Research Article

Stability and Existence of Periodic Solutions for Cellular Neural Networks with State Dependent Delays on Time Scales

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We study delayed cellular neural networks on time scales. Without assuming the boundedness of the activation functions, we establish the exponential stability and existence of periodic solutions. The results in this paper are completely new even in case of the time scale $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$ and improve some of the previously known results.

1. Introduction

Consider the following cellular neural networks with state-dependent delays on time scales:

$$x_i^\Delta(t) = -c_i(t)x_i^\sigma(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x(t)) + \sum_{j=1}^{n} b_{ij}(t)f_j\left(x_j\left(\xi_{ij}(t, x(t))\right)\right) + I_i(t), \quad t \in \mathbb{T}, \quad (1.1)$$

where $i = 1, 2, \ldots, n$, $\mathbb{T}$ is an ω-periodic time scale which has the subspace topology inherited from the standard topology on $\mathbb{R}$, $x_i^\sigma(t) = x_i(\sigma(t))$, $\sigma(t)$ will be defined in the next section, $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$, n corresponds to the number of units in the neural network, $x_i(t)$ corresponds to the state of the ith unit at time t, $f_j(x_j(t))$ denotes the output of the jth unit on ith unit at time t, b_{ij} denotes the strength of the jth unit on the ith unit at time $\xi_{ij}(t, x(t))$, I_i denotes the external bias on the ith unit at time t, $t - \xi_{ij}(t, x(t))$ corresponds to the transmission delay along the axon of the jth unit, c_i represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs.

It is well known that the cellular neural networks have been successfully applied to signal processing, pattern recognition, optimization, and associative memories, especially in image processing and solving nonlinear algebraic equations. They have been widely studied both in theory and applications [1–3]. Many results for the existence of their periodic solutions and the exponential convergence properties for cellular neural networks have been reported in the literatures. See, for instance, [4–17] and references cited therein.

In fact, continuous and discrete systems are very important in implementation and applications. It is well known that the theory of time scales has received a lot of attention which was introduced by Stefan Hilger in order to unify continuous and discrete analysis. Therefore, it is meaningful to study dynamic systems on time scales which can unify differential and difference systems see [18–28].

When $\mathbb{T} = \mathbb{R}$, $\xi_{ij}(t, \cdot) \equiv t - \tau_{ij}(t)$, (1.1) reduces to

$$x_i'(t) = -c_i(t)x_i(t) + \sum_{j=1}^n a_{ij}(t)f_j\big(x_j(t)\big) + \sum_{j=1}^n b_{ij}(t)f_j\big(x_j\big(t - \tau_{ij}(t)\big)\big) + I_i(t), \qquad (1.2)$$

where $i = 1, 2, \ldots, n$. By using Mawhin's continuation theorem and Liapunov functions, the authors [6, 14] obtained the existence and stability of periodic solutions of (1.2), respectively.

Furthermore, (1.1) also covers discrete system (for when $\mathbb{T} = \mathbb{Z}$, $\xi_{ij}(n, \cdot) \equiv n - \tau_{ij}(n)$; see [15])

$$x_i(n + 1) = x_i(n)e^{-c_i(n)h} + \theta_i(h)\sum_{j=1}^m a_{ij}(n)f_j\big(x_j(n)\big)$$

$$+ \theta_i(h)\sum_{j=1}^m b_{ij}(n)f_j\big(x_j\big(n - \tau_{ij}(n)\big)\big) + \theta_i(h)I_i(n), \qquad (1.3)$$

where $\theta_i(h) = (1 - e^{-c_i(n)h})/c_i(n)$, $i = 1, 2, \ldots, m$. In [15], the author firstly obtained the discrete-time analogue of (1.3) by the semidiscretization technique [29, 30], and then some sufficient conditions for the existence and global asymptotical stability of periodic solutions of (1.3) were established by using Mawhin's continuation theorem and Liapunov functions.

However, in [5, 13–15], the activation functions f_j, $j = 1, 2, \ldots, n$ are assumed to be bounded. Our main purpose of this paper is to establish the stability and existence of periodic solutions of (1.1) without assuming the boundedness of the activation functions.

For the sake of convenience, we denote

$$\overline{c}_i = \max_{t \in [0, \omega]_{\mathbb{T}}} |c_i(t)|, \qquad \underline{c}_i = \min_{t \in [0, \omega]_{\mathbb{T}}} |c_i(t)|, \qquad \overline{a}_{ij} = \max_{t \in [0, \omega]_{\mathbb{T}}} |a_{ij}(t)|,$$

$$\overline{b}_{ij} = \max_{t \in [0, \omega]_{\mathbb{T}}} |b_{ij}(t)|, \quad \overline{I}_i = \max_{t \in [0, \omega]_{\mathbb{T}}} |I_i(t)|, \quad i, j = 1, 2, \ldots, n. \qquad (1.4)$$

