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*Research Article*

# Oscillation Results for Second-Order Nonlinear Damped Dynamic Equations on Time Scales

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This paper is concerned with second-order nonlinear damped dynamic equations on time scales of the following more general form $(p(t)k_1(x(t),x^{\Delta}(t)))^{\Delta}+r(t)k_2(x(t),x^{\Delta}(t))x^{\Delta}(t)+f(t,x(\sigma(t)))=0$. New oscillation results are given to handle some cases not covered by known criteria. An illustrative example is also presented.

## 1. Introduction

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{T}$ a time scale, that is, a nonempty closed subset of $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. The theory of time scales was introduced by Hilger in his Ph.D. thesis [1] in 1988, and for a comprehensive treatment of the subject, see [2]. Much recent attention has been concerned with the oscillation of dynamic equations on time scales; see, for example, [1–15]. In [9], Došlý and Hilger studied the second-order dynamic equation

$$\left(p(t)x^{\Delta}(t)\right)^{\Delta}+q(t)x(\sigma(t))=0. \tag{1}$$

The authors gave a necessary and sufficient condition for the oscillation of all solutions of (1) on time scales. In [7, 8], Del Medico and Kong used the Riccati transformation as

$$u(t)=\frac{p(t)x^{\Delta}(t)}{x(t)} \tag{2}$$

and obtained some sufficient conditions for oscillation of (1). In [14], Wang considered the nonlinear second-order damped differential equation

$$\begin{aligned}&\left(a(t)\psi(x(t))k\left(x'(t)\right)\right)'+p(t)k\left(x'(t)\right)\\&\quad+q(t)f(x(t))=0,\qquad t\geq t_0,\end{aligned} \tag{3}$$

and established new oscillation criteria. In [13], Tiryaki and Zafer considered the second-order nonlinear differential equation with nonlinear damping

$$\left(r(t)k_1\left(x,x'\right)\right)'+p(t)k_2\left(x,x'\right)x'+q(t)f(x)=0 \tag{4}$$

and gave interval oscillation criteria of (4). In [10], Huang and Wang considered the second-order nonlinear dynamic equation

$$\left(p(t)x^{\Delta}(t)\right)^{\Delta}+f(t,x(\sigma(t)))=0. \tag{5}$$

The authors gave some new oscillation criteria of (5) and extended the results in [7, 8]. In [11], Qiu and Wang studied the second-order nonlinear dynamic equation

$$\left(p(t)\psi(x(t))x^{\Delta}(t)\right)^{\Delta}+f(t,x(\sigma(t)))=0. \tag{6}$$

By employing the Riccati transformation as

$$u(t)=A(t)\frac{p(t)\psi(x(t))x^{\Delta}(t)}{x(t)}+B(t), \tag{7}$$

where $A\in C^1_{rd}(\mathbb{T},(0,\infty))$, $B\in C^1_{rd}(\mathbb{T},\mathbb{R})$, the authors established interval oscillation criteria for (6). And in [12], Qiu and Wang obtained some new Kamenev-type oscillation criteria for dynamic equations of the following more general form:

$$\left(p(t)\psi(x(t))k\circ x^{\Delta}(t)\right)^{\Delta}+f(t,x(\sigma(t)))=0, \quad (8)$$

by using the transformation

$$u(t)=A(t)\frac{p(t)\psi(x(t))k\circ x^{\Delta}(t)}{x(t)}+B(t). \quad (9)$$

In this paper, we consider second-order nonlinear damped dynamic equations of the form

$$\left(p(t)k_1\left(x(t),x^{\Delta}(t)\right)\right)^{\Delta}+r(t)k_2\left(x(t),x^{\Delta}(t)\right)x^{\Delta}(t) + f(t,x(\sigma(t)))=0 \quad (10)$$

on a time scale $\mathbb{T}$. We will employ functions of the form $H(t,s)$ and a generalized Riccati transformation as (7) and (9) which was used in [14, 15] and derive oscillation criteria for (10) in Section 2. An example is presented to demonstrate the obtained results in the final section.

*Definition 1.* A solution $x$ of (10) is said to have a generalized zero at $t^*\in\mathbb{T}$ if $x(t^*)x(\sigma(t^*))\le 0$, and it is said to be nonoscillatory on $\mathbb{T}$ if there exists $t_0\in\mathbb{T}$ such that $x(t)x(\sigma(t))>0$ for all $t>t_0$. Otherwise, it is oscillatory. Equation (10) is said to be oscillatory if all solutions of (10) are oscillatory.

