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Research Article

Existence of Nonoscillatory Solutions for a Third-Order Nonlinear Neutral Delay Differential Equation

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The aim of this paper is to study the solvability of a third-order nonlinear neutral delay differential equation of the form $\{\alpha(t)[\beta(t)(x(t)+p(t)x(t-\tau))']'\}' + f(t, x(\sigma_1(t)), x(\sigma_2(t)), \ldots, x(\sigma_n(t))) = 0$, $t \geq t_0$. By using the Krasnoselskii's fixed point theorem and the Schauder's fixed point theorem, we demonstrate the existence of uncountably many bounded nonoscillatory solutions for the above differential equation. Several nontrivial examples are given to illustrate our results.

1. Introduction and Preliminaries

In recent years, the study of the oscillation, nonoscillation, asymptotic behaviors and existence of solutions for various kinds of first- and second-order neutral delay differential equations and systems of differential equations have attracted much attention, for example, see [1–12] and the references therein. Dorociaková and Olach [2] discussed the existence of nonoscillatory solutions and asymptotic behaviors for the first-order delay differential equation

$$x'(t) + p(t)x(t) + q(t)x(\tau(t)) = 0, \quad t \geq 0. \tag{1.1}$$

Elbert [3] and Huang [5] established a few oscillation and nonoscillation criteria for the second-order linear differential equation

$$x''(t) + q(t)x(t) = 0, \quad t \geq 0, \tag{1.2}$$

where $q \in C([0,+\infty),\mathbb{R}^+)$. Tang and Liu [10] studied the existence of bounded oscillation for the second-order linear delay differential equation of unstable type

$$x''(t) = p(t)x(t-\tau), \quad t \geq t_0, \tag{1.3}$$

where $\tau > 0$, $p \in C([t_0,+\infty),\mathbb{R}^+)$ and $p(t) \not\equiv 0$ on any interval of length τ. In view of the Banach fixed point theorem, Kulenović and Hadžiomerspahić [7] deduced the existence of a nonoscillatory solution for the second-order linear neutral delay differential equation with positive and negative coefficients

$$[x(t) + cx(t-\tau)]'' + Q_1(t)x(t-\sigma_1) - Q_2(t)x(t-\sigma_2) = 0, \quad t \geq t_0, \tag{1.4}$$

where $c \in \mathbb{R}\backslash\{-1,1\}$, $\tau > 0$, $\sigma_1,\sigma_2 \in [0,+\infty)$, $Q_1,Q_2 \in C([t_0,+\infty),\mathbb{R}^+)$. Qin et al. [9] and Yang et al. [11] developed several oscillation criteria for the second-order differential equation

$$\left[r(t)\left(x(t) + p(t)x(t-\tau)\right)'\right]' + q(t)f(x(t-\delta)) = 0, \quad t \geq t_0, \tag{1.5}$$

where τ and δ are nonnegative constants, $r,p,q \in C([t_0,+\infty),\mathbb{R})$, and $f \in C(\mathbb{R},\mathbb{R})$. By utilizing the Krasnoselskii's fixed point theorem, Zhou [12] discussed the existence of nonoscillatory solutions of the second-order nonlinear neutral differential equation

$$\left[r(t)\left(x(t) + p(t)x(t-\tau)\right)'\right]' + \sum_{i=1}^{m} Q_i(t)f(x_i(t-\sigma_i)) = 0, \quad t \geq t_0, \tag{1.6}$$

where $m \geq 1$ is an integer, $\tau > 0$, $\sigma_i \geq 0$, $r,p,Q_i \in C([t_0,+\infty),\mathbb{R})$, and $f_i \in C(\mathbb{R},\mathbb{R})$ for $i \in \{1,2,\ldots,m\}$.

However, the existence of nonoscillatory solutions of third-order neutral differential equations received much less attention, moreover, the results in [7, 11, 12] only figured out the existence of a nonoscillatory solution of (1.3)–(1.5), respectively.

Motivated by the papers mentioned above, in this paper, we investigate the following third-order nonlinear neutral delay differential equation

$$\left\{\alpha(t)\left[\beta(t)\left(x(t) + p(t)x(t-\tau)\right)'\right]'\right\}' + f(t, x(\sigma_1(t)), x(\sigma_2(t)), \ldots, x(\sigma_n(t))) = 0, \quad t \geq t_0, \tag{1.7}$$

where $n \geq 1$ is an integer, $\tau > 0$, α, $\beta \in C([t_0,+\infty), \mathbb{R}^+ \setminus \{0\})$, $p \in C([t_0,+\infty),\mathbb{R})$, and $f \in C([t_0,+\infty) \times \mathbb{R}^n,\mathbb{R})$. By applying the Krasnoselskii's fixed point theorem and the Schauder's fixed point theorem, we obtain some sufficient conditions for the existence of uncountably many bounded nonoscillatory solutions of (1.7).

Throughout this paper, we assume that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $C([t_0, +\infty), \mathbb{R})$ denotes the Banach space of all continuous and bounded functions on $[t_0, +\infty)$ with the norm $\|x\| = \sup_{t \geq t_0} |x(t)|$ for each $x \in C([t_0, +\infty), \mathbb{R})$ and

$$A(N, M) = \{x \in C([t_0, +\infty), \mathbb{R}) : N \leq x(t) \leq M,\ t \geq t_0\} \quad \text{for } M > N > 0. \tag{1.8}$$

It is easy to see that $A(N, M)$ is a bounded closed and convex subset of $C([t_0, +\infty), \mathbb{R})$.

By a solution of (1.7), we mean a function $x \in C([t_1 - \tau, +\infty), \mathbb{R})$ for some $t_1 \geq t_0$ such that $x(t) + p(t)x(t - \tau)$, $\beta(t)(x(t) + p(t)x(t - \tau))'$ and $\alpha(t)[\beta(t)(x(t) + p(t)x(t - \tau))']'$ are continuously differentiable in $[t_1, +\infty)$ and such that (1.7) is satisfied for $t \geq t_1$. As is customary, a solution of (1.7) is called oscillatory if it has arbitrarily large zeros, and otherwise it is said to be *nonoscillatory*.

Definition 1.1 (see [6]). A family F of functions in $C([t_0, +\infty), \mathbb{R})$ is *equicontinuous* on $[t_0, +\infty)$ if for any $\varepsilon > 0$, the interval $[t_0, +\infty)$ can be decomposed into a finite number of subintervals $I_1, I_2, \ldots, I_n$ such that

$$|f(x) - f(y)| \leq \varepsilon, \quad \forall f \in F, x, y \in I_i, i \in \{1, 2, \ldots, n\}. \tag{1.9}$$

Lemma 1.2 (see Krasnoselskii's fixed point theorem, [4]). *Let X be a Banach space. Let Ω be a bounded closed convex subset of X and S_1 and S_2 mappings from Ω into X such that $S_1 x + S_2 y \in \Omega$ for every pair $x, y \in \Omega$. If S_1 is a contraction and S_2 is completely continuous, then the equation $S_1 x + S_2 x = x$ has at least one solution in Ω.*

Lemma 1.3 (see Schauder's fixed point theorem, [4]). *Let Ω be a nonempty closed convex subset of a Banach space X. Let $S : \Omega \to \Omega$ be a continuous mapping such that $S\Omega$ is a relatively compact subset of X. Then S has at least one fixed point in Ω.*

2. Main Results

Now we study those conditions under which (1.7) possesses uncountably many bounded nonoscillatory solutions.

Theorem 2.1. *Assume that there exist constants M, N, c_1, c_2, T_0 and a function $h \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying*

$$\min\{c_1, c_2\} \geq 0, \quad c_1 + c_2 < 1, \quad 0 < N < (1 - c_1 - c_2)M, \tag{2.1}$$

$$-c_2 \leq p(t) \leq c_1, \quad t \geq T_0 > t_0; \tag{2.2}$$

$$|f(t, u_1, u_2, \ldots, u_n)| \leq h(t), \quad t \in [t_0, +\infty), \qquad u_i \in [N, M], \quad i \in \{1, \ldots, n\}; \tag{2.3}$$

$$\int_{t_0}^{+\infty} \int_{v}^{+\infty} \int_{s}^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv < +\infty. \tag{2.4}$$

Then (1.7) *possesses uncountably many bounded nonoscillatory solutions in* $A(N, M)$.

