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Research Article

Study of a Forwarding Chain in the Category of Topological Spaces between T_0 and T_2 with respect to One Point Compactification Operator

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In the following text, we want to study the behavior of one point compactification operator in the chain $\Xi :=$ {Metrizable, Normal, T_2, KC, SC, US, T_1, T_D, T_{UD}, T_0, Top} of subcategories of category of topological spaces, Top (where we denote the subcategory of Top, containing all topological spaces with property P , simply by P). Actually we want to know, for $P \in \Xi$ and $X \in P$, the one point compactification of topological space X belongs to which elements of Ξ. Finally we find out that the chain {Metrizable, T_2, KC, SC, US, T_1, T_D, T_{UD}, T_0, Top} is a forwarding chain with respect to one point compactification operator.

1. Introduction

The concept of forwarding and backwarding chains in a category with respect to a given operator has been introduced for the first time in [1] by the first author. The matter has been motivated by the following sentences in [1]: "*In many problems, mathematicians search for theorems with weaker conditions or for examples with stronger conditions. In other words they work in a subcategory $\mathscr{D}$ of a mathematical category, namely, $\mathscr{C}$, and they want to change the domain of their activity (theorem, counterexample, etc.) to another subcategory of $\mathscr{C}$ like $\mathscr{K}$ such that $\mathscr{K} \subseteq \mathscr{D}$ or $\mathscr{D} \subseteq \mathscr{K}$ according to their need.*" Most of us have the memory of a theorem and the following question of our professors: "Is the theorem valid with weaker conditions for hypothesis or stronger conditions for result?" The concept of forwarding, backwarding, or stationary chains of subcategories of a category $\mathscr{C}$ tries to describe this phenomenon.

In this text, Top denotes the category of topological spaces. Whenever P is a topological property, we denote the subcategory of Top containing all the topological spaces with property P, simply by P. For example, we denote the category of all metrizable spaces by Metrizable.

We want to study the chain {Metrizable, Normal, T_2, KC, SC, US, T_1, T_D, T_{UD}, T_0, Top} of subcategories of Top in the point of view of forwarding, backwarding, and stationary chains' concept with respect to one point compactification or Alexandroff compactification operator.

Remark 1. Suppose $\leq$ is a partial order on A. We call $B \subseteq A$

(i) a *chain*, if for all $x, y \in B$, we have $x \leq y \vee y \leq x$;

(ii) *cofinal*, if for all $x \in A$, there exists $y \in B$ such that $x \leq y$.

In the following text, by a chain of subcategories of category $\mathscr{C}$, we mean a chain under "$\subseteq$" relation (of subclasses of $\mathscr{C}$). We recall that if $\mathscr{M}$ is a chain of subcategories of category $\mathscr{C}$ such that $\bigcup\mathscr{M}$ is closed under (multivalued) operator ψ, then we call $\mathscr{M}$

(i) a *forwarding chain with respect to* ψ; if for all $C \in \mathscr{M}$, we have $\psi((\bigcup\mathscr{M}) \setminus C) \cap C = \varnothing$ (i.e., $\psi((\bigcup\mathscr{M}) \setminus C) \subseteq (\bigcup\mathscr{M}) \setminus C)$;

(ii) a *full-forwarding chain with respect to* ψ; if it is a forwarding chain with respect to ψ and for all distinct $C_1, C_2, C_3 \in \mathscr{M}$, we have

$$C_1 \subseteq C_2 \subseteq C_3 \Longrightarrow \left(\exists X \in C_2 \setminus C_1 \ \psi(X) \in C_3 \setminus C_2\right), \quad (1)$$

where, for multivalued function ψ, by $\psi(X) \in C_3 \setminus C_2$, we mean that at least one of the values of $\psi(X)$ belongs to $C_3 \setminus C_2$;

(iii) a *backwarding chain with respect to* ψ; if for all $C \in \mathscr{M}$, we have $\psi(C) \subseteq C$;

(iv) a *full-backwarding chain with respect to* ψ; if it is a backwarding chain with respect to ψ and for any distinct $C_1, C_2, C_3 \in \mathscr{M}$, we have

$$C_1 \subseteq C_2 \subseteq C_3 \Longrightarrow \left(\exists X \in C_3 \setminus C_2 \ \psi(X) \in C_2 \setminus C_1\right), \quad (2)$$

where, for multivalued function ψ, by $\psi(X) \in C_2 \setminus C_1$, we mean that at least one of the values of $\psi(X)$ belongs to $C_2 \setminus C_1$;

(v) a *stationary chain with respect to* ψ if it is both forwarding and backwarding chains with respect to ψ.

Basic properties of forwarding, backwarding, full-forwarding, full-backwarding, and stationary chains with respect to given operators have been studied in [1]. We refer the interested reader to [2] for standard concepts of the Category Theory.

We recall that by $\mathbb{N}$ we mean the set of all natural numbers $\{1, 2, \ldots\}$; also $\omega = \{0, 1, 2, \ldots\}$ is the least infinite ordinal (cardinal) number and Ω is the least infinite uncountable ordinal number. Here ZFC and GCH (generalized continuum hypothesis) are assumed (note: by GCH for infinite cardinal number α, there is not any cardinal number β with $\alpha < \beta < 2^\alpha$, i.e., $\alpha^+ = 2^\alpha$).

We call a collection $\mathscr{F}$ of subsets of X a filter over X if $\varnothing \notin \mathscr{F}$; for all $A, B \in \mathscr{F}$ we have $A \cap B \in \mathscr{F}$; for all $A \in \mathscr{F}$ and $B \subseteq X$ with $A \subseteq B$ we have $B \in \mathscr{F}$. If $\mathscr{F}$ is a maximal filter over X (under $\subseteq$ relation), then we call it an ultrafilter over X. If for all $A \in \mathscr{F}$, we have $\mathrm{card}(A) = \mathrm{card}(X)$; then we call $\mathscr{F}$ a uniform ultrafilter over X.

We end this section by the following two examples.

(I) For $n \in \mathbb{N}$, let $C_n := [0, 1/n]$ and $\psi : [0,1] \to [0,1]$ with $\psi(0) = 0$ and $\psi(x) = n^2 x + 1 - n$ for $x \in C_n \setminus C_{n+1} = (1/(n+1), 1/n]$. Then $\{C_n : n \in \mathbb{N}\}$ is full-forwarding with respect to ψ [1, Example 2.2].

(II) Let ON denote the class of all ordinal numbers; CN denotes the class of all cardinal numbers; for every set A by $|A|$ we mean cardinal number of A, and for each cardinal number $\alpha \in \mathrm{ON}$, $D_\alpha = \{\gamma \in \mathrm{ON} : |\gamma| = \alpha\}$, $C_\alpha = \{\gamma \in \mathrm{ON} : |\gamma| < \alpha\}$. Define $\psi : \mathrm{ON} \to \mathrm{ON}$ with $\psi(\gamma) = \gamma - \alpha$ for $\gamma \in D_\alpha$ and $\alpha \in \mathrm{CN}$. Then $\{C_\alpha : \alpha \geq \omega\}$ is a full-backwarding chain of subclasses of ON with respect to ψ [1, Example 2.3].

2. Basic Definitions in Separation Axioms

In this section we bring our basic definitions in Top.

Convention 1. Henceforth in the topological space X suppose $\infty \notin X$. So (see [3, 4])

$$\mathscr{B} := \{U \subseteq X : U \text{ is an open subset of } X\} \cup \{V \cup \{\infty\} : V \subseteq X,\ X \setminus V \text{ is compact and closed}\} \quad (3)$$

is a topological basis on $X \cup \{\infty\}$. The space $X \cup \{\infty\}$ with topological basis $\mathscr{B}$ is called one point compactification or Alexandroff compactification of X.

Let

$$A(X) := \begin{cases} \text{one point compactification of } X & X \text{ is not compact,} \\ X & X \text{ is compact.} \end{cases} \quad (4)$$

By the operator A, in this text we mean the above mentioned operator.

