(t). \tag{44}$$

Integrate (44) from 0 to t:

$$V(t) - V(0) \le \eta \int_0^t w^T(t) w(t) \, dt = \eta \|w(t)\|_2 \le \eta w_{\max}. \tag{45}$$

Substituting (41) into (45) with $V(0) = 0$, we get

$$x^T(t) P_S(m_s) x(t) \le V(0) + \eta w_{\max} = \rho, \tag{46}$$

which is equivalent to

$$\overline{x}^T(t) \overline{x}(t) \le \rho, \tag{47}$$

where $\overline{x}(t) = P_S^{1/2}(m_s) x(t)$.

Noting that

$$\begin{aligned}
&\max \left| \{z_{o2}(t)\}_k \right|^2 \\
&= \max \left\| \overline{x}^T(t) P_S^{-1/2}(m_s) \left\{ \overline{C}_2(m_s) \right\}_k^T \right. \\
&\quad \left. \times \left\{ \overline{C}_2(m_s) \right\}_k P_S^{-1/2}(m_s) \overline{x}(t) \right\|_2 \\
&\le \rho \theta_{\max} \left(P_S^{-1/2}(m_s) \left\{ \overline{C}_2(m_s) \right\}_k^T \left\{ \overline{C}_2(m_s) \right\}_k P_S^{-1/2}(m_s) \right),
\end{aligned} \tag{48}$$

where $\theta_{\max}$ stands for the maximum eigenvalue, we can therefore guarantee constraints mentioned in (19) as long as

$$\rho P_S^{-1/2}(m_s) \left\{ C_2(m_s) \right\}_k^T \left\{ C_2(m_s) \right\}_k P_S^{-1/2}(m_s) < I, \tag{49}$$

which is equivalent to (22) by applying Schur complement.
□

Before giving a convex expression which can be solved by LMI Toolbox, we firstly perform transformation to inequalities (20), (21), and (22).

Decompose matrix $W(m_s)$ for convenience in the following form:

$$\begin{aligned}
W(m_s) &= \begin{bmatrix} X(m_s) & Y(m_s) \\ U(m_s) & V(m_s) \end{bmatrix}, \\
W^{-1}(m_s) &= \begin{bmatrix} M(m_s) & G(m_s) \\ H(m_s) & L(m_s) \end{bmatrix}.
\end{aligned} \tag{50}$$

According to the literature [25], we assume that both $U(m_s)$ and $H(m_s)$ are invertible without loss of generality.

Denote

$$\Delta(m_s) = \begin{bmatrix} I & M(m_s) \\ 0 & H(m_s) \end{bmatrix}, \tag{51}$$

and perform the congruence transformation to inequalities (20), (21), and (22) by

$$\begin{aligned}
J_1(m_s) &= \text{diag} \left\{ \Delta^T(m_s), \Delta^T(m_s), \Delta^T(m_s), I \right\}, \\
J_2(m_s) &= \text{diag} \left\{ \Delta^T(m_s), \Delta^T(m_s), I, I \right\}, \\
J_3(m_s) &= \text{diag} \left\{ I, \Delta^T(m_s) \right\},
\end{aligned} \tag{52}$$

respectively.

Denote

$$\begin{aligned}
\widehat{Q}(m_s) &= \Delta^T(m_s) Q(m_s) \Delta(m_s), \\
\widehat{P}(m_s) &= \Delta^T(m_s) P(m_s) \Delta(m_s), \\
\widehat{P}_S(m_s) &= \Delta^T(m_s) P_S(m_s) \Delta(m_s), \\
\widehat{A}(m_s) &= \Delta^T(m_s) W^T(m_s) \overline{A}(m_s) \Delta(m_s) \\
&= \begin{bmatrix} X^T(m_s) A(m_s) + \widehat{B}_K(m_s) C & \widehat{A}_K(m_s) \\ A(m_s) + B(m_s) \widehat{D}_K(m_s) C & A(m_s) M(m_s) + B(m_s) \widehat{C}_K(m_s) \end{bmatrix}, \\
\widehat{B}(m_s) &= \Delta^T(m_s) W^T(m_s) \overline{B}(m_s) \\
&= \begin{bmatrix} X^T(m_s) B_1(m_s) \\ B_1(m_s) \end{bmatrix}, \\
\widehat{C}_1(m_s) &= \overline{C}_1(m_s) \Delta(m_s) \\
&= \begin{bmatrix} C_1(m_s) + D_1(m_s) \widehat{D}_K(m_s) C & C_1(m_s) M(m_s) + D_1(m_s) \widehat{C}_K(m_s) \end{bmatrix}, \\
\widehat{C}_2(m_s) &= \overline{C}_2(m_s) \Delta(m_s) = \begin{bmatrix} C_2(m_s) & C_2(m_s) M(m_s) \end{bmatrix},
\end{aligned}$$