Throughout this paper, we assume that

(H1) $\xi_{ij}(\cdot, \cdot) \in C(\mathbb{T} \times \mathbb{R}^n, [0, \infty)_{\mathbb{T}})$ is ω-periodic with respect to its first argument and satisfies $\xi_{ij}(t, x) \leq \tau$ for all $(t, x) \in \mathbb{T} \times \mathbb{R}^n$, $c_i \in C(\mathbb{T}, (0, +\infty))$, $a_{ij}, b_{ij}, I_i \in C(\mathbb{T}, \mathbb{R})$, $i, j = 1, 2, \ldots, n$ are ω-periodic functions;

(H2) $f_j \in C(\mathbb{R}, \mathbb{R})$, and there exists a positive number M_j such that $|f_j(x) - f_j(y)| \leqslant M_j|x - y|$ for all $x, y \in \mathbb{R}$, $j = 1, 2, \ldots, n$.

The initial conditions of system (1.1) are of the following form:

$$x_i(s) = \phi_i(s), \quad i = 1, 2, \ldots, n, \ s \in [-\tau, 0]_{\mathbb{T}}, \qquad \tau = \max_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n} \max_{t \in [0, \omega]_{\mathbb{T}}} |\tau_{ij}(t)|, \qquad (1.5)$$

where $\phi_i \in C([-\tau, 0]_{\mathbb{T}}, \mathbb{R})$, $i = 1, 2, \ldots, n$.

The organization of this paper is as follows. In Section 2, we introduce some lemmas and definitions and state some preliminary results needed in later sections, which will be used in latter sections. In Section 3, we will study the existence of periodic solutions of system (1.1) by using the method of coincidence degree. In Section 4, we will derive sufficient conditions to ensure that the periodic solutions of (1.1) are globally exponentially stable. In Section 5, an example is also provided to illustrate the effectiveness of the main results in Sections 3 and 4. The conclusions are drawn in Section 6.

2. Preliminaries

In this section, we will introduce some notations and definitions and state some preliminary results.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$ and the graininess $\mu : \mathbb{T} \rightarrow \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \qquad \mu(t) = \sigma(t) - t. \qquad (2.1)$$

A point $t \in \mathbb{T}$ is called left dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left scattered if $\rho(t) < t$, right dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is right dense continuous provided that it is continuous at right-dense point in $\mathbb{T}$, and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be continuous function on $\mathbb{T}$. The set of continuous functions $f : \mathbb{T} \rightarrow \mathbb{R}$ will be denoted by $C(\mathbb{T})$.

For $y : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^\Delta(t)$, to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\left| [y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s] \right| < \varepsilon |\sigma(t) - s| \quad \forall s \in U. \qquad (2.2)$$

If y is continuous, then y is right-dense continuous, and if y is delta differentiable at t, then y is continuous at t.

Let y be right dense continuous. If $Y^\Delta(t) = y(t)$, then we define the delta integral by $\int_a^t y(s) \Delta s = Y(t) - Y(a)$.

Definition 2.1 (see [31]). We say that a time scale $\mathbb{T}$ is periodic if there exists $p > 0$ such that if $t \in \mathbb{T}$, then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is called the period of the time scale.

Definition 2.2 (see [31]). Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period p. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period ω if there exists a natural number n such that $\omega = np$, $f(t + \omega) = f(t)$ for all $t \in \mathbb{T}$, and ω is the smallest number such that $f(t + \omega) = f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $\omega > 0$ if ω is the smallest positive number such that $f(t + \omega) = f(t)$ for all $t \in \mathbb{T}$.

If $\mathbb{T}$ is ω periodic, then $\sigma(t + \omega) = \sigma(t) + \omega$ and $\mu(t)$ is an ω-periodic function.

Definition 2.3 (see [32]). A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided that $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$, where $\mu(t) = \sigma(t) - t$ is the graininess function. The set of all regressive rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+$ is given by $\{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0\}$ for all $t \in \mathbb{T}$. Let $p \in \mathcal{R}$. The exponential function is defined by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta\tau \right), \tag{2.3}$$

where $\xi_{h(z)}$ is the so-called cylinder transformation.

Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, and we define

$$p \oplus q := p + q + \mu pq, \qquad \ominus p := -\frac{p}{1 + \mu p}, \qquad p \ominus q := p \oplus (\ominus q). \tag{2.4}$$

Then the generalized exponential function has the following properties.

Lemma 2.4 (see [32]). *Assume that $p, q : \mathbb{T} \to \mathbb{R}$ are two regressive functions, and then*

(1) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;

(2) $e_p(t, s) = 1/e_p(s, t) = e_{\ominus p}(s, t)$;

(3) $e_p(t, s)e_p(s, r) = e_p(t, r)$.

