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CLASSIFICATION OF SOLUTIONS OF DELAY DIFFERENCE EQUATIONS

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ABSTRACT. In this paper we study the classification of solutions of delay difference equation

$$\begin{cases}\Delta^2 y_n = P_n y_{n-m} \\ y_n = A_n \text{ for } n = N-(m+1), \cdots N-1\end{cases}$$

where $A_n, n = N-(m+1), \cdots, N-1$ are given, m is a nonnegative integer.

KEY WORDS AND PHRASES. Delay difference equations, oscillation, bounded solutions.
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1. INTRODUCTION. The problem of oscillation and nonoscillation of solutions of delay difference equations has been receiving a lot of attention for the last few years. Erbe and Zhang ([1]-[3]), Lalli, Zhang and Zhao ([8], [9]), Ladas, Philos and Sficas ([6], [7]), have done some extensive works on this topic. A survey on the oscillation of delay difference equations could be found in the monograph by Gyori and Ladas [5].

In this paper we consider the second order delay difference equations of the form:

$$\Delta^2 y_n = P_n y_{n-m} \tag{1.1}$$

where Δ denotes the forward difference operator: $\Delta y_n = y_{n+1} - y_n, m$ is a nonnegative integer.

By a solution of equation (1.1) we mean a sequence $\{y_n\}$ which is defined for $n \geq N-(m+1)$ and which satisfies equations (1.1) for all $n \geq N$. Clearly if

$$y_n = A_n, \text{ for } n = N-(m+1), N-m, \cdots, N \tag{1.2}$$

are given, then equation (1.1) has a unique solution satisfying the initial conditions (1.2), where N is an initial point.

A nontrivial solution $\{y_n\}$ of equation (1.1) is said to be oscillatory if for every $N > 0$ there exists an $n \geq N$ such that $y_n y_{n+1} \leq 0$. Otherwise it is called nonoscillatory.

Set $E_N = \{N-(m+1), N-m, \cdots, N-1\}$, if

$$y_n = A_n, n \in E_N \tag{1.3}$$

are given, then the solutions depend on the parameter $y_N = \xi$. We are concerning with the classification of solutions of equation (1.1) with (1.3).

2. MAIN RESULTS.

We always assume that $P_n \geq 0$ and P_n does not identically equal to zero in equation (1.1). We denote S the set of all solutions of (1.1). Since $P_n \geq 0$, it is easy to see that

$$S = S^{+\infty} \bigcup S^{-\infty} \bigcup S^{k} \bigcup S^{-k} \bigcup S^{o} \bigcup S^{\sim}$$

where

$$S^{+\infty} = \{\{y_n\} \in S: lim_{n\to\infty} y_n = +\infty\}$$

$$S^{-\infty} = \{\{y_n\} \in S: lim_{r\to\infty} y_n = -\infty\}$$

$$S^{k} = \{\{y_n\} \in S: 0 < \lim_{n\to\infty} y_n < +\infty\}$$

$$S^{-k} = \{\{y_n\} \in S: 0 > \lim_{n\to\infty} y_n > -\infty\}$$

$$S^{o} = \{\{y_n\} \in S: y_n \text{ nontrivial}, \lim_{n\to\infty} y_n = 0 \text{ monotonically}\}$$

$$S^{\sim} = \{\{y_n\} \in S: y_n \text{ is oscillatory}\}.$$

LEMMA 2.1 If

$$y_i \geq 0 \text{ on } E_N, \; y_N > y_{N-1}$$

then $y_n \in S^{+\infty}$. If

$$y_i \leq 0 \text{ on } E_N, y_N < y_{N-1}$$

than $y_n \in S^{-\infty}$.

PROOF. From (1.1), we have

$$\Delta y_{N+n} - \Delta y_{N-1} = \sum_{i=N-1}^{N+(n-1)} P_i y_{i-m}, \tag{2.1}$$

Summing it in n we have

$$y_{N+n} = y_{N-1} + n\Delta y_{N-1} + \sum_{i=0}^{n-1} \sum_{j=N-1}^{N+n-1} P_j y_{j-m}. \tag{2.2}$$

The conclusions of Lemma 2.1 follow from (2.2).

From (2.2), the following is also true.

LEMMA 2.2. If

$$\lim_{n\to\infty} \sum_{i=N-1}^{n+N-2} (n+N-1-i)P_i = \infty, \tag{2.3}$$

then

$$y_i \geq 0, \; i \in E_N, \; y_N \geq y_{N-1}$$

imply that $\{y_n\} \in S^{+\infty}$, and if

$$y_i \leq 0, \; i \in E_N, \; y_N \leq y_{N-1}$$

imply that $\{y_n\} \in S^{-\infty}$.

