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Research Article

Modified Block Pulse Functions for Numerical Solution of Stochastic Volterra Integral Equations

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We present a new technique for solving numerically stochastic Volterra integral equation based on modified block pulse functions. It declares that the rate of convergence of the presented method is faster than the method based on block pulse functions. Efficiency of this method and good degree of accuracy are confirmed by a numerical example.

1. Introduction

The numerical study and simulation of stochastic Volterra integral equations (SVIEs) have been an active field of research for the past years [1–7]. Most SVIEs do not have analytic solutions and hence it is of great importance to provide numerical schemes. Numerical schemes to stochastic differential equations (SDEs) have been well developed [8–12]. However, there are still few papers discussing the numerical solutions for stochastic Volterra integral equations.

Study in economics, sociology, and various biological and medical models leads to the stochastic Volterra integral equations. These systems are dependent on a noise source, on a Gaussian white noise, so that modeling such phenomena naturally requires the use of various stochastic Volterra integral equations.

In this paper, we consider the linear stochastic Volterra integral equation:

$$u(t) = u_0(t) + \int_0^t k_1(s,t)\, u(s)\, ds + \int_0^t k_2(s,t)\, u(s)\, dB(s) \qquad t \in [0,T], \tag{1}$$

where $u(t)$, $u_0(t)$, $k_1(s,t)$, and $k_2(s,t)$, for $s,t \in [0,T)$, are the stochastic processes defined on the same probability space $(\Omega, \mathscr{F}, P)$ with a filtration $\{\mathscr{F}_t,\ t \geq 0\}$ that is increasing and right continuous and $\mathscr{F}_0$ contains all P-null sets. $u(t)$ is unknown random function and $B(t)$ is a standard Brownian motion defined on the probability space and $\int_0^t k_2(s,t)u(s)dB(s)$ is the Itô integral. Numerous papers have been focusing on existence solution of (1) [13–15].

The paper [3] solves stochastic Volterra integral equations by block pulse functions (BPFs) and [4] applies this method for solving m-dimensional stochastic Itô Volterra integral equations. However, BPFs are very common in use; it seems that their convergence is weak. Maleknejad and Rahimi apply in [16] εModified Block Pulse Functions (εMBPFs) to solve Volterra integral equation of the first kind numerically. Here, we use this method for solving SVIEs.

This paper is organized as follows. In the rest of this section we describe some general concepts concerning the block pulse functions and epsilon modified block pulse functions and some concepts related to stochastic and Itô integral. Section 2 is devoted to stochastic integration operational matrix. In Section 3, the method is employed to solve stochastic integral equations. Section 4 discusses error analysis of this method. Section 5 gives numerical example. Finally, Section 6 provides the conclusion of this work.

1.1. Block Pulse Functions. BPFs have been variously studied [16–18] and applied for solving different problems. The goal of this section is to recall notations and definition of the BPFs that are used in the next sections.

The block pulse functions are defined on the time interval $[0, T)$ by

$$\psi_i(t) = \begin{cases} 1 & (i-1)\frac{T}{m} \le t < i\frac{T}{m}, \\ 0 & \text{elsewhere}, \end{cases} \tag{2}$$

where $i = 1, \ldots, m$ and for convenience we put $h = T/m$.

The block pulse functions on $[0, T)$ have the following properties:

(1) disjointness: for $i, j = 1, \ldots, m$

$$\psi_i(t)\psi_j(t) = \delta_{ij}\psi_i(t), \tag{3}$$

where δ_{ij} is Kronecker delta;

(2) orthogonality:

$$\int_0^T \psi_i(t)\psi_j(t)\,dt = \delta_{ij}h; \tag{4}$$

(3) completeness: for every $f \in L^2([0, T))$ when m approaches infinity, Parseval's identity holds:

$$\int_0^T f^2(t)\,dt = \lim_{m\to\infty} \sum_{i=1}^{m} (f_i)^2 \|\psi_i(t)\|^2, \tag{5}$$

where

$$f_i = \frac{1}{h}\int_0^T f(t)\psi_i(t)\,dt. \tag{6}$$

Also the Fourier coefficients f_i and the block pulse functions depend on m. The set of block pulse functions may be written as a vector $\Psi(t)$ of dimension m:

$$\Psi(t) = [\psi_1(t), \ldots, \psi_m(t)]^T \quad t \in [0, T). \tag{7}$$

From the above representation and disjointness property, it follows that

$$\Psi(t)\Psi^T(t) = \begin{pmatrix} \psi_1(t) & 0 & 0 & \cdots & 0 \\ 0 & \psi_2(t) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \psi_m(t) \end{pmatrix}_{m\times m}.$$

$$\Psi^T(t)\Psi(t) = 1,$$

$$\Psi(t)\Psi^T(t)F = D_F\Psi(t), \tag{8}$$

where F is an m-dimensional vector and $D_F = \text{diag}(F)$. Let G be an $m \times m$ matrix so that

$$\Psi^T(t)G\Psi(t) = \widehat{G}^T\Psi(t), \tag{9}$$

where $\widehat{G}$ is a vector with elements equal to the diagonal entries of G.

