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Research Article

Asymptotic Principal Values and Regularization Methods for Correlation Functions with Reflective Boundary Conditions

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We introduce a concept of asymptotic principal values which enables us to handle rigorously singular integrals of higher-order poles encountered in the computation of various quantities based on correlation functions of a vacuum. Several theorems on asymptotic principal values are proved, and they are expected to become bases for investigating and developing some classes of regularization methods for singular integrals. We make use of these theorems for analyzing mutual relations between some regularization methods, including a method naturally derived from asymptotic principal values. It turns out that the concept of asymptotic principal values and the theorems for them are quite useful in this type of analysis, providing a suitable language to describe what is discarded and what is retained in each regularization method.

1. Introduction

Physics of quantum vacuum fluctuations is one of the intriguing research topics expected to be developed through the interplay between theories, experiments, and practical applications.

Investigations of quantum vacuum fluctuations even stimulate the border area between physics and mathematics. As a typical example of this sort, we often encounter singular integrals in computing several quantities based on correlation functions of a vacuum in question. The occurrence of singularity or divergence is often a signal of surpassing the border of validity of a model by too much extrapolation. Furthermore, it could originate from deeper physical processes for which satisfactory consistent mathematics is still unavailable. How to handle singular integrals can be then a challenging topic, requiring both mathematical analysis and physical considerations.

Faced with singular integrals, we need to resort to some regularization method to get a finite result. The aim of this paper is to give an organized mathematical basis underlying some typical regularization methods and make clear their mutual relations. We introduce below a concept of *asymptotic principal values* which can be a key tool to analyze some classes of regularization methods. We then prove several theorems on the asymptotic principal values useful for studying regularization procedures.

There are still various uncertainties to clear up in regularization methods, reflecting our lack of mathematical basis for handling infinities. In this situation, we cannot expect any universal regularization method, but we need to customize the method by try and error depending on the problem in question. It is far from the aim of this paper to judge which method is better than the others. It just tries to present a concrete mathematical basis for further considerations and developments of better regularization methods.

The organization of the paper is as follows. In Section 2, we present one simple example where singular integrals of a higher-order pole emerge. The origin of singularities in this example is physically clear and we can get some idea on how these integrals should be regularized. In Section 3, we introduce *asymptotic principal values* which describe precisely how singular integrals behave, which can be useful for investigating and developing several regularization methods. We then prove several useful theorems on the asymptotic principal values. In Section 4, we analyze some typical regularization methods by means of theorems prepared in the previous section. Section 5 is devoted for a summary and several discussions.

2. Typical Example of Singular Integrals

Let us consider as an example the measurement of the electromagnetic vacuum fluctuations in a half-space bounded by a perfectly reflecting infinite mirror. Recently the switching effect [1] and the smearing effect due to the quantum spread of a probe particle [2] have been analyzed by studying the measurement process of the Brownian particle released in this environment.

We thus take a model introduced in [3] and reanalyzed in [1, 2]. Suppose that a flat, infinitely spreading mirror of perfect reflectivity is placed on the xy-plane ($z = 0$). Then let us investigate the quantum vacuum fluctuations of the electromagnetic field inside the half-space $z > 0$ by releasing a classical charged probe-particle with mass m and charge e in the environment. We can estimate the quantum fluctuations of the vacuum through the velocity dispersions of the probe-particle released in the environment.

When the velocity of the probe-particle is much smaller than the light velocity c, the motion for the particle is described by

$$m\frac{d\vec{v}}{dt} = e\vec{E}(\vec{x}, t), \tag{1}$$

where $\vec{E}(\vec{x}, t)$ is the electric field.

Within the time-period when the particle does not move so much, (1) along with the initial condition $\vec{v}(0) = \vec{v}_0$ is solved approximately as

$$\vec{v}(t) \simeq \vec{v}_0 + \frac{e}{m}\int_0^t \vec{E}(\vec{x}, t')dt'. \tag{2}$$

For simplicity, let us consider only the "sudden-switching" case; the measurement is switched on abruptly, stably continued for τ [sec] before switched off abruptly. It is mathematically described by a step-like switching function without any switching tails. The velocity dispersions of the particle, $\langle \Delta v_i^2 \rangle$ $(i = x, y, z)$, are then given by

$$\langle \Delta v_i^2(\vec{x}, \tau)\rangle = \frac{e^2}{m^2}\int_0^\tau dt' \int_0^\tau dt'' \langle E_i(\vec{x}, t')E_i(\vec{x}, t'')\rangle_R, \tag{3}$$