Remark 2. We call a topological space X (if $A \subseteq X$, by A', we mean the set of all limit points of A in X)

(i) T_0; if for all distinct $x, y \in X$, there exist open neighborhood U of x and open neighborhood V of y such that $x \notin V$ or $y \notin U$;

(ii) T_{UD}; if for all $x \in X$, $\{x\}'$ is a union of closed subsets of X;

(iii) T_D; if for all $x \in X$, $\{x\}'$ is a closed subset of X;

(iv) T_1; if for all $x \in X$, $\{x\}$ is a closed subset of X;

(v) US if any convergent sequence has a unique limit;

(vi) SC; if for any convergent sequence $(x_n : n \in \omega)$ to $x \in X$, $\{x_n : n \in \omega\} \cup \{x\}$ is a closed subset of X;

(vii) KC if any compact subset of X is closed;

(viii) T_2 (or Hausdorff); if for all distinct $x, y \in X$ there exist open neighborhood U of x and open neighborhood V of y with $U \cap V = \varnothing$;

(ix) normal; if it is T_2 and for every disjoint closed subsets A, B of X, there exist disjoint open subsets U, V of X with $A \subseteq U$ and $B \subseteq V$;

(x) k-space; if for all $A \subseteq X$, A is closed if and only if for all closed compact subset K of X, $A \cap K$ is closed.

Regarding [5], we have $T_2 \subseteq \mathrm{KC} \subseteq \mathrm{SC} \subseteq \mathrm{US} \subseteq T_1$. Also by [6] we have $T_1 \subseteq T_D \subseteq T_{UD} \subseteq T_0$; therefore.

Metrizable $\subseteq$ Normal $\subseteq T_2 \subseteq \mathrm{KC} \subseteq \mathrm{SC} \subseteq \mathrm{US} \subseteq T_1 \subseteq T_D \subseteq T_{UD} \subseteq T_0 \subseteq$ Top.

In this section, we want to study the operator A on the above chain. However, it has been proved in [1, Lemma 3.1 and Corollary 3.2] that the chain $T_1 \subseteq T_D \subseteq T_{UD} \subseteq T_0$ is stationary with respect to the operator A; therefore, the main interest is on Metrizable $\subseteq$ Normal $\subseteq$ T_2 $\subseteq$ KC $\subseteq$ SC $\subseteq$ US $\subseteq$ T_1.

Note 1. A topological space X is KC if and only if $\{U \subseteq X : U$ is an open subset of $X\} \cup \{V \cup \{\infty\} : V \subseteq X$ and $X \setminus V$ is a compact subset of $X\}$ is a topological basis on X.

Remark 3. Suppose X is noncompact space and $A(X) = X \cup \{\infty\}$ is one point compactification of X. We have the following.

(1) If X is KC, then $A(X)$ is US (therefore $A(X)$ is T_1 too) [7, Theorem 4].

(2) If X is KC, then $A(X)$ is KC if and only if X is a k-space [7, Theorem 5].

(3) $A(X)$ is T_2 if and only if X is T_2 and locally compact [4]; thus $A(X)$ is T_2 if and only if it is normal.

(4) $A(X)$ is an embedding of X.

(5) If $A(X)$ is KC, then X is KC too (hint: if K is a compact subset of X, then K is a compact subset of $A(X)$ by (2). If $A(X)$ is KC, then K is a closed subset of $A(X)$, and again by (2), K is a closed subset of X, so X is KC).

(6) A T_2 space is a k-space if it is either first countable or locally compact so every metrizable space is k-space [3, 7].

(7) X is T_1 if and only if $A(X)$ is T_1 [3]. Moreover if X is T_0 (and noncompact), then

(8) X is T_D if and only if $A(X)$ is T_D [1, Lemma 3.1].

(9) X is T_{UD} if and only if $A(X)$ is T_{UD} [1, Lemma 3.1].

(10) $A(X)$ is T_0 [1, Lemma 3.1].

For topological spaces X, Y, by $X \sqcup Y$, we mean topological disjoint union of X and Y.

Lemma 4. *Let X_1 is a compact topological space, X_2 is a noncompact topological space, then $A(X_1 \sqcup X_2) = X_1 \sqcup A(X_2)$.*

Proof. Suppose $X_1 \cap X_2 = \varnothing$ and U is an open subset of $A(X_1 \sqcup X_2) = X_1 \cup X_2 \cup \{\infty\}$. Using the following cases, U is an open subset of $X_1 \sqcup A(X_2)(= X_1 \cup X_2 \cup \{\infty\})$ too.

(i) Consider $\infty \notin U$. In this case, U is an open subset of $X_1 \sqcup X_2$, so $U_1 := U \cap X_1$ is an open subset of X_1 and $U_2 := U \cap X_2$ is an open subset of not only X_2 but also $A(X_2)$ using the definition of one point compactification. Then the set $U_1 \cup U_2$ is an open subset of $X_1 \sqcup A(X_2)$, since $U = U_1 \cup U_2$, U is an open subset of $X_1 \sqcup A(X_2)$.

(ii) Consider $\infty \in U$. In this case, $(X_1 \cup X_2) \setminus U$ is a closed compact subset of $X_1 \sqcup X_2$. Since X_1 and X_2 are two closed subsets of $X_1 \sqcup X_2$, $((X_1 \cup X_2) \setminus U) \cap X_1 = X_1 \setminus U$ is a closed subset of X_1 and $((X_1 \cup X_2) \setminus U) \cap X_2 = X_2 \setminus U$ is a closed subset of X_2 and $(X_1 \cup X_2) \setminus U$, so $X_2 \setminus U$ is a closed compact subset of X_2. Therefore, $U_1 := X_1 \setminus (X_1 \setminus U) = X_1 \cap U$ is an open subset of X_1 and $U_2 := A(X_2) \setminus (X_2 \setminus U) = A(X_2) \cap U (= (X_2 \cap U) \cup \{\infty\})$ is an open subset of $A(X_2)$. Then the set $U_1 \cup U_2$ is an open subset of $X_1 \sqcup A(X_2)$, since $U = U_1 \cup U_2$, U is an open subset of $X_1 \sqcup A(X_2)$. Conversely, if V is an open subset of $X_1 \sqcup A(X_2)$, then, using the following cases, V is an open subset of $A(X_1 \sqcup X_2)$ too.

(iii) Consider $\infty \notin V$. In this case, $V_1 := V \cap X_1$ is an open subset of X_1. Also $V_2 := V \cap A(X_2) = V \cap X_2$ is an open subset of $A(X_2)$ and X_2. Thus, $V = V_1 \cup V_2$ is an open subset of $X_1 \sqcup X_2$; hence it is an open subset of $A(X_1 \sqcup X_2)$.

(iv) Consider $\infty \in V$. In this case, $V_1 := V \cap X_1$ is an open subset of X_1 by Remark 3(4). Using the compactness of X_1, $X_1 \setminus V_1$ is a closed compact subset of X_1. Also $V_2 := V \cap A(X_2)$ is an open subset of $A(X_2)$ containing ∞; thus $X_2 \setminus V_2$ is a closed compact subset of X_2. Since $X_1 \setminus V_1$ and $X_2 \setminus V_2$ are two closed compact subsets of $X_1 \sqcup X_2$, $(X_1 \setminus V_1) \cup (X_2 \setminus V_2) = (X_1 \cup X_2) \setminus V$ is a closed compact subset of $X_1 \sqcup X_2$ too. Hence V is an open subset of $A(X_1 \sqcup X_2)$. □

Lemma 5. *If Y is a closed subset of X, then $A(X)$ is an embedding of $A(Y)$.*

Proof. If Y is compact, then $A(Y) = Y$ and by Remark 3(4) we are done. If Y is not compact, $X \setminus Y$ is an open subset of X and $A(X)$; thus $Y \cup \{\infty\}$ is a closed compact subset of $A(X)$. Suppose $F \subseteq Y \cup \{\infty\}$; we prove that F is a closed subset of $Y^* := Y \cup \{\infty\}$ as a subspace of $A(X)$ if and only if F is a closed subset of $A(Y) = Y \cup \{\infty\}$ as one point compactification of Y. However, we mention that $Y \cup \{\infty\}$ in both topologies is an embedding of Y by Remark 3(4).

First, suppose F is a closed subset of Y^*. Using the following two cases, F is a closed subset of $A(Y)$ too.

