Hindawi Publishing Corporation
Journal of Mathematics
Volume 2013, Article ID 475190, 5 pages
http://dx.doi.org/10.1155/2013/475190

Research Article

On Minimal Fuzzy Ideals of Semigroups

Madad Khan,[1] Feng Feng,[2] and M. Nouman Aslam Khan[3]

[1] *Department of Mathematics, COMSATS Institute of Information Technology, Abbottabad 22060, Pakistan*
[2] *Department of Applied Mathematics, Xi'an University of Posts and Telecommunications, Xi'an 710121, China*
[3] *School of Chemical & Materials Engineering (SCME), National University of Sciences & Technology, H-12, Islamabad 44000, Pakistan*

Correspondence should be addressed to Feng Feng; fengnix@hotmail.com

Received 18 September 2012; Accepted 27 September 2012

Academic Editor: Baoding Liu

The present paper contains the sufficient condition of a fuzzy semigroup to be a fuzzy group using fuzzy points. The existence of a fuzzy kernel in semigroup is explored. It has been shown that every fuzzy ideal of a semigroup contains every minimal fuzzy left and every minimal fuzzy right ideal of semigroup. The fuzzy kernel is the class sum of minimal fuzzy left (right) ideals of a semigroup. Every fuzzy left ideal of a fuzzy kernel is also a fuzzy left ideal of a semigroup. It has been shown that the product of minimal fuzzy left ideal and minimal fuzzy right ideal of a semigroup forms a group. The representation of minimal fuzzy left (right) ideals and also the representation of intersection of minimal fuzzy left ideal and minimal fuzzy right ideal are shown. The fuzzy kernel of a semigroup is basically the class sum of all the minimal fuzzy left (right) ideals of a semigroup. Finally the sufficient condition of fuzzy kernel to be completely simple semigroup has been proved.

1. Introduction

Zadeh in 1965 introduced the fundamental concept of a fuzzy set in his paper [1] which provides a useful mathematical tool for describing the behavior of systems that are too complex or ill-defined to admit precise mathematical analysis by classical methods. The literature in fuzzy set theory and its practicability has been functioning quickly uptil now. The applications of these concepts can now be seen in a variety of disciplines like artificial intellegence, computer science, control engineering, expert systems, operation research, management science, and robotics.

Mordeson [2] has demonstrated the basic exploration of fuzzy semigroups. He also gave a theoratical exposition of fuzzy semigroups and their application in fuzzy coding, fuzzy finite state machines, and fuzzy languages. The role of fuzzy theory in automata and daily language has extensively been discussed in [2].

Fuzzy sets are considered with respect to a nonempty set S. The main idea is that each element x of S is assigned a membership grade $f(x)$ in $[0,1]$, with $f(x) = 0$ corresponding to nonmembership, $0 < f(x) < 1$ to partial membership, and $f(x) = 1$ to full membership. Mathematically, a fuzzy subset f of a set S is a function from S into the closed interval $[0,1]$. The fuzzy theory has provided generalization in many fields of mathematics such as algebra, topology, differential equation, logic, and set theory. The fuzzy theory has been studied in the structure of groups and groupoids by Rosenfeld [3], where he has defined the fuzzy subgroupoid and the fuzzy left (right, two-sided) ideals of a groupoid S. He has used the characteristic mapping

$$C_A = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \tag{1}$$

of a subset A of a groupoid S to show that A is subgroupoid or left (right, two-sided) ideal of S if and only if C_A is the fuzzy subgroupoid or fuzzy left (right, two-sided) ideal of S. Fuzzy semigroups were first considered by Kuroki [4] in which he studied the bi-ideals in semigroups. He also considered the fuzzy interior ideal of a semigroup, characterization of groups, union of groups, and semilattices of groups by means of fuzzy bi-ideals. N. Kehayopulu, X. Y. Xie, and M. Tsingelis have produced several papers on fuzzy ideals in semigroup.