by noting that $\langle E_i(\vec{x}, t)\rangle_R = 0$. Here $\langle E_i(\vec{x}, t') E_i(\vec{x}, t'')\rangle_R$ $(i = x, y, z)$ are the renormalized two-point correlation functions of the electric field (the suffix "R" is for "renormalized"). Now $\langle E_i(\vec{x}, t')E_i(\vec{x}, t'')\rangle_R (i = x, y, z)$ are computed [4] as

$$\begin{aligned}
\langle E_z(\vec{x}, t')E_z(\vec{x}, t'')\rangle_R &= \frac{1}{\pi^2}\frac{1}{\left(T^2 - (2z)^2\right)^2}, \\
\langle E_x(\vec{x}, t')E_x(\vec{x}, t'')\rangle_R &= \left\langle E_y(\vec{x}, t')E_y(\vec{x}, t'')\right\rangle_R \\
&= -\frac{1}{\pi^2}\frac{T^2 + 4z^2}{\left(T^2 - (2z)^2\right)^3},
\end{aligned} \tag{4}$$

where $T := t' - t''$. (We set $c = \hbar = 1$ hereafter throughout the paper.)

It is obvious that the integral in (3) is regular when $\tau < 2z$, but singular when $\tau > 2z$, reflecting the singularity at $|T| = 2z$ inherent in the correlation functions $\langle E_i(\vec{x}, t')E_i(\vec{x}, t'')\rangle_R$ given in (4).

For the present purpose, it suffices to show only the result of $\langle \Delta v_z^2 \rangle$ for $\tau > 2z$ [1]

$$\langle \Delta v_z^2\rangle = \frac{e^2}{32\pi^2 m^2}\left\{\frac{\tau}{z^3}\ln\left(\frac{\tau + 2z}{\tau - 2z}\right)^2 + \frac{8(1 - 2z/\tau)}{z^2\rho} + O(\rho)\right\} \tag{5}$$

$$\sim \frac{e^2}{4\pi^2 m^2 z^2}\left(1 + \frac{1}{\rho}\right) \quad (\text{for } \tau \gg 2z), \tag{6}$$

where $\rho (> 0)$ is a dimension-free asymptotic parameter for handling the singular integral properly (see Section 3 for details). Accordingly the above expression should be understood as an asymptotic expression as $\rho \sim 0$.

This result is derived by a formula for an *asymptotic principal value*, the rigorous definition of which shall be given in the next section,

$$\wp_{(\rho)}\int_0^1 dx \; \frac{1 - x}{(x^2 - \sigma^2)^2} = \frac{1}{8\sigma^3}\ln\left(\frac{1 + \sigma}{1 - \sigma}\right)^2 + \frac{1 - \sigma}{2\sigma^2\rho} + O(\rho), \tag{7}$$

for $0 < \sigma < 1$. One can derive (7) with the help of Theorem 1; a more direct derivation is also found in *Appendix C* of [1].

Leaving the rigorous treatment of singular integrals for the next section, we here focus on the physical reason why the singularity of correlation functions occurs at $|T| = 2z$. Due to the mirror reflections of signals with the light velocity, the values of the electric field at the two world-points (t', x, y, z) and (t'', x, y, z) are expected to be strongly correlated when $|t' - t''| = 2z$. When the measuring time τ is short enough (shorter than the travel-time of the signal $2z$), then it always follows $|t' - t''| < 2z$, so that these correlations are not captured by the probe. When the measuring time is long enough ($\tau > 2z$), however, these strong correlations accumulate in the velocity fluctuations of the particle at z. Therefore it is expected that the resulting singular term of the form A/ρ ($A > 0$) contains information on the reflecting boundary.

On the other hand, typical regularization procedures [5] correspond to discarding such a singular term (e.g., the $1/\rho$ term in (6)) in effect. It should be clarified when this type of regularization is valid and when not. We shall discuss on this point in more detail in Section 4.

It turns out that the model given here is too simplified and should be modified taking into account the switching effect [1] and the smearing effect due to the quantum spread of the probe-particle [2]. However, it suffices for the present purpose of giving some example of singular integrals.

3. Basic Formulas for Handling Singular Integrals

In view of the example in the previous section, it is clear that we sometimes need to estimate a singular integral whose integrand possesses a higher-order pole. In order to investigate various regularization methods later, we first need some concrete quantity corresponding to a singular integral for which all the information is retained and nothing is discarded. Then, the following asymptotic definition of a singular integral may be relevant.

Definition 1. Let $f(x)$ be an arbitrary real function defined around an interval $[a,b]$, differentiable at $x = c$ $(a < c < b)$ sufficiently many times. For a positive integer n, then, let us introduce an *asymptotic principal value* of order ρ defined by

$$\begin{aligned}\wp_{(c,\rho)}(f,n) &:= \wp_{(\rho)}\int_a^b \frac{f(x)}{(x-c)^n}dx \\ &:= \left\{\int_a^{c-\rho} + \int_{c+\rho}^b\right\}\frac{f(x)}{(x-c)^n}dx,\end{aligned} \tag{8}$$

where ρ is a sufficiently small positive parameter.

