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Research Article

Invariant Solutions for Nonhomogeneous Discrete Diffusion Equation

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One-dimensional optimal systems for nonhomogeneous discrete heat equation with different source terms are calculated. By utilizing these optimal systems invariant solutions are found. Also generating solutions are calculated, using the elements of the symmetry algebra.

1. Introduction

Mathematics provides models, for real life phenomena, to precisely understand the underlying laws governing them. Dynamic changes in a physical phenomenon are usually modeled by differential equations. This means that it is supposed that the changes are continuous and these are taking place in a continuous domain. But there are many real life situations where either the domain or the phenomenon itself or the both are not continuous. These situations occur but are not limited to the fields of biology, physics (classical and quantum), geometry, mathematical design, finance and so forth. Discrete situations are better modeled by difference equations, in contrast with the differential equations. Therefore, difference equations make their appearance in almost every branch of mathematics and their importance cannot be overemphasized.

Differences and their calculus are as old as is the differential calculus, but apart from its importance and usefulness the theory and methods of solving difference equations are not as developed as those of differential equations. Since differential equation is a limiting case of difference equation; therefore, it is natural to extend methods available for solving differential equations to the respective difference equations, and the same is done most of the times [1–3].

Solving differential equations by exploiting symmetry properties of their solution spaces is one of the standard methods, introduced by the Norwegian mathematician Sophus Lie (1842–1899). In contrast with adhoc techniques this method provides an algorithmic and unified procedure to solve almost all types of differential equations, linear or nonlinear [4, 5].

At present, the theory of difference equations is at the same stage as was the theory of differential equations at the time of *S.* Lie. A number of adhoc approaches are available to solve difference equations, for example, substitution, nonlinear functional relation, Schroder's generation function, Maeda's method, or the theory of integrable maps. But a number of authors have attempted, following Lie's method for differential equations, to develop a unified integration procedure based on invariance properties of difference equation [6, 7]. It is worth noting that the symmetry properties are not only helpful to find solution but play an important role to understand the physical phenomenon more deeply. In this direction, a number of researchers have attempted to apply symmetries to analyze different discrete physical phenomena [6–19].

The discrete diffusion equation is widely used in many contexts [20, 21]. For instance, it has been applied to the area of population growth where one wishes to model geographic spread in addition to growth in number. In the area of physics it is used to model ionic diffusion on a lattice. It has also been used in digital filtering in the form of diffusion filtering [10]. Due to its wide utility, a number of researchers have analyzed different aspects of the discrete diffusion process. Recently, in [6], Levi et al. have attempted to generalize the

Lie infinitesimal formulism for calculation of symmetries of difference equations and calculated generalized symmetries of diffusion type difference equations. The symmetry group, which they have found for the discrete equation, is a minimal extension of the Lie point symmetries of the corresponding differential equation. They also found that in the case of linear discrete equation the symmetry algebra is isomorphic to the continuous limit.

Symmetries are used not only to integrate but also to analyze the solution space for more insight into the physical system at hand. So, once symmetries are given it is natural to ask for further analysis and find solutions of the equation. There is limited literature on the analysis of solutions of diffusion type difference equations via symmetries. In this paper one-dimensional optimal systems are obtained for the symmetry algebra of diffusion difference equations with different source terms found in [6]. We have also calculated invariant solutions corresponding to each representative of the one-dimensional optimal system, wherever they exist. Using elements of the symmetry algebra generating solutions are also obtained. Moreover, all those cases in which no group invariant solution exist are mentioned.

The rest of the paper is arranged as follows. In the next two sections we give some preliminaries and summarize the results of [6], for completeness. In Section 3 one-dimensional optimal systems are obtained for all cases and corresponding group invariant solutions are given. Section 4 is devoted to obtain generating solutions corresponding to each symmetry generator. A brief conclusion is given in Section 5.

