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Research Article

Active Optimal Control of the KdV Equation Using the Variational Iteration Method

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The optimal pointwise control of the KdV equation is investigated with an objective of minimizing a given performance measure. The performance measure is specified as a quadratic functional of the final state and velocity functions along with the energy due to open- and closed-loop controls. The minimization of the performance measure over the controls is subjected to the KdV equation with periodic boundary conditions and appropriate initial condition. In contrast to standard optimal control or variational methods, a direct control parameterization is used in this study which presents a distinct approach toward the solution of optimal control problems. The method is based on finite terms of Fourier series approximation of each time control variable with unknown Fourier coefficients and frequencies. He's variational iteration method for the nonlinear partial differential equations is applied to the problem and thus converting the optimal control of lumped parameter systems into a mathematical programming. A numerical simulation is provided to exemplify the proposed method.

1. Introduction

A modal for planar, unidirectional waves propagating in shallow water was originally introduced by Korteweg and de Vries in 1895 [1]. The modal is expressed by a third-order nonlinear partial differential equation called KdV equation. The KdV equation has been at the center of naval science studies and other physical phenomena such as weakly nonlinear long waves for the last 150 years. Therefore, solving and controlling the behavior of the KdV equation have great implications.

Review of new techniques such as variational approaches, parameter-expanding methods, and parameterized perturbation method for nonlinear problems is presented by He in [2], and a detailed study of He's approaches is given in [3]. In the literature, there are a considerable number of numerical and theoretical aspects of the KdV equation. A survey of results for the KdV equation is given in [4]. Existence and uniqueness of the solution of the KdV equation are given in different forms [5]. Numerical solution of the KdV equation is studied by using modified Bernstein polynomials in [6], Galerkin B-spline finite element method in [7], homotopy perturbation method to find solitary-wave solutions of the combined Korteweg de Vries-Modified Korteweg de Vries Equation in [8], and variational iteration method (VIM) in [9, 10]. Although, the numerical solution of the KdV equation has been studied in depth, to the best knowledge of the author, optimal control aspect of the problem did not attract many researchers. Control applications of nonlinear dispersive wave equations in [11], exact controllability and stability of the KdV equation in [12], theoretical aspect of boundary controllability of KdV on a bounded domain in [13], and stability and numerical aspect of the boundary control of the KdV equation in [14] are discussed.

In this paper, we consider an active control of nonlinear waves expressed by the KdV equation with periodic boundary conditions and initial conditions. To control water waves in a uniform channel, point-wise control actuators in the spatial domain, and linear displacement and displacement-slope feedback controls are implemented in the KdV equation. The dynamic response of the system is measured by performance index functionals that consist of weighted sum of the energy at the terminal time with the total effort of open- and closed-loop controls. The objective of the control problem is to minimize the dynamic response of the system with minimum expenditure of the modified energy. The parameterization of the actuators uses a finite term of orthogonal (or nonorthogonal) functions with unknown coefficients and the solution of state function is expressed as an iterative function with a Lagrange multiplier known as VIM. Thereby, the optimal control problem becomes a mathematical programming for unknown coefficients to be computed optimally while state solution is obtained iteratively. The compact solution of state function is expressed analytically in terms of unknown terms due to applied controls. To compare the effects of different controls, first the open-loop control is applied to the system before both open- and closed-loop controls are applied, or closed-loop control with an optimal actuator is applied.

The computational and graphical results show that the present method has a desired robustness. Moreover, it is observed that closed-loop control with an optimal actuator applied to the system reduces the energy substantially and controls the behavior of the elongation and velocity of waves.