$$\widehat{W}(m_s) = \Delta^T(m_s) W(m_s) \Delta(m_s)$$

$$= \begin{bmatrix} X^T(m_s) & Z(m_s) \\ I & M(m_s) \end{bmatrix}, \tag{53}$$

then we get the following theorem.

Theorem 2. *For the given parameters $\gamma, \eta, \rho > 0$, if there exist symmetric matrix $\widehat{P}(m_s)$, positive definite symmetric matrices* $\widehat{P}_S(m_s)$, $\widehat{Q}(m_s)$, *and general matrices* $\widehat{A}_K(m_s)$, $\widehat{B}_K(m_s)$, $\widehat{C}_K(m_s)$, $\widehat{D}_K(m_s)$, $\widehat{W}(m_s)$, $M(m_s)$, $X(m_s)$, $Z(m_s)$ *satisfying*

$$\begin{bmatrix} \mathrm{sym}\left(\widehat{W}(m_s)\right) & -\widehat{A}(m_s) + \widehat{P}_S(m_s) & -\widehat{W}^T(m_s) & -\widehat{B}(m_s) \\ * & -\widehat{P}_S(m_s) & 0 & 0 \\ * & * & -\widehat{P}_S(m_s) & 0 \\ * & * & * & -\eta I \end{bmatrix} < 0, \tag{54}$$

$$\begin{bmatrix} -\omega_1\omega_2\widehat{Q}(m_s) - \mathrm{sym}\left(\widehat{A}(m_s)\right) & \widehat{P}(m_s) - j\omega_c\widehat{Q}(m_s) + \widehat{W}^T(m_s) & -\widehat{B}(m_s) & \widehat{C}_1^T(m_s) \\ \widehat{P}(m_s) + j\omega_c\widehat{Q}(m_s) + \widehat{W}(m_s) & -\widehat{Q}(m_s) & 0 & 0 \\ -\widehat{B}^T(m_s) & 0 & -\gamma^2 I & 0 \\ \widehat{C}_1(m_s) & 0 & 0 & -I \end{bmatrix} < 0, \tag{55}$$

$$\begin{bmatrix} -I & \sqrt{\rho}\{\widehat{C}_2(m_s)\}_k \\ * & -\widehat{P}_S(m_s) \end{bmatrix} < 0, \quad k = 1, 2, \tag{56}$$

then a dynamic output feedback controller exists, which satisfies the requirements of (I), (II) and (III) with $w_{\max} = \rho/\eta$.

The corresponding controller in the form of (12) can be given by

$$D_K(m_s) = \widehat{D}_K(m_s),$$

$$C_K(m_s) = \left(\widehat{C}_K(m_s) - D_K(m_s) CM(m_s)\right) H^{-1}(m_s),$$

$$B_K(m_s) = U^{-T}(m_s)\left(\widehat{B}_K(m_s) - X^T(m_s) B(m_s) D_K(m_s)\right),$$

$$A_K(m_s)$$

$$= U^{-T}(m_s)\left[\widehat{A}_K(m_s) - X^T(m_s) A(m_s) M(m_s)\right.$$

$$- X^T(m_s) B(m_s) D_K(m_s) CM(m_s)$$

$$- U^T(m_s) B_K(m_s) CM(m_s)$$

$$\left. - X^T(m_s) B(m_s) C_K(m_s) H(m_s)\right]$$

$$\times H^{-1}(m_s). \tag{57}$$

Though a parameter-dependent controller can be designed via Theorem 2, it is difficult to obtain targeted matrices in real time as m_s varies. Therefore, we give a tractable LMI-based theorem as follows.