## 2. Main Results

In this section, we establish some oscillation criteria for (10). Our work is based on the application of the Riccati transformation. Throughout this paper we will assume that $\sup\mathbb{T}=\infty$ and

(C1) $p\in C_{rd}(\mathbb{T},(0,\infty))$;

(C2) $r\in C_{rd}(\mathbb{T},\mathbb{R})$;

(C3) $k_1,k_2\in C(\mathbb{R}^2,\mathbb{R})$, and there exist $\alpha_1\ge\alpha_2>0$ and $\alpha_3>0$ such that $0<\alpha_2 vk_1(u,v)\le k_1^2(u,v)\le\alpha_1 vk(u,v)$ and $\alpha_3k_1^2(u,v)\le uvk_2(u,v)$ for all $(u,v)\in(\mathbb{R}\setminus\{0\})^2$;

(C4) for $p,r,\alpha_1,\alpha_3$ above, we always have $\alpha_1\alpha_3r(t)+p(t)>0$;

(C5) $f\in C(\mathbb{T}\times\mathbb{R},\mathbb{R})$.

Preliminaries about time scale calculus can be found in [3–6] and are omitted here. For simplicity, we denote $(a,b)\cap\mathbb{T}=(a,b)_{\mathbb{T}}$ throughout this paper, where $a,b\in\mathbb{R}$ and $[a,b]_{\mathbb{T}}$, $[a,b)_{\mathbb{T}}$, $(a,b]_{\mathbb{T}}$ are denoted similarly.

Now, we give the first theorem.

**Theorem 2.** *Assume that (C1)–(C5) hold and that there exists a function $q\in C_{rd}(\mathbb{T},\mathbb{R})$ such that $uf(t,u)\ge q(t)u^2$. Also, suppose that $x(t)$ is a solution of (10) satisfying $x(t)>0$ for $t\in[t_0,\infty)_{\mathbb{T}}$ with $t_0\in\mathbb{T}$. For $t\in[t_0,\infty)_{\mathbb{T}}$, define*

$$u(t)=A(t)\frac{p(t)k_1\left(x(t),x^{\Delta}(t)\right)}{x(t)}+B(t), \quad (11)$$

*where $A\in C_{rd}^1(\mathbb{T},(0,\infty))$, $B\in C_{rd}^1(\mathbb{T},\mathbb{R})$, and $(\alpha_1A-(\alpha_1-\alpha_2)A^{\sigma})p+\alpha_1\alpha_2\alpha_3rA^{\sigma}>0$ for $t\in[t_0,\infty)_{\mathbb{T}}$. Then, $u(t)$ satisfies*

$$\mu(t)u(t)-\mu(t)B(t)+\alpha_1A(t)p(t)>0, \quad (12)$$

$$u^{\Delta}(t)+\Phi_0(t)+\frac{\Phi_1(t)u^2(t)-\Phi_2(t)u(t)+\Phi_3(t)}{\alpha_1A(t)p(t)(\mu(t)u(t)-\mu(t)B(t)+\alpha_1A(t)p(t))}\le 0, \quad (13)$$

*where*

$$\Phi_0(t)=A^{\sigma}(t)\left(q(t)-\left(\frac{B(t)}{A(t)}\right)^{\Delta}\right),$$
$$\Phi_1(t)=\left(\alpha_1A(t)-(\alpha_1-\alpha_2)A^{\sigma}(t)\right)p(t)+\alpha_1\alpha_2\alpha_3r(t)A^{\sigma}(t),$$
$$\Phi_2(t)=\left((2\alpha_2-\alpha_1)A^{\sigma}(t)+\alpha_1A(t)\right)p(t)B(t)+\alpha_1^2p^2(t)A^{\Delta}(t)A(t)+2\alpha_1\alpha_2\alpha_3r(t)A^{\sigma}(t)B(t),$$
$$\Phi_3(t)=\alpha_2\left(\alpha_1\alpha_3r(t)+p(t)\right)A^{\sigma}(t)B^2(t),$$
$$A^{\sigma}(t)=A(\sigma(t)). \quad (14)$$

*Proof.* By (C3) we see that $x^{\Delta}$ and $k_1(x,x^{\Delta})$ are both positive or both negative or both zero. When $x^{\Delta}>0$, which implies that $k_1(x,x^{\Delta})>0$, it follows that

$$\begin{aligned}\mu u-\mu B+\alpha_1Ap&\ge\mu A\frac{pk_1^2\left(x,x^{\Delta}\right)}{xk_1\left(x,x^{\Delta}\right)}+\alpha_2Ap\\&\ge\alpha_2\mu Ap\frac{x^{\Delta}k_1\left(x,x^{\Delta}\right)}{xk_1\left(x,x^{\Delta}\right)}+\alpha_2Ap\\&=\alpha_2Ap\frac{x^{\sigma}}{x}>0.\end{aligned} \quad (15)$$