The expansion of a function $f(t)$ over $[0, T)$ with respect to $\psi_i(t)$, $i = 1, \ldots, m$, is given by

$$f(t) \simeq \sum_{i=1}^{m} f_i\psi_i(t) = F^T\Psi(t) = \Psi^T(t)F, \tag{10}$$

where $F = [f_1, \ldots, f_m]^T$ and f_i is defined by (6).

Let $k(s, t) \in L^2([0, T_1) \times [0, T_2))$. It is expanded with respect to BPFs as

$$k(s, t) \simeq \Psi^T(s)K\Lambda(t), \tag{11}$$

where $\Psi(s)$ and $\Lambda(t)$ are m_1- and m_2-dimensional BPFs vectors, respectively, and K is the $m_1 \times m_2$ block pulse coefficient matrix with the below k_{ij}, $i = 1, \ldots, m_1$, $j = 1, \ldots, m_2$:

$$k_{ij} = \frac{m_1 m_2}{T_1 T_2}\int_0^{T_1}\int_0^{T_2} k(s, t)\psi_i(s)\lambda_j(t)\,ds\,dt. \tag{12}$$

For convenience, we put $m_1 = m_2 = m$.

Now, integration operational matrix is considered and computed:

$$\int_0^t \psi_i(s)\,ds = \begin{cases} 0 & 0 \le t \le (i-1)h, \\ t-(i-1)h & (i-1)h \le t \le ih, \\ h & ih \le t < 1. \end{cases} \tag{13}$$

Since $t - ih$ is equal to $h/2$ at midpoint of $[ih, (i+1)h)$, we can approximate $t-(i-1)h$, for $(i-1)h \le t < ih$ by $h/2$. Therefore

$$\int_0^t \psi_i(s)\,ds \simeq \left(0, \ldots, 0, \frac{h}{2}, h, \ldots, h\right)\Psi(t), \tag{14}$$

where the ith component is $h/2$. As a result

$$\int_0^t \Psi(s)\,ds \simeq Q\Psi(t), \tag{15}$$

where Q is operational matrix of integration that is given by

$$Q = \frac{h}{2}\begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{16}$$

So,

$$\int_0^t f(s)\,ds \simeq \int_0^t F^T\Psi(s)\,ds \simeq F^TQ\Psi(t). \tag{17}$$

1.2. Epsilon Modified Block Pulse Functions (EMBPFs). A set of epsilon modified block pulse functions $\theta_i(t)$, $i = 0, 1, \ldots, m$, on the interval $[0, T)$ are defined as

$$\theta_0(t) = \begin{cases} 1 & t \in \left[0, \frac{T}{m} - \varepsilon\right) = I_0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\theta_i(t) = \begin{cases} 1 & t \in \left[\frac{iT}{m} - \varepsilon, \frac{(i+1)T}{m} - \varepsilon\right) = I_i, \\ 0 & \text{otherwise}, \end{cases} \tag{18}$$

for $i = 1, .., m-1$, and

$$\theta_m(t) = \begin{cases} 1 & t \in [T-\varepsilon, T) = I_m, \\ 0 & \text{otherwise}. \end{cases} \quad (19)$$

Similar to BPFs, the most important properties of EMBPFs are

(1) disjointness:

$$\theta_i(t)\theta_j(t) = \begin{cases} \theta_i(t) & i = j, \\ 0 & i \neq j, \end{cases} \quad (20)$$

where $i, j = 0, \ldots, m$;

(2) orthogonality: if we put $h = T/m$,

$$\int_0^T \theta_i(t)\theta_j(t)\,dt = h\delta_{ij}, \quad i, j = 1, \ldots, m-1; \quad (21)$$

(3) completeness:

$$\int_0^T f^2(t)\,dt = \sum_{i=0}^{\infty} f_i^2 \left\|\theta_i(t)\right\|^2, \quad (22)$$

where

$$f_i = \frac{1}{\Delta(I_i)} \int_0^T f(t)\theta_i(t)\,dt \quad (23)$$

and $\Delta(I_i)$ is length of interval I_i.