Theorem 3. *For the given parameters γ, η, and $\rho > 0$, if there exist symmetric matrix $\widehat{P}_i$, positive definite symmetric matrices* $\widehat{P}_{Si}, \widehat{Q}_i$, *and general matrices* $\widehat{A}_{Ki}, \widehat{B}_{Ki}, \widehat{C}_{Ki}, \widehat{D}_{Ki}, \widehat{W}_i, M_i, X_i, Z_i$ *($i = 1, 2$), satisfying*

$$\begin{bmatrix} J_1 & J_2 \\ * & J_3 \end{bmatrix} < 0, \quad 1 \le i \le j \le 2, \tag{58}$$

$$\begin{bmatrix} K_1 + jK_2 & K_3 \\ * & K_4 \end{bmatrix} < 0, \quad 1 \le i \le j \le 2, \tag{59}$$

$$\begin{bmatrix} -I & \sqrt{\rho}\{\widehat{C}_{2ij} + \widehat{C}_{2ji}\}_k \\ * & -\widehat{P}_{Si} - \widehat{P}_{Sj} \end{bmatrix} < 0, \quad k = 1, 2; \ 1 \le i \le j \le 2, \tag{60}$$

where

$$J_1 = \begin{bmatrix} \mathrm{sym}\left(\widehat{W}_i + \widehat{W}_j\right) & -\widehat{A}_{ij} - \widehat{A}_{ji} + \widehat{P}_{Si} + \widehat{P}_{Sj} \\ * & -\widehat{P}_{Si} - \widehat{P}_{Sj} \end{bmatrix},$$

$$J_2 = \begin{bmatrix} -\widehat{W}_i^T - \widehat{W}_j^T & -\widehat{B}_{ij} - \widehat{B}_{ji} \\ 0 & 0 \end{bmatrix},$$

$$J_3 = \begin{bmatrix} -\widehat{P}_{Si} - \widehat{P}_{Sj} & 0 \\ * & -2\eta I \end{bmatrix},$$

$$K_1 = \begin{bmatrix} -\omega_1\omega_2\left(\widehat{Q}_i + \widehat{Q}_j\right) - \mathrm{sym}\left(\widehat{A}_{ij} + \widehat{A}_{ji}\right) & \widehat{P}_i + \widehat{P}_j + \widehat{W}_i^T + \widehat{W}_j^T \\ \widehat{P}_i + \widehat{P}_j + \widehat{W}_i + \widehat{W}_j & -\widehat{Q}_i - \widehat{Q}_j \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 0 & -\omega_c\left(\widehat{Q}_i + \widehat{Q}_j\right) \\ \omega_c\left(\widehat{Q}_i + \widehat{Q}_j\right) & 0 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} -\widehat{B}_{ij} - \widehat{B}_{ji} & \widehat{C}_{1ij}^T + \widehat{C}_{1ji}^T \\ 0 & 0 \end{bmatrix},$$

$$K_4 = \begin{bmatrix} -2\gamma^2 I & 0 \\ 0 & -2I \end{bmatrix},$$

$$\widehat{A}_{ij} = \begin{bmatrix} X_i^T A_j + \widehat{B}_{Ki} C & \widehat{A}_{Ki} \\ A_i + B_i \widehat{D}_{Kj} C & A_i M_j + B_i \widehat{C}_{Kj} \end{bmatrix},$$

$$\widehat{B}_{ij} = \begin{bmatrix} X_i^T B_{1j} \\ B_{1i} \end{bmatrix},$$

$$\widehat{C}_{1ij} = \begin{bmatrix} C_{1i} + D_{1i} \widehat{D}_{Kj} C & C_{1i} M_j + D_{1i} \widehat{C}_{Kj} \end{bmatrix},$$

$$\widehat{C}_{2ij} = \begin{bmatrix} C_{2i} & C_{2i} M_j \end{bmatrix},$$

$$\widehat{W}_i = \begin{bmatrix} X_i^T & Z_i \\ I & M_i \end{bmatrix},$$

$$(61)$$

then a dynamic output feedback controller exists, which satisfies the requirements of (I), (II) and (III) with $w_{\max} = \rho/\eta$.