Proof. Set $k \in (c_1 M + N, (1-c_2)M)$. From (2.4), we pick up $T > T_0$ such that

$$\int_T^{+\infty}\int_v^{+\infty}\int_s^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv < \min\{(1-c_2)M-k, k-c_1M-N\}. \tag{2.5}$$

Define two mappings S_{1k} and $S_{2k} : A(N,M) \to C([t_0,+\infty),\mathbb{R})$ by

$$(S_{1k}x)(t) = \begin{cases} k - p(t)x(t-\tau), & t \ge T, \\ (S_{1k}x)(T), & t_0 \le t < T, \end{cases} \tag{2.6}$$

$$(S_{2k}x)(t) = \begin{cases} \displaystyle\int_t^{+\infty}\int_v^{+\infty}\int_s^{+\infty}\frac{1}{\alpha(s)\beta(v)}f(u,x(\sigma_1(u)),\ldots,x(\sigma_n(u)))du\,ds\,dv, & t \ge T, \\ (S_{2k}x)(T), & t_0 \le t < T \end{cases} \tag{2.7}$$

for $x \in A(N,M)$.

Firstly, we prove that $S_{1k}(A(N,M)) + S_{2k}(A(N,M)) \subseteq A(N,M)$. By (2.1)–(2.3) and (2.5)–(2.7), we get that

$$\begin{aligned}(S_{1k}x)(t) + (S_{2k}y)(t) &\le k + c_2M + \int_T^{+\infty}\int_v^{+\infty}\int_s^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv \\ &\le k + c_2M + \min\{(1-c_2)M-k, k-c_1M-N\} \\ &\le M, \quad x,y \in A(N,M),\ t \ge T, \\ (S_{1k}x)(t) + (S_{2k}y)(t) &\ge k - c_1M - \int_T^{+\infty}\int_v^{+\infty}\int_s^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv \\ &\ge k - c_1M - \min\{(1-c_2)M-k, k-c_1M-N\} \\ &\ge N, \quad x,y \in A(N,M),\ t \ge T,\end{aligned} \tag{2.8}$$

which infer that $S_{1k}(A(N,M)) + S_{2k}(A(N,M)) \subseteq A(N,M)$ for any $x, y \in A(N,M)$.

Secondly, we show that S_{1k} is a contraction mapping. By (2.1), (2.2), and (2.6), we deduce that

$$\left|(S_{1k}x)(t) - (S_{1k}y)(t)\right| \le |p(t)|\left|x(t-\tau) - y(t-\tau)\right| \le (c_1+c_2)\|x-y\| \tag{2.9}$$

for $x, y \in A(N,M)$ and $t \ge T$, which gives that

$$\|S_{1k}x - S_{1k}y\| \le (c_1+c_2)\|x-y\|. \tag{2.10}$$

Thirdly, we show that S_{2k} is completely continuous. Let $x \in A(N,M)$ and $\{x_m\}_{m\geq 1} \subseteq A(N,M)$ with $\lim_{m\to+\infty} x_m = x$. By (2.7), we obtain that

$$\begin{aligned}
&|(S_{2k}x_m)(t) - (S_{2k}x)(t)| \\
&\quad \leq \int_T^{+\infty}\int_v^{+\infty}\int_s^{+\infty} \frac{1}{\alpha(s)\beta(v)} \big| f(u, x_m(\sigma_1(u)), \dots, x_m(\sigma_n(u))) \\
&\qquad\qquad - f(u, x(\sigma_1(u)), \dots, x(\sigma_n(u)))\big| du\, ds\, dv, \\
&\qquad\qquad t \geq T,\ m \geq 1.
\end{aligned} \tag{2.11}$$

Using (2.3) and (2.4), we conclude that

$$\begin{aligned}
&\big| f(u, x_m(\sigma_1(u)), \dots, x_m(\sigma_n(u))) - f(u, x(\sigma_1(u)), \dots, x(\sigma_n(u)))\big| \leq 2h(u), \\
&\qquad u \in [T, +\infty),\ m \geq 1, \\
&\int_s^{+\infty} \big| f(u, x_m(\sigma_1(u)), \dots, x_m(\sigma_n(u))) - f(u, x(\sigma_1(u)), \dots, x(\sigma_n(u)))\big| du \leq 2\int_s^{+\infty} h(u)du, \\
&\qquad s \in [T, \infty),\ m \geq 1, \\
&\int_v^{+\infty}\int_s^{+\infty} \frac{1}{\alpha(s)} \big| f(u, x_m(\sigma_1(u)), \dots, x_m(\sigma_n(u))) - f(u, x(\sigma_1(u)), \dots, x(\sigma_n(u)))\big| du\, ds \\
&\quad \leq 2\int_v^{+\infty}\int_s^{+\infty} \frac{h(u)}{\alpha(s)} du\, ds, \quad v, s \in [T, \infty),\ m \geq 1.
\end{aligned} \tag{2.12}$$

In light of (2.11)–(2.12),

$$\big| f(u, x_m(\sigma_1(u)), \dots, x_m(\sigma_n(u))) - f(u, x(\sigma_1(u)), \dots, x(\sigma_n(u)))\big| \longrightarrow 0 \quad \text{as } m \longrightarrow +\infty, \tag{2.13}$$

and the Lebesgue dominated convergence theorem, we conclude that

$$\lim_{m\to+\infty} \|S_{2k}x_m - S_{2k}x\| = 0, \tag{2.14}$$

which means that S_{2k} is continuous in $A(N,M)$.

Now we show that S_{2k} is completely continuous. By virtue of (2.3), (2.4), and (2.7), we get that

$$\|S_{2k}x\| \leq \int_T^{+\infty}\int_v^{+\infty}\int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv, \quad x \in A(N,M). \tag{2.15}$$

That is, $S_{2k}(A(N,M))$ is uniform bounded. It follows from (2.4) that for each $\varepsilon > 0$, there exists $T^* > T$ such that

$$\int_{T^*}^{+\infty}\int_{v}^{+\infty}\int_{s}^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv < \frac{\varepsilon}{2}. \tag{2.16}$$

In view of (2.3), (2.7), and (2.16), we infer that

$$\begin{aligned}
&|(S_{2k}x)(t_2) - (S_{2k}x)(t_1)| \\
&\quad \le \int_{t_2}^{+\infty}\int_{v}^{+\infty}\int_{s}^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv + \int_{t_1}^{+\infty}\int_{v}^{+\infty}\int_{s}^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv \\
&\quad < \varepsilon, \quad x \in A(N,M), t_2 > t_1 \ge T^*.
\end{aligned} \tag{2.17}$$

From (2.3) and (2.7), we get that

$$\begin{aligned}
&|(S_{2k}x)(t_2) - (S_{2k}x)(t_1)| \\
&\quad \le \int_{t_1}^{t_2}\int_{v}^{+\infty}\int_{s}^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv, \quad x \in A(N,M),\ T \le t_1 \le t_2 \le T^*,
\end{aligned} \tag{2.18}$$

which together with (2.4) ensures that there exists $\delta > 0$ satisfying

$$|(S_{2k}x)(t_2) - (S_{2k}x)(t_1)| < \varepsilon, \quad x \in A(N,M),\ t_1, t_2 \in [T, T^*] \text{ with } |t_2 - t_1| < \delta. \tag{2.19}$$

Clearly,

$$|(S_{2k}x)(t_2) - (S_{2k}x)(t_1)| = 0 < \varepsilon, \quad x \in A(N,M),\ t_0 \le t_1 \le t_2 \le T. \tag{2.20}$$

That is, $S_{2k}(A(N,M))$ is equicontinuous on $[t_0, +\infty)$. Consequently, S_{2k} is completely continuous. By Lemma 1.2, there is $x_0 \in A(N,M)$ such that $S_1x_0 + S_2x_0 = x_0$, which is a bounded nonoscillatory solution of (1.7).

Lastly, we demonstrate that (1.7) possesses uncountably many bounded nonoscillatory solutions in $A(N,M)$. Let $k_1, k_2 \in (c_1M + N, (1 - c_2)M)$ and $k_1 \ne k_2$. For each $j \in \{1,2\}$, we choose $T_j > T_0$ and the mappings S_{jk_1} and S_{jk_2} satisfying (2.5)–(2.7) with k and T replaced by k_j and T_j, respectively, and some $T_3 > \max\{T_1, T_2\}$ such that

$$\int_{T_3}^{+\infty}\int_{v}^{+\infty}\int_{s}^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv < \frac{|k_1 - k_2|}{2}. \tag{2.21}$$

Obviously, there are $x, y \in A(N,M)$ such that $S_{1k_1}x + S_{2k_1}x = x$ and $S_{1k_2}y + S_{2k_2}y = y$, respectively. That is, x and y are two bounded nonoscillatory solutions of (1.7) in $A(N,M)$.