Lemma 2.5 (see [32]). *Assume that $f, g : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^k$. Then*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)). \tag{2.5}$$

Lemma 2.6 (see [32]). *If a, $b \in \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$, and $f, g \in C(\mathbb{T}, \mathbb{R})$, then*

(1) $\int_a^b [\alpha f(t) + \beta g(t)] \Delta t = \alpha \int_a^b f(t) \Delta t + \alpha \int_a^b g(t) \Delta t$;

(2) *if $f(t) \geq 0$ for all $a \leq t < b$, then $\int_a^b f(t) \Delta t \geq 0$*;

(3) *if $|f(t)| \leq g(t)$ on $[a, b) := \{t \in \mathbb{T} : a \leq t < b\}$, then $|\int_a^b f(t) \Delta t| \leq \int_a^b g(t) \Delta t$*.

Lemma 2.7 (see [33]). *Let t_1, $t_2 \in [0, \omega]_{\mathbb{T}}$. If $x : \mathbb{T} \to \mathbb{R}$ is ω periodic, then*

$$x(t) \leqslant x(t_1) + \int_0^\omega \left| x^\Delta(s) \right| \Delta s, \qquad x(t) \geqslant x(t_2) - \int_0^\omega \left| x^\Delta(s) \right| \Delta s. \tag{2.6}$$

Definition 2.8. The periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))^T$ of system (1.1) is said to be exponentially stable if there exists a positive constant α with $\ominus\alpha \in \mathcal{R}^+$ such that for every $\delta \in [-\tau, 0]_\mathbb{T}$, and there exists $N = N(\delta) \geqslant 1$ such that the solution $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ of system (1.1) through $(\delta, x(\delta))$ satisfies

$$\sum_{i=1}^{n} |x_i(t) - x_i^*(t)| \leqslant N \|\phi(\delta) - x^*(\delta)\| e_{\ominus\alpha}(t, \delta), \quad t \in \mathbb{T}^+ := [0, \infty)_\mathbb{T}, \tag{2.7}$$

where $\|\phi(\delta) - x^*(\delta)\| = \sum_{i=1}^{n} \max_{\delta \in [-\tau, 0]_\mathbb{T}} |\phi_i(\delta) - x_i^*(\delta)|$.

In order to show that there exists at least one ω-periodic solution of system (1.1), we need the following concepts and result which are cited from [34].

Let $\mathbb{X}$, $\mathbb{Y}$ be Banach spaces, $L : \mathrm{Dom}\ L \subset \mathbb{X} \to \dim \mathbb{Y}$ be a linear mapping, and $N : \mathbb{X} \to \mathbb{Y}$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \mathrm{Ker}\ L = \mathrm{co\ dim\ Im}\ L < +\infty$ and $\mathrm{Im}\ L$ is closed in $\mathbb{Y}$. If L is a Fredholm mapping of index zero and there exists continuous projector $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ such that $\mathrm{Im}\ P = \mathrm{Ker}\ L, \mathrm{Ker}\ Q = \mathrm{Im}(I - Q)$, it follows that mapping $L|_{\mathrm{Dom}\ L \cap \mathrm{Ker}\ P} : (I - P)\mathbb{X} \to \mathrm{Im}\ L$ is invertible. We denote the inverse of that mapping by K_P. If Ω is an open bounded subset of $\mathbb{X}$, the mapping N will be called L compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to \mathbb{X}$ is compact.

Lemma 2.9 (see [34]). *Let $\mathbb{X}$, $\mathbb{Y}$ be two Banach spaces and $\Omega \subset \mathbb{X}$ be open bounded and symmetric with $0 \in \Omega$. Suppose that $L : D(L) \subset \mathbb{X} \to \mathbb{Y}$ is a linear Fredholm operator of index zero with $D(L) \cap \overline{\Omega} \neq \emptyset$, and $N : \overline{\Omega} \to \mathbb{Y}$ is L compact. Further, one also assumes that*

(H) $Lx - Nx \neq \lambda(-Lx - N(-x))$ for all $x \in D(L) \cap \partial\Omega, \lambda \in (0, 1]$.

Then equation $Lx = Nx$ has at least one solution on $D(L) \cap \overline{\Omega}$.

Definition 2.10 (see [35]). A real matrix $P = (p_{ij})_{n \times n}$ is said to be a nonsingular M matrix if $p_{ij} \leq 0, i, j = 1, 2, \dots, n, i \neq j$, and all successive principal minors of P are positive.

For $A = (a_{ij})_{m \times n}, B = (b_{ij})_{m \times n} \in \mathbb{R}^{mn}, A \geq B(A > B)$ means that each pair of corresponding elements of A and B such that $a_{ij} \geq b_{ij}\ (a_{ij} > b_{ij})$.

Lemma 2.11 (see [35]). *Assume that P is a nonsingular M matrix and $Py \leq h$, then $y \leq P^{-1}h$.*

3. Existence of Periodic Solutions

In this section, by Lemma 2.9, we will study the existence of at least one periodic solution of system (1.1).