2. Preliminaries

Let $u(x)$ be a continuous scalar function of p independent variables $x = (x_1, x_2, \dots, x_i, \dots, x_p)$ evaluated at finitely many points on a uniformly distributed lattice with spacing $\sigma_i > 0$. Then partial finite differences are defined by

$$\Delta_{x_i} u(x) = \frac{1}{\sigma_i} \begin{Bmatrix} u\left(x_1, x_2, \dots, x_{i-1}, x_i + \sigma_i, x_{i+1}, \dots, x_p\right) \\ -u\left(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_p\right) \end{Bmatrix}. \tag{1}$$

It is sometimes convenient to express finite difference operator in terms of shift operator defined as follows:

$$T_{x_i} u(x) = u\left(x_1, x_2, \dots, x_{i-1}, x_i + \sigma_i, x_{i+1}, \dots, x_p\right). \tag{2}$$

Consequently,

$$\Delta_{x_i} u(x) = \frac{1}{\sigma_i}\left(T_{x_i} - 1\right) u(x). \tag{3}$$

For example, for the factorial function $x^{(n)} = x(x-h)(x-2h)(x-3h)\cdots[x-(n-1)h]$ (2) gives

$$T_{x_i} x^{(n)} = \sigma_i n x^{(n-1)} + x^{(n)}. \tag{4}$$

Also the action of $T_{x_i}^{-1}$ on $x^{(n)}$ is given by

$$T_{x_i}^{-1} x^{(n)} = -\sigma_i n x^{(n-1)} + x^{(n)}. \tag{5}$$

Consider a partial difference equation in the following form:

$$E\left(x, T^{\alpha} u(x), T^{\beta_i} \Delta_{x_i} u(x), T^{\gamma_{ij}} \Delta_{x_i} \Delta_{x_j} u(x), \dots\right) = 0, \tag{6}$$

where the operator T^{α} operates on a function $u(x)$ as follows:

$$T^{\alpha} u(x) = T_{x_1}^{\alpha_1} T_{x_2}^{\alpha_2} \cdots T_{x_p}^{\alpha_p} u(x), \tag{7}$$

with $m_i \le \alpha_i \le n_i$, $i = 1, 2, \dots, p$, and m_i, n_i are some fixed integers. The shift operators T^{β_i}, $T^{\gamma_{ij}}$ are defined likewise.

A discrete vector field in its evolutionary form, $X_Q = Q(\partial/\partial u)$ with characteristic Q given by

$$Q = \sum_i \xi_i\left(x, T^{\alpha} u\right) T^{b} \Delta_{x_i} u - \Phi\left(x, T^{c} u\right) \tag{8}$$

is an infinitesimal generalized symmetry generator of the difference equation (6) if it satisfies the infinitesimal symmetry criterion:

$$X_Q^{(n)} E\left(x, T^{\alpha} u(x), T^{\beta_i} \Delta_{x_i} u(x), T^{\gamma_{ij}} \Delta_{x_i} \Delta_{x_j} u(x), \dots\right)\Big|_{E(x, T^{\alpha} u(x), T^{\beta_i} \Delta_{x_i} u(x), T^{\gamma_{ij}} \Delta_{x_i} \Delta_{x_j} u(x), \dots) = 0} = 0. \tag{9}$$

Here, $X_Q^{(n)}$ is the discrete analog of the *nth* prolongation of the vector field X_Q. Prolonged vector field acts on the nth extended lattice and therefore has the following form:

$$\begin{aligned} X_Q^{(n)} &= \sum_{\alpha} T^{\alpha} Q \frac{\partial}{\partial T^{\alpha}} u + \sum_{\beta_i} T^{\beta_i} Q^{x_i} \frac{\partial}{\partial T^{\beta_i}} \Delta_{x_i} u \\ &\quad + \sum_{\gamma_{ij}} T^{\gamma_{ij}} Q^{x_i x_j} \frac{\partial}{\partial T^{\gamma_{ij}}} \Delta_{x_i} \Delta_{x_j} u + \cdots . \end{aligned} \tag{10}$$