In order to prove that (1.7) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$, we prove only that $x \neq y$. In fact, by (2.6) and (2.7), we gain that for $t \geq T_3$

$$x(t) = k_1 - p(t)x(t-\tau) + \int_t^{+\infty}\int_v^{+\infty}\int_s^{+\infty} \frac{1}{\alpha(s)\beta(v)} f(u, x(\sigma_1(u)), \ldots, x(\sigma_n(u)))du\, ds\, dv,$$
$$y(t) = k_2 - p(t)y(t-\tau) + \int_t^{+\infty}\int_v^{+\infty}\int_s^{+\infty} \frac{1}{\alpha(s)\beta(v)} f(u, y(\sigma_1(u)), \ldots, y(\sigma_n(u)))du\, ds\, dv, \tag{2.22}$$

which together with (2.1)–(2.3) imply that

$$\begin{aligned}\left|x(t) - y(t)\right| &\geq |k_1 - k_2| - \left|p(t)\right|\left|x(t-\tau) - y(t-\tau)\right| - \int_t^{+\infty}\int_v^{+\infty}\int_s^{+\infty} \frac{1}{\alpha(s)\beta(v)} \\ &\quad \times \left|f(u, x(\sigma_1(u)), \ldots, x(\sigma_n(u))) - f(u, y(\sigma_1(u)), \ldots, y(\sigma_n(u)))\right| du\, ds\, dv \\ &\geq |k_1 - k_2| - (c_1 + c_2)\|x - y\| \\ &\quad - 2\int_{T_3}^{+\infty}\int_v^{+\infty}\int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv, \quad t \geq T_3,\end{aligned} \tag{2.23}$$

which means that

$$\|x - y\| \geq \frac{1}{1 + c_1 + c_2}\left(|k_1 - k_2| - 2\int_{T_3}^{+\infty}\int_v^{+\infty}\int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv\right) > 0, \tag{2.24}$$

by (2.1) and (2.21). That is, $x \neq y$. This completes the proof. □

Theorem 2.2. *Assume that there exist constants M, N, c_1, c_2, T_0 and a function $h \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying* (2.3), (2.4) *and*

$$0 < (1 - c_2)N < (1 - c_1)M; \tag{2.25}$$

$$0 \leq c_2 \leq p(t) \leq c_1 < 1, \quad t \geq T_0 > t_0. \tag{2.26}$$

Then (1.7) *possesses uncountably many bounded nonoscillatory solutions in* $A(N, M)$.

Proof. Set $k \in (c_1 M + N, M + c_2 N)$. It follows from (2.4) that there exists $T > T_0$ such that

$$\int_T^{+\infty}\int_v^{+\infty}\int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv < \min\{M + c_2 N - k, k - c_1 M - N\}. \tag{2.27}$$

Define two mappings S_{1k} and $S_{2k} : A(N, M) \to C([t_0, +\infty), \mathbb{R})$ by (2.6) and (2.7), respectively. By virtue of (2.3), (2.6), (2.7), (2.26), and (2.27), we obtain that

$$\begin{aligned}(S_{1k}x)(t) + (S_{2k}y)(t) &\leq k - p(t)x(t-\tau) + \int_T^{+\infty}\int_v^{+\infty}\int_s^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv \\ &\leq k - c_2N + \min\{M + c_2N - k, k - c_1M - N\} \\ &\leq M, \quad x, y \in A(N, M),\ t \geq T,\end{aligned} \tag{2.28}$$

$$\begin{aligned}(S_{1k}x)(t) + (S_{2k}y)(t) &\geq k - p(t)x(t-\tau) - \int_T^{+\infty}\int_v^{+\infty}\int_s^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv \\ &\geq k - c_1M - \min\{M + c_2N - k, k - c_1M - N\} \\ &\geq N, \quad x, y \in A(N, M),\ t \geq T,\end{aligned} \tag{2.29}$$

which yield that $S_{1k}(A(N, M)) + S_{2k}(A(N, M)) \subseteq A(N, M)$.

By a similar argument used in the proof of Theorem 2.1, we gain that S_{1k} is a contraction mapping S_{2k} is completely continuous, and (1.7) possesses uncountably many nonoscillatory solutions. This completes the proof. □

Theorem 2.3. *Assume that there exist constants M, N, c_1, c_2, T_0 and a function $h \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying* (2.3), (2.4), (2.25), *and*

$$-1 < -c_1 \leq p(t) \leq -c_2 \leq 0, \quad t \geq T_0 > t_0. \tag{2.30}$$

Then (1.7) *possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$.*

Proof. Set $k \in ((1 - c_2)N, (1 - c_1)M)$. It follows from (2.4) that there exists $T > T_0$ satisfying

$$\int_T^{+\infty}\int_v^{+\infty}\int_s^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv < \min\{(1 - c_1)M - k, k - (1 - c_2)N\}. \tag{2.31}$$

Let S_{1k} and $S_{2k} : A(N, M) \to C([t_0, +\infty), \mathbb{R})$ be defined by (2.6) and (2.7), respectively. In view of (2.3), (2.6), (2.7), (2.30), and (2.31), we obtain that

$$\begin{aligned}(S_1x)(t) + (S_2y)(t) &\leq k - p(t)x(t-\tau) + \int_T^{+\infty}\int_v^{+\infty}\int_s^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv \\ &\leq k + c_1M + \min\{(1 - c_1)M - k, k - (1 - c_2)N\} \\ &\leq M, \quad x, y \in A(N, M),\ t \geq T, \\ (S_1x)(t) + (S_2y)(t) &\geq k - p(t)x(t-\tau) - \int_T^{+\infty}\int_v^{+\infty}\int_s^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv \\ &\geq k + c_2N - \min\{(1 - c_1)M - k, k - (1 - c_2)N\} \\ &\geq N, \quad x, y \in A(N, M),\ t \geq T,\end{aligned} \tag{2.32}$$

which mean that $S_{1k}(A(N, M)) + S_{2k}(A(N, M)) \subseteq A(N, M)$.

The rest of the proof is similar to the proof of Theorem 2.1 and is omitted. This completes the proof. □

Theorem 2.4. *Assume that there exist constants M and N with $M > N > 0$ and a function $h \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying* (2.3) *and* (2.4)*. If $p(t) = 1$ for each $t \in [t_0, +\infty)$, then* (1.7) *possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$.*

Proof. Set $k \in (N, M)$. It follows from (2.4) that there exists $T \geq t_0 + \tau$ satisfying

$$\int_T^{+\infty} \int_v^{+\infty} \int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv < \min\{M - k, k - N\}. \tag{2.33}$$

Define a mapping $S_k : A(N, M) \to C([t_0, +\infty), \mathbb{R})$ by

$$(S_k x)(t) = \begin{cases} k + \displaystyle\sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} \int_v^{+\infty} \int_s^{+\infty} \frac{1}{\alpha(s)\beta(v)} f(u, x(\sigma_1(u)), \ldots, x(\sigma_n(u))) du\, ds\, dv, & t \geq T, \\ (S_k x)(T), & t_0 \leq t < T \end{cases} \tag{2.34}$$

for $x \in A(N, M)$.

Firstly, we prove that $S_k(A(N, M)) \subseteq A(N, M)$. By virtue of (2.3), (2.33) and (2.34), we obtain that

$$\begin{aligned} (S_k x)(t) &\leq k + \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} \int_v^{+\infty} \int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv \\ &\leq k + \int_T^{+\infty} \int_v^{+\infty} \int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv \\ &\leq M, \quad x \in A(N, M),\ t \geq T, \\ (S_k x)(t) &\geq k - \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} \int_v^{+\infty} \int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv \\ &\geq k - \int_T^{+\infty} \int_v^{+\infty} \int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv \\ &\geq N, \quad x \in A(N, M),\ t \geq T, \end{aligned} \tag{2.35}$$

which imply that $S_k(A(N, M)) \subseteq A(N, M)$.

