

## Research Article

# The Existence and Stability of Solutions for Vector Quasiequilibrium Problems in Topological Order Spaces

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In a topological sup-semilattice, we established a new existence result for vector quasiequilibrium problems. By the analysis of essential stabilities of maximal elements in a topological sup-semilattice, we prove that for solutions of each vector quasi-equilibrium problem, there exists a connected minimal essential set which can resist the perturbation of the vector quasi-equilibrium problem.

## 1. Introduction

Vector equilibrium problems can unify many nonlinear problems such as vector optimization, vector variational inequality [1], and vector complementarity problems [2]. Recently, not only vector equilibrium problems [3–7] but also vector quasiequilibrium problems [8–14] and the system of vector quasi-equilibrium problems have attracted much attention [15–19].

Topological vector spaces provide the usual mathematical framework in the study of many problems. To avoid the linear feature, sup-semilattices may be good choices. In fact, some results like the existence of KKM points in topological spaces were established in topological sup-semilattices [20], where a two-tuple  $(X, \leq)$  is said to be a sup-semilattice, if  $X$  is a partially ordered set with the partial ordering  $\leq$ , in which every pair  $(x, x')$  has a least upper bound  $x \vee x'$ .

The aim of this paper is to study the existence and essential stability of vector quasi-equilibrium problems in topological sup-semilattices. In order to achieve this, firstly, we give a stability result in relation to maximal elements in a topological sup-semilattice. Secondly, a new existence result for vector quasi-equilibrium problems is established, and we show that each vector quasi-equilibrium problem has a connected minimal essential set in its solution set.

## 2. Preliminaries

Let  $(X, \leq)$  be a sup-semilattice. If  $x$  and  $x'$  are two elements in  $(X, \leq)$  and  $x \leq x'$ , the set  $[x, x'] = \{y \in X : x \leq y \leq x'\}$

is called an order interval. Let  $A, A'$  be two nonempty finite subsets of  $X$ . Then the set  $\Delta A = \bigcup_{x \in A} [x, \sup A]$  is well defined and has the properties:  $A \subseteq \Delta A$  and  $\Delta A \subseteq \Delta A'$  if  $A \subseteq A'$ .

*Definition 1* (see [20]). A subset  $E \subseteq X$  is  $\Delta$ -convex, if for any nonempty finite subset  $A \subseteq E$ , we have  $\Delta A \subseteq E$ .  $E$  being a  $\Delta$ -convex set is equivalent to the following conditions:

- (a) if  $x, x' \in E$ , then its least upper bound  $x \vee x' \in E$ .
- (b) if  $x, x' \in E$  and  $x \leq x'$ , then the order interval  $[x, x'] \subseteq E$ .

It is easy to check that the intersection of two  $\Delta$ -convex sets is  $\Delta$ -convex as well.

A topological space  $X$  is said to be a topological sup-semilattice if  $X$  is equipped with a sup-semilattice as its partial ordering denoted by  $\leq$ , for which  $f : X \times X \rightarrow X$  with  $(x, x') \rightarrow x \vee x'$  is a continuous function.

Let  $Y$  be a topological vector space and  $\theta$  the zero element in  $Y$ . A subset  $C \subset Y$  is called a cone if, for any  $y \in C$  and real number  $t > 0$ ,  $ty \in C$ . A cone  $C$  is convex if  $C$  is a convex set. If  $C \cap -C = \{\theta\}$ , it is called a pointed cone.

*Definition 2* (see [18]). Let  $X$  be a topological sup-semilattice,  $Y$  a topological vector space with a cone  $C \subset Y$ ,  $\varphi : X \rightarrow Y$  a vector-valued function.

- (a)  $\varphi : X \rightarrow Y$  is  $C_\Delta$ -quasiconcave if, for any nonempty two points subset  $A = \{x_1, x_2\} \subset X$  and  $y \in Y$ ,  $\varphi(A) \subset y + C \Rightarrow \varphi(\Delta A) \subset y + C$ .