When $x^{\Delta}<0$, which implies that $k_1(x,x^{\Delta})<0$, it follows that

$$\begin{aligned}\mu u-\mu B+\alpha_1Ap&=\mu Ap\frac{k_1^2\left(x,x^{\Delta}\right)}{xk_1\left(x,x^{\Delta}\right)}+\alpha_1Ap\\&\ge\mu Ap\frac{\alpha_1x^{\Delta}k_1\left(x,x^{\Delta}\right)}{xk_1\left(x,x^{\Delta}\right)}+\alpha_1Ap\\&=\alpha_1Ap\frac{x^{\sigma}}{x}\ge\alpha_2Ap\frac{x^{\sigma}}{x}>0.\end{aligned} \quad (16)$$

When $x^{\Delta}=0$, which implies that $k_1(x,x^{\Delta})=0$ and $x=x^{\sigma}$, it follows that

$$\mu u-\mu B+\alpha_1Ap=\alpha_1Ap\ge\alpha_2Ap\frac{x^{\sigma}}{x}>0. \quad (17)$$

Hence, we always have

$$\mu u-\mu B+\alpha_1 Ap>0,$$
$$\frac{x}{x^\sigma}\geq\frac{\alpha_2 Ap}{\mu u-\mu B+\alpha_1 Ap}, \tag{18}$$

so (12) holds. Then differentiating (11) and using (10), it follows that

$$\begin{aligned}
u^\Delta &= A^\Delta\left(\frac{pk_1(x,x^\Delta)}{x}\right)+A^\sigma\left(\frac{pk_1(x,x^\Delta)}{x}\right)^\Delta+B^\Delta\\
&=\frac{A^\Delta}{A}(u-B)\\
&\quad+A^\sigma\frac{(pk_1(x,x^\Delta))^\Delta x-pk_1(x,x^\Delta)x^\Delta}{xx^\sigma}+B^\Delta\\
&=\frac{A^\Delta}{A}u+B^\Delta-\frac{A^\Delta}{A}B-A^\sigma\frac{f(t,x^\sigma)}{x^\sigma}\\
&\quad-A^\sigma\frac{rk_2(x,x^\Delta)x^\Delta}{x^\sigma}-A^\sigma\frac{pk_1(x,x^\Delta)x^\Delta}{xx^\sigma}\\
&\leq\frac{A^\Delta}{A}u+A^\sigma\left(\frac{B}{A}\right)^\Delta-A^\sigma q\\
&\quad-A^\sigma r\frac{k_2(x,x^\Delta)xx^\Delta}{xx^\sigma}-A^\sigma p\frac{k_1^2(x,x^\Delta)}{\alpha_1 xx^\sigma}\\
&\leq\frac{A^\Delta}{A}u-\Phi_0-\alpha_3 A^\sigma r\frac{k_1^2(x,x^\Delta)}{x^2}\frac{x}{x^\sigma}\\
&\quad-\frac{1}{\alpha_1}A^\sigma p\frac{k_1^2(x,x^\Delta)}{x^2}\frac{x}{x^\sigma}\\
&\leq\frac{A^\Delta}{A}u-\Phi_0-\left(\alpha_3 r+\frac{p}{\alpha_1}\right)A^\sigma\left(\frac{u-B}{Ap}\right)^2\\
&\quad\times\frac{\alpha_2 Ap}{\mu u-\mu B+\alpha_1 Ap}\\
&=\frac{A^\Delta}{A}u-\Phi_0-\frac{\alpha_2(\alpha_1\alpha_3 r+p)}{\alpha_1 p}\frac{A^\sigma}{A}\frac{(u-B)^2}{\mu u-\mu B+\alpha_1 Ap}\\
&=\frac{-\Phi_1 u^2+\Phi_2 u-\Phi_3}{\alpha_1 Ap(\mu u-\mu B+\alpha_1 Ap)}-\Phi_0,
\end{aligned} \tag{19}$$

so (13) holds. Theorem 2 is proved. □

*Remark 3.* In Theorem 2, the condition $(\alpha_1 A-(\alpha_1-\alpha_2)A^\sigma)p+\alpha_1\alpha_2\alpha_3 rA^\sigma>0$ ensures that the coefficient of $u^2$ in (13) is always negative. The condition is obvious and easy to be fulfilled. For example, when $A^\Delta(t)\leq 0$ for all $t\in[t_0,\infty)_{\mathbb{T}}$, we have $A^\sigma=A+\mu A^\Delta\leq A$; by (C4) we see that