Secondly, we show that S_k is continuous in $A(N,M)$. Let $x \in A(N,M)$ and $\{x_m\}_{m\geq 1} \subseteq A(N,M)$ with $\lim_{m\to+\infty} x_m = x$. By (2.34), we get that

$$\begin{aligned}
&|(S_k x_m)(t) - (S_k x)(t)| \\
&\quad \leq \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} \int_v^{+\infty} \int_s^{+\infty} \frac{1}{\alpha(s)\beta(v)} \big| f(u, x_m(\sigma_1(u)), \ldots, x_m(\sigma_n(u))) \\
&\qquad - f(u, x(\sigma_1(u)), \ldots, x(\sigma_n(u))) \big| du\, ds\, dv \\
&\quad \leq \int_T^{+\infty} \int_v^{+\infty} \int_s^{+\infty} \frac{1}{\alpha(s)\beta(v)} \big| f(u, x_m(\sigma_1(u)), \ldots, x_m(\sigma_n(u))) \\
&\qquad - f(u, x(\sigma_1(u)), \ldots, x(\sigma_n(u))) \big| du\, ds\, dv, \quad t \geq T,\ m \geq 1.
\end{aligned} \tag{2.36}$$

In view of (2.12), (2.13), (2.36), and the Lebesgue dominated convergence theorem, we deduce that

$$\lim_{m\to+\infty} \|S_k x_m - S_k x\| = 0, \tag{2.37}$$

which means that S_k is continuous in $A(N,M)$.

Thirdly, we show that $S_k(A(N,M))$ is relatively compact. From (2.3), (2.33), and (2.34), we gain that

$$\|S_k x\| \leq k + \int_T^{+\infty} \int_v^{+\infty} \int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv \leq 2M, \quad x \in A(N,M), \tag{2.38}$$

which means that $S_k(A(N,M))$ is uniform bounded.

Let $\varepsilon > 0$. It follows from (2.4) that there exist $\overline{T} > T^* > T$ such that

$$\int_{T^*}^{+\infty} \int_v^{+\infty} \int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv < \frac{\varepsilon}{2}, \tag{2.39}$$

$$\int_{\overline{T}}^{+\infty} \int_v^{+\infty} \int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv < \frac{\varepsilon}{4}. \tag{2.40}$$

By (2.3), (2.34), and (2.39), we deduce that

$$\begin{aligned}
|(S_k x)(t_2) - (S_k x)(t_1)| &\leq \sum_{i=1}^{\infty} \left(\int_{t_2+(2i-1)\tau}^{t_2+2i\tau} + \int_{t_1+(2i-1)\tau}^{t_1+2i\tau} \right) \int_v^{+\infty} \int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv \\
&\leq 2 \int_{T^*}^{+\infty} \int_v^{+\infty} \int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv \\
&< \varepsilon, \quad x \in A(N,M),\ t_2 > t_1 \geq T^*.
\end{aligned} \tag{2.41}$$

Choose an integer $z \ge 1$ with $T + (2z+1)\tau \ge \overline{T}$, and put

$$A = \max\left\{\int_v^{+\infty}\int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds : v \in [T+\tau, T^* + (2z-1)\tau]\right\},$$
$$B = \max\left\{\int_v^{+\infty}\int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds : v \in [T+2\tau, T^* + 2z\tau]\right\},$$
$$\delta = \min\left\{\frac{\varepsilon}{1+4Az}, \frac{\varepsilon}{1+4Bz}\right\}. \tag{2.42}$$

Equation (2.4) means that $\max\{A,B\} < +\infty$. It follows from (2.3), (2.34), and (2.40)–(2.42) that for $x \in A(N,M)$, $t_1, t_2 \in [T, T^*]$ with $t_1 \le t_2 \le t_1 + \delta$

$$\begin{aligned}
&|(S_k x)(t_2) - (S_k x)(t_1)| \\
&\quad \le \left|\sum_{i=1}^{z}\left(\int_{t_2+(2i-1)\tau}^{t_2+2i\tau} - \int_{t_1+(2i-1)\tau}^{t_1+2i\tau}\right)\right. \\
&\qquad \left.\times \int_v^{+\infty}\int_s^{+\infty} \frac{1}{\alpha(s)\beta(v)} f(u, x(\sigma_1(u)), \dots, x(\sigma_n(u)))du\, ds\, dv\right| \\
&\qquad + \sum_{i=z+1}^{\infty}\left(\int_{t_2+(2i-1)\tau}^{t_2+2i\tau} + \int_{t_1+(2i-1)\tau}^{t_1+2i\tau}\right)\int_v^{+\infty}\int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv \\
&\quad \le \left|\sum_{i=1}^{z}\left(\int_{t_2+(2i-1)\tau}^{t_1+(2i-1)\tau} + \int_{t_1+(2i-1)\tau}^{t_2+2i\tau} + \int_{t_1+2i\tau}^{t_1+(2i-1)\tau}\right)\right. \\
&\qquad \left.\times \int_v^{+\infty}\int_s^{+\infty} \frac{1}{\alpha(s)\beta(v)} f(u, x(\sigma_1(u)), \dots, x(\sigma_n(u)))du\, ds\, dv\right| \\
&\qquad + 2\int_{\overline{T}}^{+\infty}\int_v^{+\infty}\int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv \\
&\quad \le \sum_{i=1}^{z}\left(\int_{t_1+(2i-1)\tau}^{t_2+(2i-1)\tau} + \int_{t_1+2i\tau}^{t_2+2i\tau}\right)\int_v^{+\infty}\int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv + \frac{\varepsilon}{2} \\
&\quad \le Az(t_2 - t_1) + Bz(t_2 - t_1) + \frac{\varepsilon}{2} \\
&\quad < \varepsilon.
\end{aligned} \tag{2.43}$$

It is not difficult to verify that

$$|(S_k x)(t_2) - (S_k x)(t_1)| = 0 < \varepsilon, \quad x \in A(N,M),\ t_0 \le t_1 \le t_2 \le T. \tag{2.44}$$

Therefore $S_k(A(N,M))$ is equicontinuous on $[t_0,+\infty)$, and consequently S_k is relatively compact. By Lemma 1.3, there is $x_0 \in A(N,M)$ such that $Sx_0 = x_0$, which together with (2.34) yields that for $t \geq T+\tau$

$$x_0(t) = k + \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} \int_{v}^{+\infty} \int_{s}^{+\infty} \frac{1}{\alpha(s)\beta(v)} f(u, x_0(\sigma_1(u)), \ldots, x_0(\sigma_n(u)))du\, ds\, dv,$$
$$x_0(t-\tau) = k + \sum_{i=1}^{\infty} \int_{t+2(i-1)\tau}^{t+(2i-1)\tau} \int_{v}^{+\infty} \int_{s}^{+\infty} \frac{1}{\alpha(s)\beta(v)} f(u, x_0(\sigma_1(u)), \ldots, x_0(\sigma_n(u)))du\, ds\, dv, \tag{2.45}$$

which mean that

$$x_0(t) + x_0(t-\tau) = 2k + \int_{t}^{+\infty} \int_{v}^{+\infty} \int_{s}^{+\infty} \frac{1}{\alpha(s)\beta(v)} f(u, x_0(\sigma_1(u)), \ldots, x_0(\sigma_n(u)))du\, ds\, dv. \tag{2.46}$$

It is easy to show that x_0 is a bounded nonoscillatory solution of (1.7).