2. Problem Formulation

We consider the KdV equation

$$\begin{aligned} &u_t(x,t) + 6\alpha u(x,t)u_x(x,t) + \gamma u_{xxx}(x,t) + C_1 u(x,t) + C_2 u_x(x,t) \\ &\quad = \sum_{i=1}^{n} f_i(t)\delta(x - x_i) \quad \text{for } (x,t) \in \Omega \times \left[0, t_f\right], \end{aligned} \tag{2.1}$$

subject to the following periodic boundary conditions and initial condition:

$$\begin{aligned} &\frac{\partial^i u}{\partial x^i}(x,t) = \frac{\partial^i u}{\partial x^i}(x,t), \quad i = 0,1,2, \text{ for } (x,t) \in \partial\Omega \times \left[0, t_f\right], \\ &u(x,0) = g(x), \end{aligned} \tag{2.2}$$

where $u(x,t)$ describes the elongation of the wave at a point (x,t), and $\Omega = [a,b]$; terminal time is t_f; δ is the dirac delta function; α and γ are constants that are determined by the nature of a physical application; C_1 and C_2 are feedback gains; n is the number of control actuators placed in the interior of the spatial domain, and $f_i(t)$ is the applied force for $t \in [0,t_f]$ that will be obtained optimally.

Remarks 2.1. Let the set of Lebesue integrable functions be given as

$$\mathcal{L}^2(\Omega) = \left\{ f : \Omega \longrightarrow \mathbb{R} \mid \left(\int_\Omega (f(x))^2 dx \right)^{1/2} < \infty \right\}, \tag{2.3}$$

with the inner product $\langle f,g \rangle_{\mathcal{L}^2(\Omega)}$ and the norm $\|f\|_{\mathcal{L}^2(\Omega)}$ being

$$\langle f,g \rangle_{\mathcal{L}^2(\Omega)} = \int_\Omega f(x)g(x)dx, \qquad \|f\|^2_{\mathcal{L}^2(\Omega)} = \langle f,f \rangle, \tag{2.4}$$

and a Hilbert space $\mathscr{H}(\Omega)$ is $\mathcal{L}^2(\Omega)$ with the inner product.

Our goal in this paper is to reduce the dynamical response of nonlinear waves modelled by the KdV equation by implementing open- and closed-loop controls. The natural attempt is to minimize the energy due to waves that should be achieved with limited expenditure of control energy. The weighted performance of the system based on energy at the terminal time and total control effort is measured by the following performance functional:

$$\mathcal{J}\left(\vec{F};C_1,C_2;t_f\right) = \mathcal{E}\left(\vec{F};C_1,C_2;t_f\right) + \mathcal{E}_o\left(\vec{F};t_f\right) + \mathcal{E}_c(C_1,C_2;t_f), \tag{2.5}$$

where

$$\mathcal{E}\left(\vec{F};C_1,C_2;t_f\right) = \int_0^a \left(\epsilon_1 u^2(x,t_f) + \epsilon_2 u_t^2(x,t_f)\right)dx, \qquad \mathcal{E}_o\left(\vec{F};t_f\right) = \sum_{i=1}^n \int_0^{t_f} \mu_i f_i^2(t)dt,$$
$$\mathcal{E}_c(C_1,C_2;t_f) = \int_0^{t_f} \int_0^a \left(\epsilon_3[C_1 u(x,t)]^2 + \epsilon_4[C_2 u_x(x,t)]^2\right)dx\,dt, \tag{2.6}$$

in which $\vec{F} = (f_1,\dots,f_n)$, $\epsilon_j \geq 0$ for $j = 1,\dots,4$ such that $\sum_{j=1}^4 \epsilon_j \neq 0$ and $\mu_i \geq 0$, $i = 1,\dots,n$ are weighting factors. In (2.5), $\mathcal{E}(\vec{F};C_1,C_2;t_f)$ is the energy at the terminal time, and $\mathcal{E}_o(\vec{F};t_f)$ and $\mathcal{E}_c(C_1,C_2;t_f)$ are energies for the open-loop and closed-loop controls duration over $[0,t_f]$.

Here we are considering three optimal control problems.

(P1) The first is to find optimal $\vec{F}^0(t) \in L^2([0,t_f])$ for fixed real valued C_1 and C_2 such that

$$\mathcal{J}_o := \mathcal{J}\left(\vec{F}^0;C_1,C_2;t_f\right) = \min_{\vec{F}(t)\in L^2([0,t_f])} \mathcal{J}\left(\vec{F};C_1,C_2;t_f\right), \tag{2.7}$$

where $\mathcal{J}_o$ is the optimum open-loop performance, and subscript $(-_o)$ indicates open-loop optimization.