The corresponding controller in the form of (12) can be given by

$$D_K(m_s) = \widehat{D}_K(m_s),$$

$$C_K(m_s) = \left(\widehat{C}_K(m_s) - D_K(m_s) CM(m_s)\right) H^{-1}(m_s),$$

$$B_K(m_s) = U^{-T}(m_s)\left(\widehat{B}_K(m_s) - X^T(m_s) B(m_s) D_K(m_s)\right),$$

$$A_K(m_s) = U^{-T}(m_s)\Big[\widehat{A}_K(m_s) - X^T(m_s) A(m_s) M(m_s)$$

$$- X^T(m_s) B(m_s) D_K(m_s) CM(m_s)$$

$$- U^T(m_s) B_K(m_s) CM(m_s)$$

$$- X^T(m_s) B(m_s) C_K(m_s) H(m_s)\Big]$$

$$\times H^{-1}(m_s),$$

$$(62)$$

where

$$\left(\widehat{A}_K(m_s), \widehat{B}_K(m_s), \widehat{C}_K(m_s),\right.$$

$$\left. \widehat{D}_K(m_s), M(m_s), X(m_s), Z(m_s)\right)$$

$$= \sum_{i=1}^{2} \lambda_i \left(\widehat{A}_{Ki}, \widehat{B}_{Ki}, \widehat{C}_{Ki}, \widehat{D}_{Ki}, M_i, X_i, Z_i\right),$$

$$(63)$$

and λ_i $(i = 1, 2)$ *can be calculated by* (11).

Proof. we just prove that inequality (58) is sufficient to inequality (54) for simplicity.

Denote

$$\Psi(m_s) = \begin{bmatrix} \mathrm{sym}\left(\widehat{W}(m_s)\right) & -\widehat{A}(m_s) + \widehat{P}_S(m_s) & -\widehat{W}^T(m_s) & -\widehat{B}(m_s) \\ * & -\widehat{P}_S(m_s) & 0 & 0 \\ * & * & -\widehat{P}(m_s) & 0 \\ * & * & * & -\eta I \end{bmatrix}, \quad (64)$$

which stands for the left of inequality (54).

Inequality (58) is equivalent to

$$\Psi_{ij} + \Psi_{ji} < 0, \quad 1 \le i < j \le 2,$$

$$\Psi_{ii} < 0, \quad i = 1, 2,$$

$$(65)$$

where

$$\Psi_{ij} = \begin{bmatrix} \mathrm{sym}\left(\widehat{W}_i\right) & -\widehat{A}_{ij} + \widehat{P}_{Si} & -\widehat{W}_i^T & -\widehat{B}_{ij} \\ * & -\widehat{P}_{Si} & 0 & 0 \\ * & * & -\widehat{P}_{Si} & 0 \\ * & * & * & -\eta I \end{bmatrix},$$

$$(66)$$

$$1 \le i \le j \le 2.$$

Assume that

$$\left(\widehat{A}_K(m_s), \widehat{B}_K(m_s), \widehat{C}_K(m_s),\right.$$

$$\left. \widehat{D}_K(m_s), M(m_s), X(m_s), Z(m_s)\right)$$

$$= \sum_{i=1}^{2} \lambda_i \left(\widehat{A}_{Ki}, \widehat{B}_{Ki}, \widehat{C}_{Ki}, \widehat{D}_{Ki}, M_i, X_i, Z_i\right),$$

$$(67)$$

and then we get

$$\Psi(m_s) = \sum_{i=1}^{2}\sum_{j=1}^{2} \lambda_i \lambda_j \Psi_{ij} = \sum_{i=1}^{2} \lambda_i^2 \Psi_{ii} + \lambda_1 \lambda_2 \left(\Psi_{12} + \Psi_{21}\right), \quad (68)$$

which is negative definite by inequality (65), that is,

$$\Psi(m_s) < 0. \quad (69)$$

$\square$

Remark 4. $U(m_s)$, $H(m_s)$ should be chosen to meet the definition, that is,

$$U^T(m_s) H(m_s) = Z(m_s) - X^T(m_s) M(m_s). \quad (70)$$

However, the value of $U(m_s)$ and $H(m_s)$ can be chosen variously for the given $Z(m_s)$, $H(m_s)$, and $M(m_s)$. In this paper, we use the singular value decomposition approach.