$$\begin{aligned}
&(\alpha_1 A-(\alpha_1-\alpha_2)A^\sigma)p+\alpha_1\alpha_2\alpha_3 rA^\sigma\\
&\geq(\alpha_1 A^\sigma-(\alpha_1-\alpha_2)A^\sigma)p+\alpha_1\alpha_2\alpha_3 rA^\sigma\\
&=\alpha_2 A^\sigma(\alpha_1\alpha_3 r+p)>0.
\end{aligned} \tag{20}$$

Let $D_0=\{s\in\mathbb{T}:s\geq 0\}$ and $D=\{(t,s)\in\mathbb{T}^2:t\geq s\geq 0\}$. For any function $f(t,s)$: $\mathbb{T}^2\to\mathbb{R}$, denote by $f_2^\Delta$ the partial derivatives of $f$ with respect to $s$. For $E\subset\mathbb{R}$, denote by $L(E)$ the space of functions which are integrable on any compact subset of $E$. Define

$$\begin{aligned}
(\mathscr{A},\mathscr{B})=\Big\{&(A,B):A(s)\in C_{rd}^1(D_0,(0,\infty)),\\
&B(s)\in C_{rd}^1(D_0,\mathbb{R}),\\
&(\alpha_1 A(s)-(\alpha_1-\alpha_2)A^\sigma(s))p(s)\\
&+\alpha_1\alpha_2\alpha_3 r(s)A^\sigma(s)>0,\\
&\alpha_1 A(s)p(s)\pm\mu(s)B(s)>0,s\in D_0\Big\};\\
\mathscr{H}=\Big\{&H(t,s)\in C^1(D,[0,\infty)):\\
&H(t,t)=0,\\
&H(t,s)>0,H_2^\Delta(t,s)\leq 0,t>s\geq 0\Big\}.
\end{aligned} \tag{21}$$

These function classes will be used throughout this paper. Now, we are in a position to give the second theorem.

**Theorem 4.** *Assume that (C1)–(C5) hold and that there exists a function $q\in C_{rd}(\mathbb{T},\mathbb{R})$ such that $uf(t,u)\geq q(t)u^2$. Also, suppose that there exist $(A,B)\in(\mathscr{A},\mathscr{B})$ and $H\in\mathscr{H}$ such that $M(t,\cdot)\in L([0,\rho(t)]_{\mathbb{T}})$ and for any $t_0\in\mathbb{T}$,*

$$\begin{aligned}
\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\Bigg[&\int_{t_0}^t H(t,\sigma(s))\Phi_0(s)\Delta s-\int_{t_0}^{\rho(t)}M(t,s)\Delta s\\
&+H_2^\Delta(t,\rho(t))\\
&\times(\alpha_1 A(\rho(t))p(\rho(t))\\
&-\mu(\rho(t))B(\rho(t)))\Bigg]=\infty,
\end{aligned} \tag{22}$$

*where $\Phi_0$ is defined as before, and*

$$\begin{aligned}
M(t,s)&=\frac{\Phi_4^2(t,s)}{4\alpha_1 A(s)p(s)\min\{\Phi_5(t,s),\Phi_6(t,s)\}},\\
\Phi_4(t,s)&=\alpha_1 p(s)H(t,s)A(s)B(s)\\
&\quad+((2\alpha_2-\alpha_1)p(s)+2\alpha_1\alpha_2\alpha_3 r(s))\\
&\quad\times H(t,\sigma(s))A^\sigma(s)B(s)\\
&\quad+\alpha_1^2 p^2(s)A(s)(H(t,s)A(s))^{\Delta_s},\\
\Phi_5(t,s)&=\alpha_2 H(t,\sigma(s))A^\sigma(s)(\alpha_1\alpha_3 r(s)+p(s))\\
&\quad\times(\alpha_1 A(s)p(s)+\mu(s)B(s)),
\end{aligned}$$

$$\Phi_6(t,s) = \big(\alpha_1 p(s) H(t,s) A(s) - (\alpha_1-\alpha_2) p(s) H(t,\sigma(s)) A^\sigma(s) + \alpha_1\alpha_2\alpha_3 r(s) H(t,\sigma(s)) A^\sigma(s)\big) \times \big(\alpha_1 A(s) p(s) - \mu(s) B(s)\big). \tag{23}$$

*Then,* (10) *is oscillatory.*

*Proof.* Assume that (10) is not oscillatory. Without loss of generality we may assume that there exists $t_0 \in [0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$. Let $u(t)$ be defined by (11). Then by Theorem 2, (12) and (13) hold.