With defining $\Theta_{m+1}(t) = [\theta_0(t), \ldots, \theta_m(t)]^T$, we have

$$\Theta_{m+1}(t)\Theta_{m+1}^T(t) = \begin{pmatrix} \theta_0(t) & 0 & 0 & \cdots & 0 \\ 0 & \theta_1(t) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \theta_m(t) \end{pmatrix}_{m+1\times m+1},$$

$$\Theta_{m+1}^T(t)\Theta_{m+1}(t) = 1,$$

$$\Theta_{m+1}(t)\Theta_{m+1}^T(t)F = D_F\Theta_{m+1}(t),$$

$$\Theta_{m+1}^T(t)G\Theta_{m+1}(t) = \widehat{G}^T\Theta_{m+1}(t). \quad (24)$$

Similar to BPFs,

$$\int_0^t \Theta_{m+1}(s)\,ds \simeq P\Theta_{m+1}(t), \quad (25)$$

where the operational matrix P of EMBPFs is given by

$$P = \begin{pmatrix} \frac{h-\varepsilon}{2} & h-\varepsilon & \cdots & h-\varepsilon & h-\varepsilon \\ 0 & \frac{h}{2} & h & \cdots & h \\ 0 & 0 & \frac{h}{2} & \cdots & h \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\varepsilon}{2} \end{pmatrix}_{m+1\times m+1}, \quad (26)$$

and we have the following approximation:

$$\int_0^t f(s)\,ds \simeq \int_0^t F^T\Theta_{m+1}(s)\,ds \simeq F^T P\Theta_{m+1}(s). \quad (27)$$

1.3. Stochastic Concepts of Itô Integral

Definition 1 (Brownian motion process). A real-valued stochastic process $\{B(t),\ t \geq 0\}$ is called Brownian motion, if it satisfies the following properties:

(i) independence of increments: $B(t) - B(s)$, $t > s$, is independent of the past, that is, of $B(u)$, $0 \leq u \leq s$, or of $\mathcal{F}_s$, the σ-field generated by $B(u)$, $u \leq s$;

(ii) normal increments: $B(t) - B(s)$ has normal distribution with mean 0 and variance $t - s$;

(iii) continuity of paths: $B(t)$, $t \geq 0$, are continuous functions of t.

Definition 2. Let $\{N(t)\}_{t\geq 0}$ be an increasing family of σ-algebras of subsets of Ω. A process $g(t,\omega)$ from $[0,\infty)\times\Omega$ to R^n is called $N(t)$-adapted if for each $t \geq 0$ the function $\omega \mapsto g(t,\omega)$ is $N(t)$-measurable [19].

Definition 3 (see [19]). Let $\nu = \nu(S,T)$ be the class of functions $f(t,\omega) : [0,\infty)\times\Omega \rightarrow R$ such that

(i) $(t,\omega) \mapsto f(t,\omega)$ is $B\times\mathcal{F}$-measurable, where B denotes the Borel σ-algebra on $[0,\infty)$ and $\mathcal{F}$ is the σ-algebra on Ω;

(ii) $f(t,\omega)$ is $\mathcal{F}_t$-adapted, where $\mathcal{F}_t$ is the σ-algebra generated by the random variables $B(s)$, $s \leq t$;

(iii) $E[\int_S^T f^2(t,\omega)dt] < \infty$.

Definition 4 (the Itô integral, [19]). Let $f \in \nu(S,T)$; then the Itô integral of f (from S to T) is defined by

$$\int_S^T f(t,\omega)\,dB(t)(\omega) = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega)\,dB(t)(\omega), \quad \left(\text{limit in } L^2(P)\right), \quad (28)$$

where ϕ_n is a sequence of elementary functions such that

$$E\left[\int_S^T \left(f(t,\omega)-\phi_n(t,\omega)\right)^2 dt\right] \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \quad (29)$$

Theorem 5 (the Itô isometry). *Let $f \in \nu(S,T)$; then*

$$E\left[\left(\int_S^T f(t,\omega)\,dB(t)(w)\right)^2\right] = E\left[\int_S^T f^2(t,\omega)\,d(t)\right]. \quad (30)$$

Proof. See [19]. □

Definition 6 (1-dimensional Itô processes, [19]). Let $B(t)$ be 1-dimensional Brownian motion on $(\Omega,\mathcal{F},P)$. A 1-dimensional Itô process (stochastic integral) is a stochastic process $X(t)$ on $(\Omega,\mathcal{F},P)$ of the form

$$X(t) = X(0) + \int_0^t u(s,\omega)\,ds + \int_0^t v(s,\omega)\,dB(s), \quad (31)$$

or

$$dX(t) = u\,dt + v\,dB(t), \quad (32)$$

where

$$P\left[\int_0^t v^2(s,\omega)\,ds < \infty, \forall t \geq 0\right] = 1,$$
$$P\left[\int_0^t |u(s,\omega)|\,ds < \infty, \forall t \geq 0\right] = 1. \quad (33)$$

Theorem 7 (the 1-dimensional Itô formula). *Let $X(t)$ be an Itô process given by* (1) *and $g(t,x) \in C^2([0,\infty)\times R)$; then*

$$Y(t) = g(t,X(t)) \quad (34)$$

is again an Itô process, and

$$dY(t) = \frac{\partial g}{\partial t}(t,X(t))\,dt + \frac{\partial g}{\partial x}(t,X(t))\,dX(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X(t))\,(dX(t))^2, \quad (35)$$

where $(dX(t))^2 = (dX(t))(dX(t))$ is computed according to the rules

$$dt\cdot dt = dt\cdot dB(t) = dB(t)\cdot dt = 0,$$
$$dB(t)\cdot dB(t) = dt. \quad (36)$$

Proof. See [19].