(i) Consider $\infty \in F$. In this case, $U := Y^* \setminus F = Y \setminus F$ is an open subset of Y; therefore it is an open subset of $A(Y)$, so $F = A(Y) \setminus U$ is a closed subset of $A(Y)$.

(ii) Consider $\infty \notin F$. In this case, F is a closed subset of $A(X)$ since it is a closed subset of Y^* and Y^* is closed in $A(X)$. Therefore, $U := A(X) \setminus F$ is an open subset of $A(X)$ with $\infty \in U$. So $A(X) \setminus U$ is a closed compact subset of X. Therefore, $(A(X) \setminus U) \cap Y^* = F$ is a closed compact subset of Y^*. Since $(A(X) \setminus U) \cap Y^* = (A(X) \setminus U) \cap Y$, F is a closed compact subset of Y, so F is closed in $A(Y)$. Conversely, suppose F is a closed subset of $A(Y)$. Using the following two cases, F is a closed subset of Y^* too.

(iii) Consider $\infty \in F$. In this case, $U := A(Y) \setminus F = Y \setminus F$ is an open subset of Y; therefore, there exists an open subset V of X with $V \cap Y = U$. V is an open subset of $A(X)$ too; thus $V \cap Y^*$ is an open subset of Y^*; therefore $Y^* \setminus (V \cap Y^*) = Y^* \setminus (V \cap Y) = Y^* \setminus U = F$ is a closed subset of Y^*.

(iv) Consider $\infty \notin F$. In this case, F is a closed compact subset of $A(Y)$ with $\infty \notin F$; thus F is a closed compact subset of Y. Hence, F is a closed compact subset of X, and $U = A(X) \setminus F$ is an open subset of $A(X)$. Therefore, $U \cap Y^* = Y^* \setminus F$ is an open subset of Y^*, so F is a closed subset of Y^*. □

Lemma 6. *Suppose* $\mathscr{C} \in$ {*Metrizable, Normal,* T_2, KC, SC, US, T_1, T_D, T_{UD}, T_0, *Top*}; *also consider topological spaces* X, Y. *We have the following.*

(1) $X \sqcup Y \in \mathscr{C}$ *if and only if* $X, Y \in \mathscr{C}$.

(2) *Consider two closed subsets* A, B *of* X *with* $A \cup B = X$ *and* $A \cap B = \{t\}$. *So* $A, B \in \mathscr{C}$ *if and only if* $X \in \mathscr{C}$.

Proof. (1) has a formal proof, so we deal with (2). If $X \in \mathscr{C}$ and E is a closed subspace of X, then $E \in \mathscr{C}$. Suppose $A, B \in \mathscr{C}$; A, B are closed subspaces of X with $A \cap B = \{t\}$ and $A \cup B = X$. We prove $X \in \mathscr{C}$.

First, note the fact that if V is an open subset of A (resp. B) with $t \notin V$, then V is an open subset of X, since V is an open subset of $A \setminus \{t\}$ and $A \setminus \{t\} (= X \setminus B)$ is an open subset of X. Now consider the following cases for $\mathscr{C}$.

(i) Consider $\mathscr{C}$ = Metrizable. If A, B are metrizable subspaces of X, then there exist metrics d_1, d_2, respectively, on A, B such that $d_1, d_2 \leq 1$, the metric topology induced from d_1 on A is subspace topology on A induced from X, and the metric topology induced from d_2 on B is subspace topology on B induced from X. Define $d : X \times X \to [0, +\infty)$ with

$$d(x, y) = \begin{cases} d_1(x, y) & x, y \in A, \\ d_2(x, y) & x, y \in B, \\ 2 & \text{otherwise}. \end{cases} \quad (5)$$

Then the metric topology induced from d on X coincides with X's original topology.

(ii) Consider $\mathscr{C} = T_2$. Suppose A, B are Hausdorff subspaces of X and $x, y \in X$ are two distinct points of X. Consider the following cases:

(1) $x \in X \setminus A = B \setminus \{t\}$ and $y \in X \setminus B = A \setminus \{t\}$; in this case, $B \setminus \{t\}$ and $A \setminus \{t\}$ are disjoint open neighborhoods of, respectively, x and y;

(2) $x, y \in A$; there exist disjoint open subsets U_1, U_2 of A with $x \in U_1$ and $y \in U_2$. Suppose $t \notin U_1$; thus U_1 is an open subset of X. There exists an open subset U of X with $U \cap A = U_2$. Hence, U_1, U are disjoint open subsets of X with $x \in U_1$ and $y \in U$.

Using the above cases, X is Hausdorff.

(iii) Consider $\mathscr{C}$ = Normal. If A, B are normal subspaces of X, then A, B are Hausdorff and, using the case "$\mathscr{C} = T_2$", X is Hausdorff. Now suppose E, F are disjoint closed subsets of X; also we may suppose $t \notin E$. Let $E_A := E \cap A$, $E_B := E \cap B$, $F_A := F \cap A$. and $F_B := F \cap B$. There are disjoint open subsets U_E, U_F of A containing, respectively, E_A, F_A. Also there are disjoint open subsets V_E, V_F of B containing, respectively, E_B, F_B. There are open subsets U, V of X with $U_F = A \cap U$ and $V_F = V \cap B$. Let $W_E := (U_E \setminus \{t\}) \cup (V_E \setminus \{t\})$ and $W_F := U \cup V$; then W_E, W_F are disjoint open subsets of X containing, respectively, E, F.

(iv) Consider $\mathscr{C}$ = KC. Suppose A, B are KC and K is a compact subset of X. Since A, B are closed, $A \cap K$, $B \cap K$ are compact too. Since $A \cap K$ is a compact subset of A and A is KC, $A \cap K$ is a closed subset of A. Since $A \cap K$ is a closed subset of A and A is a closed subset of X, $A \cap K$ is closed subset of X. Similarly, $B \cap K$ is a closed subset of X. Thus $K = (A \cap K) \cup (B \cap K)$ is a closed subset of X and X is KC.

(v) Consider $\mathscr{C}$ = SC. Suppose A, B are SC and $(x_n : n \in \omega)$ is a sequence in X converging to x. Using the following cases, $\{x_n : n \in \omega\} \cup \{x\}$ is a closed subset of X.

(1) Consider $x \neq t$. Suppose $x \in A \setminus \{t\}$. In this case, $A \setminus \{t\}$ is an open neighborhood of x in X, so there exists $N \in \omega$ such that $x_n \in A \setminus \{t\}$ for all $n \geq N$. Hence $(x_n : n \geq N)$ is a converging sequence to x in A. Since A is SC, $\{x_n : n \geq N\} \cup \{x\}$ is a closed subset of A. Therefore, $\{x_n : n \geq N\} \cup \{x\}$ is a closed subset of X. For each $n \in \omega$ if $x_n \in B$ (resp. $x_n \in A$), $\{x_n\}$ is a closed subset of B (resp. A) since B (resp. A) is SC and in particular T_1. Thus for all $n \in \omega$, $\{x_n\}$ is a closed subset of X. By closeness of $\{x_n : n \leq N\}$ and $\{x_n : n \geq N\} \cup \{x\}$ in X, the set $\{x_n : n \in \omega\} \cup \{x\}$ is closed in X.

(2) Consider $x = t$ and there exists $N \in \omega$ such that $\{x_n : n \geq N\} \subseteq A$ or $\{x_n : n \geq N\} \subseteq B$. Suppose there exists $N \in \omega$ with $\{x_n : n \geq N\} \subseteq A$. In this case. $(x_n : n \geq N)$ is a converging sequence to x in A, and, using the same argument as in the second paragraph of the case "$x \neq t$", $\{x_n : n \in \omega\}$ is closed in X.

(3) Consider none of the above two cases. In this case, $(x_n : n \in \omega)$ converges to t and it has two subsequences $(x_{n_k} : k \in \omega)$ and $(x_{m_k} : k \in \omega)$ such that $\{x_{n_k} : k \in \omega\} \subseteq A$, $\{x_{m_k} : k \in \omega\} \subseteq B$, and $\{n_k : k \in \omega\} \cup \{m_k : k \in \omega\} = \omega$. Using item (2), $\{x_{n_k} : k \in \omega\} \cup \{x\}$ and $\{x_{m_k} : k \in \omega\} \cup \{x\}$ are two closed subsets of X; thus $\{x_n : k \in \omega\} \cup \{x\} = \{x_{n_k} : k \in \omega\} \cup \{x\} \cup \{x_{m_k} : k \in \omega\} \cup \{x\}$ is a closed subset of X.