Here, the coefficient functions $Q^{x_i}, Q^{x_i x_j}, \dots$ are the discrete total variations of Q given by

$$Q^{x_i} = \Delta_{x_i}^{T} Q, \qquad Q^{x_i x_j} = \Delta_{x_i}^{T} \Delta_{x_j}^{T} Q, \tag{11}$$

where the total variation Δ_x^T acts on functions of $x, u, \Delta_x u, \dots$ as follows:

$$\begin{aligned} &\Delta_{x_i}^{T} f\left(x, u(x), \Delta_x u(x), \dots\right) \\ &\quad = \frac{1}{\sigma_i}\left\{f\left(\left(x_1, x_2, \dots, x_i + \sigma_i, \dots, x_p\right),\right.\right. \end{aligned}$$

$$
\begin{aligned}
& u\left(x_1, x_2, \ldots, x_i+\sigma_i, \ldots, x_p\right), \\
& \quad\left(\Delta_x u\right)\left(x_1, x_2, \ldots, x_i+\sigma_i, \ldots, x_p\right), \ldots) \\
& -f\left(\left(x_1, x_2, \ldots, x_p\right),\right. \\
& \quad u\left(x_1, x_2, \ldots, x_p\right), \\
& \quad\left.\left(\Delta_x u\right)\left(x_1, x_2, \ldots, x_p\right), \ldots)\right\}.
\end{aligned} \tag{12}
$$

3. Symmetry Algebra of Nonhomogeneous Discrete Heat Equation

In this section we summarize the results of [6]. The difference equation

$$\Delta_t u - \Delta_{xx} u + g\left(x, t, T_x, T_t\right) u = 0 \tag{13}$$

is a discrete heat or diffusion equation. Here u is a function of space variable x, time t and partial shifts T_x and T_t with respect to x and t, respectively. Let the following be the generalized symmetry generator in the evolutionary form:

$$X_Q = \left(\xi \Delta_x u + \tau \Delta_t u + f u\right) \frac{\partial}{\partial u} = Q \frac{\partial}{\partial u}. \tag{14}$$

If (14) is admitted by (13) then the infinitesimal symmetry criterion gives

$$
\begin{aligned}
& X_Q^{(2)}\left(\Delta_t u - \Delta_{xx} u + g\left(x, t, T_x, T_t\right) u\right)\Big|_{\Delta_{xx} u = \Delta_t u + g(x,t,T_x,T_t) u} \\
& \quad = 0,
\end{aligned} \tag{15}
$$

where

$$
\begin{aligned}
X_Q^{(2)} = {} & Q \frac{\partial}{\partial u} + Q^x \frac{\partial}{\partial u_x} + Q^t \frac{\partial}{\partial u_t} + Q^{xx} \frac{\partial}{\partial u_{xx}} \\
& + Q^{xt} \frac{\partial}{\partial u_{xt}} + Q^{tt} \frac{\partial}{\partial u_{tt}}
\end{aligned} \tag{16}
$$

is the second prolongation of X_Q. Using (16) in (15) one gets the following set of determining equations:

$$\Delta_x(\tau) = 0, \tag{17}$$

$$-\Delta_t(\tau) T_t + 2\Delta_x(\xi) T_x + [\tau, g] = 0, \tag{18}$$

$$-\Delta_t(\xi) T_t + \Delta_{xx}(\xi) T_x^2 + 2\Delta_x(f) T_x + [\xi, g] = 0, \tag{19}$$

$$
\begin{aligned}
& -\Delta_t(f) T_t + \Delta_{xx}(f) T_x^{\ 2} + 2\Delta_x(\xi) T_x g + \xi \Delta_x(g) T_x \\
& \quad + \tau \Delta_t(g) T_t + [f, g] = 0.
\end{aligned} \tag{20}
$$

To obtain the symmetry generator (14) one needs to solve the set of equations (17) to (20) for unknown functions ξ, τ, and f. In [6] these functions have been found for three special source terms $g(x, t, T_x, T_t)$. Here, we give the infinitesimal symmetry generators obtained in [6] and refer to the paper for details.

3.1. Free Heat Equation. In this case $g(x, t, T_x, T_t) = 0$ and (13) takes the form

$$\Delta_t u - \Delta_{xx} u = 0. \tag{21}$$

The set of equations (17) to (20) are then solved to obtain the following symmetry algebra:

$$
\begin{aligned}
& X_{1e} = \Delta_t u \frac{\partial}{\partial u}, \qquad X_{2e} = \Delta_x u \frac{\partial}{\partial u}, \\
& X_{3e} = u \frac{\partial}{\partial u}, \\
& X_{4e} = \left(2t \Delta_t u T_t^{-1} + x T_x^{-1} \Delta_x u\right) \frac{\partial}{\partial u}, \\
& X_{5e} = \left(2t \Delta_x u T_t^{-1} + x T_x^{-1} u\right) \frac{\partial}{\partial u}, \\
& X_{6e} = \Big(t^2 T_t^{-2} \Delta_t u - t \sigma_t T_t^{-2} \Delta_t u + x t T_t^{-1} T_x^{-1} \Delta_x u \\
& \qquad + \frac{1}{4} x^2 T_x^{-2} u - \frac{1}{4} x \sigma_x T_x^{-2} u + \frac{1}{2} t T_t^{-1} u\Big) \frac{\partial}{\partial u}.
\end{aligned} \tag{22}
$$