Moreover, $\|\cdot\|$ is notation of

$$\|f(t)\|^2 = \int_0^1 |f(t)|^2 dt. \quad (37)$$

□

Lemma 8 (the Gronwall inequality). *Let $\alpha,\beta \in [t_0,T] \rightarrow R$ be integral with*

$$0 \leq \alpha(t) \leq \beta(t) + L\int_{t_0}^t \alpha(s)\,ds \quad (38)$$

for $t \in [t_0,T]$, where $L > 0$. Then

$$\alpha(t) \leq \beta(t) + L\int_{t_0}^t e^{L(t-s)}\beta(s)\,ds, \quad t \in [t_0,T]. \quad (39)$$

For more details see [19, 20].

2. Stochastic Integral Operational Matrix for EMBPFs

In this section stochastic integral operational matrix for EMBPFs is considered. For finding vector form of $\int_0^t \theta_i(s)dB(s)$, with EMBPFs, the Itô integral of each single EMBPF $\theta_i(t)$ can be computed as follows. It is clear that the integrals are stochastic and nondeterministic:

$$\int_0^t \theta_0(s)\,dB(s) = \begin{cases} B(t)-B(0) & 0 \leq t < h-\varepsilon, \\ B(h-\varepsilon)-B(0) & h-\varepsilon \leq t < T. \end{cases}$$

$$\int_0^t \theta_i(s)\,dB(s) = \begin{cases} 0 & 0 \leq t < ih-\varepsilon, \\ B(t)-B(ih-\varepsilon) & ih-\varepsilon \leq t < (i+1)h-\varepsilon, \\ B((i+1)h-\varepsilon)-B(ih-\varepsilon) & (i+1)h-\varepsilon \leq t < T, \end{cases} \quad (40)$$

for $i = 1,\ldots,m$, and

$$\int_0^t \theta_m(s)\,dB(s) = \begin{cases} 0 & 0 \leq t < T-\varepsilon, \\ B(t)-B(T-\varepsilon) & T-\varepsilon \leq t < T. \end{cases} \quad (41)$$

We approximate

(1) $B(t)-B(ih-\varepsilon)$, by $B((i+0.5)h-\varepsilon)-B(ih-\varepsilon)$, at midpoint of $[ih-\varepsilon,(i+1)h-\varepsilon)$;

(2) $B(t)-B(0)$ by $B((h-\varepsilon)/2)$ in $\theta_0(t)$ at midpoint of $[0,h-\varepsilon)$;

(3) $B(t)-B(T-\varepsilon)$ by $B(T-(\varepsilon/2))-B(T-\varepsilon)$ in $\theta_m(t)$, at midpoint of $[T-\varepsilon,T)$.

As a result, vector form of $\int_0^t \theta_i(s)dB(s)$, with EMBPFs, is given by

$$\int_0^t \theta_0(s)\,dB(s) \simeq \left(B\left(\frac{h-\varepsilon}{2}\right), B(h-\varepsilon), \dots, B(h-\varepsilon)\right)\Theta(t),$$
$$\int_0^t \theta_i(s)\,dB(s) \simeq (0,0,\dots,0,B((i+0.5)h-\varepsilon)-B(ih-\varepsilon), B((i+1)h-\varepsilon)-B(ih-\varepsilon),\dots, B((i+1)h-\varepsilon)-B(ih-\varepsilon))\Theta(t), \quad (42)$$

in which the $(i+1)$th component is $B((i+0.5)h-\varepsilon)-B(ih-\varepsilon)$,

$$\int_0^t \theta_m(s)\,dB(s) \simeq \left(0,0,\dots,B\left(T-\frac{\varepsilon}{2}\right)-B(T-\varepsilon)\right)\Theta(t). \quad (43)$$