For simplicity in the following, we let $H_\sigma = H(t,\sigma(s))$, $H = H(t,s)$, and $H_2^\Delta = H_2^\Delta(t,s)$ and omit the arguments in the integrals. For $s \in \mathbb{T}$, $H_\sigma - H = \mu H_2^\Delta$.

Multiplying (13), where $t$ is replaced by $s$, by $H_\sigma$ and integrating it with respect to $s$ from $t_0$ to $t$ with $t \in \mathbb{T}$ and $t \ge \sigma(t_0)$, we obtain

$$\int_{t_0}^{t} H_\sigma \Phi_0 \Delta s \le -\int_{t_0}^{t} \left( H_\sigma u^\Delta + H_\sigma \frac{\Phi_1 u^2 - \Phi_2 u + \Phi_3}{\alpha_1 Ap(\mu u - \mu B + \alpha_1 Ap)} \right) \Delta s, \tag{24}$$

where $\Phi_1$, $\Phi_2$, $\Phi_3$ are defined as before.

Noting that $H(t,t) = 0$, by the integration by parts formula we have

$$\begin{aligned}
\int_{t_0}^{t} H_\sigma \Phi_0 \Delta s &\le H(t,t_0) u(t_0) + \int_{t_0}^{t} \left( H_2^\Delta u - H_\sigma \frac{\Phi_1 u^2 - \Phi_2 u + \Phi_3}{\alpha_1 Ap(\mu u - \mu B + \alpha_1 Ap)} \right) \Delta s \\
&\le H(t,t_0) u(t_0) + \int_{t_0}^{t} \left( H_2^\Delta u - H_\sigma \frac{\Phi_1 u^2 - \Phi_2 u}{\alpha_1 Ap(\mu u - \mu B + \alpha_1 Ap)} \right) \Delta s \\
&= H(t,t_0) u(t_0) + \int_{\rho(t)}^{t} H_2^\Delta u \Delta s + \int_{t_0}^{\rho(t)} \left( H_2^\Delta u - H_\sigma \frac{\Phi_1 u^2 - \Phi_2 u}{\alpha_1 Ap(\mu u - \mu B + \alpha_1 Ap)} \right) \Delta s.
\end{aligned} \tag{25}$$

Since $H_2^\Delta \le 0$ on $D$, from (12) we see that, for $t \ge \sigma(t_0)$,

$$\begin{aligned}
\int_{\rho(t)}^{t} H_2^\Delta u \Delta s &= H_2^\Delta(t,\rho(t)) u(\rho(t)) \mu(\rho(t)) \\
&\le -H_2^\Delta(t,\rho(t)) \big(\alpha_1 A(\rho(t)) p(\rho(t)) - \mu(\rho(t)) B(\rho(t))\big).
\end{aligned} \tag{26}$$

Since $H_2^\Delta \le 0$ on $D$, we see that $H_\sigma \le H$. For $t \ge \sigma(t_0)$, $s \in [t_0,\rho(t))_{\mathbb{T}}$, from $(\alpha_1 A - (\alpha_1-\alpha_2)A^\sigma)p + \alpha_1\alpha_2\alpha_3 r A^\sigma > 0$ and (C4), we have

$$\begin{aligned}
&\alpha_1 pHA - (\alpha_1-\alpha_2) pH_\sigma A^\sigma + \alpha_1\alpha_2\alpha_3 r H_\sigma A^\sigma \\
&\quad \ge \alpha_1 pH_\sigma A - (\alpha_1-\alpha_2) pH_\sigma A^\sigma + \alpha_1\alpha_2\alpha_3 r H_\sigma A^\sigma \\
&\quad = \big((\alpha_1 A - (\alpha_1-\alpha_2)A^\sigma)p + \alpha_1\alpha_2\alpha_3 r A^\sigma\big) H_\sigma > 0.
\end{aligned} \tag{27}$$