A nonempty subset A of a semigroup S is a left (right) ideal of S if $SA \subseteq A (AS \subseteq A)$ and is ideal if A is both left and right ideal of S. An ideal A of S is called minimal ideal of S if A does not properly contains any other ideal of S. If the intersection K of all the ideals of a semigroup S is nonempty then we shall call K the kernel of S. Huntington in [5] has shown the simplified definition of a semigroup to be a group. We have used this concept in fuzzy semigroups and have shown that the product of minimal fuzzy left ideal and minimal fuzzy right ideal forms a group. Clifford in [6] had given the representation of minimal left (right) ideal of a semigroup. We have used this ideas to extend the fuzzy ideal theory of minimal fuzzy ideal in semigroups and have given a representation of fuzzy kernel in terms of minimal fuzzy left (right) ideals of a semigroup.

For a semigroup S, $F(S)$ will denote the collection of all fuzzy subsets of S. Let f and g be two fuzzy subsets of a semigroup S. The operation "$\circ$" in $F(S)$ is defined by;

$$(fog)(x) = \begin{cases} \bigvee_{x=yz} \{f(y) \wedge g(z)\}, & \text{if } \exists\, y,\ z \in S, \\ & \text{such that } x = yz, \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

For simplicity, we will denote $F(S)$ as S.

2. Sufficient Condition of a Semigroup to Be a Group

The very first definition of fuzzy point is given by Pu and Liu [7]. Let A be a nonempty set and $\lambda \in [0,1]$. A fuzzy point a_λ of A means that a_λ is a fuzzy subset of A defined by

$$a_\lambda(x) = \begin{cases} \lambda, & \text{if } x = a, \\ 0, & \text{if } x \neq a. \end{cases} \quad (3)$$

Each fuzzy point a_λ of A is the fuzzy subset of a set A [7]. Every fuzzy set A can be expressed as the union of all the fuzzy points in A. It is to be noted that for any x and y in A, λ, and μ in $[0,1]$ the fuzzy points x_λ and y_μ have the inclusion relation that is $x_\lambda \subseteq y_\mu$ if and only if $x = y$ and $\lambda \leq \mu$.

In the following text for $\lambda, \mu, t, \gamma, \delta, \alpha, \beta$, and ρ in $[0,1]$, we denote $a_\lambda, b_\mu, x_t, x_\mu, e_\alpha, f_\beta$, and c_ρ as the fuzzy points of different fuzzy subsets of S.

Theorem 1. *If in a semigroup S, the following conditions hold:*

(i) *for any fuzzy points a_λ and b_μ of S there exists x_γ in S, such that $a_\lambda \circ x_\gamma = b_\mu$,*

(ii) *for any fuzzy points a_λ and b_μ of S there exists y_γ in S, such that $y_\gamma \circ a_\lambda = b_\mu$, then $(S, \circ)$ is a fuzzy group.*

Proof. First of all we will show that the fuzzy points x_γ and y_γ existing in the hypothesis are uniquely determined. To show that x_γ in condition (i) of the theorem is unique, let us assume on contrary that there exists two fuzzy points x_γ and z_δ such that, $a_\lambda \circ x_\gamma = b_\mu$ and $a_\lambda \circ z_\delta = b_\mu$. Thus,

$$a_\lambda \circ x_\gamma = a_\lambda \circ z_\delta. \quad (4)$$

The condition (ii) in the theorem implies that for the fuzzy points x_γ and z_δ of S there exists a fuzzy point c_γ of S such that $z_\delta \circ c_\gamma = x_\gamma$. Hence (4) implies that,

$$a_\lambda \circ (z_\delta \circ c_\gamma) = a_\lambda \circ z_\delta, \quad (5)$$

$$(a_\lambda \circ z_\delta) \circ c_\gamma = a_\lambda \circ z_\delta. \quad (6)$$

Now, we will show that if for any fuzzy point a_λ of S, $a_\lambda \circ x_\gamma = a_\lambda$ then for each fuzzy point b_μ of S, $b_\mu \circ x_\gamma = b_\mu$. By the condition (ii) for fuzzy points a_λ and b_μ of S there exists a fuzzy point y_λ of S such that $y_\gamma \circ a_\lambda = b_\mu$. Now by hypothesis,

$$\begin{aligned} y_\gamma \circ (x_\gamma \circ a_\lambda) &= b_\mu, \\ (y_\gamma \circ x_\gamma) \circ a_\lambda &= b_\mu, \\ b_\mu \circ a_\lambda &= b_\mu. \end{aligned} \quad (7)$$

Therefore (6) implies that $z_\delta \circ c_\gamma = z_\delta$, and so $x_\gamma = z_\delta$. It has been shown that the left cancellative law holds in S.