Theorem 3.1. *Suppose that (H1)-(H2) holds, $E = (e_{ij})_{n \times n}$ is a nonsingular M matrix, where*

$$e_{ij} = z \begin{cases} r_i + d_{ii} & i = j, \\ d_{ij} & i \neq j, \end{cases} \tag{3.1}$$

and $r_i = \underline{c}_i(1 - \omega\bar{c}_i)/(1 + \underline{c}_i\omega)$, $d_{ij} = -(\bar{a}_{ij} + \bar{b}_{ij})M_j$, $i, j = 1, 2, \ldots, n$, then system (1.1) has at least one ω-periodic solution.

Proof. Let

$$\mathbb{X} = \mathbb{Y} = \left\{ x = (x_1, x_2, \ldots, x_n)^T \in C(\mathbb{T}, \mathbb{R}) : x(t + \omega) = x(t), t \in \mathbb{T} \right\} \tag{3.2}$$

with the norm defined by $\|x\| = \max_{1 \le i \le n} |x_i|_0$, where $|x_i|_0 = \max_{t \in [0, \omega]_{\mathbb{T}}} |x_i(t)|$, then $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces.

Set

$$Lx = x^{\Delta}(t), \quad Px = Qx = \frac{1}{\omega} \int_0^{\omega} x(t)\Delta t, \quad x \in \mathbb{X}, \tag{3.3}$$

and $\mathbb{N} : \mathbb{X} \to \mathbb{Y}$

$$Nx_i(t) = -c_i(t)x_i^{\sigma}(t) + \sum_{j=1}^n a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^n b_{ij}(t)f_j(x_j(\xi_{ij}(t, x(t)))) + I_i(t), \tag{3.4}$$

where $i = 1, 2, \ldots, n$. Obviously, Ker $L = \mathbb{R}^n$, Im $L = \{x \in \mathbb{X} : \int_0^{\omega} x(s)\Delta s = 0\}$ is closed in $\mathbb{Y}$ and

$$\dim \text{Ker } L = n = \text{codim Im } L. \tag{3.5}$$

Hence, L is a Fredholm mapping of index zero. Furthermore, similar to the proof of Theorem 3.4 in [21], one can easily show that N is L compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset \mathbb{X}$. Corresponding to the operator equation $Lx - Nx = \lambda(-Lx - N(-x))$, $\lambda \in (0, 1]$, we have

$$x_i^{\Delta}(t) = \frac{1}{1 + \lambda}\left[-c_i(t)x_i^{\sigma}(t) + \sum_{j=1}^n a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^n b_{ij}(t)f_j(x_j(\xi_{ij}(t, x(t)))) + I_i(t) \right]$$

$$- \frac{\lambda}{1 + \lambda}\left[c_i(t)x_i^{\sigma}(t) + \sum_{j=1}^n a_{ij}(t)f_j(-x_j(t)) + \sum_{j=1}^n b_{ij}(t)f_j(-x_j(\xi_{ij}(t, x(t)))) + I_i(t) \right],$$

$$\tag{3.6}$$

where $i = 1, 2, \ldots, n$.

Suppose that $x = (x_1, x_2, \ldots, x_n)^T$ is a solution of system (3.6) for some $\lambda \in (0, 1]$. In view of (3.6) and (H2), we have

$$
\begin{aligned}
\int_0^\omega \left| x_i^\Delta(t) \right| \Delta t &\leq \int_0^\omega \left| c_i(t) x_i^\sigma(t) \right| \Delta t + \int_0^\omega |I_i(t)| \Delta t \\
&\quad + \frac{1}{1+\lambda} \int_0^\omega \left| \left[\sum_{j=1}^n \overline{a}_{ij} f_j(x_j(t)) + \sum_{j=1}^n \overline{b}_{ij} f_j(x_j(\xi_{ij}(t, x(t)))) \right] \right| \Delta t \\
&\quad + \frac{\lambda}{1+\lambda} \int_0^\omega \left| \left[\sum_{j=1}^n \overline{a}_{ij} f_j(-x_j(t)) + \sum_{j=1}^n \overline{b}_{ij} f_j(-x_j(\xi_{ij}(t, x(t)))) \right] \right| \Delta t \\
&\leq \int_0^\omega \overline{c}_i \left| x_i^\sigma(t) \right| \Delta t + \int_0^\omega \left[\sum_{j=1}^n \overline{a}_{ij} \left(M_j |x_j(t)| + |f_j(0)| \right) \right. \\
&\qquad\qquad \left. + \sum_{j=1}^n \overline{b}_{ij} \left(M_j |x_j(\xi_{ij}(t, x(t)))| + |f_j(0)| \right) + \overline{I}_i \right] \Delta t \\
&\leq \overline{c}_i \omega |x_i|_0 + \omega \sum_{j=1}^n \left(\overline{a}_{ij} + \overline{b}_{ij} \right) M_j |x_j|_0 + \omega \left[\sum_{j=1}^n \left(\overline{a}_{ij} + \overline{b}_{ij} \right) |f_j(0)| + \overline{I}_i \right] \\
&\leq \overline{c}_i \omega |x_i|_0 + \omega \sum_{j=1}^n \left(\overline{a}_{ij} + \overline{b}_{ij} \right) M_j |x_j|_0 + \omega \alpha_i, \quad i = 1, 2, \ldots, n,
\end{aligned}
\tag{3.7}
$$

where $\alpha_i = \sum_{j=1}^n (\overline{a}_{ij} + \overline{b}_{ij}) |f_j(0)| + \overline{I}_i$.

