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Research Article

The Stationary Distribution and Extinction of Generalized Multispecies Stochastic Lotka-Volterra Predator-Prey System

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This paper is concerned with the existence of stationary distribution and extinction for multispecies stochastic Lotka-Volterra predator-prey system. The contributions of this paper are as follows. (a) By using Lyapunov methods, the sufficient conditions on existence of stationary distribution and extinction are established. (b) By using the space decomposition technique and the continuity of probability, weaker conditions on extinction of the system are obtained. Finally, a numerical experiment is conducted to validate the theoretical findings.

1. Introduction

The dynamic relationship between the predators and the preys has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [1]. The classic predator-prey model is the Lotka-Volterra model, governed by the following differential equation:

$$\dot{x} = x(a - by), \qquad \dot{y} = y(-c + fx), \tag{1}$$

where $x(t)$ and $y(t)$ denote the prey and predator population size, respectively, at time t. For the prey component, the parameters a and b are the fixed growth and mortality rates, respectively. For the predator component, the parameters c and f are the fixed growth and mortality rates, respectively. Since then, variants of the two-species Lotka-Volterra system have been frequently investigated to describe population dynamics with predator-prey relations; see, for example, [2–4].

Recently, the multispecies predator-prey systems have received a great deal of research attention since they took the differences among individual growth and mortality into account (see [5–8]). In order to understand the nature of the competitive interactions and relationships between predator and prey, Yang and Xu [8] considered the following periodic m-prey and $n-m$-predator Lotka-Volterra differential system with periodic coefficients:

$$\begin{aligned} dx_i &= x_i\left(b_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j\right)dt, \quad i = 1, \ldots, m, \\ dx_i &= x_i\left(-b_i(t) + \sum_{j=1}^{m} a_{ij}(t)x_j - \sum_{j=m+1}^{n} a_{ij}(t)x_j\right)dt, \\ &\qquad i = m+1, \ldots, n, \end{aligned} \tag{2}$$

where $x_i(t)$, $i = 1, \ldots, m$, denotes the density of prey species at time t and $x_i(t)$, $i = m+1, \ldots, n$, denotes the density of predator species at time t. Under the assumption that $b_i(t) > 0$ $(i = 1, \ldots, m)$, $b_i(t) \ge 0$ $(i = m+1, \ldots, n)$, $a_{ii}(t) > 0$ $(i = 1, \ldots, n)$, $a_{ij} \ge 0$ $(i \ne j)$ are continuous periodic functions with a common periodic $T > 0$, a set of sufficient conditions on the existence and global attractiveness of the periodic solution to system (2) are obtained. Recently, Chen and Shi [5] further considered the almost periodic case of more complicated systems than system (2) under the almost periodic case. By constructing a suitable Lyapunov function, they obtained a set of sufficient conditions which guarantees the existence of a unique globally attractive positive almost periodic solution to the corresponding system.

On the other hand, from the biological point of view, population systems in the real world are inevitably affected by environmental noise. In the past decades, the dynamics of stochastic populations and related topic have received a great deal of research attention (see [9–21]), since they have been successfully used in a variety of application fields, including biology (see [22–28]), epidemiology (see [29, 30]), and neural networks (see [31–33]). More recently, the asymptotic properties of stochastic predator-prey systems have received a lot of attention; the readers can refer to [10, 11, 34] and the references therein. For example, the dynamics of the density dependent stochastic predator-prey system with different functional response have been studied by Ji and Jiang in [10, 11]. Vasilova [34] has investigated a stochastic Gilpin-Ayala predator-prey model with time-dependent delay, and certain asymptotic results regarding the long-time behavior of trajectories of the solution and sufficient criteria for extinction of species for a special case of the considered system are given.

In this paper, considering the effect of environmental noise, we introduce stochastic perturbation into the growth rate of the prey and the predator in system (2) and assume that parameters b_i and a_{ij} are constant. Then we obtain the following m-prey and $n-m$-predator stochastic Lotka-Volterra system with constant coefficients:

$$\begin{aligned} dx_i &= x_i\left(b_i-\sum_{j=1}^{n}a_{ij}x_j\right)dt+\sigma_i x_i dB_i(t), \\ &\qquad\qquad i=1,\ldots,m, \\ dx_i &= x_i\left(-b_i+\sum_{j=1}^{m}a_{ij}x_j-\sum_{j=m+1}^{n}a_{ij}x_j\right)dt \\ &\quad+\sigma_i x_i dB_i(t), \quad i=m+1,\ldots,n, \end{aligned} \tag{3}$$

where $B(t)=(B_1(t),B_2(t),\ldots,B_n(t))$ is an n-dimensional Brownian motion and σ_i^2 will be called the noise intensity. Throughout this paper, we always assume that the following hypothesis holds:

$$\begin{aligned} &b_i>0, \quad i=1,\ldots,m, \\ &b_i\ge 0, \quad i=m+1,\ldots,n, \\ &a_{ii}>0, \quad i=1,\ldots,n, \\ &a_{ij}\geqslant 0 \quad (i\ne j). \end{aligned} \tag{4}$$