For $t \ge \sigma(t_0)$, $s \in [t_0,\rho(t))_{\mathbb{T}}$, and $u(s) \le 0$, from (27) we have

$$\begin{aligned}
&H_2^\Delta u - H_\sigma \frac{\Phi_1 u^2 - \Phi_2 u}{\alpha_1 Ap(\mu u - \mu B + \alpha_1 Ap)} \\
&\quad = \frac{-[\alpha_1 pHA - (\alpha_1-\alpha_2) pH_\sigma A^\sigma + \alpha_1\alpha_2\alpha_3 r H_\sigma A^\sigma] u^2 + \Phi_4 u}{\alpha_1 Ap(\mu u - \mu B + \alpha_1 Ap)} \\
&\quad = -\frac{\alpha_1 pHA - (\alpha_1-\alpha_2) pH_\sigma A^\sigma + \alpha_1\alpha_2\alpha_3 r H_\sigma A^\sigma}{\alpha_1 Ap(\mu u - \mu B + \alpha_1 Ap)} u^2 + \frac{\Phi_4}{\alpha_1 Ap(\alpha_1 Ap - \mu B)} u - \frac{\Phi_4}{\alpha_1 Ap(\alpha_1 Ap - \mu B)} \frac{\mu u^2}{\mu u - \mu B + \alpha_1 Ap} \\
&\quad = -\frac{\Phi_5}{\alpha_1 Ap(\alpha_1 Ap - \mu B)(\mu u - \mu B + \alpha_1 Ap)} u^2 + \frac{\Phi_4}{\alpha_1 Ap(\alpha_1 Ap - \mu B)} u \\
&\quad \le -\frac{\Phi_5}{\alpha_1 Ap(\alpha_1 Ap - \mu B)^2} u^2 + \frac{\Phi_4}{\alpha_1 Ap(\alpha_1 Ap - \mu B)} u \\
&\quad = -\frac{\Phi_5}{\alpha_1 Ap(\alpha_1 Ap - \mu B)^2} \left( u - \frac{(\alpha_1 Ap - \mu B)\Phi_4}{2\Phi_5} \right)^2 + \frac{\Phi_4^2}{4\alpha_1 Ap \Phi_5} \\
&\quad \le \frac{\Phi_4^2}{4\alpha_1 Ap \min\{\Phi_5, \Phi_6\}} = M.
\end{aligned} \tag{28}$$

For $t \ge \sigma(t_0)$, $s \in [t_0,\rho(t))_{\mathbb{T}}$, and $u(s) > 0$, from (27) we have

$$\begin{aligned}
&H_2^\Delta u - H_\sigma \frac{\Phi_1 u^2 - \Phi_2 u}{\alpha_1 Ap(\mu u - \mu B + \alpha_1 Ap)} \\
&\quad = \frac{-[\alpha_1 pHA - (\alpha_1-\alpha_2) pH_\sigma A^\sigma + \alpha_1\alpha_2\alpha_3 r H_\sigma A^\sigma] u^2 + \Phi_4 u}{\alpha_1 Ap(\mu u - \mu B + \alpha_1 Ap)} \\
&\quad = -\frac{\alpha_1 pHA - (\alpha_1-\alpha_2) pH_\sigma A^\sigma + \alpha_1\alpha_2\alpha_3 r H_\sigma A^\sigma}{\alpha_1 Ap(\mu u - \mu B + \alpha_1 Ap)} \times \left( u - \frac{\Phi_4}{2(\alpha_1 pHA - (\alpha_1-\alpha_2) pH_\sigma A^\sigma + \alpha_1\alpha_2\alpha_3 r H_\sigma A^\sigma)} \right)^2
\end{aligned}$$

$$+\frac{\Phi_4^2}{4\alpha_1 A p \Phi_6}$$
$$\leq \frac{\Phi_4^2}{4\alpha_1 A p \min\{\Phi_5, \Phi_6\}} = M. \quad (29)$$

Therefore, for all $t \geq \sigma(t_0)$, $s \in [t_0, \rho(t))_{\mathbb{T}}$, we have

$$H_2^{\Delta} u - H_{\sigma} \frac{\Phi_1 u^2 - \Phi_2 u}{\alpha_1 A p (\mu u - \mu B + \alpha_1 A p)} \leq M. \quad (30)$$

Then, from (25), (26), and (30) we obtain that, for $t \in \mathbb{T}$ and $t > \sigma(t_0)$,

$$\int_{t_0}^{t} H_{\sigma} \Phi_0 \Delta s \leq H(t, t_0) u(t_0) + \int_{t_0}^{\rho(t)} M \Delta s - H_2^{\Delta}(t, \rho(t)) (\alpha_1 A(\rho(t)) p(\rho(t)) - \mu(\rho(t)) B(\rho(t))). \quad (31)$$