The asymptotic principal value is a generalization of the standard Cauchy principal value, corresponding to $\lim_{\rho\downarrow 0}\wp_{(c,\rho)}(f,1)$, in two ways. First, the order of singularity n can be greater than 1. Second, only the asymptotic behavior as $\rho \sim 0$ is concerned and the convergence for the limit $\rho \to 0$ is not necessarily required. In other words, we focus on how the integral behaves near $\rho \sim 0$ rather than the $\rho \to 0$ limit itself. In this sense, all the information is retained and no infinities are discarded in defining the asymptotic principal value.

Let us now introduce another asymptotic quantity.

Definition 2. With the same premises as in *Definition 1*, we define

$$\wp_{(\rho)}\left[\frac{f(x)}{(x-c)^n}\right]_a^b := \left[\frac{f(x)}{(x-c)^n}\right]_a^{c-\rho} + \left[\frac{f(x)}{(x-c)^n}\right]_{c+\rho}^b. \tag{9}$$

It is easily shown that

$$\wp_{(\rho)}\left[\frac{f(x)}{(x-c)^n}\right]_a^b = \left[\frac{f(x)}{(x-c)^n}\right]_a^b - \frac{1}{\rho^n}\{f\}_{(n)}^{(c,\rho)} \tag{10}$$

with

$$\{f\}_{(n)}^{(c,\rho)} := f(c+\rho) - (-)^n f(c-\rho). \tag{11}$$

By a Taylor-expansion in ρ, it is obvious that $\{f\}_{(n)}^{(c,\rho)}$ is $O(\rho)$ for an even n (when $f'(c) \neq 0$), so that the term $(1/\rho^n)\{f\}_{(n)}^{(c,\rho)}$ in (10) is $O(1/\rho^{n-1})$ and singular in the $\rho \to 0$ limit. Similarly, $\{f\}_{(n)}^{(c,\rho)}$ is $O(1)$ for an odd n (when $f(c) \neq 0$), so that $(1/\rho^n)\{f\}_{(n)}^{(c,\rho)}$ is $O(1/\rho^n)$ and singular as $\rho \to 0$.

The following definition is just for making formulas below concise.

Definition 3. We have

$$\zeta_{(c,\rho)}(f,n) := \frac{1}{n}\wp_{(\rho)}\left[\frac{f(x)}{(x-c)^n}\right]_a^b. \tag{12}$$

With these preparations, let us start with the following lemma.

Lemma 1. *For any function $f(x)$ differentiable at $x = c$ and for any integer n (≥ 2), it follows that*

$$\wp_{(c,\rho)}(f,n) = \frac{1}{n-1}\wp_{(c,\rho)}(f',n-1) - \zeta_{(c,\rho)}(f,n-1). \tag{13}$$

Proof. Noting that

$$\frac{1}{(x-c)^n} = -\frac{1}{n-1}\left(\frac{1}{(x-c)^{n-1}}\right)', \tag{14}$$

we have

$$\begin{aligned}\wp_{(c,\rho)}(f,n) &:= -\frac{1}{n-1}\wp_{(\rho)}\int_a^b\left(\frac{1}{(x-c)^{n-1}}\right)' f(x)dx \\ &= -\frac{1}{n-1}\left\{\wp_{(\rho)}\left[\frac{f(x)}{(x-c)^{n-1}}\right]_a^b\right. \\ &\qquad \left. - \wp_{(\rho)}\int_a^b\frac{f'(x)}{(x-c)^{n-1}}dx\right\},\end{aligned} \tag{15}$$

where the partial-integral has been performed to get the last line. Then the equality follows. □

We now prove a formula which relates a multipole integral with a simple-pole integral.

Theorem 1. *For any function $f(x)$ differentiable sufficiently many times at $x = c$ and for any integer n (≥ 2), it follows that*

$$\begin{aligned}\wp_{(c,\rho)}(f,n) &= \frac{1}{(n-1)!}\wp_{(c,\rho)}\left(f^{(n-1)},1\right) \\ &\quad - \sum_{k=1}^{n-1}\frac{(n-k)!}{(n-1)!}\zeta_{(c,\rho)}\left(f^{(k-1)},n-k\right).\end{aligned} \tag{16}$$

Proof. (1°) For $n = 2$, the claimed equality reduces to

$$\wp_{(c,\rho)}(f,2) = \wp_{(c,\rho)}(f',1) - \zeta_{(c,\rho)}(f,1), \tag{17}$$

which clearly holds due to Lemma 1.