Similarly, we can show that for any fuzzy points a_λ and b_μ of S there exists a unique fuzzy point y_γ in S, such that $y_\gamma \circ a_\lambda = b_\mu$. Similarly, we can show that S is a right cancellative.

Hence the cancellation law holds in a semigroup S and thus for each fuzzy point a_λ of S we have, $a_\lambda \circ S = S$ a nd $S \circ a_\lambda = S$. Now there exists fuzzy points e_α and f_β of S such that $f_\beta \circ a_\lambda = a_\lambda$ and $a_\lambda \circ e_\alpha = a_\lambda$. Let x_γ be any arbitrary fuzzy point of S then x_γ is in $S \circ a_\lambda$, that is there exists b_μ in S such that $x_\gamma = b_\mu \circ a_\lambda$. Thus, $x_\gamma \circ e_\alpha = (b_\mu \circ a_\lambda) \circ e_\alpha = b_\mu \circ (a_\lambda \circ e_\alpha) = b_\mu \circ a_\lambda = x_\gamma$, which implies that e_α is the right identity in S. Similarly we can show that f_β is the left identity in S. Now since f_β is the left identity so $f_\beta \circ e_\alpha = e_\alpha$ and e_α is the right identity so $f_\beta \circ e_\alpha = f_\beta$. Hence $f_\beta = e_\alpha$, which implies that S has an identity element e_α in S. Now for e_α in S, there exists x_γ and y_ρ in S such that $x_\gamma \circ a_\lambda = e_\alpha$ and $a_\lambda \circ y_\rho = e_\alpha$. Now

$$\begin{aligned} (x_\gamma \circ a_\lambda) \circ y_\rho &= e_\alpha \circ y_\rho, \\ x_\gamma \circ (a_\lambda \circ y_\rho) &= y_\rho, \\ x_\gamma \circ e_\alpha &= y_\rho, \\ x_\gamma &= y_\rho. \end{aligned} \quad (8)$$

Hence there exists an inverse element for each fuzzy point of S. The uniqueness of identity and inverse elements can be proved easily. □

The following corollary can be easily asserted from the proof of Theorem 1.

Corollary 2. *A fuzzy semigroup S is a fuzzy group if and only if $a_\lambda \circ S = S$ and $S \circ a_\lambda = S$ for all fuzzy points a_λ of S.*

3. Minimal Fuzzy Ideals of a Semigroup

The nonempty intersection of fuzzy ideals of semigroup S is called the fuzzy kernel of a semigroup S. The fuzzy ideal f of

S is called minimal fuzzy ideal if f contains no proper fuzzy ideal of S. A semigroup S is called a fuzzy simple if it does not contain any proper fuzzy ideal of S.

The following lemmas are in our previous knowledge [2].

Lemma 3. *For any nonempty subsets A and B of semigroup S, we have $A \subseteq B$ if and only if $C_A \subseteq C_B$.*

Lemma 4. *Let A be a nonempty subset of a semigroup S, then A is an ideal of S if and only if C_A is a fuzzy ideal of S.*

Lemma 5. *Let f and g be fuzzy ideals of a semigroup S, then $f \circ g$ and $g \circ f$ are also fuzzy ideals of S.*

Lemma 6. *Let f and g be fuzzy ideals of a semigroup S, then $f \cap g$ and $g \cap f$ are also fuzzy ideals of S.*

Theorem 7. *A nonempty subset A of a semigroup S is minimal ideal if and only if C_A is minimal fuzzy ideal of S.*

Proof. Let A be a minimal ideal of S, then by Lemma 4, C_A is a fuzzy ideal of S. Suppose that C_A is not minimal fuzzy ideal of S, then there exists some fuzzy ideal C_B of S such that, $C_B \subseteq C_A$. Hence by Lemma 3, $B \subseteq A$, where B is an ideal of S. This is a contradiction to the fact that A is minimal ideal of S. Thus C_A is the minimal fuzzy ideal of S. Conversely, let C_A be the minimal fuzzy ideal of S, then by Lemma 4, A is the ideal of S. Suppose that A is not minimal ideal of S, then there exists some ideal B of S such that $B \subseteq A$. Now by Lemma 3, $C_B \subseteq C_A$, where C_B is the fuzzy ideal of S. This is a contradiction to the fact that C_A is the minimal fuzzy ideal of S. Thus A is the minimal ideal of S. □