Hence,

$$\frac{1}{H(t,t_0)} \left[ \int_{t_0}^{t} H(t, \sigma(s)) \Phi_0(s) \Delta s - \int_{t_0}^{\rho(t)} M(t,s) \Delta s + H_2^{\Delta}(t, \rho(t)) \times (\alpha_1 A(\rho(t)) p(\rho(t)) - \mu(\rho(t)) B(\rho(t))) \right] \leq u(t_0) < \infty, \quad (32)$$

which contradicts (22) and completes the proof. □

*Remark 5.* If we change the condition $(\alpha_1 A - (\alpha_1 - \alpha_2) A^{\sigma}) p + \alpha_1 \alpha_2 \alpha_3 r A^{\sigma} > 0$ in the definition of $(\mathscr{A}, \mathscr{B})$ to a stronger one $A^{\Delta}(t) \leq 0$, (27) in the proof of Theorem 4 will be changed to

$$\begin{aligned} &\alpha_1 p H A - (\alpha_1 - \alpha_2) p H_{\sigma} A^{\sigma} + \alpha_1 \alpha_2 \alpha_3 r H_{\sigma} A^{\sigma} \\ &\quad \geq \alpha_1 p H_{\sigma} A^{\sigma} - (\alpha_1 - \alpha_2) p H_{\sigma} A^{\sigma} + \alpha_1 \alpha_2 \alpha_3 r H_{\sigma} A^{\sigma} \\ &\quad = \alpha_2 H_{\sigma} A^{\sigma} (\alpha_1 \alpha_3 r + p) > 0. \end{aligned} \quad (33)$$

Then the definition of $M$ can be simplified as

$$M(t,s) = (\Phi_4^2(t,s)) \times (4\alpha_1 \alpha_2 p(s) H(t, \sigma(s)) A^{\sigma}(s) A(s) \times (\alpha_1 \alpha_3 r(s) + p(s)) \min\{\Phi_7(s), \Phi_8(s)\})^{-1}, \quad (34)$$

where

$$\begin{aligned} \Phi_7(s) &= \alpha_1 A(s) p(s) - \mu(s) B(s), \\ \Phi_8(s) &= \alpha_1 A(s) p(s) + \mu(s) B(s). \end{aligned} \quad (35)$$

When $(A, B) = (1, 0)$, Theorem 4 can be simplified as Corollary 6.

**Corollary 6.** *Assume that (C1)–(C5) hold and that there exists a function $q \in C_{rd}(\mathbb{T}, \mathbb{R})$ such that $uf(t,u) \geq q(t)u^2$. Also, suppose that there exists $H \in \mathscr{H}$ such that, for any $t_0 \in \mathbb{T}$,*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \times \left[ \int_{t_0}^{t} H(t, \sigma(s)) q(s) \Delta s - \frac{\alpha_1^2}{4\alpha_2} \int_{t_0}^{\rho(t)} \frac{(p(s) H_2^{\Delta}(t,s))^2}{(\alpha_1 \alpha_3 r(s) + p(s)) H(t, \sigma(s))} \Delta s + \alpha_1 H_2^{\Delta}(t, \rho(t)) p(\rho(t)) \right] = \infty. \quad (36)$$

*Then, (10) is oscillatory.*

When $r(t) \equiv 0$, (10) will be simplified as

$$\left(p(t) k_1 \left(x(t), x^{\Delta}(t)\right)\right)^{\Delta} + f(t, x(\sigma(t))) = 0. \quad (37)$$

Then Theorem 4 can be simplified as Corollary 7.

**Corollary 7.** *Assume that (C1)–(C5) hold and that there exists a function $q \in C_{rd}(\mathbb{T}, \mathbb{R})$ such that $uf(t,u) \geq q(t)u^2$. Also, suppose that there exist $(A, B) \in (\mathscr{A}, \mathscr{B})$ and $H \in \mathscr{H}$ such that, for any $t_0 \in \mathbb{T}$,*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \left[ \int_{t_0}^{t} H(t, \sigma(s)) \Phi_0(s) \Delta s - \int_{t_0}^{\rho(t)} M_1(t,s) \Delta s + H_2^{\Delta}(t, \rho(t)) \times (\alpha_1 A(\rho(t)) p(\rho(t)) - \mu(\rho(t)) B(\rho(t))) \right] = \infty, \quad (38)$$