(2°) Let us assume that the equality holds for a function $F(x)$ and for $n = m$ $(m \geq 2)$, that is,

$$\begin{aligned}\wp_{(c,\rho)}(F,m) &= \frac{1}{(m-1)!}\wp_{(c,\rho)}\left(F^{(m-1)},1\right) \\ &\quad - \sum_{k=1}^{m-1}\frac{(m-k)!}{(m-1)!}\zeta_{(c,\rho)}\left(F^{(k-1)},m-k\right).\end{aligned} \tag{18}$$

Now applying Lemma 1 for $n = m + 1$, we have

$$\begin{aligned}
&\wp_{(c,\rho)}(f, m+1) \\
&= \frac{1}{m}\,\wp_{(c,\rho)}(f', m) - \zeta_{(c,\rho)}(f, m) \\
&= \frac{1}{m}\left\{\frac{1}{(m-1)!}\,\wp_{(c,\rho)}\left(f^{(m)}, 1\right) - \sum_{k=1}^{m-1}\frac{(m-k)!}{(m-1)!}\,\zeta_{(c,\rho)}\left(f^{(k)}, m-k\right)\right\} \\
&\quad - \zeta_{(c,\rho)}(f, m),
\end{aligned} \tag{19}$$

where the assumed equation (18) for $F(x) = f'(x)$ has been used to get the second equality. Rearranging the summation, the last equality reduces to

$$\begin{aligned}
&\wp_{(c,\rho)}(f, m+1) \\
&= \frac{1}{m!}\,\wp_{(c,\rho)}\left(f^{(m)}, 1\right) - \sum_{k=1}^{m}\frac{((m+1)-k)!}{m!}\,\zeta_{(c,\rho)}\left(f^{(k-1)}, (m+1)-k\right).
\end{aligned} \tag{20}$$

Thus the claimed equation (16) holds for $n = m + 1$.

(3°) By the mathematical induction, (16) holds for any integer n ($n \geq 2$). □

Based on Theorem 1, it is natural to introduce a quantity $\widetilde{\wp}_{(c,\rho)}(f, n)$, which is a simple-pole part plus a regular part of $\wp_{(c,\rho)}(f, n)$, putting aside singular contributions from higher-order poles.

Definition 4. We define

$$\begin{aligned}
\widetilde{\wp}_{(c,\rho)}(f, n) :=& \frac{1}{(n-1)!}\,\wp_{(c,\rho)}\left(f^{(n-1)}, 1\right) \\
&- \sum_{k=1}^{n-1}\frac{(n-k-1)!}{(n-1)!}\left[\frac{f^{(k-1)}(x)}{(x-c)^{n-k}}\right]_a^b.
\end{aligned} \tag{21}$$

The "mild part" $\widetilde{\wp}_{(c,\rho)}(f, n)$ of $\wp_{(c,\rho)}(f, n)$ shall be important in the discussion of regularization methods in Section 4.

We now have a formula which enables us to separate singular contributions from a multipole integral.

Theorem 2. *For any function $f(x)$ differentiable sufficiently many times at $x = c$ and for any integer n (≥ 2), it follows that*

$$\begin{aligned}
\wp_{(c,\rho)}(f, n) =& \widetilde{\wp}_{(c,\rho)}(f, n) \\
&+ \sum_{k=1}^{n-1}\frac{(n-k-1)!}{(n-1)!}\,\frac{1}{\rho^{n-k}}\left\{f^{(k-1)}\right\}_{(n-k)}^{(c,\rho)}.
\end{aligned} \tag{22}$$

Proof. It is straightforward to show this formula due to Theorem 1 along with (12) and (10). □

Lemma 2. *For any function $f(x)$ differentiable at $x = c$ and for a positive integer n, it follows that*

$$\wp_{(c,\rho)}(f, n+1) = \frac{1}{n}\,\partial_c\,\wp_{(c,\rho)}(f, n) + \frac{1}{n\rho^n}\{f\}_n^{(c,\rho)}. \tag{23}$$

Proof. We compute $\partial_c\,\wp_{(c,\rho)}(f, n)$ directly as

$$\begin{aligned}
\partial_c\,\wp_{(c,\rho)}(f, n) &= \partial_c\left(\left\{\int_a^{c-\rho} + \int_{c+\rho}^b\right\}\frac{f(x)}{(x-c)^n}dx\right) \\
&= n\wp_{(c,\rho)}(f, n+1) - \frac{1}{\rho^n}\{f\}_n^{(c,\rho)},
\end{aligned} \tag{24}$$

where the second term in the last line comes from the c-derivative applied to the upper and the lower limit of the integral region. Thus the claimed equation follows. □