Remark 5. Based on [26], for real matrices S_1 and S_2, $S_1 + jS_2 < 0$ is equivalent to $\begin{bmatrix} S_1 & S_2 \\ -S_2 & S_1 \end{bmatrix} < 0$. Thusly, (59) can be converted into real matrix inequality by defining

$$S_1 = \begin{bmatrix} K_1 & K_3 \\ * & K_4 \end{bmatrix}, \qquad S_2 = \begin{bmatrix} K_2 & 0 \\ 0 & 0 \end{bmatrix}. \quad (71)$$

Remark 6. The matrices of dynamic output feedback controller $A_K(m_s)$, $B_K(m_s)$, $C_K(m_s)$, $D_K(m_s)$, and Lyapunov matrix $P_S(m_s)$ are dependent nonlinearly on parameter m_s. For instance, as can be seen in the above deduction, matrix $D_K(m_s)$ in Theorem 3 can be formulated as

$$
\begin{aligned}
D_K(m_s) &= \widehat{D}_K(m_s) = \sum_{i=1}^{2} \lambda_i \widehat{D}_{Ki} = \lambda_1 \widehat{D}_{K1} + \lambda_2 \widehat{D}_{K2} \\
&= \frac{1/m_s - 1/m_{s\min}}{1/m_{s\max} - 1/m_{s\min}} \widehat{D}_{K1} \\
&\quad + \frac{1/m_{s\max} - 1/m_s}{1/m_{s\max} - 1/m_{s\min}} \widehat{D}_{K2},
\end{aligned}
\tag{72}
$$

where $\widehat{D}_{K1}$ and $\widehat{D}_{K2}$ are the corresponding matrix solutions of LMIs in Theorem 3, and the value of matrix $D_K(m_s)$ may vary with the change of m_s.

Remark 7. Inequalities from (58) to (60) can be simplified from 12 LMIs to 4 LMIs if the following matrix variables are chosen as $\widehat{P}_1 = \widehat{P}_2$, $\widehat{P}_{S1} = \widehat{P}_{S2}$, $\widehat{Q}_1 = \widehat{Q}_2$, $\widehat{A}_{K1} = \widehat{A}_{K2}$, $\widehat{B}_{K1} = \widehat{B}_{K2}$, $\widehat{C}_{K1} = \widehat{C}_{K2}$, $\widehat{D}_{K1} = \widehat{D}_{K2}$, $\widehat{W}_1 = \widehat{W}_2$, $M_1 = M_2$, $X_1 = X_2$, and $Z_1 = Z_2$. However, this simplification will keep the matrices for the designed controller constant for all m_s, and an invariant Lyapunov function, rather than a parameter-dependent one, has to be used for the whole domain, which will bring a larger conservativeness than Theorem 3.

3.2. Full Frequency Design.

Based on the theorem of [27], we formulate Theorem 8 without proof for simplicity.

Theorem 8. *For the given parameter $\gamma > 0$, if there exist positive definite symmetric matrices Y_{Ci}, X_{Ci} and general matrices $\widehat{A}_{Ci}, \widehat{B}_{Ci}, \widehat{C}_{Ci}, \widehat{D}_{Ci}$, $(i = 1, 2)$, satisfying*

$$
\begin{bmatrix}
\mathrm{sym}\left(\overline{A}_{eij} + \overline{A}_{eji}\right) & \overline{B}_{eij} & \overline{C}_{e1ij}^T + \overline{C}_{e1ji}^T \\
* & -2\gamma^2 I & 0 \\
* & * & -2I
\end{bmatrix} < 0,
$$

$$
1 \le i \le j \le 2,
\tag{73}
$$

$$
\begin{bmatrix}
-2I & \sqrt{\rho}\left\{\overline{C}_{e2ij} + \overline{C}_{e2ji}\right\}_k \\
* & -\overline{P}_{Ci} - \widehat{P}_{Cj}
\end{bmatrix} < 0,
$$

$$
k = 1, 2; \quad 1 \le i \le j \le 2,
\tag{74}
$$

where

$$
\overline{A}_{eij} = \begin{bmatrix}
A_i X_{Cj} + B_i \widehat{C}_{Cj} & A_i + B_i \widehat{D}_{Cj} C \\
\widehat{A}_{Ci} & Y_{Ci} A_j + \widehat{B}_{Ci} C
\end{bmatrix},
$$

$$
\overline{B}_{eij} = \begin{bmatrix}
B_{1i} \\
Y_{Ci} B_{1j}
\end{bmatrix},
$$

$$
\overline{C}_{e1ij} = \begin{bmatrix} C_{1i} X_{Cj} + D_{1i} \widehat{C}_{Cj} & C_{1i} + D_{1i} \widehat{D}_{Cj} C \end{bmatrix},
$$

$$
\overline{C}_{e2ij} = \begin{bmatrix} C_{2i} X_{Cj} & C_{2i} \end{bmatrix},
$$

$$
\overline{P}_{C_i} = \begin{bmatrix} X_{Ci} & I \\ I & Y_{Ci} \end{bmatrix},
\tag{75}
$$

then a dynamic output feedback controller exists, which satisfies (I) and (III), and $\|G(j\omega)\|_\infty < \gamma$ with $w_{\max} = \rho/\eta$.