*where*

$$\begin{aligned} M_1(t,s) &= \frac{\Phi_9^2(t,s)}{4\alpha_1 A(s) \min\{\Phi_{10}(t,s), \Phi_{11}(t,s)\}}, \\ \Phi_9(t,s) &= \alpha_1 H(t,s) A(s) B(s) + (2\alpha_2 - \alpha_1) H(t, \sigma(s)) A^{\sigma}(s) B(s) + \alpha_1^2 p(s) A(s) (H(t,s) A(s))^{\Delta_s}, \\ \Phi_{10}(t,s) &= \alpha_2 H(t, \sigma(s)) A^{\sigma}(s) (\alpha_1 A(s) p(s) + \mu(s) B(s)), \\ \Phi_{11}(t,s) &= (\alpha_1 H(t,s) A(s) - (\alpha_1 - \alpha_2) H(t, \sigma(s)) A^{\sigma}(s)) \times (\alpha_1 A(s) p(s) - \mu(s) B(s)). \end{aligned} \quad (39)$$

*Then, (37) is oscillatory.*

*Remark 8.* When $r(t) \equiv 0$, $k_1(u,v) = \Psi(u)k(v)$, and (C3) is replaced by

(C6) $\psi \in C(\mathbb{R},(0,\eta])$, where $\eta$ is a fixed positive constant;

(C7) $k \in C(\mathbb{R},\mathbb{R})$, and there exists $\gamma_1 \geq \gamma_2 > 0$ such that $0 < \gamma_2 yk(y) \leq k^2(y) \leq \gamma_1 yk(y)$ for all $y \neq 0$.

Theorem 4 is reduced to [12, Theorem 4].

## 3. Example

In this section, we will give an example to demonstrate Corollary 7.

*Example 1.* Consider the equations

$$\left[\frac{1}{t^2}\frac{2+x^2(t)}{1+x^2(t)}x^{\Delta}(t)\right]^{\Delta} + t^2(2+\sin t)\,x(\sigma(t)) = 0, \quad (40)$$

$$\left[\frac{1}{t^2}\frac{1+2x^2(t)\left(x^{\Delta}(t)\right)^2}{1+x^2(t)\left(x^{\Delta}(t)\right)^2}x^{\Delta}(t)\right]^{\Delta} + t^2(2+\sin t)\,x(\sigma(t)) = 0, \quad (41)$$

where $p(t) = 1/t^2$, $r(t) \equiv 0$, $q(t) = t^2$, $k_1(u,v) = ((2+u^2)/(1+u^2))v$ in (40), and $k_1(u,v) = ((1+2u^2v^2)/(1+u^2v^2))v$ in (41), so we have both $\alpha_1 = 2$, $\alpha_2 = 1$. Letting $H(t,s) = (t-s)^2$, we have

(1) $\mathbb{T} = [1,\infty)$, $(A,B) = (s^2, 1/s^2)$,

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\left[\int_{t_0}^{t} H(t,\sigma(s))\,\Phi_0(s)\,\Delta s - \int_{t_0}^{\rho(t)} M_1(t,s)\,\Delta s + H_2^{\Delta}(t,\rho(t)) \times \left(\alpha_1 A(\rho(t))\,p(\rho(t)) - \mu(\rho(t))\,B(\rho(t))\right)\right] \\
&= \limsup_{t\to\infty}\frac{1}{(t-1)^2}\left[\int_1^t (t-s)^2\left(s^4+\frac{4}{s^3}\right)ds - \int_1^t \frac{\left(t-s+4ts-8s^2\right)^2}{4s^4}ds\right] \\
&= \infty.
\end{aligned} \quad (42)$$

That is, (38) holds. By Corollary 7 we see that (40) and (41) are oscillatory. Consider

(2) $\mathbb{T} = \mathbb{N}$, $(A,B) = (1,0)$,

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\left[\int_{t_0}^{t} H(t,\sigma(s))\,\Phi_0(s)\,\Delta s - \int_{t_0}^{\rho(t)} M_1(t,s)\,\Delta s + H_2^{\Delta}(t,\rho(t))\left(\alpha_1 A(\rho(t))\,p(\rho(t)) - \mu(\rho(t))\,B(\rho(t))\right)\right] \\
&= \limsup_{t\to\infty}\frac{1}{(t-1)^2}\left[\int_1^t (t-s-1)^2 s^2\,\Delta s - \int_1^{t-1}\frac{(2t-2s-1)^2}{s^2(t-s-1)^2}\Delta s - \frac{2}{(t-1)^2}\right] \\
&= \limsup_{n\to\infty}\frac{1}{(n-1)^2}\left[\sum_{k=1}^{n-1}(n-k-1)^2k^2 - \sum_{k=1}^{n-2}\frac{(2n-2k-1)^2}{k^2(n-k-1)^2} - \frac{2}{(n-1)^2}\right] \\
&= \infty.
\end{aligned} \quad (43)$$

That is, (38) holds. By Corollary 7 we see that (40) and (41) are oscillatory.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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