(b)  $\varphi : X \rightarrow Y$  is said to be  $C_\Delta$ -quasiconcave-like if, for any  $x_1, x_2 \in X$ ,  $\varphi(\Delta\{x_1, x_2\}) \in \varphi(x_1) + C$  or  $\varphi(\Delta\{x_1, x_2\}) \in \varphi(x_2) + C$ .

*Remark 3.* In general cases,  $C_\Delta$ -quasiconcave,  $C_\Delta$ -quasiconcave-like, and usual quasiconcave functions are independent of each other. See examples in [18]. Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ ,  $C = -\mathbb{R}^+$ . Then the partial order on  $\mathbb{R}$  is “ $\leq$ ” (less than or equal to); hence, the  $C_\Delta$ -quasiconcave,  $C_\Delta$ -quasiconcave-like, and usual quasiconcave property of  $\varphi$  coincide (the usual quasiconcave function  $\varphi$  means that for any  $x_1, x_2, y \in \mathbb{R}$ ,  $\varphi(x_1) \leq y$  and  $\varphi(x_2) \leq y \Rightarrow \varphi(\lambda x_1 + (1 - \lambda)x_2) \leq y$ , for all  $\lambda \in [0, 1]$ ).

**Lemma 4** (see [18]). *Let  $X$  be a topological sup-semilattice,  $Y$  a Hausdorff locally convex topological vector space with a closed, convex, and pointed cone  $C \subset Y$ . If the vector-valued function  $\varphi : X \rightarrow Y$  is  $C_\Delta$ -quasiconcave or  $C_\Delta$ -quasiconcave-like, then the set  $A = \{x : \varphi(x) \in \text{int } C\}$  is  $\Delta$ -convex.*

Now we introduce the vector quasiequilibrium problem (VQEP) that we will consider in this paper.

Let  $X$  be a topological sup-semilattice and  $Y$  a topological vector space.  $C \subset Y$  is a closed, convex, and pointed cone with  $\text{int } C \neq \emptyset$ .  $\varphi : X \times X \rightarrow Y$  is a vector-valued function, and  $G$  is a multivalued mapping on  $X$ . The vector quasi-equilibrium problem  $\phi$  with  $\phi = (X, Y, C, \varphi, G)$  is to find  $\bar{x} \in X$ , such that

$$\bar{x} \in G(\bar{x}) : \varphi(\bar{x}, y) \notin \text{int } C, \quad \forall y \in G(\bar{x}). \quad (1)$$

Let  $G(x) = X$ , for all  $x \in X$ ; then the VQEP is just a vector equilibrium problem  $(X, Y, C, \varphi)$  (VEP). That is to find  $\bar{x} \in X$ , such that

$$\varphi(\bar{x}, y) \notin \text{int } C, \quad \forall y \in X. \quad (2)$$

*Definition 5* (see [21, 22]). A vector-valued function  $\varphi : X \rightarrow Y$  is said to be  $C$ -continuous on  $X$  if, for each  $x \in X$  and any open neighborhood  $V(\theta)$  of  $\theta$  in  $Y$ , there exists an open neighborhood  $O(x)$  of  $x$  in  $X$  such that

$$\forall x' \in O(x), \quad \varphi(x') \in \varphi(x) + V(\theta) + C. \quad (3)$$

*Remark 6.* For a function  $\varphi : X \rightarrow \mathbb{R}$ ,  $\varphi$  is  $\mathbb{R}_+$ -continuous on  $X$  if and only if  $\varphi$  is lower semicontinuous on  $X$ .

A maximal element version of the Browder fixed point theorem in a topological sup-semilattice can be found in [18]. We limit it in a metric space as the following lemma.