(vi) Consider $\mathscr{C}$ = US. If A, B are US and X is not US, consider converging sequence $(x_n : n \in \omega)$ in X to x, y with $x \neq y$. Let $x \neq t$; we may suppose $x \in A$. The set $A \setminus \{t\} (= X \setminus B)$ is an open neighborhood of x. Thus there exists $N \in \omega$ with $\{x_n : n \geq N\} \subseteq A \setminus \{t\}$ and $y \in \overline{\{x_n : n \geq N\}} \subseteq A$. So $(x_n : n \geq N)$ is a converging sequence to x, y in A and $x \neq y$; thus A is not US which is a contradiction.

(vii) Consider $\mathscr{C} = T_1$. Suppose A and B are T_1; let $x \in X$. We may suppose $x \in A$. Since A is T_1, $\{x\}$ is a closed subset of A. Since A is a closed subset of X and $\{x\}$ is a closed subset of A, $\{x\}$ is a closed subset of X.

(viii) Use similar methods for the rest of the cases of $\mathscr{C}$. □

Lemma 7. *Suppose* $\mathscr{C} \in \{$*Metrizable, Normal,* T_2, KC, SC, US, T_1, T_D, T_{UD}, T_0, *Top*$\}$; *also consider topological spaces* X, Y. *We have* $A(X \sqcup Y) \in \mathscr{C}$ *if and only if* $A(X), A(Y) \in \mathscr{C}$.

Proof. By Lemma 6 and Lemma 4, it is clear if X or Y is compact. So we may suppose X and Y are two disjoint noncompact topological spaces. Since X and Y are two open subset of $X \sqcup Y$, two sets $X^* := A(X \sqcup Y) \setminus Y (= X \cup \{\infty\})$ and $Y^* := A(X \sqcup Y) \setminus X (= Y \cup \{\infty\})$ are two closed subsets of $A(X \sqcup Y)$ with $X^* \cup Y^* = A(X \sqcup Y)$. By Lemma 6, $A(X \sqcup Y) \in \mathscr{C}$ if and only if $X^*, Y^* \in \mathscr{C}$. By Lemma 5, X^* is homeomorphic to $A(X)$ and Y^* is homeomorphic to $A(Y)$; hence $A(X \sqcup Y) \in \mathscr{C}$ if and only if $A(X), A(Y) \in \mathscr{C}$. □

Lemma 8. *In topological space* X, *if* X *is SC, then* $A(X)$ *is US.*

Proof. Let X be a noncompact SC space. Suppose $(x_n : n \in \omega)$ is a sequence in $A(X) = X \cup \{\infty\}$ converging to $x, y \in A(X)$. We have the following cases.

(i) Consider $x, y \in X$. In this case, X is an open neighborhood of x, y in $A(X)$; hence there exists $N \in \omega$ such that $x_n \in X$ for all $n \geq N$. Therefore, $(x_n : n \geq N)$ is a converging sequence in X to x, y. Since X is SC, X is US and $x = y$.

(ii) Consider $x \in X$, $y = \infty$. In this case, there exists $N \in \omega$ such that $x_n \in X$ for all $n \geq N$. Therefore, $(x_n : n \geq N)$ is a converging sequence in X to x. Thus $\{x_n : n \geq N\} \cup \{x\}$ is a closed subset of X. So $\{x_n : n \geq N\} \cup \{x\}$ is a compact closed subset of X and $V := A(X) \setminus (\{x_n : n \geq N\} \cup \{x\})$ is an open neighborhood of $\infty (= y)$ which is a contradiction by $x_n \notin V$ for all $n \geq N$ and by converging $(x_n : n \in \omega)$ to y. So this case does not occur.

Using the above cases, we have $x = y$, and $A(X)$ is US. □

3. The Main Table

See Figure 1; then we have Table 1 which we prove in this Section and where:

The mark "$\surd$" indicates that in the corresponding case, there exists $X \in P$ such that $A(X) \in Q$, and the mark "—" indicates that in the corresponding case for all $X \in P$ we have $A(X) \notin Q$.

Let

$$\begin{aligned} \mathtt{E} &:= \{C_1, C_2 \setminus C_1, C_3 \setminus C_2, C_4 \setminus C_3, \\ &\qquad C_5 \setminus C_4, C_6 \setminus C_5, C_7 \setminus C_6\}, \\ \mathtt{F} &:= \{C_8 \setminus C_7, C_9 \setminus C_8, C_{10} \setminus C_9, C_{11} \setminus C_{10}\}. \end{aligned} \tag{6}$$

By Remark 3(7) in Table 1, the mark "—" for cases in which "$P \in \mathtt{E}, Q \in \mathtt{F}$" or "$P \in \mathtt{F}, Q \in \mathtt{E}$" is evident. However, it has been proved in [1, Lemma 3.1 and Corollary 3.2] that the chain $T_1 \subseteq T_D \subseteq T_{UD} \subseteq T_0$ is stationary with respect to the operator A, so corresponding marks of the cases in which $P, Q \in \mathtt{F}$ are obtained. Thus it remains to discuss cases in which $P, Q \in \mathtt{E}$.

Since the subspace of a metrizable (resp. T_2, SC, and US) space is metrizable (resp. T_2, SC, and US) using Remark 3(4) and (5), if $A(X)$ is, respectively, metrizable T_2, KC, SC, or US, then X is too. Hence we obtain "—" for the following cases too (choose P and Q from the same rows of Table 2).

Proof (proof of the rest of the cells of Figure 1).

First Row. Here we have $P = C_1$ and the following cases for Q.

(i) Consider $Q = C_1$. Consider two spaces $X := (0, 1)$ (with induced metric from Euclidean space $\mathbb{R}$) and $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ (with induced metric from Euclidean space $\mathbb{R}^2$); then $A(X)$ is homeomorphic to $\mathbb{S}^1$; moreover $X, \mathbb{S}^1 \in C_1$; therefore $X, A(X) \in C_1$.

(ii) Consider $Q = C_2 \setminus C_1$. Consider $X := (0, 1)$ with discrete topology. $A(X)$ is compact Hausdorff, so it is normal. If $\mathscr{D}$ is a topological basis for $A(X)$, then for all $t \in (0, 1)$ we have $\{t\} \in \mathscr{D}$. Therefore $\mathscr{D}$ is uncountable and compact space $A(X)$ is not metrizable. Thus $X \in C_1$ and $A(X) \in C_2 \setminus C_1$.

(iii) Consider $Q = C_3 \setminus C_2$. Use Remark 3(3).

(iv) Consider $Q = C_4 \setminus C_3$. Consider X as the set of all rational numbers as a subspace of Euclidean space $\mathbb{R}$. Since X is not locally compact, by Remark 3(3), $A(X)$ is not Hausdorff. Suppose M is a compact subset of $A(X)$); in order to show that $A(X)$ is KC, we show M is a closed subset of $A(X)$. We have the following two cases.

Case 1. If $\infty \notin M$, then M is a compact subset of X; since X is a metric space, M is a closed subset of X too. Therefore, $A(X) \setminus M$ is an open subset of $A(X)$. Hence, M is a closed subset of $A(X)$.