(P2) Secondly, we seek optimal feedback gains, $C_1, C_2 \in \mathbb{R}$, for optimal actuators $\overrightarrow{F}^0(t)$ obtained in (P1) such that

$$\mathcal{J}_c := \mathcal{J}\left(\overrightarrow{F}^0; C_1^0, C_2^0; t_f\right) = \min_{C_1, C_2} \mathcal{J}\left(\overrightarrow{F}^0; C_1, C_2; t_f\right), \tag{2.8}$$

where $\mathcal{J}_c$ is the optimum closed-loop performance, and subscript $(-_c)$ indicates closed-loop optimization.

(P3) Finally, we seek optimal feedback gains, $C_1^0, C_2^0 \in \mathbb{R}$, and optimal actuators, $\overrightarrow{F}^0(t)$, such that

$$\mathcal{J}\left(\overrightarrow{F}^0; C_1^0, C_2^0; t_f\right) = \min_{\overrightarrow{F}(t) \in L^2([0, t_f]); C_1, C_2} \mathcal{J}\left(\overrightarrow{F}; C_1, C_2; t_f\right). \tag{2.9}$$

In problems (2.7), (2.8) and (2.9), $u(x,t)$ is subject to (2.1) and (2.2).

3. Solving Optimal Control Problems

The distributed parameter control problems in (2.7)–(2.9) are transformed into an iterative parameter control problem in which a parameterization of actuators is introduced as a direct method. The transformation is done by VIM. Optimal control problems are then transformed into a mathematical programming problem that consists of unknowns due to parameterization of actuators and the feedback parameters at each search of optimal values the solution of the KdV is obtained iteratively. In the following subsections, the parametrization of actuators is first introduced. It is followed by the solution technique for KdV with a brief introduction to VIM.

3.1. Control Parameterization

A direct control parameterization is introduced using finite terms of a Fourier series approximation. In the approximation, each actuator is expressed as a finite sum of a Fourier series with unknown Fourier coefficients and frequencies. Actuators in (2.1) are given by

$$\begin{aligned} f_i(t) &= \sum_{k=1}^{N} \alpha_{ik} \cos(\zeta_{ik} t) + \beta_{ik} \sin(\zeta_{ik} t) \\ &= \overrightarrow{\alpha}_i^T \cos\left(\overrightarrow{\zeta}_i t\right) + \overrightarrow{\beta}_i^T \sin\left(\overrightarrow{\zeta}_i t\right) \end{aligned} \tag{3.1}$$

where $\overrightarrow{\alpha}_i = (\alpha_{i1}, \ldots, \alpha_{iN})^T$, $\overrightarrow{\beta}_i = (\beta_{i1}, \ldots, \beta_{iN})^T$, and $\overrightarrow{\zeta}_i = (\zeta_{i1}, \ldots, \zeta_{iN})^T$, $i = 1, \ldots, n$. It follows from (3.1) that

$$\overrightarrow{\kappa}_i = \left(\overrightarrow{\alpha}_i, \overrightarrow{\beta}_i, \overrightarrow{\zeta}_i\right), \quad i = 1, \ldots, n \tag{3.2}$$

needs to be calculated optimally in (P1) and (P3).

3.2. Application of He's Variational Iteration Method

It is observed that the VIM converges to the exact solutions, and VIM is used for many nonlinear partial differential equations successfully such as [9, 10, 15]. Although the concern about the convergence rate depends on the accurate calculation of the Lagrange multiplier [16], the nonlinear terms in nonlinear problems are taken as restricted variations in order to determine the Lagrange multipliers. A brief introduction to VIM is given in this section, but readers are referred to a recent review of the method by He in [16, 17] and references therein.