The corresponding controller in the form of (12) can be given by

$$
D_K(m_s) = \widehat{D}_C(m_s),
\tag{76}
$$

$$
C_K(m_s) = \left(\widehat{C}_C(m_s) - \widehat{D}_C(m_s) C X_C(m_s)\right) M_C^{-T}(m_s),
\tag{77}
$$

$$
B_K(m_s) = N_C^{-1}(m_s)\left(\widehat{B}_C(m_s) - Y_C(m_s) B(m_s) \widehat{D}_C(m_s)\right),
\tag{78}
$$

$$
\begin{aligned}
&A_K(m_s) \\
&= N_C^{-1}(m_s)\Big[\widehat{A}_C(m_s) - Y_C(m_s) A(m_s) X_C(m_s) \\
&\quad - Y_C(m_s) B(m_s) \widehat{D}_C(m_s) C X_C(m_s) \\
&\quad - N_C(m_s) B_K(m_s) C X_C(m_s) \\
&\quad - Y_C(m_s) B(m_s) C_K(m_s) M_C^T\Big] \\
&\quad \times M_C^{-T}(m_s),
\end{aligned}
\tag{79}
$$

where

$$
\begin{aligned}
&\left(\widehat{A}_C(m_s), \widehat{B}_C(m_s), \widehat{C}_C(m_s), \widehat{D}_C(m_s), X_C(m_s), Y_C(m_s)\right) \\
&= \sum_{i=1}^{2} \lambda_i \left(\widehat{A}_{Ci}, \widehat{B}_{Ci}, \widehat{C}_{Ci}, \widehat{D}_{Ci}, X_{Ci}, Y_{Ci}\right),
\end{aligned}
\tag{80}
$$

and λ_i $(i = 1, 2)$ can be calculated by (11).

The same approach mentioned in Remark 4 is used to obtain the value of $N_C(m_s)$ and $M_C(m_s)$, which should meet the following equation:

$$
N_C(m_s) M_C^T(m_s) = I - Y_C(m_s) X_C(m_s).
\tag{81}
$$

4. A Design Example

In this section, we will show how to apply the above theorems to design finite frequency controller and full frequency controller for a specific suspension system. The parameter for the suspension is shown in Table 1.

We first choose the following parameters: $\rho = 1$, $\eta = 10000$, $z_{\max} = 0.1\,\mathrm{m}$, $\omega_1 = 8\pi\,\mathrm{rad/s}$, and $\omega_2 = 16\pi\,\mathrm{rad/s}$. Then, the matrices in (12) can be calculated by applying Theorem 3

TABLE 1: The model parameters of active suspensions.

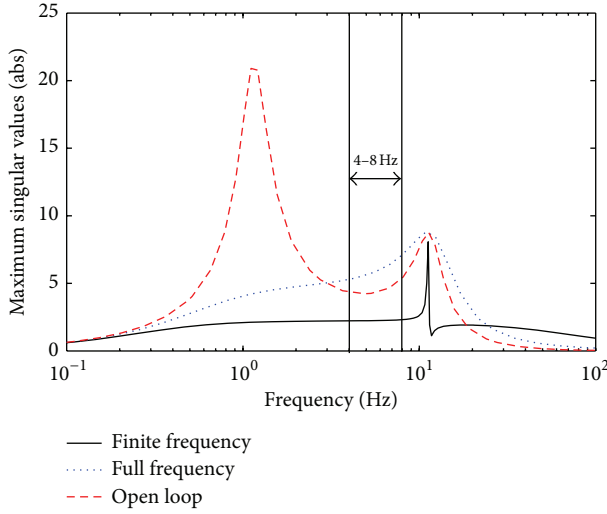| k_s	k_t	c_s	c_t	$m_{s\,max}$	$m_{s\,min}$	m_s	m_u
18 kN/m	200 kN/m	1 kN/m	10 N/m	384 kg	254 kg	320 kg	40 kg

FIGURE 2: The curves of maximum singular values.

and Remark 7 for finite frequency controller and Theorem 8 for full frequency controller (shown in **Appendix B**) with the least guaranteed H_∞ performances $\gamma = 2.54$ for Theorem 3, $\gamma = 2.73$ for Remark 6, and $\gamma = 10.98$ for Theorem 8.