**Lemma 7** (see [18]). *Let  $(X, d)$  be a compact sup-semilattice with path connected interval, where  $d$  is the metric on  $X$ ,  $S : X \rightarrow 2^X$  a multivalued map on  $X$  with the conditions: (i) for all  $x \in X$ ,  $S(x)$  is  $\Delta$ -convex; (ii) for all  $y \in X$ ,  $S^{-1}(y) = \{x \in X : y \in S(x)\}$  is open in  $X$ ; (iii) for all  $x \in X$ ,  $x \notin S(x)$ . Then there exists an  $\bar{x} \in X$ , such that  $S(\bar{x}) = \emptyset$ .*

*Remark 8.* The existence of a metric space with a sup-semilattice can be guaranteed. For instance, let  $x^j = (x_1^j, \dots, x_i^j, \dots, x_n^j) \in \mathbb{R}^n$ ,  $j = 1, 2$ , if  $x^1 \leq x^2$  means that  $x^2 \in$

$x^1 + \mathbb{R}_+^n$ , then  $x^1 \vee x^2 = \bar{x}$ , where  $\bar{x}_i = \max\{x_i^1, x_i^2\}$ . Clearly,  $(\mathbb{R}^n, \leq)$  with the usual Euclidean metric is a topological sup-semilattice.

Let  $M$  denote the collection of  $S$  satisfying all the conditions of Lemma 7. For any  $S_1, S_2 \in M$ , define the metric between  $S_1$  and  $S_2$  as

$$\rho(S_1, S_2) = \sup_{y \in X} h(X \setminus S_1^{-1}(y), X \setminus S_2^{-1}(y)), \quad (4)$$

where  $h$  is the Hausdorff metric induced by  $d$ . For each  $S \in M$  and each  $y \in X$ , since  $y \notin S(y)$ , we have  $y \in X \setminus S^{-1}(y)$ , that is,  $X \setminus S^{-1}(y) \neq \emptyset$ . Noting that  $X \setminus S^{-1}(y)$  is closed, the metric  $\rho$  on  $M$  is well defined. Then  $(M, \rho)$  is a metric space.

For each  $S \in M$ , denote by  $F(S)$  the set of all maximal elements of  $S$ . Then  $F$  defines a multivalued mapping from  $M$  to  $X$  and  $F(S) = \bigcap_{y \in X} (X \setminus S^{-1}(y))$ .

*Definition 9.* For each  $S \in M$ , a set  $e(S)$  is called an essential set of  $F(S)$  if it satisfies the following conditions:

- (1)  $e(S)$  is closed subset of  $F(S)$ .
- (2) For any open set  $U \supset e(S)$ , there exists an open neighborhood  $O(S)$  of  $S \in M$  such that  $U \cap F(S') \neq \emptyset$ , for any  $S' \in O(S)$ .

A set  $m(S)$  is called a minimal essential set of  $F(S)$  if it is a minimal element of all essential sets ordered by set inclusion in  $F(S)$ . A connected component in  $F(S)$  is called an essential component, if it includes at least one minimal essential set of  $F(S)$ .

We recall some notions about multi-valued mappings. Let  $G : Y \rightarrow 2^P$  be a multi-valued mapping, where  $Y, P$  are two topological vector spaces. Then (i)  $G$  is said to be upper semicontinuous at  $y \in Y$ , if for each open set  $U \supset G(y)$ , there exists an open neighborhood  $O(y)$  of  $y$  such that  $U \supset G(y')$  for any  $y' \in O(y)$ . (ii)  $G$  is lower semi-continuous at  $y \in Y$ , if for each open set  $U \cap G(y) \neq \emptyset$ , there exists an open neighborhood  $O(y)$  of  $y$  such that  $U \cap G(y') \neq \emptyset$  for any  $y' \in O(y)$ .

*Remark 10.* For each  $S \in M$ , a set  $e(S) \subset F(S)$  is essential if  $F$  is lower semi-continuous at  $S$ . If  $F$  is upper semi-continuous at  $S$ , then  $F(S)$  itself is an essential set. For any two closed sets  $A, B \subset F(S)$  with  $A \subset B$ , if  $A$  is essential, then  $B$  is also essential. For each  $S \in M$  and each  $y \in X$ , if  $S^{-1}(y)$  is open, then  $X \setminus S^{-1}(y)$  is closed; hence,  $F(S)$  is closed because we have  $F(S) = \bigcap_{y \in X} X \setminus S^{-1}(y)$ ; consequently,  $F(S)$  is compact.