Integrating both sides of (3.6) from 0 to ω, we obtain that

$$
\begin{aligned}
\int_0^\omega c_i(t) x_i^\sigma(t) \Delta t &= \frac{1}{1+\lambda} \int_0^\omega \left[\sum_{j=1}^n a_{ij}(t) f_j(x_j(t)) + \sum_{j=1}^n b_{ij}(t) f_j(x_j(\xi_{ij}(t, x(t)))) + I_i(t) \right] \Delta t \\
&\quad - \frac{\lambda}{1+\lambda} \int_0^\omega \left[\sum_{j=1}^n a_{ij}(t) f_j(-x_j(t)) + \sum_{j=1}^n b_{ij}(t) f_j(-x_j(\xi_{ij}(t, x(t)))) + I_i(t) \right] \Delta t,
\end{aligned}
\tag{3.8}
$$

where $i = 1, 2, \ldots, n$. Then we have from (3.7) that

$$
\begin{aligned}
\left| \int_0^\omega c_i(t) x_i^\sigma(t) \Delta t \right| &\leq \int_0^\omega \left[\sum_{j=1}^n |a_{ij}(t) f_j(x_j(t))| + \sum_{j=1}^n |b_{ij}(t) f_j(x_j(\xi_{ij}(t, x(t))))| + |I_i(t)| \right] \Delta t \\
&\leq \omega \sum_{j=1}^n \left(\overline{a}_{ij} + \overline{b}_{ij} \right) M_j |x_j|_0 + \omega \alpha_i, \quad i = 1, 2, \ldots, n.
\end{aligned}
\tag{3.9}
$$

From Lemma 2.7, for any t_1^i, $t_2^i \in [0, \omega]_{\mathbb{T}}$, $i = 1, 2, \ldots, n$, we have

$$\int_0^\omega c_i(t) x_i^\sigma(t) \Delta t \leqslant \int_0^\omega c_i(t) x_i\left(t_1^i\right) \Delta t + \int_0^\omega c_i(t) \left(\int_0^\omega \left|x_i^\Delta(t)\right| \Delta t\right) \Delta t,$$

$$\int_0^\omega c_i(t) x_i^\sigma(t) \Delta t \geqslant \int_0^\omega c_i(t) x_i\left(t_2^i\right) \Delta t - \int_0^\omega c_i(t) \left(\int_0^\omega \left|x_i^\Delta(t)\right| \Delta t\right) \Delta t, \quad i = 1, 2, \ldots, n. \tag{3.10}$$

Dividing by $\int_0^\omega c_i(t) \Delta t$ on the two sides of the inequalities above, we obtain that

$$-x_i\left(t_1^i\right) \leqslant -\frac{1}{\int_0^\omega c_i(t) \Delta t} \int_0^\omega c_i(t) x_i^\sigma(t) \Delta t + \int_0^\omega \left|x_i^\Delta(t)\right| \Delta t, \tag{3.11}$$

$$x_i\left(t_2^i\right) \leqslant \frac{1}{\int_0^\omega c_i(t) \Delta t} \int_0^\omega c_i(t) x_i^\sigma(t) \Delta t + \int_0^\omega \left|x_i^\Delta(t)\right| \Delta t, \quad i = 1, 2, \ldots, n. \tag{3.12}$$

Let $|x_i(t_0^i)| = \max_{t \in [0, \omega]_{\mathbb{T}}} |x_i(t)|$, $i = 1, 2, \ldots, n$. If for some $i = 1, 2, \ldots, n$, $x_i(t_0^i) \geqslant 0$, we choose $t_2^i = t_0^i$. Hence $\max_{t \in [0, \omega]_{\mathbb{T}}} |x_i(t)| = x_i(t_2^i)$. From (3.12), we have

$$\underline{c}_i |x_i|_0 \leqslant \underline{c}_i \left(\frac{1}{\int_0^\omega c_i(t) \Delta t} \int_0^\omega c_i(t) x_i^\sigma(t) \Delta t + \int_0^\omega \left|x_i^\Delta(t)\right| \Delta t\right)$$

$$\leqslant \frac{1}{\omega} \left|\int_0^\omega c_i(t) x_i^\sigma(t) \Delta t\right| + \underline{c}_i \int_0^\omega \left|x_i^\Delta(t)\right| \Delta t. \tag{3.13}$$

If for some $i = 1, 2, \ldots, n$, $x_i(t_0^i) \leqslant 0$, we choose $t_1^i = t_0^i$. Hence $\max_{t \in [0, \omega]_{\mathbb{T}}} |x_i(t)| = -x_i(t_1^i)$. From (3.11), we also have (3.13).