Theorem 3. *For any function $f(x)$ differentiable sufficiently many times at $x = c$ and for an integer n (≥ 2), it follows that*

$$\widetilde{\wp}_{(c,\rho)}(f, n) = \frac{1}{(n-1)!}\,{\partial_c}^{n-1}\,\wp_{(c,\rho)}(f, 1). \tag{25}$$

Proof. Due to Theorem 2, the claimed equation (25) is equivalent to

$$\begin{aligned}
\wp_{(c,\rho)}(f, n) =& \frac{1}{(n-1)!}\,{\partial_c}^{n-1}\,\wp_{(c,\rho)}(f, 1) \\
&+ \sum_{k=1}^{n-1}\frac{(n-k-1)!}{(n-1)!}\,\frac{1}{\rho^{n-k}}\left\{f^{(k-1)}\right\}_{(n-k)}^{(c,\rho)}.
\end{aligned} \tag{26}$$

Thus it suffices to show (26).

(1°) Let us consider the case $n = 2$, where the R.H.S. (right-hand side) of (26) becomes

$$\frac{\partial}{\partial c}\left(\left\{\int_a^{c-\rho} + \int_{c+\rho}^b\right\}\frac{f(x)}{x-c}dx\right) + \frac{1}{\rho}\{f(c-\rho) + f(c+\rho)\}. \tag{27}$$

In this expression, the c-derivative applied to the upper and the lower limit of the integral region yields a term which exactly cancels the second term. As a result, the above expression reduces to $\wp_{(c,\rho)}(f, 2)$, that is, the L.H.S. (left-hand side) of (26). Thus (26) holds for $n = 2$.

(2°) Let us now assume that (26) holds for $n = m$ ($m \geq 2$), that is,

$$\begin{aligned}
\wp_{(c,\rho)}(f, m) =& \frac{1}{(m-1)!}\,{\partial_c}^{m-1}\,\wp_{(c,\rho)}(f, 1) \\
&+ \sum_{k=1}^{m-1}\frac{(m-k-1)!}{(m-1)!}\,\frac{1}{\rho^{m-k}}\left\{f^{(k-1)}\right\}_{(m-k)}^{(c,\rho)}.
\end{aligned} \tag{28}$$

Due to Lemma 2, then, it becomes

$$\begin{aligned}
&\wp_{(c,\rho)}(f,m+1) \\
&\quad = \frac{1}{m}\partial_c \wp_{(c,\rho)}(f,m) + \frac{1}{m\rho^m}\{f\}_m^{(c,\rho)} \\
&\quad = \frac{1}{m}\partial_c \Bigg\{ \frac{1}{(m-1)!}\ \partial_c^{\,m-1}\ \wp_{(c,\rho)}(f,1) \\
&\qquad\qquad + \sum_{k=1}^{m-1} \frac{(m-k-1)!}{(m-1)!}\ \frac{1}{\rho^{m-k}} \left\{f^{(k-1)}\right\}_{(m-k)}^{(c,\rho)} \Bigg\} \\
&\qquad + \frac{1}{m\rho^m}\{f\}_m^{(c,\rho)},
\end{aligned} \tag{29}$$

where (28) has been used to get the last line. Noting that the relation

$$\partial_c\{f\}_m^{(c,\rho)} = \{f'\}_m^{(c,\rho)}, \tag{30}$$

which obviously holds from (11), it reduces to

$$\begin{aligned}
\wp_{(c,\rho)}(f,m+1) = {}& \frac{1}{m!}\ \partial_c^{\,m}\ \wp_{(c,\rho)}(f,1) \\
&+ \sum_{k=1}^{m} \frac{(m-k)!}{m!}\ \frac{1}{\rho^{m-k+1}} \left\{f^{(k-1)}\right\}_{(m-k+1)}^{(c,\rho)}.
\end{aligned} \tag{31}$$

Thus (26) holds for $n = m + 1$.

(3°) By the mathematical induction, (26) holds for any integer n ($n \geq 2$). Thus the claimed formula (25) has been shown. □

4. Typical Regularization Methods and Their Mutual Relations

Based on the results in the previous section, let us now come back to the problem of regularization methods for singular integrals.

Let us consider a typical singular integral

$$I = \int_a^b \frac{f(x)}{(x-c)^n}dx \tag{32}$$

for any function $f(x)$ differentiable sufficiently many times at $x = c$ ($a < c < b$) and for a positive integer n.