Then, the curves of maximum singular values of systems using open-loop, finite frequency (parameter-dependent), and full frequency controllers are shown in **Figure 2**. Compared with the other two curves (the open-loop system and the system with full frequency controller), the system with finite frequency controller has the least H_∞ norm of the three systems in 4–8 Hz, which indicates that the finite frequency controller has a better effect on the attenuation of targeted frequency disturbance.

In order to examine the performance of finite frequency controller, we assume that the disturbance is in the form of

$$w(t) = \begin{cases} A \sin(2\pi f t) & t \in [0, T] \\ 0 & t \notin [0, T], \end{cases} \tag{82}$$

where A, f stand for the amplitude and the frequency of disturbance, respectively, and $T = 1/f$.

Suppose that A is 0.4 m and f is 5 Hz, and then the body acceleration and the relative constraints responses to this disturbance are shown in Figures 3, 4, and 5, respectively.

It is obvious that the body acceleration response for finite frequency controller decreases faster with respect to time than the other two controllers, and at the same time, both the relative dynamic tire load response and relative suspension stroke response are within the allowable range, which satisfy requirement (III), namely, $|\{z_{o2}(t)\}_k| < 1$, $k = 1, 2$.

5. Conclusion

In this paper, we manage to design a dynamic output feedback controller for active suspensions with practical
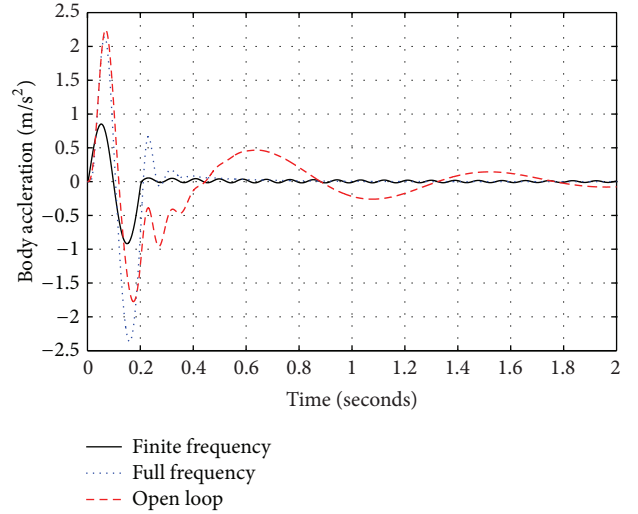

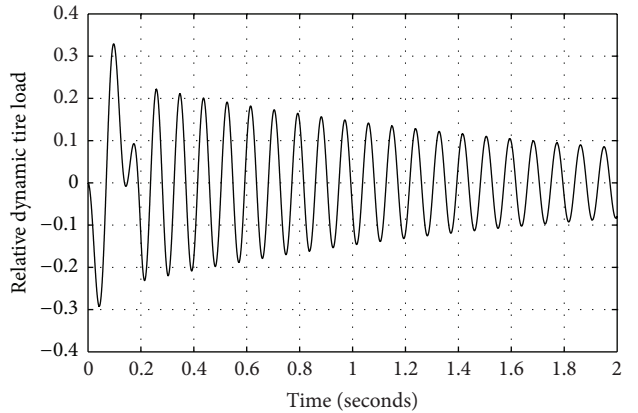FIGURE 3: The time response of body acceleration.

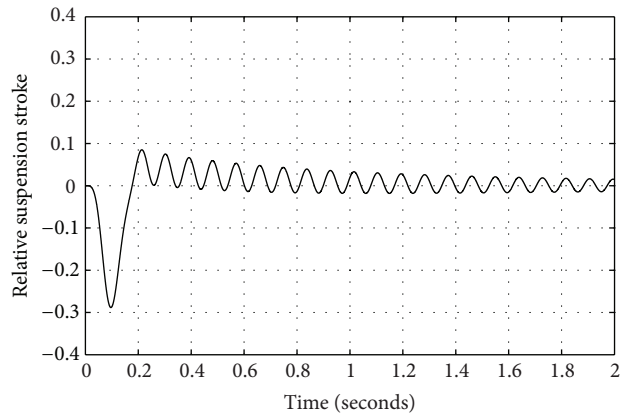FIGURE 4: The time response of relative dynamic tire load.