By using (3.7) and (3.9) and (3.13), for $i = 1, 2, \ldots, n$, we obtain that

$$\underline{c}_i |x_i|_0 \leqslant \sum_{j=1}^n \left(\overline{a}_{ij} + \overline{b}_{ij}\right) M_j |x_j|_0 + \alpha_i + \overline{c}_i \omega |x_i|_0 + \underline{c}_i \omega \sum_{j=1}^n \left(\overline{a}_{ij} + \overline{b}_{ij}\right) M_j |x_j|_0 + \underline{c}_i \omega \alpha_i, \tag{3.14}$$

so,

$$\frac{\underline{c}_i(1 - \omega \overline{c}_i)}{(1 + \underline{c}_i \omega)} |x_i|_0 \leqslant \sum_{j=1}^n \left(\overline{a}_{ij} + \overline{b}_{ij}\right) M_j |x_j|_0 + \alpha_i, \tag{3.15}$$

namely,

$$r_i |x_i|_0 - \sum_{j=1}^n d_{ij} |x_j|_0 \leqslant \alpha_i, \quad i = 1, 2, \ldots, n. \tag{3.16}$$

Denote the following:

$$|x|_0 = \left(|x_1|_0, |x_2|_0, \ldots, |x_n|_0\right)^T, \quad C = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T. \tag{3.17}$$

Thus (3.16) is rewritten in the matrix form

$$E|x|_0 \leqslant C. \tag{3.18}$$

From the conditions of Theorem 3.1, E is a nonsingular M matrix, therefore

$$|x|_0 \leqslant E^{-1}C := (B_1, B_2, \ldots, B_n)^T, \tag{3.19}$$

that is, $|x_i|_0 \leqslant B_i$, $i = 1, 2, \ldots, n$.

Take $A = \max_{1 \leq i \leq n} B_i + 1$ and

$$\Omega = \{x \in \mathbb{X} : \|x\| < A\}. \tag{3.20}$$

It is clear that Ω satisfies all the requirements in Lemma 2.9, and condition (H) is satisfied. In view of all the discussions above, we conclude from Lemma 2.9 that system (1.1) has at least one ω-periodic solution. This completes the proof. $\square$

From Theorem 3.1, when $\mathbb{T} = \mathbb{R}$, $\xi_{ij}(t, \cdot) \equiv t - \tau_{ij}(t)$, we have the following.

Corollary 3.2. *Under assumptions of Theorem 3.1, system* (1.2) *has at least one ω-periodic solution.*

When $\mathbb{T} = \mathbb{Z}$, $\xi_{ij}(n, \cdot) \equiv n - \tau_{ij}(n)$, from Theorem 3.1, we have the following.

Corollary 3.3. *Under assumptions of Theorem 3.1, system* (1.3) *has at least one ω-periodic solution.*

4. Global Exponential Stability of the Periodic Solution

In this section, we will establish some results for the global exponential stability of the periodic solution of (1.1). The following lemma is essential.

Lemma 4.1 (see [36]). *Let $p : \mathbb{T} \to \mathbb{R}$ be rd-continuous and regressive. Suppose that $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous, $\delta \in \mathbb{T}$, and $y_0 \in \mathbb{R}$. Then y is the unique solution of the initial value problem*

$$y^\Delta(t) + p(t)y^\sigma(t) = h(t), \qquad y(\delta) = y_0, \tag{4.1}$$

if and only if

$$y(t) = e_{\ominus p}(t, \delta)y_0 + \int_\delta^t e_{\ominus p}(t, s)h(s)\Delta s. \tag{4.2}$$

Theorem 4.2. *Assume (H_1), (H_2), and (H_4) hold. Furthermore, suppose that*

(H_5) *for $t \in [0, \omega]_\mathbb{T}$, $i = 1, 2, \ldots, n$, $c_i(t) - \widehat{M} > 0$, where $\widehat{M} = \max_{1 \leq i \leq n}\{\sum_{j=1}^n \overline{a}_{ij}M_j + \sum_{j=1}^n \overline{b}_{ij}M_j\}$*

holds. Then the ω-periodic solution of system (1.1) *is globally exponentially stable.*