Case 2. If $\infty \in M$, we claim that $X \setminus M$ is an open subset of X and so an open subset of $A(X)$; otherwise (since X is metrizable) there exists a one-to-one sequence $(x_n : n \in \omega)$ in $X \setminus (X \setminus M) (= X \cap M)$ converging to a point $x \in X \setminus M$ (in metric space X). For all $m \in \omega$, $\{x_n : n \geq m\} \cup \{x\}$ is a compact closed subset of X, and $U_m := A(X) \setminus (\{x_n : n \geq m\} \cup \{x\})$ is an open subset of $A(X)$. Since $x \notin M$, $M \subseteq \bigcup\{U_m : m \geq 0\}$. Using the compactness of M, there exists $m \geq 1$ such that $M \subseteq U_0 \cup U_1 \cup \cdots \cup U_m$. Since $x_m \in M \setminus (U_0 \cup U_1 \cup \cdots \cup U_m) = M \setminus U_m$, we have $M \nsubseteq U_0 \cup U_1 \cup \cdots \cup U_m$ which

C_1 := Metrizable spaces
C_2 := Normal spaces
C_3 := T_2 spaces
C_4 := KC spaces
C_5 := SC spaces
C_6 := US spaces
C_7 := T_1 spaces
C_8 := T_D spaces
C_9 := T_{UD} spaces
C_{10} := T_0 spaces
C_{11} := Top

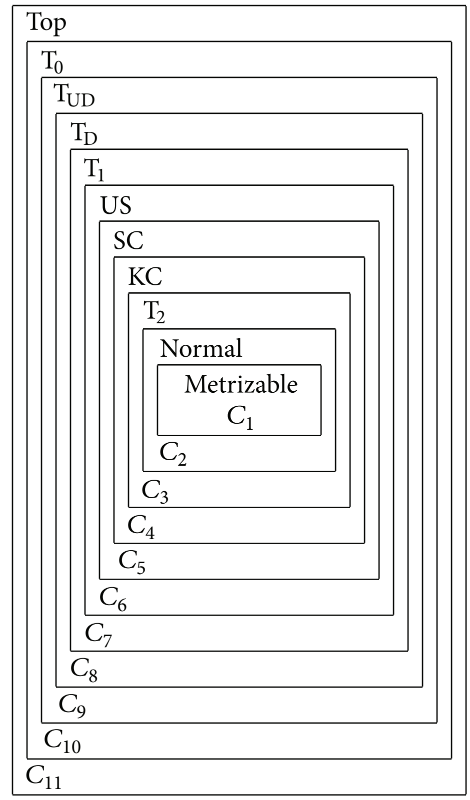

Figure 1: Consider the previous classes of topological spaces (in order to be more convenient, note the right diagram).

Table 1

P	Q										
	C_1	$C_2 \setminus C_1$	$C_3 \setminus C_2$	$C_4 \setminus C_3$	$C_5 \setminus C_4$	$C_6 \setminus C_5$	$C_7 \setminus C_6$	$C_8 \setminus C_7$	$C_9 \setminus C_8$	$C_{10} \setminus C_9$	$C_{11} \setminus C_{10}$
C_1	√	√	—	√	—	—	—	—	—	—	—
$C_2 \setminus C_1$	—	√	—	√	√	√	—	—	—	—	—
$C_3 \setminus C_2$	—	√	—	√	√	√	—	—	—	—	—
$C_4 \setminus C_3$	—	—	—	√	√	√	—	—	—	—	—
$C_5 \setminus C_4$	—	—	—	—	√	√	—	—	—	—	—
$C_6 \setminus C_5$	—	—	—	—	—	√	√	—	—	—	—
$C_7 \setminus C_6$	—	—	—	—	—	—	√	—	—	—	—
$C_8 \setminus C_7$	—	—	—	—	—	—	—	√	—	—	—
$C_9 \setminus C_8$	—	—	—	—	—	—	—	—	√	—	—
$C_{10} \setminus C_9$	—	—	—	—	—	—	—	—	—	√	—
$C_{11} \setminus C_{10}$	—	—	—	—	—	—	—	—	—	—	√

Table 2

P	Q	Reason of omitting this case
$C_2 \setminus C_1, C_3 \setminus C_2, C_4 \setminus C_3, C_5 \setminus C_4, C_6 \setminus C_5, C_7 \setminus C_6$	C_1	If $A(X)$ is metrizable, then X is metrizable too
$C_4 \setminus C_3, C_5 \setminus C_4, C_6 \setminus C_5, C_7 \setminus C_6$	$C_2 \setminus C_1, C_3 \setminus C_2$	If $A(X)$ is T_2, then X is T_2 too
$C_5 \setminus C_4, C_6 \setminus C_5, C_7 \setminus C_6$	$C_4 \setminus C_3$	If $A(X)$ is KC, then X is KC too
$C_6 \setminus C_5, C_7 \setminus C_6$	$C_5 \setminus C_4$	If $A(X)$ is SC, then X is SC too
$C_7 \setminus C_6$	$C_6 \setminus C_5$	If $A(X)$ is US, then X is US too

is a contradiction. Thus $X \setminus M$ is an open subset of X, and $M = A(X) \setminus (X \setminus M)$ is a closed subset of $A(X)$.

Finally we have $X \in C_1$ and $A(X) \in C_4 \setminus C_3$.

(v) Consider $Q = C_5 \setminus C_4$. If X is a metric space, then it is a k-space and, by Remark 3(2), $A(X)$ is KC; hence $A(X) \notin C_5 \setminus C_4$.

(vi) Consider $Q = C_6 \setminus C_5$ or $C_7 \setminus C_6$. We claim that if X is a metric space, then $A(X)$ is SC. First, note the fact that, by Remark 3(1), $A(X)$ is US and hence T_1. Suppose $(x_n : n \in \omega)$ is a sequence in $A(X)$ converging to $x \in A(X)$; we show that $\{x_n : n \in \omega\} \cup \{x\}$ is a closed subset of $A(X)$. Consider the following cases.

Case 1. $x \neq \infty$; in this case, X is an open neighborhood of x, thus there exists $m \in \omega$ such that $x_n \in X$ for all $n \geq m$ and $(x_n : n \geq m)$ converges to x in metric space X (by Remark 3(4), X as a subspace of $A(X)$ has its original topology). Thus $\{x_n : n \geq m\} \cup \{x\}$ is a closed compact subset of X; therefore $A(X) \setminus (\{x_n : n \geq m\} \cup \{x\})$ is an open subset of $A(X)$. Finally $\{x_n : n \geq m\} \cup \{x\}$ is a closed subset of $A(X)$ and since $A(X)$ is T_1, $\{x_n : n \geq m\} \cup \{x\} \cup \{x_0, \ldots, x_m\} = \{x_n : n \in \omega\} \cup \{x\}$ is a closed subset of $A(X)$ too.

Case 2. $x = \infty$ and $\{x_n : n \in \omega\}$ is finite. In this case, $\{x_n : n \in \omega\} \cup \{x\}$ is a finite subset of (T_1 space) $A(X)$ and it is closed.

Case 3. $x = \infty$ and $\{x_n : n \in \omega\}$ is infinite. In this case, we may assume $x_n \in X$ for all $n \in \omega$. If $\{x_n : n \in \omega\}$ is not a closed subset of X, then there exists a subsequence $(x_{n_k} : k \in \omega)$ of $(x_n : n \in \omega)$ converging to $y \in X \setminus \{x_n : n \in \omega\}$. Thus $(x_{n_k} : k \in \omega)$ converges to y in $A(X)$ too (use Remark 3(4)). Since $(x_{n_k} : k \in \omega)$ converges to $y, x(= \infty)$ and $A(X)$ is US, we have $y = x$ which is a contradiction with $y \in X = A(X) \setminus \{\infty\} = A(X) \setminus \{x\}$. Therefore $\{x_n : n \in \omega\}$ is a closed subset of X, so

$$\begin{aligned} X \setminus \{x_n : n \in \omega\} &= X \setminus (\{x_n : n \in \omega\} \cup \{x\}) \\ &= A(X) \setminus (\{x_n : n \in \omega\} \cup \{x\}) \end{aligned} \tag{7}$$

is an open subset of X and $A(X)$. Finally $\{x_n : n \in \omega\} \cup \{x\}$ is a closed subset of $A(X)$.

Using the above three cases, $\{x_n : n \in \omega\} \cup \{x\}$ is a closed subset of $A(X)$ and we are done.

Second Row. Here we have $P = C_2 \setminus C_1$ and the following cases for Q.

(i) Consider $Q = C_2 \setminus C_1$. Suppose Ω is the least uncountable ordinal number. Consider $X = \Omega + 1$ (with order topology). Since X is well ordered, it is normal. However, X is not metrizable and $A(X) = X$.

(ii) Consider $Q = C_3 \setminus C_2$. If X and $A(X)$ are T_2, then, by Remark 3(3), $A(X)$ is normal.