Theorem 8. *If a semigroup S contains a minimal fuzzy ideal f of S then f is the fuzzy kernel of S.*

Proof. Let g be any fuzzy ideal of S then for a minimal fuzzy ideal f of S, $g \circ f \subseteq g \cap f$. Thus $g \cap f$ is non-empty. Since by Lemma 6, $g \cap f$ is a fuzzy ideal of S and $g \cap f \subseteq f$, which implies that $g \cap f = f$. But then $f = g \cap f \subseteq g$, so f contains in every fuzzy ideal of S and hence is a fuzzy kernel of S. □

Theorem 9. *If a semigroup has fuzzy kernel f then f is a simple subsemigroup of S.*

Proof. Since f is a fuzzy ideal of S, so f is a fuzzy subsemigroup of S, since $f \circ f \subseteq S \circ f \subseteq f$. To show f is simple, let g be any fuzzy ideal of f, then $f \circ g \circ f$ is fuzzy ideal of S, since $S \circ (f \circ g \circ f) = (S \circ f) \circ g \circ f \subseteq f \circ g \circ f$ and $(f \circ g \circ f) \circ S = f \circ g \circ (f \circ S) \subseteq f \circ g \circ f$. Also $f \circ g \circ f \subseteq f \circ f \subseteq f$, but by Theorem 8, f is minimal fuzzy ideal of S because every fuzzy kernel of S, if exists, is a minimal fuzzy ideal of S. Hence $f \circ g \circ f = f$. Also $f \circ g \circ f \subseteq g \circ f \subseteq g$, which implies that $f \subseteq g$. Thus $f = g$, implies that f is simple subsemigroup of S. □

Lemma 10. *If f is a minimal left ideal of a semigroup S and c_λ be any fuzzy point of S for λ in $[0,1]$ then $f \circ c_\lambda$ is also a minimal fuzzy left ideal of S.*

Proof. Since for a minimal fuzzy left ideal f of S, $S \circ (f \circ c_\lambda) = (S \circ f) \circ c_\lambda \subseteq f \circ c_\lambda$, which implies that $f \circ c_\lambda$ is a fuzzy left ideal of S. Suppose g is the fuzzy left ideal of $f \circ c_\lambda$ and let $h = \{a_\mu \in f : a_\mu \circ c_\lambda \subseteq g \text{ for } \mu \in [0,1]\}$. Let for t in $[0,1]$, b_t be a fuzzy point of S then for $a_\mu \circ c_\lambda$ in h, $b_t \circ a_\mu$ is in f and so $b_t \circ a_\mu \circ c_\lambda \subseteq g$. Now $b_t \circ a_\mu \subseteq h$, which implies that $S \circ h \subseteq h$. Hence h is fuzzy left ideal of S contained in minimal fuzzy left ideal f of S and so $h = f$. Thus for all a_μ in f, $a_\mu \circ c_\lambda \subseteq g$, which implies that $f \circ c_\lambda \subseteq g$. Hence $f \circ c_\lambda = g$, and so $f \circ c_\lambda$ is a minimal fuzzy left ideal of S. □

Lemma 11. *A fuzzy ideal f of a semigroup S contains every minimal fuzzy left ideal of S.*

Proof. Let g be any minimal fuzzy left ideal of S, then $f \circ g$ is fuzzy left ideal of S since, $S \circ (f \circ g) = (S \circ f) \circ g \subseteq f \circ g$. Also $f \circ g \subseteq g$ as g is a fuzzy left ideal of S. But g is minimal so $g = f \circ g \subseteq f$. Hence every minimal fuzzy left ideal g of S is contained in every fuzzy ideal of S. □

Theorem 12. *If a semigroup S contains at least one minimal fuzzy left ideal then it has a fuzzy kernel of S, where fuzzy kernel is the class sum of all the minimal fuzzy left ideals of S.*