3.2. Nonzero Potential. In this case the potential g is taken to be a nonzero function $g = g(x, T_x)$. Plugging in this assumption on g in the set of equations (17) to (20) one gets

$$\xi = \frac{1}{2} x^{(1)} \Delta_t(\tau) T_t T_x^{-1} + \alpha\left(t, T_t\right), \tag{23}$$

where $\alpha(t, T_t)$ is an arbitrary function to be known. And

$$
\begin{aligned}
f = {} & \frac{1}{8} x^{(2)} \Delta_{tt}(\tau) T_t^2 T_x^{-2} + \frac{1}{2} x^{(1)} \Delta_t \alpha T_t T_x^{-1} \\
& - \frac{1}{4} \Delta_x^{-1}\left[x^{(1)} \Delta_t(\tau) T_t T_x^{-1}, g\right] T_x^{-1} + \beta\left(t, T_t\right),
\end{aligned} \tag{24}
$$

where $\beta(t, T_t)$ is an arbitrary function to be known. To find the unknown functions f and ξ one puts a condition on the potential function g such that the commutator $[x^{(1)} \Delta_t(\tau) T_t T_x^{-1}, g]$ vanishes and (24) becomes

$$f = \frac{1}{8} x^{(2)} \Delta_{tt}(\tau) T_t^2 T_x^{-2} + \frac{1}{2} x^{(1)} \Delta_t \alpha T_t T_x^{-1} + \beta\left(t, T_t\right). \tag{25}$$

Using (25) in (20) and simplifying one gets

$$
\begin{aligned}
& -\frac{1}{8} x^{(2)} \Delta_{ttt}(\tau) T_t^3 T_x^{-2} - \frac{1}{2} x^{(1)} \Delta_{tt} \alpha T_t^2 T_x^{-1} - \Delta_t \beta\ T_t \\
& + \frac{1}{4}\ \Delta_{tt}(\tau) T_t^2 + \Delta_t(\tau) T_t g + \frac{1}{2} x^{(1)} \Delta_t(\tau) T_t T_x^{-1} \Delta_x(g) T_x \\
& + \alpha\left(t, T_t\right) \Delta_x(g) T_x = 0.
\end{aligned} \tag{26}
$$

The authors then consider the following two particular physically important models that obey the assumptions imposed on the function g.

3.2.1. Discrete Harmonic Oscillator. The discrete harmonic oscillator follows the discrete heat equation with the following potential function:

$$g\left(x,T_x\right)=k^2x^{(2)}T_x^{-2},\quad k\in R^{>0}. \tag{27}$$

Solving the set of determining equations (17) to (20) for above value of g one gets the following symmetry algebra:

$$X_{1e}=\Delta_t u\frac{\partial}{\partial u},\qquad X_{2e}=u\frac{\partial}{\partial u},$$

$$X_{3e}=E_{2k}(t)\left(\Delta_x u+x^{(1)}kT_x^{-1}u\right)\frac{\partial}{\partial u}, \tag{28}$$

$$X_{4e}=E_{-2k}(t)\left(\Delta_x u-x^{(1)}kT_x^{-1}u\right)\frac{\partial}{\partial u},$$

$$\begin{aligned}X_{5e}&=E_{4k}(t)\\&\quad\times\left(\Delta_t u+2x^{(2)}k^2T_x^{-2}u+2x^{(1)}kT_x^{-1}\Delta_x u+ku\right)\frac{\partial}{\partial u},\\X_{6e}&=E_{-4k}(t)\\&\quad\times\left(\Delta_t u+2x^{(2)}k^2T_x^{-2}u-2x^{(1)}kT_x^{-1}\Delta_x u-ku\right)\frac{\partial}{\partial u}.\end{aligned} \tag{29}$$