Therefore

$$\int_0^t \Theta(s)\,dB(s) \simeq P_S\Theta(t), \quad (44)$$

where stochastic operational matrix of integration is given by

$$P_S = \begin{pmatrix} B\left(\frac{h-\varepsilon}{2}\right) & B(h-\varepsilon) & B(h-\varepsilon) & \cdots & B(h) \\ 0 & B\left(\frac{3h}{2}-\varepsilon\right)-B(h-\varepsilon) & B(2h-\varepsilon)-B(h-\varepsilon) & \cdots & B(2h-\varepsilon)-B(h-\varepsilon) \\ 0 & 0 & B\left(\frac{5h}{2}-\varepsilon\right)-B(2h-\varepsilon) & \cdots & B(3h-\varepsilon)-B(2h-\varepsilon) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B\left(\frac{(2m-1)h}{2}-\varepsilon\right)-B((m-1)h-\varepsilon) \\ 0 & 0 & 0 & \cdots & B\left(T-\frac{\varepsilon}{2}\right)-B(T-\varepsilon) \end{pmatrix}_{m+1\times m+1}. \quad (45)$$

So, the Itô integral of every function $f(t)$ can be approximated as follows:

$$\int_0^t f(s)\,dB(s) \simeq \int_0^t F^T\Theta(s)\,dB(s) \simeq F^T P_S\Theta(t). \quad (46)$$

3. Numerical Solution of SVIEs by EMBPFs

Here, we modify the method that has been used in [16] by EMBPFs. In the below equation:

$$u(t) = u_0(t) + \int_0^t k_1(s,t)\,u(s)\,ds + \int_0^t k_2(s,t)\,u(s)\,dB(s) \quad t\in[0,T], \quad (47)$$

we approximate functions $u(t)$, $u_0(t)$, $k_1(s,t)$, and $k_2(s,t)$ by EMBPFs:

$$u(t) \simeq U^T\Theta(t) = \Theta^T(t)\,U,$$
$$u_0(t) \simeq U_0^T\Theta(t) = \Theta^T(t)\,U_0,$$
$$k_1(s,t) \simeq \Theta^T(s)\,K_1\Theta(t) = \Theta^T(t)\,K_1^T\Theta(s),$$
$$k_2(s,t) \simeq \Theta^T(s)\,K_2\Theta(t) = \Theta^T(t)\,K_2^T\Theta(s), \quad (48)$$

where the vectors U, U_0 and matrices K_1, K_2 are EMBPFs coefficient of u, u_0, k_1, and k_2, respectively.

Substituting (3) into (47) and using previous relations,

$$U^T\Theta(t) \simeq U_0^T\Theta(t) + U^T\left(\int_0^t \Theta(s)\,\Theta^T(s)\,ds\right)K_1\Theta(t) + U^T\left(\int_0^t \Theta(s)\,\Theta^T(s)\,dB(s)\right)K_2\Theta(t). \quad (49)$$

Finally

$$U^T\Theta(t) \simeq U_0^T\Theta(t) + U^T B\Theta(t) + U^T B_s\Theta(t), \quad (50)$$

where

$$B=\begin{pmatrix} k_{00}^{1}\left(\frac{h-\varepsilon}{2}\right) & k_{01}^{1}(h-\varepsilon) & k_{02}^{1}(h-\varepsilon) & \cdots & k_{0m}^{1}(h-\varepsilon)\\ 0 & k_{11}^{1}\left(\frac{h}{2}\right) & 2k_{12}^{1}h & \cdots & 2k_{1m}^{1}h\\ 0 & 0 & k_{33}^{1}\frac{h}{2} & \cdots & 2k_{3m}^{1}h\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & k_{mm}^{1}\frac{\varepsilon}{2} \end{pmatrix}_{m+1\times m+1},$$

$$B_s=\begin{pmatrix} k_{00}^{2}B\left(\frac{h-\varepsilon}{2}\right) & k_{01}^{2}B(h-\varepsilon) & k_{02}^{2}B(h-\varepsilon) & \cdots & k_{0m}^{2}B(h-\varepsilon)\\ 0 & k_{11}^{2}\left(B\left(\frac{3h}{2}-\varepsilon\right)-B(h-\varepsilon)\right) & k_{12}^{2}(B(2h-\varepsilon)-B(h-\varepsilon)) & \cdots & k_{1m}^{2}(B(2h-\varepsilon)-B(h-\varepsilon))\\ 0 & 0 & k_{22}^{2}\left(B\left(\frac{5h}{2}-\varepsilon\right)-B(2h-\varepsilon)\right) & \cdots & k_{2m}^{2}(B(3h-\varepsilon)-B(2h-\varepsilon))\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & k_{m-1,m}^{2}\left(B\left(\frac{(2m-1)h}{2}-\varepsilon\right)-B((m-1)h-\varepsilon)\right)\\ 0 & 0 & 0 & \cdots & k_{mm}^{2}\left(B\left(T-\frac{\varepsilon}{2}\right)-B(T-\varepsilon)\right) \end{pmatrix}. \tag{51}$$

Then

$$U^{T}(I-B-B_s)\simeq U_0. \tag{52}$$

With replacing $\simeq$ by $=$, we have a linear system of equations.