Proof. Let f be the union of all the minimal fuzzy left ideals of S. Since S contains at least one minimal fuzzy left ideal of S, so f is non-empty. Let $f_1, f_2, f_3, \dots$ be all the minimal fuzzy left ideals of S then, $S \circ f = S \circ (f_1 \cup f_2 \cup f_3 \cup \cdots) \subseteq S \circ f_1 \cup S \circ f_2 \cup S \circ f_3 \cup \cdots \subseteq f_1 \cup f_2 \cup f_3 \cup \cdots = f$, which implies that f is the fuzzy left ideal of S. Now it has to be shown that f is minimal. Let, for λ, μ in $[0,1]$, a_λ and b_μ be arbitrary fuzzy points of S and f, respectively. By definition of f, the fuzzy point b_μ is in some f_i for some $i \in \mathbb{N}$, such that $b_\mu \circ a_\lambda \subseteq f_i \circ a_\lambda$. Now since f_i is minimal fuzzy left ideal of S so by Lemma 10, $f_i \circ a_\lambda$ is also minimal fuzzy left ideal of S and hence contained in f. Thus, $b_\mu \circ a_\lambda \subseteq f$ and so $f \circ S \subseteq f$. Hence f is fuzzy right ideal of S and so fuzzy ideal of S. Now by Lemma 11, every fuzzy ideal contains every minimal fuzzy left ideal of S and so f being union of all minimal fuzzy left ideals of S is contained in every fuzzy ideal. Hence f is the fuzzy kernel of S. □

Theorem 13. *Let a semigroup S contain at least one minimal left ideal then every fuzzy left ideal of fuzzy kernel is also a fuzzy left ideal of S.*

Proof. Let f be the fuzzy kernel of S, then by Theorem 12, f is the class sum of all the minimal fuzzy left ideals of S. Let g be a fuzzy left ideal of f, then each fuzzy point a_λ for λ in $[0,1]$, of f belongs to some minimal fuzzy left ideal f_i, $i \in \mathbb{N}$ of S. Now $f \circ a_\lambda$ is a fuzzy left ideal of S, since $S \circ (f \circ a_\lambda) = (S \circ f) \circ a_\lambda \subseteq f \circ a_\lambda$. Also $f \circ a_\lambda \subseteq f \circ f_i \subseteq f_i$, where f_i is minimal fuzzy left ideal so $f \circ a_\lambda = f_i$. Thus $a_\lambda \subseteq f \circ a_\lambda$ implies that $f \subseteq f \circ a_\lambda$, where $f \circ a_\lambda$ is minimal fuzzy left ideal of S and so $f = f \circ a_\lambda$. Hence, f is the fuzzy left ideal of S. □

Remark 14. Every minimal fuzzy left ideal of S is a minimal fuzzy left ideal of fuzzy kernel of S and vice versa.

Theorem 15. *Let a semigroup S contain at least one minimal fuzzy left ideal of S then every fuzzy left ideal of S contains at least one minimal fuzzy left ideal of S.*

Proof. Let f be the fuzzy left ideal of S, then by Theorem 12, S has a fuzzy kernel g, where g is the class sum of all the minimal left ideals of S. Now $g \circ f$ is a fuzzy left ideal of S, since $S \circ (g \circ f) = (S \circ g) \circ f \subseteq g \circ f$, and also $g \circ f \subseteq g$. Thus $g \circ f$ is one of the minimal fuzzy left ideals of S or contains some minimal fuzzy left ideals of S. But $g \circ f$ is also contained in f. Hence f contains at least one minimal fuzzy left ideal of S. □

Remark 16. Lemmas 10 and 11 and Theorems 12, 13, and 15 are also applicable for fuzzy right ideal of a semigroup.