For a nonlinear partial differential equation of the following form

$$L[u(x,t)] + N[u(x,t)] = f(x,t) \tag{3.3}$$

where L is a linear operator, N is a nonlinear operator, and $f(x,t)$ is a nonhomogeneous term, $n-th$ order approximation to $u(x,t)$ is obtained iteratively by a correction function in t-direction

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\left(L[u_n(x,\tau)] + N[\tilde{u}_n(x,\tau)] - f(x,\tau)\right)d\tau \tag{3.4}$$

in which λ is a general Lagrange multiplier, and $\tilde{u}_n$ is a restricted variation, that is, $\delta\tilde{u}_n = 0$. The general Lagrange multiplier λ is found optimally via the variational theory [18].

For (2.1), a correctional functional is

$$\begin{aligned} u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \Bigg[& u_{nt}(x,\tau) + 6\alpha\tilde{u}_n(x,\tau)\tilde{u}_{nx}(x,\tau) + \gamma\tilde{u}_{nxxx}(x,\tau) \\ & + C_1 u_n(x,\tau) + C_2\tilde{u}_{nx}(x,\tau) - \sum_{i=1}^{n} f_i(\tau)\delta(x - x_i)\Bigg] d\tau. \end{aligned} \tag{3.5}$$

The dirac delta function in (2.1) is taken as a pseudo dirac delta function for the sake of simplicity in VIM.

If the variation is taken with respect to $u_n(x,t)$ in (3.5), the following stationary equations are obtained for $\lambda(\tau)$:

$$\begin{aligned} \delta u_{nt} &: 1 + \lambda|_{\tau=t} = 0, \\ \delta u_n &: \lambda' + C_1 = 0 \end{aligned} \tag{3.6}$$

from which it follows immediately that the Lagrange multiplier, λ, is identified as

$$\lambda(\tau) = -C_1(\tau - t) - 1. \tag{3.7}$$

Thus, optimal control problem is reduced to a variational iteration with lumped parameters:

$$\begin{aligned} u_{n+1}(x,t) = u_n(x,t) - \int_0^t &(C_1(\tau - t) + 1) \\ &\times \Big[u_{nt}(x,\tau) + 6\alpha u_n(x,\tau) u_{nx}(x,\tau) \\ &\quad + \gamma u_{nxxx}(x,\tau) + C_1 u_n(x,\tau) + C_2 u_{nx}(x,\tau) - \sum_{i=1}^{n} f_i(\tau)\delta(x - x_i) \Big] d\tau, \end{aligned} \tag{3.8}$$

where $f_i(\tau)$ is given by the parametrization (3.1). The initial guess, $u_0(x,t)$, is given by a trial function that satisfies boundary conditions.

The closed-form expression for $u_n(x,t)$ in (3.8) is found with the aid of Maple. The resulting $u_n(x,t)$ is substituted into (2.7)–(2.9) for the prospective optimization problem. To proceed with problems (P1)–(P3), necessary adjustments are done in (3.8). For (P1), $\vec{\kappa}$ given by (3.2) is calculated optimally for fixed C_1 and C_2 for which $3 \times n \times N$ unknown terms have to be determined in the solution $u(x,t)$ in (3.8). For (P3), the solution $u(x,t)$ in (3.8) consists of $3 \times n \times N + 2$ unknown due to parameterization and feedback gains that are calculated optimally. The iterative terms of $u_{n+1}(x,t)$ are obtained through Maple, and then the terms are placed in appropriate performance functional. Finally, optimization toolbox in MATLAB is used to calculate the optimal parameters. In calculations, a zero vector for $\vec{\kappa}$ as an initial guess is taken until a convergence is reached in L_2 sense, that is,

$$\| J_{o,i+1} - J_{o,i} \|^2 \longrightarrow 0, \quad \text{as } i \longrightarrow \infty. \tag{3.9}$$

The same procedure is repeated for the calculations of feedback gains. Therefore, the algorithm given in [19] is performed.

4. Numerics

In this section, the proposed technique is examined numerically. In the calculations, the following data is used: $a = 1$, $t_f = 1, \alpha = 0.0001$, $\gamma = 0.001$, $n = 1; N = 2$, $\epsilon_1 = 0.1$, $\epsilon_2 = 0.01$, $\mu_1 = 0.001$, and a nascent delta function $\delta(x) \approx e^{-x^2/p^2}/(p\sqrt{\pi})$ in (2.1). We will conduct the simulations for VIM and Adomian methods.