4.1. Regularization Method with Partial Integrals. The first method of regularization we consider is a *method of partial integrals* which is sometimes made used of. We insert an identity

$$\frac{1}{(x-c)^n} = -\frac{1}{n-1}\left(\frac{1}{(x-c)^{n-1}}\right)', \tag{33}$$

into (32) and *formally* perform a partial integral

$$\begin{aligned}
I &= -\frac{1}{n-1}\int_a^b \left(\frac{1}{(x-c)^{n-1}}\right)' f(x)dx \\
&= \frac{1}{n-1}\left\{\int_a^b \frac{f'(x)}{(x-c)^{n-1}}dx - \left[\frac{f(x)}{(x-c)^{n-1}}\right]_a^b\right\}.
\end{aligned} \tag{34}$$

In this way, the order of singularity is reduced by one. Repeating the similar procedure, the integral I is reduced to the $n = 1$ case for which the prescription of the Cauchy principal value may be applied.

Due to Theorem 2, however, it is obvious that singular terms should exist and should have been discarded by hand in the above procedure. Indeed, compared with (34) with the rigorous expression (15), it is obvious that the singularities which should reside in the second term on the R.H.S. of (34) are simply discarded by hand. Thus, in view of Theorem 2, the above method is equivalent to the simple replacement of I as

$$I \longmapsto \widetilde{\wp}_{(c,\rho)}(f,n). \tag{35}$$

There is still room, however, to regard the method of partial integrals as a shorthand prescription of what we here call the *method of infinitesimal imaginary part*, which is much more of theoretical grounds [5]. We shall consider this method in the next subsection.

4.2. Regularization Method with Infinitesimal Imaginary Part. The method of infinitesimal imaginary part is based on well-known Dirac's formula [6] for an integral kernel

$$\frac{1}{(x-c)\pm i\rho} = \wp_{(c,\rho)}\frac{1}{x-c} \mp i\pi\delta(x-c), \tag{36}$$

which is most easily shown by estimating an integral $\int_a^b dx f(x)/((x-c)\pm i\rho)$ by means of an appropriate contour-integral for a suitable function $f(x)$.

By differentiating the both-sides of (36) $n-1$ times with respect to c, and by applying Theorem 3, we get

$$\frac{1}{((x-c)\pm i\rho)^n} = \widetilde{\wp}_{(\rho,c)}(\cdot,n) \mp i\frac{\pi}{(n-1)!}\delta^{(n-1)}(x-c) \tag{37}$$

in the sense of an integral kernel. Recalling *Definition 4*, however, we see that the R.H.S. is reduced to the $n = 1$ case, for which the prescription of the Cauchy principal value may be applied.

It is notable that just the introduction of some infinitesimal imaginary part results in a tamable quantity such as $\widetilde{\wp}_{(\rho,c)}(\cdot,n)$ at the cost of the imaginary contribution of the second term on the R.H.S. of (37). Thus along with some causality arguments [5], it is often argued that the singular integral I should be interpreted as the real part of $\int(f(x)/((x-c)\pm i\rho)^n)dx$, that is,

$$I \longmapsto \Re\int \frac{f(x)}{((x-c)\pm i\rho)^n}dx. \tag{38}$$

As far as one is evaluating *real* quantities, one may further argue that the second term on the R.H.S. of (37) shall not contribute. If so, the procedure is in effect equivalent to the replacement (35). In this sense, the method of partial integrals discussed in the previous subsection may be justified provided that it is regarded as a shorthand prescription of the method of infinitesimal imaginary part.

Another way of looking at this method is to pay attention to the L.H.S. (rather than the R.H.S.) of (37). As far as computations of real quantities are concerned, then, this method is equivalent to the replacement $1/(x-c)^n$ with $((x-c)/((x-c)^2+\rho^2))^n$ along with taking the limit $\rho \to 0$ after evaluating the integral

$$I \longmapsto \lim_{\rho \to 0} \int_a^b f(x) \left(\frac{x-c}{(x-c)^2+\rho^2} \right)^n dx. \tag{39}$$

There is some subtle points in this method. One of them is to discard the imaginary part of the R.H.S. of (37) on the grounds that one is evaluating *real* quantities. Considering that the regularization has been achieved at the cost of introducing the imaginary part though tiny, the imaginary part should carry important information and some concern naturally arises whether one can discard it so freely.

Indeed, a simple example can be presented for which this kind of procedure fails. Let us consider an integral $I_1 = \int_{-1}^{1} dx$ which is purposefully regarded as

$$I_1 = \int_{-1}^{1} x \cdot \frac{1}{x} dx. \tag{40}$$

It is obvious that $I_1 = 2$. The analysis by the asymptotic principal value (see the next subsection) also results in $I_1^{(\rho)} \to 2$ in the limit $\rho \to 0$. This is because all the information is retained in the prescription of the asymptotic principal value.