3.2.2. Discrete Centrifugal Barrier. Likewise, the discrete centrifugal barrier with potential function

$$g\left(x,T_x\right)=T_x x^{(-2)}T_x \tag{30}$$

has the following symmetry algebra:

$$\begin{aligned}X_{1e}&=\Delta_t u\frac{\partial}{\partial u},\qquad X_{2e}=u\frac{\partial}{\partial u},\\X_{3e}&=\left(t^{(1)}T_t^{-1}\Delta_t u+\frac{1}{2}x^{(1)}T_x^{-1}\Delta_x u\right)\frac{\partial}{\partial u},\\X_{4e}&=\left(x^{(1)}t^{(1)}T_x^{-1}T_t^{-1}\Delta_x u+t^{(2)}T_t^{-2}\Delta_t u+\frac{1}{4}x^{(2)}T_x^{-2}u\right.\\&\qquad\left.+\frac{1}{2}T_t^{-1}t^{(1)}\tau_2 u\right)\frac{\partial}{\partial u}.\end{aligned} \tag{31}$$

We consider the following case where the potential function also depends on the variables t, T_t and yet satisfies the required condition.

3.2.3. Time-Dependent Discrete Harmonic Oscillator. Now, if the potential function considered in (27) depends on all the four variables x, t, T_x, and T_t and is given by the following

$$g\left(x,t,T_x,T_t\right)=k^2x^{(2)}T_x^{-2}+k^2t^{(2)}T_t^{-2}, \tag{32}$$

Table 1: Optimal systems for free heat equation and respective invariant solutions.

Optimal systems	Invariant solutions
X_{1e}	$u(x,t)=c_1+x^{(1)}c_2$
X_{2e}	$u(x,t)=c$
X_{3e}	No invariant solution
$X_{2e}-X_{1e}$	$u(x,t)=c_1+c_2 2^{x-t}$
X_{4e}	$w(y+1)+\frac{1}{2}yw(y)=0$ (reduced form) $u=c\int_{-\infty}^{0}\frac{t^{((x^{(2)}-x^{(1)}\sigma_x)t^{(-1)}-1)}}{\ln t^{(1)}}(2x^{(1)}-\sigma_x)t^{(-1)}e^{2t}dt$
$X_{4e}+X_{1e}$	$u(x,t)=c_1+c_2\left(x^{(2)}+x^{(1)}\sigma_x\right)$

Table 2: The optimal systems for discrete harmonic oscillator and respective invariant solutions.

Optimal systems	Invariant solutions
X_{1e}	$u(x,t)=c_1+x^{(1)}c_2-k^2T_x^{-2}\left(x^{(2)}+6x^{(1)}+6\right)$
X_{2e}	No invariant solution
X_{3e}	$u(x,t)=c(k+1)^2-\frac{k^2T_x^{-2}}{k+1}\left(x^{(2)}+\frac{2x^{(1)}}{K+1}+\frac{2}{(k+1)^2}\right)$
X_{4e}	$u(x,t)=c+k^2T_x^{-2}\left(x^{(2)}+2x^{(1)}-1\right)$
X_{5e}	$u(x,t)=c_1+x^{(1)}c_2+4t^{(1)}+\left(2+k^2T_x^{-2}\right)x^{(2)}$ $+\left(8+4k^2T_x^{-2}\right)x^{(1)}+\left(2\sigma_x-1+6k^2T_x^{-2}\right)$

Table 3: The optimal systems for discrete centrifugal barrier and respective invariant solutions.