Finally, we demonstrate that (1.7) possesses uncountably many bounded nonoscillatory solutions in $A(N,M)$. Let $k_1, k_2 \in (N,M)$ and $k_1 \neq k_2$. For each $j \in \{1,2\}$, we pick up a positive integer $T_j \geq t_0 + \tau$ and the mapping S_{k_j} satisfying (2.33) and (2.34), where k and T are replaced by k_j and T_j, respectively, and some $T_3 > \max\{T_1, T_2\}$ satisfying (2.21). Clearly, there are x and $y \in A(N,M)$ such that $S_{k_1}x = x$ and $S_{k_2}y = y$, respectively. That is, x and y are bounded nonoscillatory solutions of (1.7) in $A(N,M)$. By (2.34) we get that for $t \geq T_3$

$$x(t) = k_1 + \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} \int_{v}^{+\infty} \int_{s}^{+\infty} \frac{1}{\alpha(s)\beta(v)} f(u, x(\sigma_1(u)), \ldots, x(\sigma_n(u)))du\, ds\, dv,$$
$$y(t) = k_2 + \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} \int_{v}^{+\infty} \int_{s}^{+\infty} \frac{1}{\alpha(s)\beta(v)} f(u, y(\sigma_1(u)), \ldots, y(\sigma_n(u)))du\, ds\, dv, \tag{2.47}$$

which together with (2.3) and (2.21) yield that

$$\|x-y\| \geq |k_1 - k_2| - 2\int_{T_3}^{+\infty} \int_{v}^{+\infty} \int_{s}^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv > 0, \tag{2.48}$$

which implies that $x \neq y$. This completes the proof. □

Theorem 2.5. *Assume that there exist constants M and N with $M > N > 0$ and a function $h \in C([t_0,+\infty), \mathbb{R}^+)$ satisfying* (2.3) *and*

$$\sum_{i=1}^{\infty} \int_{t_0+i\tau}^{+\infty} \int_{v}^{+\infty} \int_{s}^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv < +\infty. \tag{2.49}$$

If $p(t) = -1$ for each $t \in [t_0,+\infty)$, then (1.7) *possesses uncountably many bounded nonoscillatory solutions in $A(N,M)$.*

Proof. Set $k \in (N, M)$. It follows from (2.49) that there exists $T > t_0$ satisfying

$$\sum_{i=1}^{\infty} \int_{T+i\tau}^{+\infty} \int_{v}^{+\infty} \int_{s}^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv < \min\{M-k, k-N\}. \tag{2.50}$$

Define a mapping $S_k : A(N,M) \to C([t_0,+\infty),\mathbb{R})$ by

$$(S_k x)(t) = \begin{cases} k - \displaystyle\sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_{v}^{+\infty} \int_{s}^{+\infty} \frac{1}{\alpha(s)\beta(v)} f(u, x(\sigma_1(u)), \dots, x(\sigma_n(u))) du\, ds\, dv, & t \geq T, \\ (S_k x)(T), & t_0 \leq t < T \end{cases} \tag{2.51}$$

for $x \in A(N,M)$.

Firstly, we prove that $S_k(A(N,M)) \subseteq A(N,M)$. By (2.3), (2.50), and (2.51), we obtain that

$$\begin{aligned} (S_k x)(t) &\leq k + \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_{v}^{+\infty} \int_{s}^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv \leq M, \quad x \in A(N,M),\ t \geq T, \\ (S_k x)(t) &\geq k - \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_{v}^{+\infty} \int_{s}^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv \geq N, \quad x \in A(N,M),\ t \geq T, \end{aligned} \tag{2.52}$$

which imply that $S(A(N,M)) \subseteq A(N,M)$.

Secondly, we show that S_k is continuous in $A(N,M)$. Let $x \in A(N,M)$ and $\{x_m\}_{m\geq 1} \subseteq A(N,M)$ with $\lim_{m\to+\infty} x_m = x$. By (2.51), we get that

$$\begin{aligned} &|(S_k x_m)(t) - (S_k x)(t)| \\ &\quad \leq \sum_{i=1}^{\infty} \int_{T+i\tau}^{+\infty} \int_{v}^{+\infty} \int_{s}^{+\infty} \frac{1}{\alpha(s)\beta(v)} \big| f(u, x_m(\sigma_1(u)), \dots, x_m(\sigma_n(u))) \\ &\qquad\qquad - f(u, x(\sigma_1(u)), \dots, x(\sigma_n(u))) \big| du\, ds\, dv, \quad t \geq T,\ m \geq 1. \end{aligned} \tag{2.53}$$

Equation (2.53) together with (2.3), (2.49) and the Lebesgue dominated convergence theorem yields that

$$\lim_{m\to+\infty} \|S_k x_m - S_k x\| = 0, \tag{2.54}$$

that is, S_k is continuous in $A(N,M)$.

Thirdly, we show that $S_k(A(N,M))$ is relatively compact. From (2.3), (2.50), and (2.51), we obtain that

$$\|S_k x\| \leq k + \sum_{i=1}^{\infty} \int_{T+i\tau}^{+\infty} \int_{v}^{+\infty} \int_{s}^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv \leq 2M, \quad x \in A(N,M), \tag{2.55}$$

which means that $S_k(A(N,M))$ is uniform bounded. It follows from (2.49) that, for each $\varepsilon > 0$, there exists $T^* > T$ such that

$$\sum_{i=1}^{\infty} \int_{T^*+i\tau}^{+\infty} \int_{v}^{+\infty} \int_{s}^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv < \frac{\varepsilon}{2}. \tag{2.56}$$

Notice that (2.3), (2.51), and (2.56) yield that

$$\begin{aligned} |(S_k x)(t_2) - (S_k x)(t_1)| &\le 2\sum_{i=1}^{\infty} \int_{T^*+i\tau}^{+\infty} \int_{v}^{+\infty} \int_{s}^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv \\ &< \varepsilon, \quad x \in A(N,M),\ t_2 > t_1 \ge T^*. \end{aligned} \tag{2.57}$$

Choose a positive integer z satisfying $T + z\tau \ge T^*$. Put

$$\begin{aligned} D &= \max\left\{ \int_{v}^{+\infty} \int_{s}^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds : v \in [T+\tau,\ T^* + z\tau] \right\}, \\ \delta &= \frac{\varepsilon}{1 + 2Dz}. \end{aligned} \tag{2.58}$$

By (2.3), (2.51), and (2.58), we gain that

$$\begin{aligned} &|(S_k x)(t_2) - (S_k x)(t_1)| \\ &\quad = \left| \sum_{i=1}^{\infty} \left(\int_{t_2+i\tau}^{+\infty} - \int_{t_1+i\tau}^{+\infty} \right) \int_{v}^{+\infty} \int_{s}^{+\infty} \frac{1}{\alpha(s)\beta(v)} f(u, x(\sigma_1(u)), \dots, x(\sigma_n(u))) du\, ds\, dv \right| \\ &\quad \le \sum_{i=1}^{\infty} \int_{t_1+i\tau}^{t_2+i\tau} \int_{v}^{+\infty} \int_{s}^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv \\ &\quad \le \sum_{i=1}^{z} \int_{t_1+i\tau}^{t_2+i\tau} \int_{v}^{+\infty} \int_{s}^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv + \sum_{i=z+1}^{\infty} \int_{t_1+i\tau}^{t_2+i\tau} \int_{v}^{+\infty} \int_{s}^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv \\ &\quad \le Dz(t_2 - t_1) + \sum_{i=1}^{\infty} \int_{T^*+i\tau}^{+\infty} \int_{v}^{+\infty} \int_{s}^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv \\ &\quad < \varepsilon, \quad x \in A(N,M),\ t_1, t_2 \in [T,\ T^*] \text{ with } t_1 \le t_2 \le t_1 + \delta. \end{aligned} \tag{2.59}$$

Clearly,

$$|(S_k x)(t_2) - (S_k x)(t_1)| = 0 < \varepsilon, \quad x \in A(N,M),\ t_0 \le t_1 \le t_2 \le T. \tag{2.60}$$

That is, $S_k(A(N,M))$ is equicontinuous on $[t_0, +\infty)$, and S_k is relatively compact. The rest argument is similar to the proof of Theorem 2.4 and is omitted. This completes the proof. □

Theorem 2.6. *Assume that there exist constants M, N, c_1, c_2, T_0 and a function $h \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying* (2.3), (2.4), *and*

$$1 < c_2 < c_1 < c_2^2, \quad 0 < \frac{c_1^2 - c_2}{c_1} N < \frac{c_2^2 - c_1}{c_2} M; \tag{2.61}$$

$$c_2 \le p(t) \le c_1, \quad t \ge T_0 \ge t_0 + \tau. \tag{2.62}$$

Then (1.7) *possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$.*

Proof. Set $k \in (c_1 N + (c_1/c_2)M, c_2 M + (c_2/c_1)N)$. It follows from (2.4) that there exists $T > T_0$ such that

$$\int_T^{+\infty} \int_v^{+\infty} \int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv < \min\left\{ c_2 M + \frac{c_2}{c_1} N - k, \frac{c_2}{c_1} k - M - c_2 N \right\}. \tag{2.63}$$

Define two mappings S_{1k} and $S_{2k} : A(N, M) \to C([t_0, +\infty), \mathbb{R})$ by

$$(S_{1k}x)(t) = \begin{cases} \dfrac{k}{p(t+\tau)} - \dfrac{x(t+\tau)}{p(t+\tau)}, & t \ge T, \\ (S_{1k}x)(T), & t_0 \le t < T, \end{cases} \tag{2.64}$$

$$(S_{2k}x)(t) = \begin{cases} \dfrac{1}{p(t+\tau)} \displaystyle\int_{t+\tau}^{+\infty} \int_v^{+\infty} \int_s^{+\infty} \frac{1}{\alpha(s)\beta(v)} f(u, x(\sigma_1(u)), \dots, x(\sigma_n(u))) du\, ds\, dv, & t \ge T, \\ (S_{2k}x)(T), & t_0 \le t < T \end{cases} \tag{2.65}$$

for $x \in A(N, M)$.