Proof. From Theorem 3.1, we see that system (1.1) has at least one ω-periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T$. Let $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ be an arbitrary solution of system (1.1). Let $y(t) = x(t) - x^*(t)$, and then we obtain that

$$
\begin{aligned}
y_i^\Delta(t) = & -c_i(t)y_i^\sigma(t) + \sum_{j=1}^n a_{ij}(t)\left[f(x_j(t)) - f\left(x_j^*(t)\right)\right] \\
& + \sum_{j=1}^n b_{ij}(t)\left[f_j(x_j(\xi_{ij}(t, x(t)))) - f_j\left(x_j^*(\xi_{ij}(t, x(t)))\right)\right], \quad i = 1, 2, \ldots, n
\end{aligned}
\tag{4.3}
$$

with initial values given by

$$
y_i(s) = \varphi_i(s) - x_i^*(s), \quad s \in [-\tau, 0]_{\mathbb{T}}, \ i = 1, 2, \ldots, n,
\tag{4.4}
$$

where each $\varphi_i \in C[(-\tau, 0]_{\mathbb{T}}, \mathbb{R}]$ is bounded.

Let $h(t) = (h_1(t), h_2(t), \ldots, h_n(t))^T$ and

$$
\begin{aligned}
h_i(t) = & \sum_{j=1}^n a_{ij}(t)\left(f(x_j(t)) - f\left(x_j^*(t)\right)\right) \\
& + \sum_{j=1}^n b_{ij}(t)\left[f_j(x_j(\xi_{ij}(t, x(t)))) - f_j\left(x_j^*(\xi_{ij}(t, x(t)))\right)\right], \quad i = 1, 2, \ldots, n.
\end{aligned}
\tag{4.5}
$$

From Lemma 4.1, then

$$
y_i(t) = e_{\ominus c_i}(t, \delta)y_0 + \int_\delta^t e_{\ominus c_i}(t, s)h_i(s)\Delta s, \quad i = 1, 2, \ldots, n, \ \delta \in (-\infty, 0]_{\mathbb{T}}, \ t \geq \delta
\tag{4.6}
$$

is the unique solution of (4.3) with initial value $y(\delta) = y_0$. From (4.6), we have

$$
\frac{|y_i(t)|}{e_{\ominus c_i}(t, \delta)} \leq |y_0| + \int_\delta^t \frac{1}{e_{\ominus c_i}(s, \delta)}|h_i(s)|\Delta s, \quad i = 1, 2, \ldots, n, \ t \geq \delta
\tag{4.7}
$$

and, according to (4.5), we can obtain that

$$
\|h\| \leq \left(\sum_{j=1}^n \overline{a}_{ij}M_j + \sum_{j=1}^n \overline{b}_{ij}M_j\right)\|y\| \leq \widehat{M}\|y\|, \quad i = 1, 2, \ldots, n, \ t \geq \delta.
\tag{4.8}
$$

Thus, by (4.7), we have

$$
\frac{\|y\|}{e_{\ominus c_i}(t, \delta)} \leq \|y_0\| + \int_\delta^t \widehat{M}\frac{\|y\|}{e_{\ominus c_i}(s, \delta)}\Delta s, \quad i = 1, 2, \ldots, n, \ t \geq \delta.
\tag{4.9}
$$

By virtue of Gronwall's inequality ([32], Corollary 6.7), we obtain

$$\frac{\|y\|}{e_{\ominus c_i}(t, \delta)} \leq \|y_0\| e_{\widehat{M}}(t, \delta), \quad i = 1, 2, \ldots, n, \ t \geq \delta. \tag{4.10}$$

Hence we have

$$\begin{aligned}
\|y\| &\leq \|y_0\| e_{\widehat{M}}(t, \delta) e_{\ominus c_i}(t, \delta) \\
&\leq \|y_0\| e_{\ominus c_i \oplus \widehat{M}}(t, \delta) \\
&= \|y_0\| e_{-(c_i - \widehat{M})/(1 + \mu c_i)}(t, \delta) \\
&\leq \|y_0\| e_{-\gamma}(t, \delta), \quad t \geq \delta,
\end{aligned} \tag{4.11}$$

where $\gamma = \min_{1 \leq i \leq n}\{(c_i - \widehat{M})/(1 + \mu c_i)\}, i = 1, 2, \ldots, n$. We can rewrite (4.10) as follows

$$\sum_{i=1}^{n} \|x_i(t) - x_i^*(t)\| \leq \|\phi(\delta) - x^*(\delta)\| e_{-\gamma}(t, \delta), \quad t \geq \delta. \tag{4.12}$$

Therefore, the ω-periodic solution of system (1.1) is globally exponentially stable. This completes the proof. □

From Theorem 4.2, when $\mathbb{T} = \mathbb{R}$, $\xi_{ij}(t, \cdot) \equiv t - \tau_{ij}(t)$, we have the following.

Corollary 4.3. *Under assumptions of Theorem 4.2, all the ω-periodic solutions of system* (1.2) *are globally exponentially stable.*

When $\mathbb{T} = \mathbb{Z}$, $\xi_{ij}(n, \cdot) \equiv n - \tau_{ij}(n)$, we have the following.