(iii) Consider $Q = C_4 \setminus C_3$. Consider X as disjoint union of $X_1 = \Omega + 1$ with order topology and $X_2 = \mathbb{Q}$ as the set of all rational numbers with induced metric from Euclidean space $\mathbb{R}$. The topological space X is normal since X_1, X_2 are normal. Moreover, X is disjoint union of X_1 and X_2, so X has nonmetrizable subspace X_1, thus X is nonmetrizable and $X \in C_2 \setminus C_1$. Since X_1 is compact, by Lemma 4 we have $A(X) = X_1 \sqcup A(X_2)$. Considering case "$P = C_1, Q = C_4 \setminus C_3$", we have $A(X_2) \in C_4 \setminus C_3$. Using Lemma 6(1), we have $X_1 \sqcup A(X_2) \in C_4 \setminus C_3$; thus $A(X) \in C_4 \setminus C_3$.

(iv) Consider $Q = C_5 \setminus C_4$. Let X be an uncountable set and $b \in X$. Consider X under Fortissimo topology with particular point b, that is, under the topology $\{U \subseteq X : b \notin U \vee (X \setminus U \text{ is countable})\}$ (see [4, counterexample 25]). For distinct $x, y \in X$, suppose $x \neq b$. Then $U = \{x\}, V = X \setminus \{x\}$ are disjoint open subsets of X and X is Hausdorff. On the other hand, if E, F are disjoint closed subsets of X, suppose $b \notin E$, then E and $X \setminus E (\supseteq F)$ are disjoint open subsets of X containing E and F. Thus X is normal.

Moreover, if W is an open neighborhood of b, then $X \setminus W$ is countable; therefore W is uncountable and $W \cap (X \setminus \{b\}) \neq \varnothing$. Therefore, $b \in \overline{X \setminus \{b\}}$. Let $(x_n : n \in \omega)$ be a sequence of elements of $X \setminus \{b\}$. The sequence $(x_n : n \in \omega)$ does not converge to b, since $X \setminus \{x_n : n \in \omega\}$ is an open neighborhood of b. The space X is not metrizable since $b \in \overline{X \setminus \{b\}}$ and there is not any sequence in $X \setminus \{b\}$ converging to b. So $X \in C_2 \setminus C_1$.

On the other hand, using the definition of one point compactification, any subset M of $A(X)$ containing ∞ is a compact subset of $A(X)$. Therefore, $A(X) \setminus \{b\}$ is a compact subset of $A(X)$, but it is not a closed subset of $A(X)$; thus $A(X)$ is not KC. We claim that $A(X)$ is SC. Suppose $(x_n : n \in \omega)$ is a sequence in $A(X)$ converging to w. We have the following cases.

Case 1. Consider $w \in X \setminus \{b\}$. In this case, $\{w\}$ is an open neighborhood of w and there exists $N \geq 1$ such that for all $n \geq N$ we have $x_n = w$. Using Remark 3(1), $A(X)$ is T_1; thus $\{x_n : n \in \omega\} \cup \{w\} = \{x_0, \ldots, x_N, w\}$ is a closed subset of $A(X)$.

Case 2. Consider $w = b$. The set $X \setminus \{x_n : n \in \omega, x_n \neq b\}$ is an open neighborhood of w; thus there exists $N \geq 1$ such that for all $n \geq N$ we have $x_n \in (X \setminus \{x_n : n \in \omega, x_n \neq b\})$ (thus for all $n \geq N$ we have $x_n = b$). Using Remark 3(1), $A(X)$ is T_1; thus $\{x_n : n \in \omega\} \cup \{w\} = \{x_0, \ldots, x_N, w\}$ is a closed subset of $A(X)$.

Case 3. Consider $w = \infty$. In this case, $A(X) \setminus (\{x_n : n \in \omega\} \cup \{w\}) = X \setminus \{x_n : n \in \omega\}$ is an open subset of X; therefore it is an open subset of $A(X)$. Thus $\{x_n : n \in \omega\} \cup \{w\}$ is a closed subset of $A(X)$.

Using the above three cases, $\{x_n : n \in \omega\} \cup \{w\}$ is a closed subset of $A(X)$, and $A(X)$ is SC.

Since $A(X)$ is SC and it is not KC, $A(X) \in C_5 \setminus C_4$.

(v) Consider $Q = C_6 \setminus C_5$. Suppose $\mathcal{F}$ is a uniform ultrafilter over $\mathbb{N}$. Consider $X = \mathbb{N} \cup \{0\} (= \omega)$ under topology $\{A \subseteq X : 0 \notin A \vee A \setminus \{0\} \in \mathcal{F}\}$. If $x, y \in X$ are distinct with $x \neq 0$, then $\{x\}, X \setminus \{x\}$ are disjoint open subsets of X containing x, y and X is Hausdorff. If E, F are disjoint closed subsets of X, suppose $0 \notin E$. Therefore, $E, X \setminus E (\supseteq F)$ are disjoint open subsets of X and X is normal. Since $\mathcal{F}$ is a uniform ultrafilter over $\mathbb{N}$, it does not contain any finite subset of X. Since all of the elements of $\mathcal{F}$ are infinite, 0 is a limit point of X and $0 \in \overline{X \setminus \{0\}}$. Consider a sequence $(x_n : n \in \omega)$ in $X \setminus \{0\}$. We have the following cases.

Case 1. $(x_n : n \in \omega)$ has a constant subsequence like $(x_{n_k} : k \in \omega)$. Since X is Hausdorff and every sequence converges to at most one point, $(x_{n_k} : k \in \omega)$ converges to its constant value and does not converge to 0. Thus $(x_n : n \in \omega)$ does not converge to 0.

Case 2. $(x_n : n \in \omega)$ does not have any constant subsequence. Suppose $(x_{n_k} : k \in \omega)$ is a one-to-one subsequence of $(x_n : n \in \omega)$. Since $\mathcal{F}$ is an ultrafilter over $X \setminus \{0\}(= \mathbb{N})$, and $\{x_{n_{2k}} : k \in \omega\} \cap \{x_{n_{2k+1}} : k \in \omega\} = \emptyset$, we have $\{x_{n_{2k}} : k \in \omega\} \notin \mathcal{F}$ or $\{x_{n_{2k+1}} : k \in \omega\} \notin \mathcal{F}$. Suppose $\{x_{n_{2k}} : k \in \omega\} \notin \mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter over $\mathbb{N}$, $\mathbb{N} \setminus \{x_{n_{2k}} : k \in \omega\} \in \mathcal{F}$. Therefore $(\mathbb{N} \setminus \{x_{n_{2k}} : k \in \omega\}) \cup \{0\}$ is an open neighborhood of 0 and $(x_n : n \in \omega)$ does not converge to 0.

Since $0 \in \overline{X \setminus \{0\}}$ and by the above two cases, there is not any sequence in $X \setminus \{0\}$ converging to 0; X is not metrizable. Thus $X \in C_2 \setminus C_1$.

Now pay attention to the following claims.

Claim 1. The sequence $(n : n \geq 1)$ converges to ∞ in $A(X)$. Suppose U is an open neighborhood of ∞ in $A(X)$. Since X is T_1, $A(X)$ is T_1 too. Therefore, $V := U \setminus \{0\}$ is an open neighborhood of ∞ in $A(X)$; thus $A(X) \setminus V = X \setminus V (\subseteq X \setminus \{0\})$ is a compact (and closed) subset of X. Also $A(X) \setminus V$ is finite, since $X \setminus \{0\}$ is discrete and $A(X) \setminus V$ is a compact subset of $X \setminus \{0\}$ (use Remark 3(4)). Suppose $N = \max(A(X) \setminus V)$. For all $n > N$, we have $n \in V \subseteq U$. Hence $(n : n \geq 1)$ converges to ∞.

Claim 2. $\{n : n \geq 1\} \cup \{\infty\}$ is not a closed subset of $A(X)$. Using the fact that $0 \in \overline{X \setminus \{0\}}$, we have $\overline{\{n : n \geq 1\} \cup \{\infty\}} = A(X)$; hence $\{n : n \geq 1\} \cup \{\infty\}$ is not a closed subset of $A(X)$.