Firstly, we show that $S_{1k}(A(N, M)) + S_{2k}(A(N, M)) \subseteq A(N, M)$. From (2.3), (2.61)–(2.65), we get that

$$\begin{aligned} &(S_{1k}x)(t) + (S_{2k}y)(t) \\ &\quad \le \frac{k}{c_2} - \frac{N}{c_1} + \frac{1}{c_2} \min\left\{ c_2 M + \frac{c_2}{c_1} N - k, \frac{c_2}{c_1} k - M - c_2 N \right\} \\ &\quad \le M, \quad x, y \in A(N, M), \ t \ge T, \\ &(S_{1k}x)(t) + (S_{2k}y)(t) \\ &\quad \ge \frac{k}{c_1} - \frac{M}{c_2} - \frac{1}{c_2} \min\left\{ c_2 M + \frac{c_2}{c_1} N - k, \frac{c_2}{c_1} k - M - c_2 N \right\} \\ &\quad \ge N, \quad x, y \in A(N, M), \ t \ge T, \end{aligned} \tag{2.66}$$

That is, $S_{1k}(A(N, M)) + S_{2k}(A(N, M)) \subseteq A(N, M)$.

Secondly, by (2.61), (2.62), and (2.64), we conclude that

$$\|S_{1k}x - S_{1k}y\| \le \frac{1}{c_2}\|x - y\|, \quad x, y \in A(N, M), \tag{2.67}$$

which implies that S_{1k} is a contraction mapping in $A(N, M)$.

Thirdly, we show that S_{2k} is completely continuous. Let $x \in A(N, M)$ and $\{x_m\}_{m\ge1} \subseteq A(N, M)$ be such that $x_m \to x$ as $m \to +\infty$. By (2.61), (2.62), and (2.65), we gain that

$$\begin{aligned}
&|(S_{2k}x_m)(t) - (S_{2k}x)(t)| \\
&\quad\le \frac{1}{p(t+\tau)}\int_{t+\tau}^{+\infty}\int_v^{+\infty}\int_s^{+\infty}\frac{1}{\alpha(s)\beta(v)}\big|f(u, x_m(\sigma_1(u)), \ldots, x_m(\sigma_n(u))) \\
&\qquad\qquad -f(u, x(\sigma_1(t)), \ldots, x(\sigma_n(t)))\big|du\,ds\,dv \\
&\quad\le \frac{1}{c_2}\int_{T}^{+\infty}\int_v^{+\infty}\int_s^{+\infty}\frac{1}{\alpha(s)\beta(v)}\big|f(u, x_m(\sigma_1(u)), \ldots, x_m(\sigma_n(u))) \\
&\qquad\qquad -f(u, x(\sigma_1(u)), \ldots, x(\sigma_n(u)))\big|du\,ds\,dv, \quad t \ge T,\ m \ge 1.
\end{aligned} \tag{2.68}$$

In view of (2.12), (2.13), (2.68), and the Lebesgue dominated convergence theorem, we obtain that

$$\lim_{m\to+\infty}\|S_{2k}x_m - S_{2k}x\| = 0, \tag{2.69}$$

that is, S_{2k} is continuous in $A(N, M)$.

For each $\varepsilon > 0$, it follows from (2.4) that there exists $T^* \ge T$ satisfying

$$\int_{T^*}^{+\infty}\int_v^{+\infty}\int_s^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv < \min\left\{\frac{c_2\varepsilon}{2}, \frac{c_2^2\varepsilon}{4c_1}\right\}. \tag{2.70}$$

From (2.3), (2.61), (2.62), (2.65), and (2.70), we gain that

$$\begin{aligned}
&|(S_{2k}x)(t_2) - (S_{2k}x)(t_1)| \\
&\quad\le \frac{1}{c_2}\int_{t_2+\tau}^{+\infty}\int_v^{+\infty}\int_s^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv + \frac{1}{c_2}\int_{t_1+\tau}^{+\infty}\int_v^{+\infty}\int_s^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv \\
&\quad\le \frac{2}{c_2}\int_{T^*}^{+\infty}\int_v^{+\infty}\int_s^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv \\
&\quad< \varepsilon, \quad x \in A(N, M),\ t_2 > t_1 \ge T^*.
\end{aligned} \tag{2.71}$$

By (2.3), (2.61), (2.62), (2.65), and (2.70), we obtain that, for $x \in A(N,M), T \le t_1 \le t_2 \le T^*$,

$$\begin{aligned}
&|(S_{2k}x)(t_2) - (S_{2k}x)(t_1)| \\
&\quad \le \frac{1}{p(t_2+\tau)} \int_{t_1+\tau}^{t_2+\tau} \int_v^{+\infty} \int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv \\
&\qquad + \left| \frac{1}{p(t_2+\tau)} - \frac{1}{p(t_1+\tau)} \right| \int_{t_1+\tau}^{+\infty} \int_v^{+\infty} \int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv \\
&\quad \le \frac{1}{c_2} \int_{t_1+\tau}^{t_2+\tau} \int_v^{+\infty} \int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv + \frac{2c_1}{c_2^2} \int_{T^*}^{+\infty} \int_v^{+\infty} \int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv,
\end{aligned} \tag{2.72}$$

which means that there exists $\delta > 0$ such that

$$|(S_{2k}x)(t_2) - (S_{2k}x)(t_1)| < \varepsilon, \quad x \in A(N,M),\ t_1, t_2 \in [T, T^*] \text{ with } |t_2 - t_1| < \delta. \tag{2.73}$$

It is easy to verify that

$$|(S_{2k}x)(t_2) - (S_{2k}x)(t_1)| = 0 < \varepsilon, \quad x \in A(N,M),\ t_0 \le t_1 \le t_2 \le T. \tag{2.74}$$

That is, $S_{2k}(A(N,M))$ is equicontinuous on $[t_0, +\infty)$, and S_{2k} is completely continuous. By Lemma 1.2, there is $x_0 \in A(N,M)$ such that $S_{1k}x_0 + S_{2k}x_0 = x_0$, which is a bounded nonoscillatory solution of (1.7).

Lastly, we demonstrate that (1.7) possesses uncountably many nonoscillatory solutions. Let $k_1, k_2 \in (c_1 N + (c_1/c_2)M, c_2 M + (c_2/c_1)N)$ and $k_1 \ne k_2$. For each $j \in \{1,2\}$, we choose $T_j > t_0$ and contraction mappings S_{jk_1} and S_{jk_2} satisfying (2.63)–(2.65) with k and T replaced by k_j and T_j, respectively, and some $T_3 > \max\{T_1, T_2\}$ satisfying

$$\int_{T_3}^{+\infty} \int_v^{+\infty} \int_s^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv < \frac{c_2|k_1 - k_2|}{2c_1}. \tag{2.75}$$