FIGURE 5: The time response of relative suspension stroke.

constraints included. This controller particularly diminishes disturbance at 4–8 Hz, which is the frequency sensitive band for human. Besides, for the reason that the controller is a parameter-dependent one, it has a smaller conservativeness than controller designed on the basis of quadratic stability and constant parameter feedback. The excellent performance of the closed-loop system with finite frequency controller has been demonstrated by simulation.

Appendices

A. Related Lemmas

Lemma A.1 (generalized KYP lemma [28]). *Let matrices Θ, F, Φ, and Ψ be given, Denote by N_ω the null space of $T_\omega F$, where $T_\omega = [I \quad -j\omega I]$. The inequality*

$$N_\omega^* \Theta N_\omega < 0, \quad \omega \in [\omega_1 \quad \omega_2] \tag{A.1}$$

holds, if and only if, there exist P, $Q > 0$ such that

$$F^* (\Phi \otimes P + \Psi \otimes Q) F < 0, \tag{A.2}$$

where

$$\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \Psi = \begin{bmatrix} -1 & j\omega_c \\ -j\omega_c & -\omega_1\omega_2 \end{bmatrix}, \tag{A.3}$$

with $\omega_c = (\omega_1 + \omega_2)/2$.

Lemma A.2 (projection lemma [25]). *Let Γ, Λ, and Θ be given, there exists a matrix satisfying*

$$\text{sym } (\Gamma\Lambda\Theta) + \Theta < 0, \tag{A.4}$$

if and only if, the following two conditions hold:

$$\Gamma^\perp \Theta (\Gamma^\perp)^T < 0, \qquad (\Lambda^T)^\perp \Theta ((\Lambda^T)^\perp)^T < 0. \tag{A.5}$$

Lemma A.3 (reciprocal projection lemma [25]). *Let P be any given positive definite matrix. The following statements are equivalent.*

(1) $\Psi + S + S^T < 0$.

(2) *The LMI problem*

$$\begin{bmatrix} \Psi + P - [W_s] & S^T + W^T \\ * & -P \end{bmatrix} < 0 \tag{A.6}$$

is feasible with respect to X.

B. Controller Matrices

The parameter matrices of the dynamic output feedback controller for Theorem 3:

$$A_K = \begin{bmatrix} -14.821 & -0.12731 & -1.1424 & 0.00255 \\ -228.44 & -136.92 & 13.701 & 0.02030 \\ 2038.1 & 151.59 & -21.603 & -0.47598 \\ 2.4991 \times 10^9 & -7.4319 \times 10^7 & 4.2012 \times 10^6 & -1.0171 \times 10^6 \end{bmatrix},$$

$$B_K = \begin{bmatrix} 99.943 & 109.08 & 44.969 \\ -3170.6 & -2956.8 & -891.61 \\ -632.00 & 1156.4 & -1125.8 \\ -1.0944 \times 10^8 & -1.8461 \times 10^9 & -7.9445 \times 10^7 \end{bmatrix},$$

$$C_K = [3814.4 \quad -686.19 \quad 1758.9 \quad 3.5961],$$

$$D_K = [-1.8676 \times 10^5 \quad -2.0429 \times 10^5 \quad -86675].$$

$$\tag{B.1}$$

The parameter matrices of the dynamic output feedback controller for Theorem 8:

$$A_K = \begin{bmatrix} -12.734 & 26.225 & 21.367 & -0.31502 \\ -36.286 & 14.954 & 36.984 & 0.69500 \\ -111.01 & -84.507 & -65.487 & -0.0771 \\ -45995 & -42118 & -29117 & -1623.6 \end{bmatrix},$$

$$B_K = \begin{bmatrix} -793.37 & -56.495 & -136.30 \\ 1770.9 & 122.15 & 113.03 \\ -90.855 & 3448.7 & -802.39 \\ 6.4417 \times 10^5 & 7.1058 \times 10^6 & -2.7515 \times 10^5 \end{bmatrix},$$

$$C_K = [518.19 \quad -604.03 \quad -590.41 \quad 10.015],$$

$$D_K = [25422 \quad 1720.8 \quad 2609.8].$$

$$\tag{B.2}$$

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