LEMMA 2.3. Assume that the solution y_n and z_n have same initial values on E_N with $\Delta y_{N-1} > \Delta z_{N-1}$. Then $y_n > z_n, \Delta y_n > \Delta z_n, n \geq N$ and

$$\lim_{n\to\infty}(y_n - z_n) = \infty. \tag{2.4}$$

PROOF. Set $x_n = y_n - z_n$, then $x_i = 0$ on E_N and $\Delta x_{N-1} > 0$. By Lemma 2.1, $\{x_n\} \in S^{+\infty}$ From (2.1) $\Delta x_n > 0$ for $n \geq N$.

LEMMA 2.4. For every given initial value on E_N, equation (1.1) has no more than one bounded solution.

PROOF. Suppose the contrary, let $\{y_n\}$, $\{z_n\}$ be two bounded solutions of (1.1) with $y_i = z_i$ on E_N and $y_N > z_N$. This implies that $|y_n - z_n|$ is bounded. On the other hand, by Lemma 2.3, (2.4) should be true. This contradiction proves Lemma 2.4.

For given $y_i = A_i$ on E_N, then the solution of (1.1) depends on the parameter $y_N = \xi \in R$. Define the sets of ξ as follows:

$$K^{+\infty} = \{\xi \in R, \{y_n\} \in S^{+\infty}\}$$

$$K^{-\infty} = \{\xi \in R, \{y_n\} \in S^{-\infty}\}$$

$$K^{o} = \{\xi \in R, \{y_n\} \in S^{o}\}$$

$$K^{\sim} = \{\xi \in R, \{y_n\} \in S^{\sim}\}$$

THEOREM 2.1. For given y_i on E_N, the sets $K^{+\infty}$ and $K^{-\infty}$ are nonempty.

PROOF. If $y_i = 0$ on E_N, the conclusion follows from Lemma 2.1. Otherwise, from (2.1) and (2.2) we can find a number $y_N = \xi$ so large that $y_i > 0, i = N, N+1, \cdots N+m$ and $\Delta y_{N+m} > 0$. Translating the initial point to $n+m$ and using Lemma 2.1 we conclude that the solution with this y_N belongs to $S^{+\infty}$. Therefore $\xi \in K^{+\infty}$. It is similar to prove that $K^{-\infty}$ is nonempty.

THEOREM 2.2. The sets $K^{-\infty}, K^{+\infty}$ are open sets which are given by nonintersecting half lines $(-\infty, \alpha)$ and $(\beta, +\infty)(\alpha \leq \beta)$. The set $F = R - (K^{+\infty} \bigcup K^{-\infty})$ is nonempty and consists of the interval $[\alpha, \beta]$, if $\alpha < \beta$, or the point α, if $\alpha = \beta$.

PROOF. Let $\{y_n\} \in S^{+\infty}$. Then there exists N' such that $y_i > 0$ and $\Delta y_i > 0$ on $E_{N'}$. By continuous dependence of solutions and their differences on the initial conditions, all solutions with y_i on E_N and $\bar{y}_N$ differ slightly from y_N are positive and have positive differences on $E_{N'}$. If the initial point is translated to the point $i = N'$, then by Lemma 2.1 all those solutions belong to $S^{+\infty}$, i.e., $K^{+\infty}$ is open. Similarly, one can prove that $K^{-\infty}$ is open. Using Lemma 2.3, the conclusions of theorem follow.

THEOREM 2.3. If $\alpha < \beta$, then each $y_N \in F$ the corresponding solution is unbounded and oscillatory.

PROOF. It is sufficient to show that every solution with $y_N \in F$ is unbounded. Suppose the contrary, $\{y_n\}$ is a bounded solution with $y_N \in F$. Let $z_N \neq y_N$. By Lemma 2.4, $\{z_n\}$ is unbounded and oscillatory. On the other hand, Lemma 2.3 shows that $|y_n - z_n| \to \infty$ as $n \to \infty$ and hence $lim_{n\to\infty} |z_n| = \infty$ which contradicts the oscillation of $\{z_n\}$.

THEOREM 2.4. If $\sum_{i=N}^{\infty} iP_i = \infty$, then every bounded solution of (1.1) either belongs to S^o or $S^{\sim}$.