Obviously, there are $x, y \in A(N,M)$ such that $S_{1k_1}x + S_{2k_1}x = x$ and $S_{1k_2}y + S_{2k_2}y = y$, respectively, and x and y are two bounded nonoscillatory solutions of (1.7) in $A(N,M)$. By (2.64) and (2.65), we gain that for $t \ge T_3$

$$\begin{aligned}
x(t) &= \frac{k_1}{p(t+\tau)} - \frac{x(t+\tau)}{p(t+\tau)} \\
&\quad + \frac{1}{p(t+\tau)} \int_{t+\tau}^{+\infty} \int_v^{+\infty} \int_s^{+\infty} \frac{1}{\alpha(s)\beta(v)} f(u, x(\sigma_1(u)), \dots, x(\sigma_n(u))) du\, ds\, dv, \\
y(t) &= \frac{k_2}{p(t+\tau)} - \frac{y(t+\tau)}{p(t+\tau)} \\
&\quad + \frac{1}{p(t+\tau)} \int_{t+\tau}^{+\infty} \int_v^{+\infty} \int_s^{+\infty} \frac{1}{\alpha(s)\beta(v)} f(u, y(\sigma_1(u)), \dots, y(\sigma_n(u))) du\, ds\, dv,
\end{aligned} \tag{2.76}$$

which together with (2.3), (2.61), and (2.62) mean that

$$\begin{aligned}
|x(t)-y(t)| &\geq \frac{1}{p(t+\tau)}|k_1-k_2| - \frac{1}{p(t+\tau)}|x(t-\tau)-y(t-\tau)| \\
&\quad - \frac{1}{p(t+\tau)}\int_{t+\tau}^{+\infty}\int_v^{+\infty}\int_s^{+\infty}\frac{1}{\alpha(s)\beta(v)} \\
&\quad \times \left|f(u,x(\sigma_1(u)),\ldots,x(\sigma_n(u))) - f(u,y(\sigma_1(u)),\ldots,y(\sigma_n(u)))\right| du\,ds\,dv \\
&\geq \frac{1}{c_1}|k_1-k_2| - \frac{1}{c_2}\|x-y\| \\
&\quad - \frac{2}{c_2}\int_{T_3}^{+\infty}\int_v^{+\infty}\int_s^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv, \quad t \geq T_3,
\end{aligned} \tag{2.77}$$

which together with (2.75) yields that

$$\|x-y\| \geq \frac{c_2}{1+c_2}\left(\frac{1}{c_1}|k_1-k_2| - \frac{2}{c_2}\int_{T_3}^{+\infty}\int_v^{+\infty}\int_s^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv\right) > 0, \tag{2.78}$$

that is, $x \neq y$. This completes the proof. □

Theorem 2.7. *Assume that there exist constants M, N, c_1, c_2, T_0 and a function $h \in C([t_0,+\infty),\mathbb{R}^+)$ satisfying* (2.3), (2.4), *and*

$$0 < (c_1-1)N < (c_2-1)M; \tag{2.79}$$

$$-\infty < -c_1 \leq p(t) \leq -c_2 < -1, \quad t \geq T_0 \geq t_0 + \tau. \tag{2.80}$$

Then (1.7) *possesses uncountably many bounded nonoscillatory solutions in $A(N,M)$.*

Proof. Set $k \in ((c_1-1)N,(c_2-1)M)$. It follows from (2.4) that there exists $T > T_0$ such that

$$\int_T^{+\infty}\int_v^{+\infty}\int_s^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv < \min\left\{(c_2-1)M-k, \frac{c_2(k+N)}{c_1} - c_2N\right\}. \tag{2.81}$$

Define two mappings S_{1k} and $S_{2k} : A(N,M) \to C([t_0,+\infty),\mathbb{R})$ by

$$(S_{1k}x)(t) = \begin{cases} \dfrac{-k}{p(t+\tau)} - \dfrac{x(t+\tau)}{p(t+\tau)}, & t \geq T, \\ (S_{1k}x)(T), & t_0 \leq t < T, \end{cases}$$

$$(S_{2k}x)(t) = \begin{cases} \dfrac{1}{p(t+\tau)}\displaystyle\int_{t+\tau}^{+\infty}\int_v^{+\infty}\int_s^{+\infty}\frac{1}{\alpha(s)\beta(v)}f(u,x(\sigma_1(u)),\ldots,x(\sigma_n(u)))du\,ds\,dv, & t \geq T, \\ (S_{2k}x)(T), & t_0 \leq t < T \end{cases} \tag{2.82}$$

for $x \in A(N, M)$. From (2.3), (2.80)–(2.82), we obtain that

$$\begin{aligned}
&(S_{1k}x)(t) + (S_{2k}y)(t) \\
&\quad \le \frac{k}{c_2} + \frac{M}{c_2} + \frac{1}{c_2}\min\left\{(c_2-1)M - k, \frac{c_2(k+N)}{c_1} - c_2N\right\} \\
&\quad \le M, \quad x, y \in A(N, M),\ t \ge T, \\
&(S_{1k}x)(t) + (S_{2k}y)(t) \\
&\quad \ge \frac{k}{c_1} + \frac{N}{c_1} - \frac{1}{c_2}\min\left\{(c_2-1)M - k, \frac{c_2(k+N)}{c_1} - c_2N\right\}, \\
&\quad \ge N, \quad x, y \in A(N, M),\ t \ge T
\end{aligned} \tag{2.83}$$

which implies that $S_{1k}(A(N, M)) + S_{2k}(A(N, M)) \subseteq A(N, M)$. The rest of the proof is similar to the proof of Theorem 2.6 and is omitted. This completes the proof. □

3. Examples

In this section we construct seven examples as applications of the results presented in Section 2.

Example 3.1. Consider the following third-order nonlinear neutral delay differential equation:

$$\begin{aligned}
&\left\{\frac{t^3}{e^t}\left[\frac{t}{\ln(1+t)}\left(x(t) + \frac{3\sin(t^2-2)+1}{7}x(t-\tau)\right)'\right]'\right\}' \\
&\quad + \frac{1}{e^t}\left(x^2\left(\frac{2t^2}{1+3t}\right) + x^2\left(t - \frac{1}{t}\right)\right) = 0, \quad t \ge t_0,
\end{aligned} \tag{3.1}$$

where $\tau > 0$ and $t_0 > 0$ are fixed. Let $n = 2$, and let M and N be two positive constants with $M > 7N$ and

$$\begin{aligned}
&\alpha(t) = \frac{t^3}{e^t}, \qquad \beta(t) = \frac{t}{\ln(1+t)}, \qquad p(t) = \frac{3\sin(t^2-2)+1}{7}, \\
&c_1 = \frac{4}{7}, \qquad c_2 = \frac{2}{7}, \qquad \sigma_1(t) = \frac{2t^2}{1+3t}, \qquad \sigma_2(t) = t - \frac{1}{t}, \qquad h(t) = \frac{2M^2}{e^t}, \\
&f(t,u,v) = \frac{u^2+v^2}{e^t}, \qquad (t,u,v) \in [t_0, +\infty) \times \mathbb{R}^2.
\end{aligned} \tag{3.2}$$

Obviously, (2.1)–(2.3) hold. On the other hand,

$$\int_{t_0}^{+\infty}\int_{v}^{+\infty}\int_{s}^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv = M^2\int_{t_0}^{+\infty}\frac{\ln(1+v)}{v^3}dv < +\infty. \tag{3.3}$$

That is, (2.4) holds. Thus Theorem 2.1 means that (1.7) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$.

Example 3.2. Consider the following third-order nonlinear neutral delay differential equation:

$$\left\{\frac{(2t+1)^3}{t}\left[\left(4t^2+4t+5\right)\left(x(t)+\frac{2t^2}{1+3t^2}x(t-\tau)\right)'\right]'\right\}'+\frac{2^{x(\sqrt{t}+1)}}{t^2}=0, \quad t\geq t_0, \tag{3.4}$$

where $\tau > 0$ and $t_0 > 0$ are fixed. Let $n = 1$, and let M and N be two positive constants with $(3t_0^2+1)M > 3(t_0^2+1)N$ and

$$\begin{aligned}
&\alpha(t)=\frac{(2t+1)^3}{t}, \qquad \beta(t)=4t^2+4t+5, \qquad p(t)=\frac{2t^2}{1+3t^2},\\
&c_1=\frac{2}{3}, \qquad c_2=\frac{2t_0^2}{1+3t_0^2}, \qquad \sigma_1(t)=\sqrt{t}+1, \qquad h(t)=\frac{2^M}{t^2},\\
&f(t,u)=\frac{2^u}{t^2}, \quad (t,u)\in[t_0,+\infty)\times\mathbb{R}.
\end{aligned} \tag{3.5}$$

Obviously, (2.3), (2.25), and (2.26) hold. Moreover,

$$\int_{t_0}^{+\infty}\int_{v}^{+\infty}\int_{s}^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv=\frac{2^M}{16}\left[\frac{1}{2(2t_0+1)}-\frac{1}{2}\left(\frac{\pi}{2}-\operatorname{arc\,tan}(2t_0+1)\right)\right]<+\infty. \tag{3.6}$$

Hence (2.4) holds. Thus Theorem 2.2 ensures that (1.7) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$.