In the study of stochastic population systems, extinction and existence of stationary distribution are two important and interesting properties, respectively, meaning that the population system will die out or the distribution of the solution converges weakly to the probability measure in the future, which have received a lot of attention (see [12, 35–37]). Then one question arises naturally: under what condition can system (3) have a stationary distribution and become extinct, respectively? This issue constitutes the first motivation of this paper.

In addition, the existing literatures (see [10, 35]) show clearly that if the noise intensity of every prey species is more than twice the corresponding intrinsic growth rate, the population will become extinct exponentially. Then one interesting question is as follows: What will happen if the noise intensity equals twice the intrinsic growth rate? Thus, the second purpose of this paper is to solve this interesting problem.

The organization of the paper is as follows. Section 2 describes some preliminaries. The main results are stated in Sections 3 and 4. In Section 3, sufficient conditions are obtained under which there is a stationary distribution to system (3). By utilizing some novel stochastic analysis techniques, sufficient criteria for ensuring the extinction of system (3) are obtained in Section 4. Section 5 provides some numerical examples to check the effectiveness of the derived results. Conclusion is made in Section 6.

2. Notation

Throughout this paper, unless otherwise specified, let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge0},\mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t)=(B_1(t),B_2(t),\ldots,B_n(t))$ be an n-dimensional Brownian motion defined on the probability space. If $A\in R^{n\times n}$ is symmetric, its largest and smallest eigenvalues are denoted by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$. Let $x^*=(x_1^*,x_2^*,\ldots,x_n^*)$ be the positive equilibrium of the corresponding deterministic predator-prey system to system (3), that is, the solution to the following equation:

$$\begin{aligned} &b_i-\sum_{j=1}^{n}a_{ij}x_j^*=0, \quad i=1,2,\ldots,m, \\ &-b_i+\sum_{j=1}^{m}a_{ij}x_j^*-\sum_{j=m+1}^{n}a_{ij}x_j^*=0, \quad i=m+1,\ldots,n. \end{aligned} \tag{5}$$

In the same way as Zhu and Yin [38] and Liu et al. [39] did, we can also show the following result on the existence of global positive solution.

Lemma 1. *Suppose that condition (4) holds; then one has the following assertions:*

(i) *For any given initial value $x_0\in R_+^n$, there is a unique solution $x(t,x_0)$ to system (3) and the solution will remain in R_+^n with probability 1; namely,*

$$\mathbb{P}\left\{x\left(t,x_0\right)\in R_+^n,\ \forall t\ge0\right\}=1, \tag{6}$$

for any $x_0\in R_+^n$.

(ii) *For any given initial value $x_0\in R_+^n$ and any $\beta>0$, almost every sample path of $x^\beta(t,x_0)$ is locally but uniformly Holder continuous.*

Lemma 2 (see [40]). *Let $f(t)$ be a nonnegative function defined on $[0,+\infty)$ such that $f(t)$ is integrable on $[0,+\infty)$ and is uniformly continuous on $[0,+\infty)$; then $\lim_{t\to\infty}f(t)=0$.*

3. Stationary Distribution

In this section, we mainly show that system (3) has a stationary distribution. Let us give a lemma that will be used in the following proof. Let $X(t)$ be a homogeneous Markov process in $E^n \subset R^n$ described by the following stochastic equation:

$$dX(t) = b(X)\,dt + \sum_{k=1}^{d} \sigma_k(X)\,dB_k(t). \tag{7}$$

The diffusion matrix is

$$A(x) = \left(a_{ij}(x)\right), \quad a_{ij}(x) = \sum_{k=1}^{d} \sigma_k^i(x)\,\sigma_k^j(x). \tag{8}$$

Lemma 3 (see [41]). *One assumes that there is a bounded open subset $G \subset E^n$ with a regular (i.e., smooth) boundary such that its closure $\overline{G} \subset E^n$, and*

(i) *in the domain G and some neighborhood therefore, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero;*

(ii) *if $x \in E^n \setminus G$, the mean time τ at which a path issuing from x reaches the set G is finite, and $\sup_{x\in K} E_x \tau < +\infty$ for every compact subset $K \in E^n$ and throughout this paper one sets $\inf \emptyset = \infty$.*