Case 1 ($u_0 = \sin(\pi x)$ and $x_1 = 0.5$). For the given data above, the nonlinear partial differential equation is solved by using VIM where only one iteration is used to ease the complexity in the calculations without compromising the convergence in the solution. The solution for one iteration is found by using Maple and the performance functions in (2.7)–(2.9) are written in MATLAB to perform fminsearch command to find unknown parameters optimally. An optimal

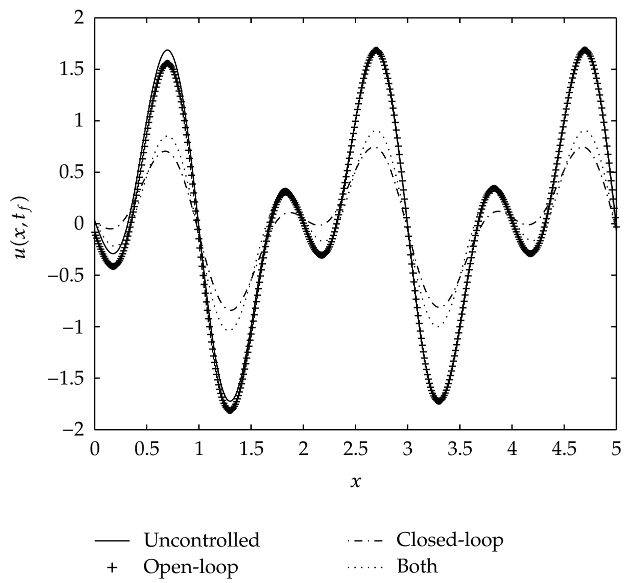

Figure 1: Uncontrolled distance functions at the terminal time.

result is obtained after three iterations with the updated initial guess obtained at the end of each iteration in the Algorithm [19].

In the problem (P1), optimal actuator (3.2) is obtained as

$$\overrightarrow{\kappa}_1 = (-10.43, 10.01; 0.6468, 1.238; 35.23, -16.89)^T. \tag{4.1}$$

In the problem (P2), optimal feedback gains are found as $C_1 = 1.209$ and $C_2 = -0.03902$ for the optimal $\overrightarrow{\kappa}_1$ given by (4.1). For problem (P3), an optimal parameters $\overrightarrow{\kappa}_1 = (2.392, -3.918; 2.123, -0.2676; 0.4523, -17.09)^T$, $C_1 = 0.8111$, and $C_2 = -0.0319$ are found.

The energy for the uncontrolled system is 0.2459 that is reduced by 0.5% when the open-loop control is applied to the system alone and by 49% when the open- and closed-loop controls are applied to the system simultaneously. 78% is the reduction in the energy when the closed-loop is applied to the system along with the optimal actuator obtained in the open-loop control. The elongation of the wave, $u(x,t)$, at a fixed point $x = 0.8$ and at the terminal time are presented for uncontrolled cases in Figures 1 and 2, respectively. The velocity of elongation of the wave, $u_t(x,t)$, at a fixed point $x = 0.8$ and at the terminal time is presented for uncontrolled cases in Figures 3 and 4, respectively.

5. Discussions and Conclusion

The active control of waves defined by the KdV equation is studied by implementing point-wise actuators in the spatial domain and linear displacement and slope-displacement feedback controls in the system. An energy-based performance measure for control is minimized to get a minimum expenditure in the total control effort. It is observed that open-loop control alone does not reduces the energy, and simultaneously applying the