On the other hand, the above-mentioned scheme makes a replacement

$$\frac{1}{x} \longmapsto \frac{x}{x^2+\rho^2} = \frac{1}{2}\left(\ln(x^2+\rho^2)\right)', \tag{41}$$

so that

$$\begin{aligned} I_1 \longrightarrow I_1^{(\rho)} &:= \int_{-1}^{1} x \cdot \frac{1}{2}\left(\ln(x^2+\rho^2)\right)' dx \\ &= \ln(1+\rho^2) - \int_0^1 \ln(x^2+\rho^2) dx, \end{aligned} \tag{42}$$

where a partial integral has been performed to get the last line. However, it is clear that $I_1^{(\rho)} \to \infty$ as $\rho \to 0$, contradicting with the obvious result $I_1 = 2$.

Quite interestingly, no contradiction occurs for $I_m = \int_{-1}^{1} x^m \cdot (1/x) dx$ with $m \geq 2$ since the second term in (42) becomes $-m\int_0^1 x^{m-1} \ln(x^2+\rho^2) dx$ so that no singularity occurs around $x \sim 0$ for $m \geq 2$. More generally, the integral of the form $\int_{-1}^{1}(f(x)/x)dx$, if treated by the above prescription, gives rise to the dominant contribution $-f'(0)\int_0^1 \ln(x^2+\rho^2)dx$ which diverges as $\rho \to 0$.

With these caveats in mind, let us now move to a new regularization method based on the asymptotic principal values.

4.3. Regularization Method with Asymptotic Principal Values. Let us finally introduce a new regularization method based on the asymptotic principal values.

For the simple example in Section 2, there has been a definite physical interpretation of the singularity in the correlation function. Furthermore, the system considered there has been a combination of quantum objects with a macroscopic mirror. Therefore it might be also probable that the deepest cause of the singularity resides in the validity issue of the model originating from too much extrapolation from the quantum side to the macroscopic situation. Indeed there is an investigation showing that the quantum fluctuations of the mirror boundary drastically decrease the singular behavior near the mirror [7]. Therefore it is reasonable to take the origin of the singularity more realistically (rather than just mathematical phenomenon), expecting that some physical processes suppress the order of singularity.

Going back to the example of the integral I in (32), then, it is possible to interpret I in the sense of an asymptotic principal value,

$$I \longmapsto \wp_{(c,\rho)}(f, n) \tag{43}$$

with the dimension-free parameter ρ being provided by the ratio of some natural cut-off scale with the system-size in question. (E.g., the ratio of the plasma wave-length of the mirror with $2z$ for the example in Section 2). The advantage of this regularization scheme is that one can explicitly analyze the ρ-dependence of the integral. For instance, one may study the influence of the quantum fluctuations of the mirror by treating ρ as a fluctuation parameter. The result for $\langle \Delta {v_z}^2 \rangle$ given in (6) along with (7) is an example of the computation by the method of asymptotic principal values.

We see that Theorem 2 is the basis for understanding the relation between the regularization methods discussed so far. The difference between the method of asymptotic principal value ($\wp_{(c,\rho)}(f,n)$) and the method of infinitesimal imaginary part ($\widetilde{\wp}_{(c,\rho)}(f,n)$) is given by the second term on the R.H.S. of (22), which is of $O(1/\rho^{n-1})$.

5. Summary and Discussions

In this paper, we have focussed on singular integrals with a higher-order pole which frequently emerge in computing quantities based on two-point correlation functions of a vacuum.

To deal with this type of singular integrals, we have introduced the concept of *asymptotic principal values*. The asymptotic principal value of order ρ, which is a generalization of the Cauchy principal value, is defined by introducing a cut-off parameter ρ, focussing solely on the asymptotic behavior of the integral as $\rho \sim 0$. In this sense, it is a rigorous object retaining all the information on the singular integral.

We have then proved several theorems on asymptotic principal values which are expected to serve as bases for studying regularization methods for singular integrals.

To see how asymptotic principal values can be made use of, we have selected three typical regularization methods and have analyzed their mutual relations with the help of theorems we have prepared. It has turned out that the concept of asymptotic principal values and related theorems are quite useful in this kind of analysis. Indeed, in terms of asymptotic principal values, it has been possible to describe without ambiguity what is discarded and what is retained in each regularization method.