**Lemma 11** (see [23]). *Let  $(X, d)$  be a metric space,  $K_1$  and  $K_2$  two nonempty compact subsets of  $X$ ,  $U_1$ , and  $U_2$  two nonempty disjoint open subsets of  $X$ . If  $h(K_1, K_2) < d(U_1, U_2)$ , then  $h(K_1, (K_1 \setminus U_2) \cup (K_2 \setminus U_1)) \leq h(K_1, K_2)$ , where  $h$  is the Hausdorff metric defined on  $X$ .*

### 3. The Stability of Maximal Elements on Topological Semilattices

**Theorem 12.**  $F : M \rightarrow 2^X$  is an upper semi-continuous mapping with compact values.

*Proof.* For each  $S \in M$ , by Remark 10,  $F(S)$  is compact. Suppose that  $F$  is not upper semi-continuous. Then there is a  $S \in M$ , an open set  $U$  with  $U \supset F(S)$  and  $S_n$ , such that  $S_n \rightarrow S$  and  $F(S_n) \not\subset U$ ,  $n = 1, 2, \dots$ . That is, there exists a point  $x_n \in F(S_n)$  such that  $x_n \notin U$ . Without loss of generality, we may assume that  $x_n \rightarrow x^*$ . Since  $S_n \rightarrow S$ , it holds that  $X \setminus S_n^{-1}(y) \rightarrow X \setminus S^{-1}(y)$ , for all  $y \in X$ . Since  $x_n \in F(S_n)$ , we have  $x_n \in X \setminus S_n^{-1}(y)$ , for all  $y \in X$ . As  $n$  gets close to infinity, we can obtain that  $x^* \in X \setminus S^{-1}(y)$ , for all  $y \in X$ , that is,  $x^* \in F(S) \subset U$ . This results in the fact that  $x_n \in U$  while  $n$  is large enough, a contradiction with  $x_n \notin U$ . Therefore,  $F$  is definitely upper semi-continuous.  $\square$

**Theorem 13.** For each  $S \in M$ , there exists at least a minimal essential set of  $F(S)$ . If  $m(S)$  is a minimal essential set of  $F(S)$ , then  $m(S)$  is connected.

*Proof.* For the existence, by Remark 10, each decreasing chain, consisting of essential subsets of  $F(S)$ , has a minimal element, which is the intersection of the chain. By the Zorn's lemma, the minimal element is just a minimal essential set. For the connectedness, by way of contradiction, suppose that  $m(S)$  is not connected. There exist two disjoint closed sets  $C_1(S), C_2(S)$  such that  $m(S) = C_1(S) \cup C_2(S)$ .

Since  $C_i(S)$  is not essential, there is an open set  $W_i$  with  $W_i \supset C_i(S)$  such that for any  $\varepsilon > 0$ , there exists a  $S_i \in M$  with  $\rho(S, S_i) < \varepsilon$  and  $F(S_i) \cap W_i = \emptyset$ ,  $i = 1, 2$ . Clearly,  $C_i(S)$  is compact, then there is an open set  $U_i$  with  $C_i(S) \subset U_i \subset W_i$ ,  $i = 1, 2$ , such that  $U_1 \cap U_2 = \emptyset$ . For  $U_1 \cup U_2 \supset m(S)$ , because  $m(S)$  is essential, there is a number  $\delta < 2d(U_1, U_2)$ , such that  $F(T) \cap (U_1 \cup U_2) \neq \emptyset$  for each  $T$  satisfying  $\rho(T, S) < \delta$ . Therefore, we can select a  $S_i \in M$  such that  $\rho(S, S_i) < \delta/4$  and  $F(S_i) \cap U_i = \emptyset$ ,  $i = 1, 2$ . Then  $\rho(S_1, S_2) < \rho(S_1, S) + \rho(S, S_2) < \delta/2 < d(U_1, U_2)$ .

Define a multi-valued mapping  $S' : M \rightarrow 2^X$  as

$$S' = \begin{cases} S_1(x), & x \in U_1, \\ S_2(x), & x \in U_2, \\ S_1(x) \cap S_2(x), & x \in X \setminus (U_1 \cup U_2). \end{cases} \quad (5)$$

We show that  $S' \in M$ .