Now if $\varepsilon_j = jh/k$, $j = 0,1,\ldots,k-1$, there will be k numerical answers $\hat{f}_{jh/k}$. Solution is approximated by

$$\overline{f}(t)=\frac{1}{k}\sum_{i=0}^{k-1}\hat{f}_{ih/k}(x). \tag{53}$$

4. Error Analysis

In this section, error analysis is studied. In the following theorems, for simplicity, we assume $T = 1$ and $h = 1/m$.

Theorem 9. *If* $\hat{f}_m(x) = \sum_{i=0}^{m} f_i\theta_i(x)$ *and* $f_i = (1/\Delta(I_i))\int_0^1 f(x)\theta_i(t)dt$, $i = 0,\ldots,m$, *then*

(1) $\delta = \int_0^1 (f(x)-\sum_{i=0}^{m} f_i\phi_i(x))^2dx$ *achieves its minimum value;*

(2) $\hat{f}_m(x)$ *approach* $f(x)$ *pointwise;*

(3) $\int_0^1 f^2(x)dx = \sum_{i=0}^{\infty} f_i^2\|\phi_i\|^2$.

Proof. See [16]. □

Theorem 10. *Assume the following.*

(1) $f(x)$ *is continuous and differentiable in* $[-h,1+h]$, *with bounded derivative; that is,* $|f'(x)| < M$.

(2) $\hat{f}_{ih/k}(x)$, $i = 0,1,\ldots,k-1$, *are correspondingly BPFs.* h/k *MBPFs,…,* $(k-1)h/k$ *MBPFs expansions of* $f(x)$ *base on* $m+1$ *EMBPFs over interval* $[0,1)$.

(3) $\overline{f}(t) = (1/k)\sum_{i=0}^{k-1}\hat{f}_{ih/k}(x)$.

Then

$$\begin{gathered}\left\|f(x)-\hat{f}_{ih/k}(x)\right\| = O(h),\\ \left\|f(x)-\overline{f}(x)\right\| = O\left(\frac{h}{k}\right)\quad in\ [h,1-h].\end{gathered} \tag{54}$$

Proof. Trapezoidal rule for integral is

$$\begin{aligned}\int_a^b f(x)\,dx &= \frac{b-a}{2}(f(a)+f(b))-\frac{(b-a)^3f''(\eta)}{12}\\ &= \frac{b-a}{2}(f(a)+f(b))+E,\quad \eta\in[a,b],\end{aligned} \tag{55}$$

where E is error of integration. Suppose $t_i = i/m = ih$ and $I_i = [t_{i-1},t_i]$. The representation error when $f(x)$ is represented by a series of BPFs over every subinterval $[t_i, t_i+h/k]$, $i = 0,\ldots,m-1$, is

$$e_i(x)=f(x)-f_i\phi_i(x)=f(x)-f_i, \tag{56}$$

where

$$f_i = \frac{1}{h}\int_{ih}^{(i+1)h} f(x)\,dx. \quad (57)$$

From (55),

$$f_i = \frac{1}{2}\left(f(t_i) + f(t_i + h)\right) + E. \quad (58)$$

It is obvious that if $f(x) = C(constant)$, then $e_i(x) = 0$.

So, this error is computed for $f(x) = x$ in interval $[t_i, t_i + h/k]$, $i = 1, \ldots, m-1$.

For this function $E = 0$, so

$$\begin{aligned} e_i(x)_{[t_i, t_i+h/k]} &= |x - f_i| = \left|x - \frac{t_i + t_{i+1}}{2}\right| \\ &= \left|x - \left(t_i + \frac{h}{2}\right)\right| \le \frac{h}{2}. \end{aligned} \quad (59)$$

Then this error with BPFs is $(h/2)M$.

Similarly, the error when $f(x)$ is represented in a series of EMBPFs over every subinterval $[t_i, t_i + h/k]$ is

$$\begin{aligned} e_i(x)_{[t_i, t_i+h/k]} &= \left|x - \left(\frac{\sum_{j=0}^{k-1}\left(t_i - (jh/k) + t_{i+1} - jh/k\right)}{2k}\right)\right| \\ &= \left|x - \left(\frac{\sum_{j=0}^{k-1}\left(t_i - jh/k + t_i + h - jh/k\right)}{2k}\right)\right| \\ &= \left|x - \left(t_i + \frac{h}{2}\right) - \frac{(k-1)h}{2k}\right| \le \frac{h}{2k}. \end{aligned} \quad (60)$$

So, the error with EMBPFs is $(h/2k)M$.