Corollary 4.4. *Under assumptions of Theorem 4.2, all the ω-periodic solutions of system* (1.3) *are globally exponentially stable.*

Remark 4.5. Corollaries 3.2 and 4.3 improve the corresponding results obtained in [4–6, 13, 14] and are different from those in [8, 11, 12, 16, 17] and the references cited therein. Therefore, the results of this paper are new, and they complement previously known results. Corollaries 3.3 and 4.4 improve the results obtained in [15].

5. An Example

Consider the following cellular neural networks with delays:

$$x_i^\Delta(t) = -c_i(t)x_i(t) + \sum_{j=1}^{2} a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{2} b_{ij}(t)f_j(x_j(t - \tau_{ij}(t))) + I_i(t), \tag{5.1}$$

where $t \in \mathbb{T}$, $i = 1, 2$, $\mathbb{T}$ is a 1-periodic time scale, $c_1(t) = 1/4 + (1/8)\sin(2\pi t)$, $c_2(t) = 1/3 + (1/6)\cos(2\pi t)$, $a_{11}(t) = \sin^2(\pi t)$, $a_{12}(t) = a_{21}(t) = (1/4)\sin^4(\pi t)$, $a_{22}(t) = |\cos(\pi t)|$,

$b_{11}(t) = 2\cos^2(\pi t)$, $b_{12}(t) = b_{21}(t) = (1/8)\cos^6(\pi t)$, $b_{22}(t) = (1/2)|\sin(\pi t)|$, $f_1(x_1) = (1/112)x + (1/56)\sin((1/2)x_1) + 3$, $f_2(x_2) = (1/126)x + (1/42)\cos((2/3)x_2) - 11$, $I_1(t) = \sin^6(\pi t)$, $I_2(t) = 3\cos(2\pi t)$, $\tau_{11}(t) = e^{\cos(2\pi t)}$, $\tau_{12}(t) = 3^{\sin^2(\pi t)}$, $\tau_{21}(t) = \ln(\sin^4(\pi t))$, $\tau_{22}(t) = 3/7 + 5^{\cos^6(\pi t)}$.

By calculating, we have $\overline{c}_1 = 3/8$, $\underline{c}_1 = 1/8$, $\overline{c}_2 = 1/2$, $\underline{c}_2 = 1/6$, $M_1 = 1/56$, $M_2 = 1/42$, $\overline{a}_{11} = 1$, $\overline{a}_{12} = \overline{a}_{21} = 1/4$, $\overline{a}_{22} = 1$, $\overline{b}_{11} = 2$, $\overline{b}_{12} = \overline{b}_{21} = 1/8$, $\overline{b}_{22} = 1/2$. It is not difficult to verify that (H1)-(H2) are satisfied.

Also by calculating, we have that

$$
E = (e_{ij})_{2 \times 2} = \begin{pmatrix} \dfrac{1}{28} & -\dfrac{1}{112} \\ -\dfrac{15}{228} & \dfrac{1}{28} \end{pmatrix}
\tag{5.2}
$$

is a nonsingular M matrix. If we take $\xi_1 = \xi_2 = 1$, we can obtain that

$$
-\xi_1 \underline{c}_1 + \sum_{j=1}^{2} \xi_j \left(\overline{a}_{1j} + \overline{b}_{1j} \right) M_j = -\frac{7}{112} < 0, \qquad -\xi_2 \underline{c}_2 + \sum_{j=1}^{2} \xi_j \left(\overline{a}_{2j} + \overline{b}_{2j} \right) M_j = -\frac{167}{1344} < 0. \tag{5.3}
$$

The condition (H3) is satisfied. By Theorems 3.1 and 4.2, we know that system (5.1) has at least one 1-periodic solution, and this solution is globally exponentially stable.

6. Conclusion

In this paper, without assuming the boundedness of the activation functions, we establish the stability and existence of periodic solutions of cellular neural networks with delays on time scales. Our results obtained in this paper are completely new even in case of the time scale $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$. Besides, our method used in this paper may be used to study other type neural networks such as BAM neural networks, high-order Hopfield neural networks, and Cohen-Grossberg neural networks.
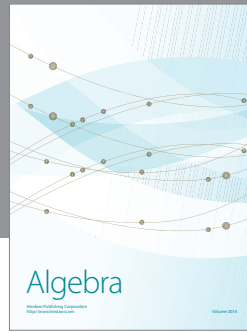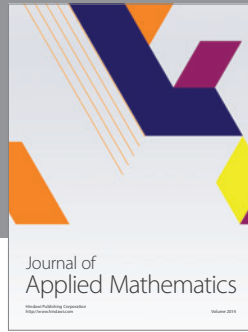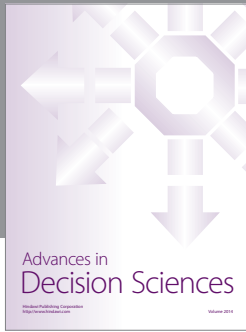
Acknowledgments

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