Theorem 17. *If a semigroup S contains at least one minimal fuzzy left ideal and one minimal fuzzy right ideal of S, then the class sum of all the minimal fuzzy left ideals of S coincide with the class sum of all the minimal fuzzy right ideals and constitutes the fuzzy kernel of S.*

Proof. By Theorem 12 the class sum of all the minimal fuzzy left ideals of S is a fuzzy kernel of S. Similarly the class sum of all the minimal fuzzy right ideals of S is also the fuzzy kernel of S. But fuzzy kernel of a semigroup S is unique since it is the intersection of all the fuzzy ideals of S. □

From now on we assume that f_R and f_L will denote the minimal fuzzy right ideal and the minimal fuzzy left ideal of a semigroup S, respectively. A fuzzy idempotent point x_λ of semigroup S is said to be under another fuzzy idempotent point y_μ if $x_\lambda \circ y_\mu = y_\mu \circ x_\lambda = x_\lambda$. The fuzzy idempotent x_λ is called fuzzy primitive if there is no fuzzy idempotent under x_λ. A simple fuzzy semigroup S is said to be completely simple if every fuzzy idempotent point of S is fuzzy primitive and for each fuzzy point a_γ of S there exists fuzzy idempotents x_λ and y_μ such that $x_\lambda \circ a_\gamma = a_\gamma \circ y_\mu = a_\gamma$.

Lemma 18. *Let a_λ be in f_R and b_μ be in $f_R \circ f_L$, then there exists a fuzzy point x_t in $f_R \circ f_L$ such that $a_\lambda \circ x_t = b_\mu$.*

Proof. Note that $a_\lambda \circ f_R$ is a fuzzy right ideal of S since, $(a_\lambda \circ f_R) \circ S = a_\lambda \circ (f_R \circ S) \subseteq a_\lambda \circ f_R$ and also $a_\lambda \circ f_R \subseteq f_R \circ f_R \subseteq f_R$. But f_R is the minimal fuzzy right ideal of S so $a_\lambda \circ f_R = f_R$. Hence $a_\lambda \circ f_R \circ f_L = f_R \circ f_L$. □

Lemma 19. *Let a_λ be in f_L and b_μ be in $f_R \circ f_L$, then there exists a fuzzy point x_t in $f_R \circ f_L$ such that $x_t \circ a_\lambda = b_\mu$.*

Proof. Since $f_L \circ a_\lambda$ is a fuzzy left ideal of S since, $S \circ (f_L \circ a_\lambda) = (S \circ f_L) \circ a_\lambda \subseteq f_L \circ a_\lambda$ and also $f_L \circ a_\lambda \subseteq f_L \circ f_L \subseteq f_L$. But f_L is the minimal fuzzy left ideal of S so $f_L \circ a_\lambda = f_L$. Hence $f_R \circ f_L \circ a_\lambda = f_R \circ f_L$. □

Theorem 20. *$f_R \circ f_L$ is a group.*

Proof. Since $(f_R \circ f_L) \circ (f_R \circ f_L) \subseteq (f_R \circ f_L) \circ f_L \subseteq f_R \circ f_L$, which implies that $f_R \circ f_L$ is a semigroup. Let a_λ and b_μ be any two fuzzy points of $f_R \circ f_L$ then a_λ and b_μ are in both f_R and f_L, since $f_R \circ f_L \subseteq f_R$ and $f_R \circ f_L \subseteq f_L$. Thus by Lemmas 18 and 19 there exists x_γ and y_ρ in S such that, $a_\lambda \circ x_\gamma = b_\mu$ and $y_\rho \circ a_\lambda = b_\mu$. Hence by Theorem 1, $f_R \circ f_L$ is a group. □

Lemma 21. *Let e_α be the identity element of the group $(f_R \circ f_L, \circ)$, then $f_R = e_\alpha \circ S$, $f_L = S \circ e_\alpha$ and $f_R \cap f_L = e_\alpha \circ S \circ e_\alpha$.*

Proof. Since identity element is in f_R since $f_R \circ f_L \subseteq f_R$. Now $(e_\lambda \circ S) \circ S = e_\lambda \circ (S \circ S) \subseteq e_\lambda \circ S$, which implies that $e_\lambda \circ S$ is a fuzzy right ideal of S and is contained in minimal fuzzy right ideal f_R of S and hence $e_\lambda \circ S = f_R$. Similarly we can show that $S \circ e_\lambda = f_L$. Now $f_R \circ f_L = (e_\lambda \circ S) \circ (S \circ e_\lambda) = e_\lambda \circ (S \circ S) \circ e_\lambda \subseteq e_\lambda \circ S \circ e_\lambda$, also $e_\lambda \circ S \circ e_\lambda \subseteq f_R \circ S \circ f_L \subseteq f_R \circ f_L$. □