Regarding Claims 1 and 2, $A(X)$ is not SC. Since X is normal, it is KC; so using Remark 3(1), $A(X)$ is US. Therefore, $A(X) \in C_6 \setminus C_5$.

(vi) Consider $Q = C_7 \setminus C_6$. If X is Normal, then it is KC and by Remark 3(1), $A(X)$ is US. Thus $A(X) \notin C_7 \setminus C_6$.

Third Row. Here we have $P = C_3 \setminus C_2$ and the following cases for Q.

(i) Consider $Q = C_2 \setminus C_1$. Suppose X is a Hausdorff locally compact nonnormal topological space. Since X is not normal, it is not metrizable and $X \in C_3 \setminus C_2$. By Remark 3(3), $A(X)$ is normal. By Remark 3(4), $A(X)$ is not metrizable. Hence $A(X) \in C_2 \setminus C_1$. Moreover $X = ((\omega + 1) \times (\Omega + 1)) \setminus \{(\omega, \Omega)\}$, where $\omega + 1$ and $\Omega + 1$ have their order topology and $(\omega + 1) \times (\Omega + 1)$ equipped with product topology (deleted Tykhonoff plank [4, counterexample 87]) is Hausdorff locally compact nonnormal topological space and is an example for this case.

(ii) Consider $Q = C_3 \setminus C_2$. Use a similar method described for $P = C_2 \setminus C_1$ and $Q = C_3 \setminus C_2$.

(iii) Consider $Q = C_4 \setminus C_3$. Consider $X = \mathbb{R}$ under topology $\{O \setminus B : O$ is an open subset of $\mathbb{R}$ in its Euclidean topology and $B \subseteq \{1/n : n \in \mathbb{N}\}\}$, then $X \in C_3 \setminus C_2$ (Smirnov's deleted sequence topology [4, counterexample 64]). Also X is first countable; therefore it is a k-space by Remark 3(6). By Remark 3(2), $A(X)$ is KC. Moreover, $A(X)$ is not Hausdorff, since X is not locally compact in 0 (use Remark 3(3)). Hence $A(X) \in C_4 \setminus C_3$.

(iv) Consider $Q = C_5 \setminus C_4$. Consider X as disjoint union of X_1 and X_2, where

(1) X_1 is an uncountable set under Fortissimo topology with particular point $b \in X_1$, that is, under topology $\{U \subseteq X_1 : b \notin U \vee (X_1 \setminus U$ is countable$)\}$ (see [4, counterexample 25] and proof of Table 1 regarding case "$P = C_2 \setminus C_1$, $Q = C_5 \setminus C_4$");

(2) $X_2 = \mathbb{R}$ under the topology $\{O \setminus B : O$ is an open subset of $\mathbb{R}$ in its Euclidean topology and $B \subseteq \{1/n : n \in \mathbb{N}\}\}$ (see Smirnov's deleted sequence topology [4, counterexample 64] and proof of Table 1 regarding case "$P = C_3 \setminus C_2$, $Q = C_4 \setminus C_3$").

Since $X_1 \in C_2 \setminus C_1$ and $X_2 \in C_3 \setminus C_2$, we have $X = X_1 \sqcup X_2 \in C_3 \setminus C_2$ by Lemma 6(1). Moreover $A(X_1) \in C_5 \setminus C_4$ and $A(X_2) \in C_4 \setminus C_3$ lead us to $A(X) = A(X_1 \sqcup X_2) \in C_5 \setminus C_4$ by Lemma 7.

(v) Consider $Q = C_6 \setminus C_5$. Consider X as disjoint union of X_1 and X_2, where we have the following.

(1) Suppose $\mathcal{F}$ is a uniform ultrafilter over $\mathbb{N}$. Consider $X_1 = \mathbb{N} \cup \{0\}(= \omega)$ under topology $\{A \subseteq X : 0 \notin A \vee A \setminus \{0\} \in \mathcal{F}\}$ (see proof of Table 1 regarding case "$P = C_2 \setminus C_1$, $Q = C_6 \setminus C_5$").

(2) X_2 is Smirnov's deleted sequence topological space (see proof of Table 1 regarding case "$P = C_3 \setminus C_2$, $Q = C_4 \setminus C_3$").

Then $X \in C_3 \setminus C_2$ by $X_1 \in C_2 \setminus C_1$ and $X_2 \in C_3 \setminus C_2$ and Lemma 6(1). Also $A(X) \in C_6 \setminus C_5$ by $A(X_1) \in C_6 \setminus C_5$, $A(X_2) \in C_4 \setminus C_3$, and Lemma 7.

(vi) Consider $Q = C_7 \setminus C_6$. If X is T_2, then it is KC and by Remark 3(1), $A(X)$ is US. Thus $A(X) \notin C_7 \setminus C_6$.

Fourth Row. Here we have $P = C_4 \setminus C_3$ and the following cases for Q.

(i) Consider $Q = C_4 \setminus C_3$. Consider W as the set of all rational numbers as a subspace of Euclidean space $\mathbb{R}$. Using the case "$P = C_1$, $Q = C_4 \setminus C_3$" for $X := A(W)$, we have $X \in C_4 \setminus C_3$. Since X is compact, we have $A(X) = X \in C_4 \setminus C_3$.

(ii) Consider $Q = C_5 \setminus C_4$. Consider uncountable set X with countable complement topology $\{U \subseteq X : U = \emptyset \vee (X \setminus U$ is countable$)\}$ [4, counterexamples 20 and 21]. Since every two nonempty open subsets of X have nonempty intersection, X is not Hausdorff. It is clear that X is T_1. Moreover, M is a compact subset of X if and only if M is finite. Therefore, every compact subset of X is closed and X is KC. So $X \in C_4 \setminus C_3$.

Now suppose E is an uncountable subset of X with uncountable complement. So E is not closed. For all compact subset M of X, the set $E \cap M$ is finite and closed. Therefore,

X is not a k-space. Using Remark 3(2), $A(X)$ is not KC. Using Remark 3(1), $A(X)$ is US; we claim that $A(X)$ is SC. Suppose $(x_n : n \in \omega)$ is a sequence in $A(X)$, converging to $x \in A(X)$. We have the following cases.

Case 1. Consider $x \in X$. In this case, $U := (X \setminus \{x_n : n \in \omega\}) \cup \{x\}$ is an open neighborhood of x in $A(X)$. So there exists $N \geq 1$ such that for all $n \geq N$ we have $x_n \in U$ and $x_n = x$. Therefore, $\{x_n : n \in \omega\} \cup \{x\} = \{x_0, x_1, \ldots, x_N, x\}$ is a (finite and) closed subset of X.

Case 2. Consider $x = \infty$. Since $A(X) \setminus (\{x_n : n \in \omega\} \cup \{x\}) = X \setminus \{x_n : n \in \omega\}$ is open in X, it is open in $A(X)$ too. Thus $\{x_n : n \in \omega\} \cup \{x\}$ is closed in $A(X)$.

By the above cases, $\{x_n : n \in \omega\} \cup \{x\}$ is closed in $A(X)$ and $A(X)$ is SC. Hence $A(X) \in C_5 \setminus C_4$.

(iii) Consider $Q = C_6 \setminus C_5$. Consider X as disjoint union of X_1 and X_2, where we have the following.

(1) Suppose $\mathcal{F}$ is a uniform ultrafilter over $\mathbb{N}$. Consider $X_1 = \mathbb{N} \cup \{0\} (= \omega)$ under topology $\{A \subseteq X : 0 \notin A \vee A \setminus \{0\} \in \mathcal{F}\}$ (see proof of Table 1 regarding case "$P = C_2 \setminus C_1, Q = C_6 \setminus C_5$").

(2) X_2 is an uncountable set with countable complement topology $\{U \subseteq X_2 : U = \varnothing \vee (X_2 \setminus U \text{ is countable})\}$ (see [4, counterexamples 20 and 21] and proof of Table 1 regarding case "$P = C_4 \setminus C_3, Q = C_5 \setminus C_4$").

Then $X \in C_4 \setminus C_3$ by $X_1 \in C_2 \setminus C_1$, $X_2 \in C_4 \setminus C_3$, and Lemma 6(1). Also $A(X) \in C_6 \setminus C_5$ by $A(X_1) \in C_6 \setminus C_5$, $A(X_2) \in C_5 \setminus C_4$, and Lemma 7.