Optimal systems	Invariant solutions
X_{1e}	$u(x,t)=c_1+x^{(1)}c_2+T_x^2\left(4x^{(-1)}-x^{(-2)}-6\right)$
X_{2e}	No invariant solution
X_{3e}	$\Delta_y w+\frac{1}{2}yw(y)=T_x x^{(-2)}T_x$ (reduced form) $u(x,t)=c\int_{-\infty}^{0}e^{((t^{(2)}/x^{(2)})-1)}dt+\frac{2t^{(1)}}{x^{(2)}}T_x^{-2}x^{(-2)}$
$X_{2e}+X_{3e}$	$\Delta_y v-H(x,t)v=-T_x^2x^{(-2)}$ Where $H(x,t)=\frac{1}{2}\frac{x^{(2)}}{t^{(2)}}\sigma_t+\frac{1}{2}\frac{x^{(2)}}{t^{(1)}}+1$

then the values of coefficient functions of infinitesimal generator (14) are given by:

$$\begin{aligned}\tau&=0,\\\xi&=E_{2k}(t)\alpha_1+E_{-2k}(t)\alpha_2,\\f&=x^{(1)}kE_{2k}(t)\alpha_1T_x^{-1}-x^{(1)}kE_{-2k}(t)\alpha_2T_x^{-1}+\beta_0.\end{aligned} \tag{33}$$

Table 4: Generating solutions for free heat equation.

Generators	Infinitesimal transformations	Generating solution
X_{1e}	$G_1(x^{(1)}, t^{(1)} + \varepsilon, u)$	$u(x,t) = f\left(x^{(1)}, t^{(1)} - \varepsilon\right)$
X_{2e}	$G_2(x^{(1)} + \varepsilon, t^{(1)}, u)$	$u(x,t) = f\left(x^{(1)} - \varepsilon, t^{(1)}\right)$
X_{3e}	$G_3(x^{(1)}, t^{(1)}, ue^{\varepsilon})$	$u(x,t) = e^{\varepsilon} f\left(x^{(1)}, t^{(1)}\right)$
X_{4e}	$G_4\left(x^{(1)}e^{\varepsilon}, t^{(1)}e^{2\varepsilon}, u\right)$	$u(x,t) = f\left(x^{(1)}e^{-\varepsilon}, t^{(1)}e^{-2\varepsilon}\right)$
X_{5e}	$G_5\left(x^{(1)} + 2t^{(1)}\varepsilon, t^{(1)}, u\exp\left(-\left(x^{(1)}\varepsilon + \varepsilon^2 t^{(1)}\right)\right)\right)$	$u(x,t) = \exp\left(\varepsilon^2 t^{(1)} - x^{(1)}\varepsilon\right) f\left(x^{(1)} - 2t^{(1)}e^{\varepsilon}, t^{(1)}\right)$
X_{6e}	$G_6\left(\frac{x^{(1)}}{1-4\varepsilon t^{(1)}}, \frac{t^{(1)}}{1-4\varepsilon t^{(1)}}, u\sqrt{1-4\varepsilon t^{(1)}}\exp\left(\frac{-x^{(2)}\varepsilon + x^{(1)}\sigma_x\varepsilon}{\sqrt{1-4\varepsilon t^{(1)}}}\right)\right)$	$u(x,t) = \frac{1}{\sqrt{4\pi t^{(1)}}}\exp\left(\frac{\left(-x^{(2)} - x^{(1)}\sigma_x\right)\varepsilon}{4t^{(1)}}\right)$

And one gets 3-dimensional subalgebra of the discrete harmonic oscillator obtained in ((28) and (29)). This is generated by the following infinitesimal generators:

$$X_{1e} = u\frac{\partial}{\partial u},$$
$$X_{2e} = \left(E_{2k}(t)\Delta_x u + x^{(1)}kE_{2k}(t)T_x^{-1}u\right)\frac{\partial}{\partial u}, \quad (34)$$
$$X_{3e} = \left(E_{-2k}(t)\Delta_x u - x^{(1)}kE_{-2k}(t)T_x^{-1}u\right)\frac{\partial}{\partial u}.$$

4. One-Dimensional Optimal System and Invariant Solution

In this section one-dimensional optimal system for each of the above cases of discrete diffusion equation is calculated. It is done in analogy with the procedure to calculate optimal systems for differential equation, laid down in [4]. These optimal systems are then used to calculate invariant solutions.

4.1. Free Heat Equation. A calculation gives eight equivalence classes of one-dimensional optimal systems represented by the following generators: $\langle X_{6e}\rangle$, $\langle X_{5e}\rangle$, $\langle X_{1e} + X_{4e}\rangle$, $\langle X_{4e}\rangle$, $\langle X_{3e}\rangle$, $\langle X_{2e} - X_{1e}\rangle$, $\langle X_{2e}\rangle$, and $\langle X_{1e}\rangle$.