Theorem 22. *For minimal fuzzy right ideal f_R and minimal fuzzy left ideal f_L of a semigroup S, $f_R \circ f_L = f_R \cap f_L$.*

Proof. We only need to show that $f_R \cap f_L \subseteq f_R \circ f_L$. Let a_λ be arbitrary fuzzy point of $f_R \cap f_L$ and let e_α be identity element of $f_R \circ f_L$ then by Lemma 18 there exists fuzzy point x_γ of $f_R \circ f_L$ such that $a_\lambda \circ x_\gamma = e_\alpha$. Let y_ρ be the inverse fuzzy element of x_γ in $f_R \circ f_L$ then, $a_\lambda \circ e_\alpha = a_\lambda \circ (x_\gamma \circ y_\rho) = (a_\lambda \circ x_\gamma) \circ y_\rho = e_\alpha \circ y_\rho = y_\rho$. Now $a_\lambda \subseteq f_R \circ f_L \subseteq f_L$, so by Lemma 21, $a_\lambda \circ e_\alpha = e_\alpha$. Hence $a_\lambda = y_\rho$ and so a_λ is in $f_R \circ f_L$. □

Lemma 23. *The identity fuzzy point e_λ of the fuzzy group $f_R \circ f_L$ is a fuzzy primitive.*

Proof. Let x_μ be a fuzzy idempotent under e_λ, then $x_\mu \circ e_\lambda = e_\lambda \circ x_\mu = x_\mu$. Further, Lemma 21 implies that x_μ is in $f_R \cap f_L$. By Theorem 22 x_μ is in the fuzzy group $f_R \circ f_L$, which can contain only one fuzzy idempotent point, namely the identity element e_λ, so $x_\mu = e_\lambda$. □

Theorem 24. *Let a semigroup S have at least one minimal fuzzy left ideal and at least one minimal fuzzy right ideal of S then the fuzzy kernel f_K of S is completely simple semigroup.*

Proof. Every fuzzy kernel of a semigroup is simple. Each fuzzy point a_μ of the fuzzy kernel f_K of S belongs to exactly one minimal fuzzy left ideal f_L and to exactly one minimal fuzzy right ideal f_R of S, otherwise f_L and f_R are not the minimal fuzzy left and minimal fuzzy right ideals of S. So the fuzzy point a_μ belongs to unique fuzzy group $f_R \circ f_L$. Thus f_K is the union of the disjoint fuzzy groups $f_R \circ f_L$. Each fuzzy idempotent point of f_K must be in one of these fuzzy groups, which can only have the fuzzy idempotent point e_λ, namely the identity point. But by Lemma 23 the identity point is fuzzy primitive. The second condition for f_K to be completely simple is straightforward as each fuzzy point a_μ of f_K is in one of the fuzzy groups $f_R \circ f_L$ having fuzzy idempotent e_λ such that $e_\lambda \circ a_\mu = a_\mu \circ e_\lambda = a_\mu$. Hence f_K is completely simple semigroup. □

References

[1] L. A. Zadeh, “Fuzzy sets,” *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] J. N. Mordeson, *Fuzzy Semigroups*, Springer, Berlin, Germany, 2003.

[3] A. Rosenfeld, "Fuzzy groups," *Journal of Mathematical Analysis and Applications*, vol. 35, no. 3, pp. 512–517, 1971.

[4] N. Kuroki, "Fuzzy bi-ideals in Semigroups," *Commentarii Mathematici Universitatis Sancti Pauli*, vol. 27, pp. 17–21, 1979.

[5] E. V. Huntington, "Simplified definition of a group," *Bulletin of the American Mathematical Society*, vol. 8, pp. 296–300, 1901/1902.

[6] A. H. Clifford, "Semigroups containing minimal ideals," *American Journal of Mathematics*, vol. 70, pp. 521–526, 1948.

[7] P. M. Pu and Y. M. Liu, "Fuzzy topology," *Journal of Mathematical Analysis and Applications*, vol. 76, pp. 512–517, 1980.