No universal regularization method is available so far and we need to carefully select or invent a suitable method depending on the problem in question. For instance, we recall the example in Section 2, where velocity dispersion of the probe, $\langle \Delta {v_z}^2 \rangle$, is sensitive to the regularization method. In particular, the result expected by the method of infinitesimal imaginary part (Section 4.2) (and the method of partial integrals (Section 4.1)) is

$$\langle \Delta {v_z}^2 \rangle \sim \frac{e^2}{4\pi^2 m^2 z^2} \quad (\text{for } \tau \gg 2z). \tag{44}$$

On the other hand, the result expected by the method of asymptotic principal values (Section 4.3) is

$$\langle \Delta {v_z}^2 \rangle \sim \frac{e^2}{4\pi^2 m^2 z^2} \left(1 + \frac{1}{\rho}\right) \quad (\text{for } \ \tau \gg 2z), \tag{45}$$

by choosing ρ in the order of the ratio of plasma wavelength and the typical size $2z$. Strictly speaking, the model in Section 2 is a too simplified one and should be modified taking into account the quantum spread of the probe-particle itself. Then the behavior of $\langle \Delta {v_z}^2 \rangle$ at late time is corrected to a more reasonable one $\langle \Delta {v_z}^2 \rangle \sim 1/\tau^2$ rather than $\sim 1/z^2$ [2].

In any case it is significant to compare the results derived by different regularization methods in more detail for approaching to a more satisfactory mathematical theory of regularization procedures.

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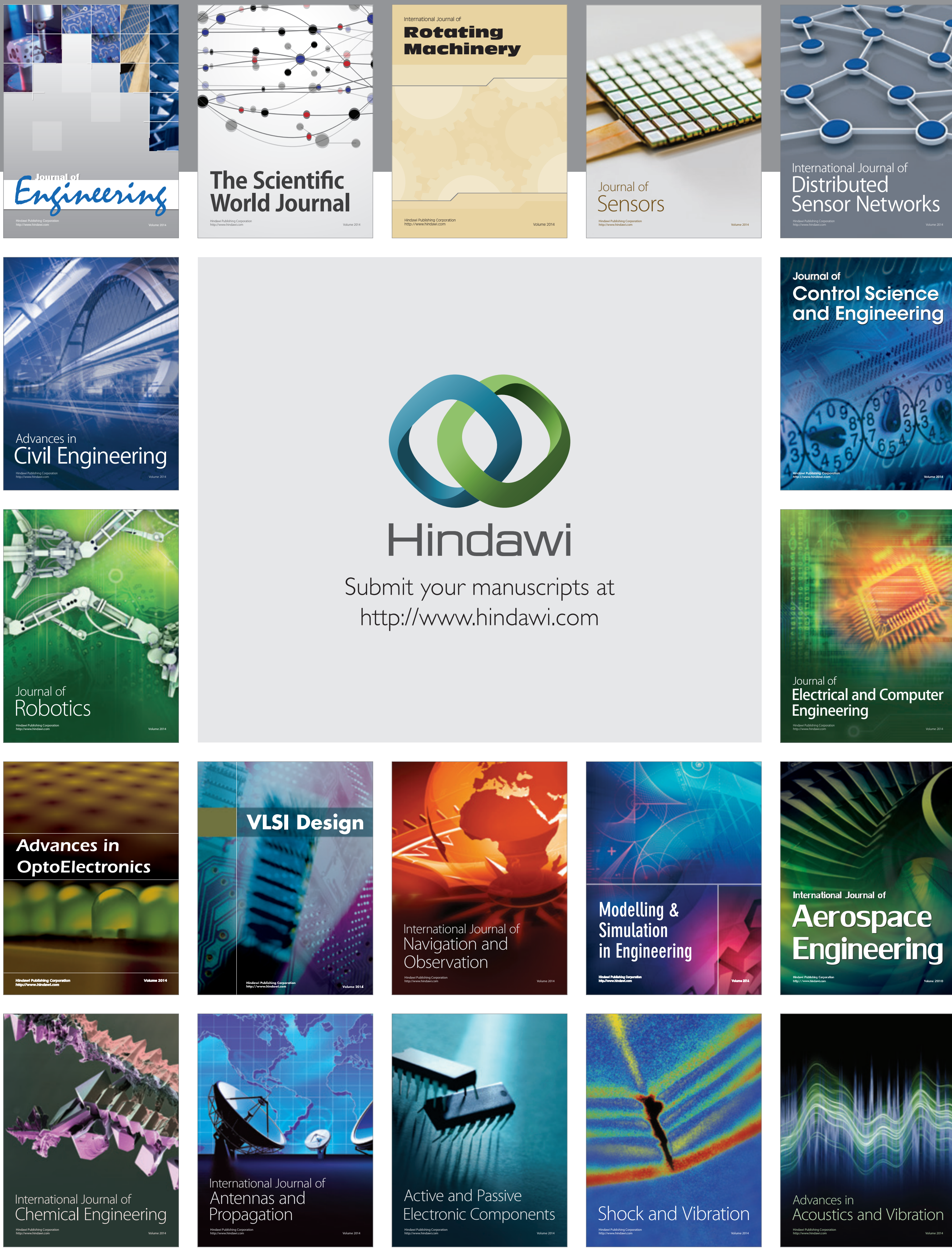
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