Example 3.3. Consider the following third-order nonlinear neutral delay differential equation:

$$\left\{e^{t-t^2}\left[\left(t^4+1\right)^{1/3}\left(x(t)+\frac{-3t-1}{5t+2}x(t-\tau)\right)'\right]'\right\}'+\frac{tx(t^3)}{e^{t^2}(1+x^2(t^2+2t))}=0, \quad t\geq 0, \tag{3.7}$$

where $\tau > 0$ is fixed and $t_0 = 0$. Let $n = 2$, and let M and N be two positive constants with $4M > 5N$ and

$$\begin{aligned}
&\alpha(t)=e^{t-t^2}, \qquad \beta(t)=\left(t^4+1\right)^{1/3}, \qquad p(t)=\frac{-3t-1}{5t+2}, \qquad c_1=\frac{3}{5},\\
&c_2=\frac{1}{2}, \qquad \sigma_1(t)=t(t+2), \qquad \sigma_2(t)=t^3, \qquad h(t)=\frac{Mt}{(1+N^2)e^{t^2}},\\
&f(t,u,v)=\frac{tv}{e^{t^2}(1+u^2)}, \quad (t,u,v)\in[t_0,+\infty)\times\mathbb{R}^2.
\end{aligned} \tag{3.8}$$

Clearly, (2.3), (2.25), and (2.30) hold, and

$$\int_{t_0}^{+\infty}\int_{v}^{+\infty}\int_{s}^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv=\frac{M}{2(1+N^2)}\int_{t_0}^{+\infty}e^{-v}\left(t^4+1\right)^{-1/3}dv<+\infty, \tag{3.9}$$

hence (2.4) holds. Therefore Theorem 2.3 guarantees that (1.7) possesses uncountably many bounded nonoscillatory solutions in $A(N,M)$.

Example 3.4. Consider the following third-order nonlinear neutral delay differential equation:

$$\left\{\frac{t^3+1}{t^2\ \text{arc tan}\,t}\left[\frac{1}{t}(x(t)+x(t-\tau))'\right]'\right\}'+\frac{x^2(t^2-t)(x(t^2-t)+3)}{t^4}=0,\quad t\geq t_0, \tag{3.10}$$

where $\tau>0$ and $t_0>0$ are fixed. Let $n=1$, and let M and N be two positive constants with $M>N$ and

$$\begin{aligned}\alpha(t)&=\frac{t^3+1}{t^2\ \text{arc tan}\,t},\qquad \beta(t)=\frac{1}{t},\ p(t)=1,\qquad \sigma_1(t)=t^2-t,\\ h(t)&=\frac{M^2(M+3)}{t^4},\qquad f(t,u)=\frac{u^2(u+3)}{t^4},\quad (t,u)\in[t_0,+\infty)\times\mathbb{R}.\end{aligned} \tag{3.11}$$

Obviously, (2.3) holds. Furthermore,

$$\int_{t_0}^{+\infty}\int_{v}^{+\infty}\int_{s}^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv=\frac{M^2(M+3)}{3}\int_{t_0}^{+\infty}\frac{v\text{arc tan}\,v}{v^3+1}dv<+\infty, \tag{3.12}$$

that is, (2.4) holds. Thus Theorem 2.4 ensures that (1.7) possesses uncountably many bounded nonoscillatory solutions in $A(N,M)$.

Example 3.5. Consider the following third-order nonlinear neutral delay differential equation:

$$\left\{t^3\left[\frac{1}{t}(x(t)-x(t-\tau))'\right]'\right\}'+\frac{x^2(t+(1/t)+\ln(1+t))}{t^3}=0,\quad t\geq t_0, \tag{3.13}$$

where $\tau>0$ and $t_0>0$ are fixed. Let $n=1$, and let M and N be two positive constants with $M>N$ and

$$\begin{aligned}\alpha(t)&=t^3,\qquad \beta(t)=\frac{1}{t},\qquad p(t)=-1,\qquad \sigma_1(t)=t+\frac{1}{t}+\ln(1+t),\\ h(t)&=\frac{M^2}{t^3},\qquad f(t,u)=\frac{u^2}{t^3},\quad (t,u)\in[t_0,+\infty)\times\mathbb{R}.\end{aligned} \tag{3.14}$$

Obviously, (2.3) holds. Notice that

$$\sum_{i=1}^{+\infty}\int_{t_0+i\tau}^{+\infty}\int_{v}^{+\infty}\int_{s}^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv=\frac{M^2}{16}\sum_{i=1}^{+\infty}\frac{1}{(t_0+i\tau)^2}<+\infty, \tag{3.15}$$

that is, (2.49) holds. Hence Theorem 2.5 implies that (1.7) possesses uncountably many bounded nonoscillatory solutions in $A(N,M)$.

Example 3.6. Consider the following third-order nonlinear neutral delay differential equation:

$$\left\{\frac{t^2}{e^{t^2}}\left[\left(1+\ln^2 t\right)\left(x(t)+\left(\frac{5\pi}{2}+\arctan t^{2/3}\right)x(t-\tau)\right)'\right]'\right\}'+\frac{t}{e^{t^2}\left(x^2(t^2+1)+1\right)}=0,\quad t\geq t_0, \tag{3.16}$$

where $\tau>0$ and $t_0>0$ are fixed. Let $n=1$, and let M and N be two positive constants with $(5/2)(9\pi-(5/2))N<3((25\pi/4)-3)M$ and

$$\begin{gathered}\alpha(t)=\frac{t^2}{e^{t^2}},\qquad \beta(t)=1+\ln^2 t,\qquad c_1=3\pi, c_2=\frac{5\pi}{2},\\ p(t)=\frac{5\pi}{2}+\arctan t^{2/3},\qquad \sigma_1(t)=t^2+1,\qquad h(t)=\frac{t}{(N^2+1)e^{t^2}},\\ f(t,u)=\frac{t}{(u^2+1)e^{t^2}},\qquad (t,u)\in[t_0,+\infty)\times\mathbb{R}.\end{gathered} \tag{3.17}$$

Clearly, (2.3), (2.61), and (2.62) hold, and

$$\int_{t_0}^{+\infty}\int_{v}^{+\infty}\int_{s}^{+\infty}\frac{h(u)}{\alpha(s)\beta(v)}du\,ds\,dv=\frac{1}{2(N^2+1)}\left(\frac{\pi}{2}-\operatorname{arc\,tan}(\ln t_0)\right)<+\infty, \tag{3.18}$$

that is, (2.4) holds also. Thus Theorem 2.6 ensures that (1.7) possesses uncountably many bounded nonoscillatory solutions in $A(N,M)$.

Example 3.7. Consider the following third-order nonlinear neutral delay differential equation

$$\left\{\frac{1}{t}\left[\frac{t^2+1}{e^{t^2}}\left(x(t)+\left(\sin\left(t-\frac{1}{t}\right)+4\cos(\ln t)-7\right)x(t-\tau)\right)'\right]'\right\}'+\frac{tx^2\left(\sqrt{t^2+1}\right)}{e^{t^2}}=0,\quad t\geq t_0, \tag{3.19}$$

where $\tau > 0$ and $t_0 > 0$ are fixed. Let $n = 1$, and let M and N be two positive constants with $M > 11N$ and

$$\alpha(t) = \frac{1}{t}, \qquad \beta(t) = \frac{t^2+1}{e^{t^2}}, \qquad c_1 = 12,\ c_2 = 2,$$
$$p(t) = \sin\left(t - \frac{1}{t}\right) + 4\cos(\ln t) - 7, \qquad \sigma_1(t) = \sqrt{t^2+1}, \tag{3.20}$$
$$h(t) = \frac{tM^2}{e^{t^2}}, \qquad f(t,u) = \frac{tu^2}{e^{t^2}}, \quad (t,u) \in [t_0, +\infty) \times \mathbb{R}.$$

Obviously, (2.3), (2.79), and (2.80) hold. Note that

$$\int_{t_0}^{+\infty}\int_{v}^{+\infty}\int_{s}^{+\infty} \frac{h(u)}{\alpha(s)\beta(v)} du\, ds\, dv = \frac{M^2}{4}\left(\frac{\pi}{2} - \operatorname{arc}\tan t_0\right) < +\infty, \tag{3.21}$$

that is, (2.4) holds. It follows from Theorem 2.7 that (1.7) possesses uncountably many bounded nonoscillatory solutions in $A(N, M)$.

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