For I_0 in $[0, h/k]$ we have

$$\begin{aligned} e_i(x) &= \left|x - \sum_{j=0}^{k-1}\frac{h - jh/k}{2k}\right| \\ &= \left|x - \left(\frac{h}{2} - \frac{(k-1)h}{4k}\right)\right| = \left|x - \left(\frac{h}{4} + \frac{h}{4k}\right)\right| \\ &= O\left(\frac{h}{4}\right). \end{aligned} \quad (61)$$

So, the error is $O(h/4)$ also for I_n.

Now,

$$\begin{aligned} \|e_i(x)\|^2 &= \int_{t_i}^{t_i+h/k} |e_i(x)|^2 dx \\ &= \int_{t_i}^{t_i+h/k} \frac{h^2}{4k^2}M^2 dx = \frac{h^3}{4k^3}M^2, \\ \|e\|^2 &= \int_0^1 e^2(x)\,dx = \int_0^1 \left(\sum_{i=1}^{m}\sum_{j=0}^{k-1} e_i(x)\right)^2 dx \\ &= \sum_{i=1}^{m}\sum_{j=0}^{k-1}\int_0^1 e_i^2(x)\,dx = \sum_{i=1}^{m}\sum_{j=0}^{k-1}\|e_i(x)\|^2 \\ &= \frac{1}{h}\cdot k \cdot \frac{h^3}{4k^3}M^2 = \frac{h^2}{4k^2}M^2. \end{aligned} \quad (62)$$

We define the representation error between $f(x, y)$ and its 2D-EMBPFs expansion, $f_{i,j}$, over every subregion D_{ij}, is defined as

$$e_{ij}(x, y) = f(x, y) - f_{ij}, \quad (63)$$

where

$$D_{ij} := \left\{(x, y) \mid t_i \le x \le t_i + \frac{h}{k},\ t_j \le x \le t_j + \frac{h}{k}\right\}. \quad (64)$$

With Taylor's expansion and similarity to the above discussion,

$$\|e(x, y)\| = \frac{h}{2k}M. \quad (65)$$

□

Theorem 11. *Assume that*

(1) $P(w \in \Omega : \|u(\omega, t)\| < C) = 1$,

(2) $\|k_i\| < C \quad i = 1, 2$.

Then

$$\sup_{0\le t\le T}\left(E(\|(u - \overline{u})\|)^2\right)^{1/2} = O\left(\frac{h}{k}\right), \quad t \in [h, 1-h]. \quad (66)$$

Proof. Consider

$$\begin{aligned} u(t) - \overline{u}(t) &= u_0(t) - \overline{u_0}(t) \\ &\quad + \int_0^t k_1(s,t)\,u(s) - \overline{k_1}(s,t)\,\overline{u(s)}ds \\ &\quad + \int_0^t k_2(s,t)\,u(s) - \overline{k_2}(s,t)\,\overline{u(s)}dB(s). \end{aligned} \quad (67)$$

Table 1: Mean, standard deviation, and confidence interval for error mean in Example 1 with $m = 4$, $k = 4$.

n	$\overline{x}_E$	s_E	%95 confidence interval for mean of E	
			Lower	Upper
30	3.5678×10^{-3}	4.5802×10^{-3}	1.9287×10^{-4}	5.2068×10^{-3}
50	5.0234×10^{-3}	7.5849×10^{-3}	2.9209×10^{-3}	7.1258×10^{-3}
100	3.3467×10^{-3}	4.7983×10^{-3}	2.4062×10^{-3}	4.2871×10^{-3}
125	4.3526×10^{-3}	6.3657×10^{-3}	3.2367×10^{-3}	5.4685×10^{-3}

Table 2: Mean, standard deviation, and confidence interval for error mean in Example 1 with $m = 8$, $k = 4$.

n	$\overline{x}_E$	s_E	%95 confidence interval for mean of E	
			Lower	Upper
30	3.0924×10^{-3}	5.1132×10^{-3}	2.6266×10^{-3}	4.9221×10^{-3}
50	2.0598×10^{-3}	6.1477×10^{-3}	3.5574×10^{-4}	3.7635×10^{-3}
100	1.9728×10^{-3}	2.2587×10^{-3}	1.5300×10^{-3}	2.4155×10^{-3}
125	1.7054×10^{-3}	2.6547×10^{-3}	1.2400×10^{-3}	2.1707×10^{-3}

So,

$$\begin{aligned} E\left(\|u-\overline{u}\|^2\right) \leq 3 & \left[E\left(\left\|(u_0-\overline{u_0})\right\|^2\right) + E\left(\left\|\int_0^t \left(k_1 u - \overline{k_1 \overline{u}}\right) ds\right\|^2\right) + E\left(\left\|\int_0^t \left(k_2 u - \overline{k_2 \overline{u}}\right) dB(s)\right\|^2\right)\right] \\ \leq 3 & \left[E\left(\left\|(u_0-\overline{u_0})\right\|^2\right) + \left(\int_0^t E\left(\left\|k_1 u - \overline{k_1 \overline{u}}\right\|^2\right) ds\right) + \int_0^t E\left(\left\|k_2 u - \overline{k_2 \overline{u}}\right\|^2\right) ds\right], \end{aligned} \tag{68}$$

by the Cauchy-Schwartz inequality, Itô isometry formula, and the linearity of Itô integrals in their integrands.