- (a) For each  $x \in X$ , since  $x \notin S_1(x)$  and  $x \notin S_2(x)$ , we have  $x \notin S'(x)$ ;
- (b) for each  $x \in X$ , because  $S_1(x)$  and  $S_2(x)$  are  $\Delta$ -convex sets, it follows that  $S'(x)$  is  $\Delta$ -convex;
- (c) for each  $y \in X$ , we have

$$\begin{aligned} S'^{-1}(y) &= (S_1^{-1}(y) \cap S_2^{-1}(y)) \cup (S_2^{-1}(y) \cap U_2) \cup (S_1^{-1}(y) \cap U_1). \end{aligned} \quad (6)$$

Noting that  $S_1^{-1}(y), S_2^{-1}(y), U_1$ , and  $U_2$  are open sets, it follows that  $S'^{-1}(y)$  is open.

Through a direct calculation,  $X \setminus S'^{-1}(y)$  can be written as

$$\begin{aligned} &((X \setminus S_1^{-1}(y)) \cap (X \setminus S_2^{-1}(y))) \\ &\cup ((X \setminus S_1^{-1}(y)) \cap (X \setminus U_2)) \\ &\cup ((X \setminus S_2^{-1}(y)) \cap (X \setminus U_1)). \end{aligned} \quad (7)$$

Take any  $x \in X \setminus S'^{-1}(y)$ . Note that if  $x \in U_1$ , then  $x \in X \setminus U_2$ ; if  $x \in U_2$ , then  $x \in X \setminus U_1$ ; if  $x \in X \setminus (U_1 \cup U_2)$ , then  $x \in (X \setminus U_1) \cap (X \setminus U_2)$ . Consequently, we can obtain that if  $x \in (X \setminus S_1^{-1}(y)) \cap (X \setminus S_2^{-1}(y))$ , then

$$\begin{aligned} x &\in ((X \setminus S_1^{-1}(y)) \cap (X \setminus U_2)) \\ &\cup ((X \setminus S_2^{-1}(y)) \cap (X \setminus U_1)). \end{aligned} \quad (8)$$

Therefore, we have

$$\begin{aligned} X \setminus S'^{-1}(y) &= ((X \setminus S_1^{-1}(y)) \cap (X \setminus U_2)) \\ &\cup ((X \setminus S_2^{-1}(y)) \cap (X \setminus U_1)). \end{aligned} \quad (9)$$

Since  $\rho(S_1, S_2) < d(U_1, U_2)$ , by Lemma 11, we have

$$\begin{aligned} &h(X \setminus S'^{-1}(y), X \setminus S_1^{-1}(y)) \\ &\leq h(X \setminus S_2^{-1}(y), X \setminus S_1^{-1}(y)) \\ &\leq \rho(S_2, S) + \rho(S, S_1) < \frac{\delta}{2}. \end{aligned} \quad (10)$$

That is,  $\rho(S', S_1) < \delta/2$ . This results in the fact that

$$\rho(S', S) < \rho(S', S_1) + \rho(S_1, S) < \delta. \quad (11)$$

Consequently, we have  $F(S') \cap (U_1 \cup U_2) \neq \emptyset$ . If there is a point  $x \in U_i \cap F(S')$ , then  $x \in U_i$  and  $x \in X \setminus S_i^{-1}(y)$ , for all  $y \in X$ , that is,  $x \in U_i$  and  $x \in F(S_i)$  which contradicts with  $F(S_i) \cap U_i = \emptyset$ ,  $i = 1, 2$ . Therefore,  $m(S)$  is connected.  $\square$

### 4. The Existence and Stability of Solutions for VQEP

This section gives an existence result in relation to VQEP in topological sup-semilattices and induces the existence of minimal essentially stable sets for each VQEP in the set of its solutions.