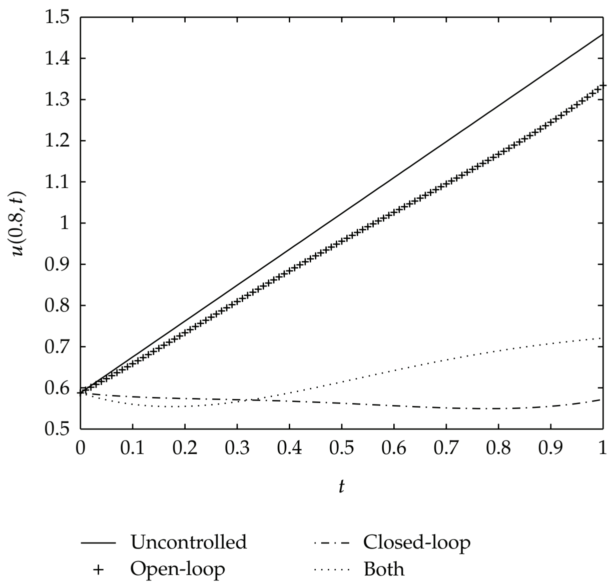

Figure 2: Uncontrolled distance functions at a fixed point.

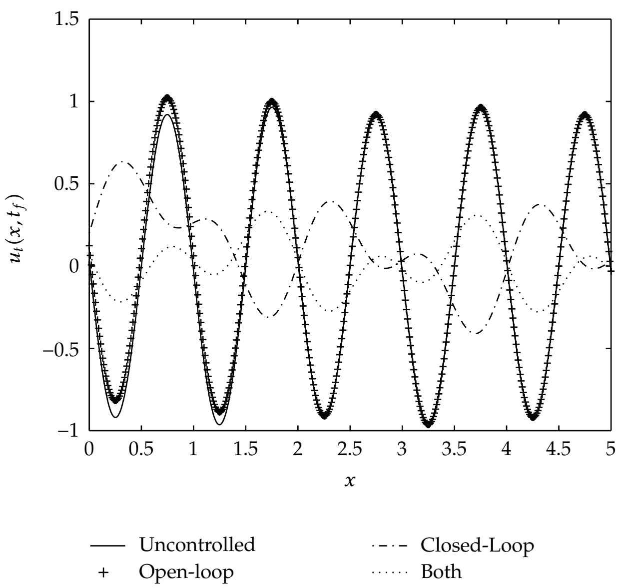

Figure 3: Uncontrolled velocity functions at the terminal time.

open- closed-loop controls reduce the energy noticeably. On the contrary, closed-loop control with optimal actuators reduces the energy substantially. In Figures 1 and 3, it is observed that when one actuator is applied to the system, the displacement and the velocity show indifferent behaviors with the uncontrolled cases, respectively, that might lead to different results if more than one actuator is placed in the spatial domain. The behavior of the system over time is presented in Figure 2 in which it becomes clear that the open-loop control for

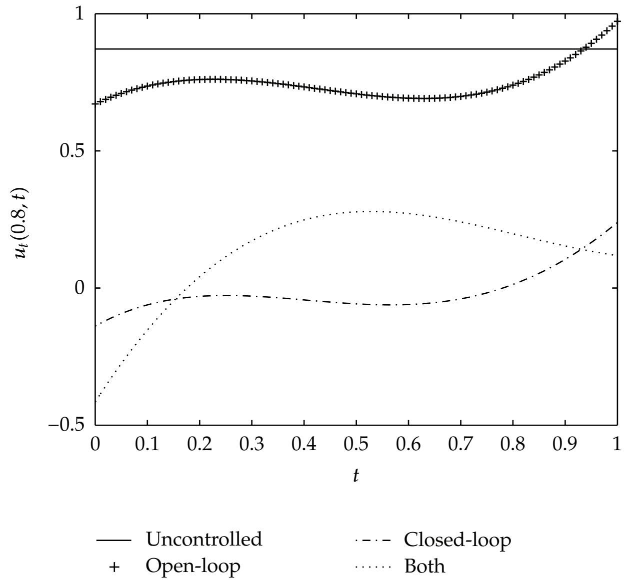

Figure 4: Uncontrolled velocity functions at a fixed point.

one actuator in the spatial domain does not improve the system. The nonlinear involvement of the optimal control problem is solved by using VIM and available toolboxes in MATLAB. Although applying the VIM is of great achievement in the optimal control problems for a nonlinear phenomena, the computational cost in the calculations increases with the number of iterations taken in VIM. It should also be noted that accurate calculation of the Lagrange multiplier in VIM reduces the number of iterations in the calculations.

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