PROOF. Let $\{y_n\}$ be a bounded positive solution of (1.1). Then

$$\Delta y_n < 0 \text{ eventually and } \lim_{n\to\infty} \Delta y_n = 0.$$

From (2.1)

$$\Delta y_{N-1} = -\sum_{i=N-1}^{\infty} P_i y_{i-m}$$

and from (2.2)

$$y_{N+n} = y_{N-1} - n\sum_{i=N-1}^{\infty} P_i y_{i-m} + \sum_{i=0}^{n-1}\sum_{j=N-1}^{N+i-1} P_j y_{j-m}$$

$$= y_{N-1} - n\sum_{i=N-1}^{\infty} P_i y_{i-m} + \sum_{i=N-1}^{N+n-2}(n+N-1-i)P_i y_{i-m}$$

$$= y_{N-1} - n\sum_{i=N-1}^{N+n-2} P_i y_{i-m} - n\sum_{i=N+n-1}^{\infty} P_i y_{i-m} + \sum_{i=N-1}^{N+n-2}(n+N-1-i)P_i y_{i-m}$$

$$= y_{N-1} - n\sum_{i=N+n-1}^{\infty} P_i y_{i-m} + \sum_{i=N-1}^{N+n-2}(N-1-i)P_i y_{i-m}$$

$$= y_{N-1} - n\sum_{i=N+n-1}^{\infty} P_i y_{i-m} + (N-1)\sum_{i=N-1}^{N+n-2} P_i y_{i-m} - \sum_{i=N-1}^{N+n-2} iP_i y_{i-m}$$

$$= y_{N-1} + (N-1)(\Delta y_{N+n-2} - \Delta y_{N-1}) - \sum_{i=N-1}^{N+n-2} ip_i y_{i-m} + n\Delta y_{N+n-1}$$

$$\leq y_{N-1} - (N-1)\Delta y_{N-1} - \sum_{i=N-1}^{N+n-2} ip_i y_{i-m}. \tag{2.5}$$

If $y_n \to l > 0$, then (2.5) lead to that $lim_{n\to\infty} y_n = -\infty$. This contradiction shows that $lim_{n\to\infty} y_n = 0$. The proof is complete.

COROLLARY 2.1. If $\sum_{i=N}^{\infty} ip_i = \infty$, then

$$R = K^{+\infty}\bigcup K^{-\infty}\bigcup K^{o}\bigcup K^{\sim} \tag{2.6}$$

and $K^{+\infty}, K^{-\infty}$ and $K^{o}\bigcup K^{\sim}$ are nonempty.

THEOREM 2.5. Assume that

$$\lim_{n\to\infty} sup \sum_{i=n-m+1}^{n} (i-(n-m))p_i > 1 \tag{2.7}$$

Then every bounded solution of (1.1) is oscillatory.

PROOF. Let $\{y_n\}$ be a bounded positive solution of (1.1). Then $\Delta y_n < 0$ eventually. Summing (1.1) from N to n, we have

$$\Delta y_{n+1} - \Delta y_N = \sum_{i=N}^{n} p_i y_{i-m}$$

Summing it from $n-m+1$ to n in N, we obtain

$$m\Delta y_{n+1} - y_{n+1} + y_{n-m+1} = \sum_{j=n-m+1}^{n}\sum_{i=j}^{n} p_i y_{i-m}$$

Hence

$$0 \leq y_{n+1} - y_{n-m+1} + \sum_{i=n-m+1}^{n} (i-(n-m))p_i y_{i-m}$$

$$\leq y_{n+1} - y_{n-m+1}(1 - \sum_{i=n-m+1}^{n} (i-(n-m))p_i)$$

which contradicts to (2.7). The proof is complete.

COROLLARY 2.2 Assume that the assumptions of Corollary 2.1 and Theorem 2.5 hold.

Then $K^{\sim}$ is nonempty.

In fact, by Corollary 2.1, $K^{o} \bigcup K^{\sim}$ is nonempty and by Theorem 2.5, K^{o} is empty Therefore $K^{\sim}$ is nonempty.

It is easy to see that if $p_i \equiv p > 0$ in (1.1), then all assumptions of Corollary 2.2 hold, therefore for any given A_n on E_N, equation (1.1) with (1.3) has at least one oscillatory solution, i.e., $K^{\sim}$ is nonempty.

EXAMPLE 2.1. Consider

$$\Delta^2 y_n = P_n y_{n-4} \tag{2.8}$$

with $y_i = (-1)^i, i = 1, \cdots, 5 P_n \equiv 1$. Then through computation if $y_6 > -0.21675$, the solution $\{y_n\} \in S^{+\infty}$, if $y_6 < -0.21676$, the solution $\{y_n\} \in S^{-\infty}$, in this case we see $\alpha = \beta$.

OPEN PROBLEM. What condition could guarantee that $\alpha < \beta$?

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