**Theorem 14.** Let  $\phi = (X, Y, C, \varphi, G)$  be a VQEP, where  $X$  is a compact topological sup-semilattice with path connected intervals,  $Y$  is a Hausdorff locally convex topological vector space, and  $G : X \rightarrow 2^X$  is a multi-valued mapping with nonempty and  $\Delta$ -convex values. If the VQEP satisfies that

- (i) for all  $x \in X$ ,  $\varphi(x, x) \notin \text{int} C$ ;
- (ii) for all  $y \in X$ ,  $x \rightarrow \varphi(x, y)$  is  $C$ -continuous;
- (iii) for all  $x \in X$ ,  $y \rightarrow \varphi(x, y)$  is  $C_\Delta$ -quasiconcave-like or  $C_\Delta$ -quasiconcave;
- (iv) for all  $y \in X$ ,  $\{x \in X : y \in G(x)\}$  is open in  $X$ ,
- (v)  $\{x \in X : x \in G(x)\}$  is closed in  $X$ ,

then the VQEP has a solution.

*Proof.* Denote  $\{x \in X : x \in G(x)\}$  by  $K$ . Let  $B : X \rightarrow 2^X$  such that  $B(x) = \{y \in X : \varphi(x, y) \in \text{int} C\}$ , for all  $x \in X$ . Define

$$S(x) = \begin{cases} B(x) \cap G(x), & x \in K, \\ G(x), & x \in X \setminus K. \end{cases} \quad (12)$$

Then for each  $x \in X$ , if  $x \in K$ , we have  $S(x) = B(x) \cap G(x)$ , by the condition (i),  $x \notin B(x)$ , hence,  $x \notin S(x)$ ; if  $x \in X \setminus K$ , from the definition of  $K$ , we have  $x \notin G(x) = S(x)$ .

Since  $y \rightarrow \varphi(x, y)$  is  $C_\Delta$ -quasiconcave-like or  $C_\Delta$ -quasiconcave, by Lemma 4, we have that  $B(x)$  is  $\Delta$ -convex. Then  $B(x) \cap G(x)$  is a  $\Delta$ -convex set, noting that  $G$  has  $\Delta$ -convex values, we have that  $S(x)$  is also  $\Delta$ -convex.

For each  $y \in X$ , we can check that

$$S^{-1}(y) = (B^{-1}(y) \cap G^{-1}(y)) \cup (G^{-1}(y) \cap (X \setminus K)). \quad (13)$$

Take a point  $x \in B^{-1}(y) = \{x \in X : \varphi(x, y) \in \text{int} C\}$ , since  $\text{int} C$  is open, there is an open set  $V(\theta)$  such that  $V(\theta) + \varphi(x, y) \subset \text{int} C$ , then, by the condition (ii), there exists an open neighborhood  $O(x)$  in  $X$  such that for all  $x' \in O(x)$ ,

$$\varphi(x', y) \in \varphi(x, y) + V(\theta) + C \subset \text{int} C + C \subset \text{int} C. \quad (14)$$

That is,  $O(x) \subset B^{-1}(y)$ ; hence,  $B^{-1}(y)$  is open. Noting that  $X \setminus K$  and  $G^{-1}(y)$  are open sets in  $X$ . We can obtain that  $S^{-1}(y)$  is also open in  $X$ .

Thus, there is an  $\bar{x} \in X$  such that  $S(\bar{x}) = \emptyset$  by Lemma 7. If  $\bar{x} \in X \setminus K$ , then  $G(\bar{x}) = \emptyset$ , a contradiction to the fact that  $G$  has nonempty values. Therefore,  $\bar{x} \in K$  and  $B(\bar{x}) \cap G(\bar{x}) = \emptyset$ , that is,  $\bar{x} \in G(\bar{x})$ ,  $\varphi(\bar{x}, y) \notin \text{int} C$ , for all  $y \in X$ .  $\square$

By Theorem 14 and its proof, we can also obtain the existence result for VQEP as the following.

**Corollary 15.** Let  $\phi = (X, Y, C, \varphi, G)$  be a VQEP, where  $X$  is a compact topological sup-semilattice with path connected intervals,  $Y$  is a Hausdorff topological vector space, and  $G : X \rightarrow 2^X$  is a multi-valued mapping with nonempty and  $\Delta$ -convex values. If the VQEP satisfies that

- (i) for all  $x \in X$ ,  $\varphi(x, x) \notin \text{int} C$ ;
- (ii) for all  $y \in X$ ,  $\{x \in X : \varphi(x, y) \in \text{int} C\}$  is open in  $X$ ;
- (iii) for all  $x \in X$ ,  $\{y \in X : \varphi(x, y) \in \text{int} C\}$  is  $\Delta$ -convex;
- (iv) for all  $y \in X$ ,  $\{x \in X : y \in G(x)\}$  is open in  $X$ ,
- (v)  $\{x \in X : x \in G(x)\}$  is closed in  $X$ ,

then the VQEP has a solution.