4.1.1. Invariant Solution. Employing the method given in [4] for calculating invariant solution for differential equations, the invariant solution corresponding to the representatives of the optimal systems given above are obtained below.

To obtain invariant solution corresponding to the subalgebra $\langle X_{2e}-X_{1e}\rangle$ we find the group invariants $u = h(x^{(1)}+t^{(1)})$ and define new variables in terms of these invariants by $y = x^{(1)} + t^{(1)}$, $u = h(y)$. This change of variables reduces the partial difference equation (21) to the ordinary 2nd order difference equation $\Delta_{yy}h - \Delta_y h = 0$. This equation is then solved to get the invariant solution $u = c_1 + c_2 2^{(x^{(1)}+t^{(1)})}$. Similarly, corresponding to X_{4e}, $u = h((x^{(2)} - x^{(1)}\sigma_x)t^{(-1)})$ is the group invariant. Substituting $y = (x^{(2)} - x^{(1)}\sigma_{(x)})t^{(-1)}$, $u = h(y)$ (21) reduces to the ordinary difference equation $\Delta_{yy}h + (1/2)y\Delta_y h = 0$. A further change of variable $\Delta_y h = w(y)$ reduces the equation to the following first-order difference equation with variable coefficient:

$$w(y+1) + \frac{1}{2}yw(y) = 0. \quad (35)$$

To solve (35) we use Laplace method given in [3]. Let $w(y) = \int_a^b t^{y-1}u(t)dt$, $yw(y) = t^y u(t)|_a^b - \int_a^b t^y Du(t)dt$, and $w(y+1) = \int_a^b t^y u(t)dt$. On substituting these values (35) becomes

$$\frac{1}{2}t^y u(t)|_a^b + \int_a^b t^y\left(u(t) - \frac{1}{2}Du(t)\right)dt = 0. \quad (36)$$

Now there are two possibilities either $u(t) - (1/2)Du(t) = 0$ and this gives $u(t) = ce^{2t}$ or $ct^y e^{2t} = 0$; therefore, $t \to 0$ or $t \to -\infty$. So, $w(y) = c\int_{-\infty}^0 t^{y-1}e^{2t}dt$, and one gets the following invariant solution:

$$u = c\int_{-\infty}^0 \frac{t^{((x^{(2)}-x^{(1)}\sigma_x)t^{(-1)}-1)}}{\ln t}\left(2x^{(1)} - \sigma_x\right)t^{(-1)}e^{2t}dt. \quad (37)$$

Similar calculations give invariant solutions corresponding to other representatives of optimal classes. The results are summarized in Table 1.

4.2. Discrete Harmonic Oscillators and Centrifugal Barrier. For the discrete harmonic oscillator and the discrete centrifugal barrier the representatives of their one-dimensional optimal systems and the corresponding invariant solutions are summarized in Tables 2 and 3.

Remark 1. The solution corresponding to X_{3e} is obtained by applying Laplace method in reduced form.

Since, in case of time-dependent harmonic oscillator, the symmetry algebra is a subalgebra of the time-independent case therefore one does not get new invariant solutions.

5. Generating Solutions

Since a symmetry of a given equation maps its solution space onto itself, so, one can find new solutions by applying symmetry transformations to a known solution of the

Table 5: Generating solutions for discrete harmonic oscillator.