(iv) Consider $Q = C_7 \setminus C_6$. If X is KC and by Remark 3(1), $A(X)$ is US. Thus $A(X) \notin C_7 \setminus C_6$.

Fifth Row. Here we have $P = C_5 \setminus C_4$ and the following cases for Q.

(i) Consider $Q = C_5 \setminus C_4$. Consider X as $\Omega + 1$ with doubling Ω [8]; that is, if $p \notin \Omega + 1$, let $X = (\Omega + 1) \cup \{p\}$ under topological basis $\{(\alpha, \beta) : \alpha$ and β are ordinal numbers with $\alpha, \beta < \Omega\} \cup \{[0, \alpha) : \alpha$ is an ordinal number with $\alpha < \omega\} \cup \{(\alpha, \Omega] : \alpha$ is an ordinal number with $\alpha < \omega\} \cup \{(\alpha, \Omega) \cup \{p\} : \alpha$ is an ordinal number with $\alpha < \omega\}$. Then $X \in C_5 \setminus C_4$ and X is compact which leads to $A(X) \in C_5 \setminus C_4$ too.

(ii) Consider $Q = C_6 \setminus C_5$. Consider X as disjoint union of X_1 and X_2, where we have the following.

(1) X_1 is $\Omega + 1$ with doubling Ω [8] as in case "$P = C_5 \setminus C_4, Q = C_5 \setminus C_4$". Then $X_1 = A(X_1) \in C_5 \setminus C_4$.

(2) For uniform ultrafilter $\mathcal{F}$ over $\mathbb{N}$, consider $X_2 = \mathbb{N} \cup \{0\} (= \omega)$ is equipped with topology $\{A \subseteq Y : 0 \notin A \vee A \setminus \{0\} \in \mathcal{F}\}$. Using case "$P = C_2 \setminus C_1$, $Q = C_6 \setminus C_5$", we have $X_2 \in C_2 \setminus C_1$ and $A(X_2) \in C_6 \setminus C_5$. By $X_1 \in C_5 \setminus C_4$, $X_2 \in C_2 \setminus C_1$, and Lemma 6(1), we have $X = X_1 \sqcup X_2 \in C_5 \setminus C_4$. By $A(X_1) \in C_5 \setminus C_4$, $A(X_2) \in C_6 \setminus C_5$, and Lemma 7, we have $A(X) = A(X_1 \sqcup X_2) \in C_6 \setminus C_5$.

(iii) Consider $Q = C_7 \setminus C_6$. Use Lemma 8.

Table 3

P	Q			
	T_2	$KC \setminus T_2$	$SC \setminus KC$	$T_1 \setminus SC$
T_2	√	√	√	√
$KC \setminus T_2$	—	√	√	√
$SC \setminus KC$	—	—	√	√
$T_1 \setminus SC$	—	—	—	√

Sixth Row. Here we have $P = C_6 \setminus C_5$ and the following cases for Q.

(i) Consider $Q = C_6 \setminus C_5$. Consider X as $(\omega + 1) \cup \{\mathcal{F}\}$, such that $\omega + 1$ has its usual order topology, $\mathcal{F}$ is a uniform ultrafilter over ω, and $\{\{\mathcal{F}\} \cup U : U \in \mathcal{F}\}$ is an open neighborhood basis for $\mathcal{F}$ [8, example 1.2]; then X is compact and $A(X) = X \in C_6 \setminus C_5$.

(ii) Consider $Q = C_7 \setminus C_6$. According to [7, example 5], there exists a US topological space X such that $A(X)$ is not US. By Remark 3(7), $A(X)$ is T_1. Hence $A(X) \in C_7 \setminus C_6$. Moreover, by Lemma 8, X is not SC; thus $X \in C_6 \setminus C_5$.

Seventh Row. Here we have $P = C_7 \setminus C_6$ and $Q = C_7 \setminus C_6$. Suppose X as an infinite set with finite complement topology $\{U \subseteq X : U = \varnothing \vee (X \setminus U \text{ is finite})\}$ [4, counterexamples 18 and 19]. Then X is compact and $A(X) = X \in C_7 \setminus C_6$. □

4. Some Observations in Figure 1

Using Figure 1, we have the following results.

(i) The collection $\{T_2, KC, SC, T_1\}$ is a full-forwarding chain with respect to A. In other words, Table 3 is valid.

In Table 3, the mark "√" indicates that in the corresponding case there exists $X \in P$ such that $A(X) \in Q$, and the mark "—" indicates that

in the corresponding case for all $X \in P$ we have $A(X) \notin Q$.

(ii) The collection {Metrizable, T_2, KC, SC, US, T_1, T_D, T_{UD}, T_0, Top} is a forwarding chain with respect to A. The collection T_1, T_D, T_{UD}, T_0, Top is a stationary chain with respect to A.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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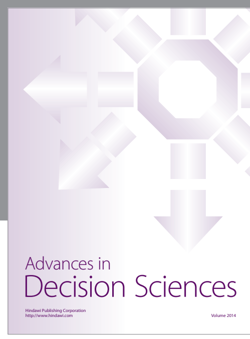
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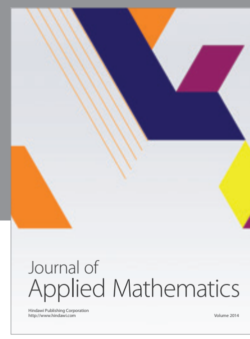
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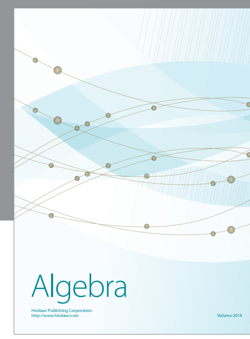
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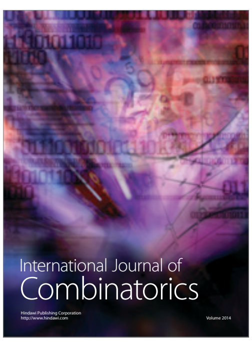
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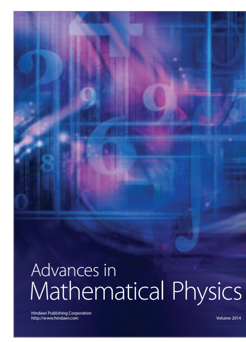
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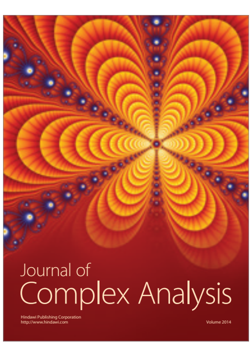
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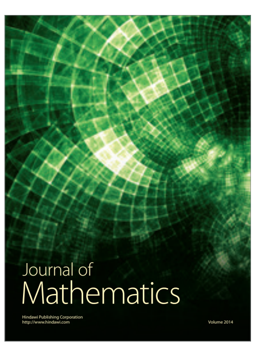
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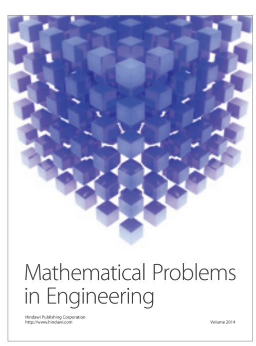
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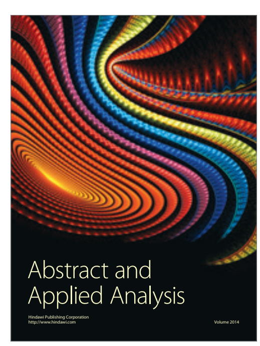
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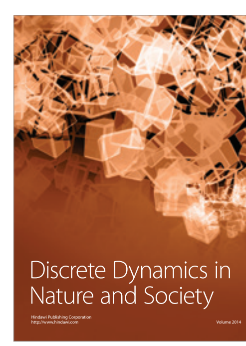
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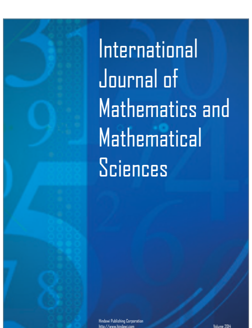
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