By Theorem 14, for the special case of VQEP without the feasible mapping  $G$ , we can obtain the existence result concerning VEP as the following.

**Corollary 16.** Let  $(X, Y, C, \varphi)$  be a vector equilibrium problem, where  $X$  is a compact topological sup-semilattice with path connected intervals,  $Y$  is a Hausdorff locally convex topological vector space. If the VEP satisfies the following conditions:

- (i) for all  $x \in X$ ,  $y \rightarrow \varphi(x, y)$  is  $C_\Delta$ -quasiconcave or  $C_\Delta$ -quasiconcave-like;
- (ii) for all  $y \in X$ ,  $x \rightarrow \varphi(x, y)$  is  $C$ -continuous;
- (iii) for all  $x \in X$ ,  $\varphi(x, x) \notin \text{int} C$ ,

then this VEP has a solution.

*Example 17.* Let  $X = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ ,  $C = -\mathbb{R}_+$ . The  $(X, \leq)$  is a sup-semilattice, in which  $x^1 \leq x^2$  means that  $x^2 \in x^1 + \mathbb{R}_+^2$ , for all  $x^1, x^2 \in X$ .

(a) For any  $x = (x_1, x_2)$ ,  $y = (y_1, y_2) \in X$ , the function  $\varphi$  is defined as

$$\varphi(x, y) = (1 - y_1)(1 - y_2) - (1 - x_1)(1 - x_2). \quad (15)$$

It can be easily checked that for each  $x \in X$ ,  $\varphi(x, \cdot)$  is  $C_\Delta$  quasiconcave and  $C_\Delta$  quasiconcave-like but not a usual quasiconcave function.

Denote by  $D$  the set  $(1 \times [0, 1]) \cup ([0, 1] \times 1)$ . For each  $x = (x_1, x_2) \in X$ , the multi-valued mapping  $G$  satisfies that

$$G(x) = \begin{cases} (x_1, 1] \times [0, 1] \cup [0, 1] \times (x_2, 1], & x \in X \setminus D, \\ (1, 1), & x \in D. \end{cases} \quad (16)$$

Note that  $G$  is not a usual convex but a  $\Delta$ -convex multi-valued mapping. For each  $y = (y_1, y_2) \in X$ , if  $y \in X \setminus (1, 1)$ , then  $G^{-1}(y) = [0, y_1) \times [0, y_2)$ ; if  $y = (1, 1)$ , then  $G^{-1}(y) = X$ . Thus,  $G^{-1}(y)$  is open in  $X$  for each  $y \in X$ . Then  $\varphi$  and  $G$  satisfy all the conditions in Theorem 14. We can find that  $\bar{x} = (1, 1)$  is the unique solution for the VQEP,  $(X, \mathbb{R}, C, \varphi, G)$ .

(b) For any  $x = (x_1, x_2)$ ,  $y = (y_1, y_2) \in X$ , let  $T(x) = B_\varepsilon(x)$ , where  $B_\varepsilon(x)$  is the  $\varepsilon$ -neighborhood of  $x$  in  $X$ ,

$$\varphi(x, y) = (1 + x_1 - y_1)(1 + x_2 - y_2), \quad (17)$$

and  $G$  is the same as the setting in (a). Then the function  $\varphi$  and the mappings  $G$  and  $T$  meet all the conditions in Theorem 14. The set of solutions for the VEP  $(X, \mathbb{R}, C, \varphi)$  is the overall  $X$  which is also the set of solutions for  $(X, \mathbb{R}, C, \varphi, T)$ . The solution of  $(X, \mathbb{R}, C, \varphi, G)$  is just one point  $\bar{x} = (1, 1)$ .