Generators	Infinitesimal transformations	Generating solution
X_{1e}	$G_1(x^{(1)}, t^{(1)} + \varepsilon, u)$	$u(x,t) = f\left(x^{(1)}, t^{(1)} - \varepsilon\right)$
X_{2e}	$G_2(x^{(1)}, t^{(1)}, ue^{\varepsilon})$	$u(x,t) = e^{\varepsilon} f\left(x^{(1)}, t^{(1)}\right)$
X_{3e}	$G_3\left(x^{(1)} + \varepsilon, t^{(1)}, u\exp\left(x^{(1)}k\varepsilon + \frac{k\varepsilon^2}{2}\right)\right)$	$u(x,t) = \exp\left(x^{(1)}k\varepsilon - \frac{k\varepsilon^2}{2}\right) f\left(x^{(1)} - \varepsilon, t^{(1)}\right)$
X_{4e}	$G_4\left(x^{(1)} + \varepsilon, t^{(1)}, u\exp\left(-x^{(1)}k\varepsilon - \frac{k\varepsilon^2}{2}\right)\right)$	$u(x,t) = \exp\left(-x^{(1)}k\varepsilon + \frac{k\varepsilon^2}{2}\right) f\left(x^{(1)} - \varepsilon, t^{(1)}\right)$
X_{5e}	$G_5\left(x^{(1)}e^{\varepsilon}, t^{(1)} + \varepsilon, u\exp\left(\exp\left(\frac{1}{2}(x^{(2)} - x^{(1)}\sigma_x)e^{4\varepsilon} + \varepsilon\right)\right)\right)$, $k = 1$	$u(x,t) = \exp\left(\exp\left(\frac{1}{2}(x^{(2)} - x^{(1)}\sigma_x) + \varepsilon\right)\right) f\left(x^{(1)}e^{2\varepsilon}, t^{(1)} - \varepsilon\right)$
X_{6e}	$G_5\left(x^{(1)}e^{-2\varepsilon}, t^{(1)} + \varepsilon, u\exp\left(\exp\left(\frac{1}{2}(x^{(2)} - x^{(1)}\sigma_x)e^{4\varepsilon} - \varepsilon\right)\right)\right)$, $k = 1$	$u(x,t) = \exp\left(\exp - \left(\frac{1}{2}\left(x^{(2)} - x^{(1)}\sigma_x\right)e^{8\varepsilon} - \varepsilon\right)\right) f\left(x^{(1)}e^{2\varepsilon}, t^{(1)} - \varepsilon\right)$

Table 6: Generating solutions for discrete centrifugal barrier.

Generators	Infinitesimal transformations	Generating solution
X_{1e}	$G_1(x^{(1)}, t^{(1)} + \varepsilon, u)$	$u(x,t) = f\left(x^{(1)}, t^{(1)} - \varepsilon\right)$
X_{2e}	$G_2(x^{(1)}, t^{(1)}t, ue^{\varepsilon})$	$u(x,t) = e^{\varepsilon} f\left(x^{(1)}, t^{(1)}\right)$
X_{3e}	$G_3\left(x^{(1)}e^{\varepsilon}, t^{(1)}e^{2\varepsilon}, u\right)$	$u(x,t) = f\left(x^{(1)}e^{-\varepsilon}, t^{(1)}e^{-2\varepsilon}\right)$
X_{4e}	$G_6\left(\frac{x^{(1)}}{1 - 4\varepsilon t^{(1)}}, \frac{t^{(1)}}{1 - 4\varepsilon t^{(1)}}, \frac{u}{\sqrt{1 - 4\varepsilon t^{(1)}}}\exp\left(\frac{\left(x^{(2)} - x^{(1)}\sigma_x\right)\varepsilon}{1 - 4\varepsilon t^{(1)}}\right)\right)$	$u(x,t) = \sqrt{1 + 4\varepsilon t^{(1)}}\exp\left(\frac{\left(x^{(2)} - x^{(1)}\sigma_x\right)\varepsilon}{1 + 4\varepsilon t^{(1)}}\right) f\left(\frac{x}{1 + 4\varepsilon t^{(1)}}, \frac{t}{1 + 4\varepsilon t^{(1)}}\right)$

equation. The solution thus obtained is called a generating solution. Here, these generating solutions are obtained by first obtaining the one parameter transformation corresponding to each infinitesimal generators obtained in Section 2. These solutions are obtained in analogy with the method given in [4] for differential equations. The results are summarized in the form of Tables 4, 5, and 6.

6. Conclusion

In this paper, in analogy with differential equations, the symmetry analysis of a nonhomogeneous discrete heat equation (13) has been carried out. The one-dimensional optimal systems are obtained for the equation with different source terms by using symmetry algebra given in [6]. These optimal systems are then used to obtain invariant solutions if they exist. Moreover, one parameter group of transformations corresponding to all infinitesimal generators of the symmetry algebra has been calculated to obtain generating solutions of the equation. Main results are summarized in Tables 1, 2, 3, 4, 5, and 6. Working on the same lines one can calculate conservation laws which we will report soon.

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