The first term is satisfied by last theorem:

$$E\left(\left\|u_0-\overline{u_0}\right\|^2\right) \leq E\left(\frac{C^2 h^2}{k^2}\right) = O\left(\frac{h^2}{k^2}\right). \tag{69}$$

Now,

$$\begin{aligned} & \left\|(k_i(s,t)u(t) - \overline{k_i}(s,t)\overline{u}(t)\right\|^2 \\ & \leq 2\left\|\left(k_i - \overline{k_i}\right)u\right\|^2 + 2\left\|\overline{k_i}\left(\overline{u}-u\right)\right\|^2 \\ & \leq C\cdot\left(\left\|k_i - \overline{k_i}\right\|^2\right) + C\cdot\left(\|(\overline{u}-u)\|^2\right). \end{aligned} \tag{70}$$

Furthermore,

$$\left\|k_i - \overline{k_i}\right\|^2 = O\left(\frac{h^2}{k^2}\right), \quad i = 1,2. \tag{71}$$

Hence

$$\begin{aligned} E\left(\|u-\overline{u}\|^2\right) \leq 3 & \left[E\left(\left\|(u_0-\overline{u_0})\right\|^2\right) + \int_0^t E\left(\left\|\left(k_1 u - \overline{k_1 \overline{u}}\right)\right\|^2 ds\right) + \int_0^t E\left(\left\|(k_2 u - \overline{k_2 \overline{u}})\right\|^2\right) ds\right] \\ & \leq C_0 E\left(\left\|u_0-\overline{u_0}\right\|^2\right) + C_1 \int_0^t E\left(\left\|k_1 - \overline{k_1}\right\|^2\right) ds \\ & + C_2 \int_0^t E\left(\left\|k_2 - \overline{k_2}\right\|^2\right) ds \\ & + C_3 \int_0^t E\left(\|(u-\overline{u})\|^2\right) ds. \end{aligned} \tag{72}$$

Then by Gronwall's inequality, we get

$$E\left(\left\|(u-\overline{u})^2\right\|\right) \leq C\frac{h^2}{k^2}. \tag{73}$$

□

5. Numerical Example

In this section, we present an example for showing the features of the EMBPFs method in this paper. Let X_i denote the EMBP coefficient of exact solution of the given example and

Table 3: Mean, standard deviation, and confidence interval for error mean in Example 1 with $m = 32$.

n	$\overline{x}_E$	s_E	%95 confidence interval for mean of E	
			Lower	Upper
30	0.02308947	0.00442835	0.02150480	0.02467413
50	0.02341165	0.00511389	0.02199415	0.02482915
100	0.02364843	0.00524000	0.02262139	0.02467548
125	0.02345691	0.00477156	0.02262042	0.02429340

let Y_i be the EMBP coefficient of computed solution by the presented method. In this example error is defined as

$$\|E\|_\infty = \max_{1\le i\le m} |X_i - Y_i| . \quad (74)$$

Example 1 (see [3]). Consider the following linear stochastic Volterra integral equation:

$$u(t) = \frac{1}{12} + \int_0^t \cos(s)\, u(s)\, ds + \int_0^t \sin(s)\, u(s)\, dB(s) \quad s, t \in [0, 0.5), \quad (75)$$

with the exact solution $u(t) = (1/12)e^{-t/4+\sin(t)+\sin(2t)/8+\int_0^t \sin(s)dB(s)}$, for $0 \le t < 0.5$.

The numerical results are shown in Tables 1 and 2. In the tables, n is the number of iterations, $\overline{x}_E$ is error mean, and s_E is standard deviation of error.

Table 3 is from [3] for comparison.

In some examples by applying BPFs when m increases, accuracy decreases, but in EMBPFs we achieve good accuracy by increasing k.

6. Conclusion

As some SVIEs cannot be solved analytically, in this paper we present a new technique for solving SVIEs numerically. Here, we consider a modification of the block pulse functions. Some theorems show that if EMBPFs are used for achieving numerical expansions with k times more precision, there is no need to increase the number of BPFs, k times, which leads to solving a system of equations with k times more equations and unknowns. But the results of BPFs solution can be combined with solutions of $k-1$ systems of equations with one more unknown and nearly achieve k times more precision. Parallel programming is so useful for this method. Efficiency of this method and good degree of accuracy are confirmed by a numerical example.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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