To study the stability of vector quasi-equilibrium problems, let  $(X, d)$  be a metric space and define the set  $M'$  as

$$M' = \{\phi = (X, Y, C, \varphi, G) : \phi \text{ satisfies all the conditions in Theorem 14}\} \quad (18)$$

For each  $\phi \in M'$ , by the proof of Theorem 14, we can find that a point  $\bar{x} \in X$  is a solution of  $\phi$  if and only if  $\bar{x}$  is a maximal element of  $S$  defined in the proof. Let  $F'(\phi)$  denote



all the solutions of  $\phi$ . Then  $F'$  is a multi-valued mapping from  $M'$  to  $X$ . For any two  $\phi_1, \phi_2$ , define the metric  $\rho'(\phi_1, \phi_2)$  between  $\phi_1$  and  $\phi_2$  as

$$\rho'(\phi_1, \phi_2) = \rho(S_1, S_2), \quad (19)$$

where  $S_1$  and  $S_2$  are multi-valued mappings corresponding to  $\phi_1$  and  $\phi_2$  in the proof of Theorem 14. Then  $(M', \rho')$  is a metric space. Instead of  $M, S$  and  $F(S)$  by  $M', \phi$  and  $F'(\phi)$  in Definition 9, we can also define essential sets  $e(\phi)$ , minimal essential sets  $m(\phi)$  of  $F'(\phi)$ , and essential component in  $F'(\phi)$ . If an essential set  $e(\phi)$  is singleton set  $\{x^*\}$ ,  $x^*$  is called an essential solution of  $\phi$ .

From Theorems 12 and 13, we have the following results.

**Theorem 18.**  $F' : M' \rightarrow 2^X$  is an upper semi-continuous mapping with compact values. For each VQEP  $\phi \in M'$ , there exists at least a connected minimal essential set  $m(\phi)$  of  $F'(\phi)$ .

*Remark 19.* For each  $\phi \in M', y \in X$ , let

$$A_\phi(y) = \{x : y \notin G(x), \text{ or } \varphi(x, y) \notin \text{int } C \text{ and } x \in G(x)\}. \quad (20)$$

For any  $\phi_1, \phi_2 \in M'$ , from the definition of the metric between  $S_1$  and  $S_2$ , then

$$\rho'(\phi_1, \phi_2) = \sup_{y \in X} h(A_{\phi_1}(y), A_{\phi_2}(y)), \quad (21)$$

which gives an overall consideration of  $\varphi$  and  $G$ . If  $\phi_1$  and  $\phi_2$  are two VEP, then

$$\begin{aligned} \rho'(\phi_1, \phi_2) &= \sup_{y \in X} h(\{x : \varphi_1(x, y) \notin \text{int } C\}, \\ &\quad x \{x : \varphi_2(x, y) \notin \text{int } C\}). \end{aligned} \quad (22)$$

For the essential stability of solutions for VQEP, clearly, the class of perturbations induced by the metric  $\rho'$  is different from the perturbation of uniform topology in [3, 14] and also different from the perturbation of best response defined in [16]. For example, the existence of essential sets of solutions for VQEP in topological vector spaces is proved in [3], and the uniform metric for two VQEP  $\phi_1 = (X, X, C, \varphi_1, G_1)$  and  $\phi_2 = (X, X, C, \varphi_2, G_2)$  is defined as

$$\begin{aligned} \rho''(\phi_1, \phi_2) &= \sup_{(x,y) \in X \times X} \|\varphi_1(x, y) - \varphi_2(x, y)\| \\ &\quad + \sup_{x \in X} h(G_1(x), G_2(x)), \end{aligned} \quad (23)$$

where  $X$  is a compact convex subset of a Banach space. Naturally, the feasible mapping  $G$  requires closed values, which is not a requirement in Theorem 14, however, where each inverse image being open is necessary.

By Theorem 18, each connected component including a connected minimal essential set of solutions is essential; that is, the existence of essential components can be induced.

**Corollary 20.** Let  $\phi \in M'$ . There is an essential component in  $F'(\phi)$ . If  $F'(\phi) = \{x^*\}$  is a singleton, then  $x^*$  is an essential solution of  $\phi$ .

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