One then has the following assertions:

(1) *The Markov process $X(t)$ has a stationary distribution $\mu(\cdot)$ with density in E^n.*

(2) *Let $f(x)$ be a function integrable with respect to the measure $\mu(\cdot)$. Then*

$$\mathbb{P}\left\{\lim_{t\to\infty}\frac{1}{t}\int_0^t f(x(s))\,ds = \int_{E^n} f(y)\,\mu(dy)\right\} = 1. \tag{9}$$

Remark 4. The proof is given by [41] in detail. Precisely, the existence of a stationary distribution with density is obtained in Theorem 4.3 on pp. 117. The ergodic property is referred to Theorem 4.2 on pp. 110. To validate (i), we can directly show that $\lambda_{\min}\{A(x)\} > 0$. To validate (ii), it suffices to prove that there is some neighborhood U and a nonnegative C^2-function V such that, for any $x \in E^n \setminus U$, $\mathscr{L}V(x)$ is negative (for details refer to [42], pp. 1163).

Theorem 5. *Let condition (4) hold and let $x(t, x_0)$ be the global solution to system (3) with any positive initial value $x_0 \in R_+^n$. Assume that there exists $c = (c_1, c_2, \ldots, c_n) \gg 0$ such that*

$$\begin{aligned}
&c_i a_{ii} - \frac{1}{2}\sum_{\substack{j\neq i\\ i=1}}^{n}\left(c_i a_{ij} + c_j a_{ji}\right) > 0, \quad i = 1, 2, \ldots, n,\\
&\frac{1}{2}\sum_{i=1}^{n} c_i x_i^* \sigma_i^2\\
&\quad < \min_{1\leqslant i\leqslant n}\left\{\left(c_i a_{ii} - \frac{1}{2}\sum_{\substack{j\neq i\\ i=1}}^{n}\left(c_i a_{ij} + c_j a_{ji}\right)\right)\left(x_j^*\right)^2\right\}.
\end{aligned} \tag{10}$$

Then there is a stationary distribution for system (3).

Proof. Let $x_i(t) = x_i(t, x_0)$ for simplicity. Applying Itô's formula to $V(x) = \sum_{i=1}^{n} c_i(x_i - x_i^* - x_i \ln(x_i/x_i^*))$ yields

$$\begin{aligned}
\mathscr{L}V = &-\sum_{i=1}^{n} c_i a_{ii}\left(x_i - x_i^*\right)^2\\
&-\sum_{i=1}^{m}\sum_{\substack{j\neq i\\ j=1}}^{m} c_i a_{ij}\left(x_i - x_i^*\right)\left(x_j - x_j^*\right)\\
&-\sum_{i=1}^{m}\sum_{j=m+1}^{n} c_i a_{ij}\left(x_i - x_i^*\right)\left(x_j - x_j^*\right)\\
&+\sum_{i=m+1}^{n}\sum_{j=1}^{m} c_i a_{ij}\left(x_i - x_i^*\right)\left(x_j - x_j^*\right)\\
&-\sum_{i=m+1}^{n}\sum_{\substack{j\neq i\\ j=m+1}}^{n} c_i a_{ij}\left(x_i - x_i^*\right)\left(x_j - x_j^*\right)\\
&+\frac{1}{2}\sum_{i=1}^{n} c_i x_i^* \sigma_i^2.
\end{aligned} \tag{11}$$

By the inequality $2ab \leqslant (a^2 + b^2)$, we have

$$\begin{aligned}
\mathscr{L}V \leqslant &-\sum_{i=1}^{n} c_i a_{ii}\left(x_i - x_i^*\right)^2 + \frac{1}{2}\sum_{i=1}^{m}\sum_{\substack{j\neq i\\ j=1}}^{m} c_i a_{ij}\left(x_i - x_i^*\right)^2\\
&+\frac{1}{2}\sum_{i=1}^{m}\sum_{\substack{j\neq i\\ j=1}}^{m} c_i a_{ij}\left(x_j - x_j^*\right)^2\\
&+\frac{1}{2}\sum_{i=m+1}^{n}\sum_{\substack{j\neq i\\ j=m+1}}^{n} c_i a_{ij}\left(x_i - x_i^*\right)^2\\
&+\frac{1}{2}\sum_{i=m+1}^{n}\sum_{\substack{j\neq i\\ j=m+1}}^{n} c_i a_{ij}\left(x_j - x_j^*\right)^2\\
&+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=m+1}^{n} c_i a_{ij}\left(x_i - x_i^*\right)^2\\
&+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=m+1}^{n} c_i a_{ij}\left(x_j - x_j^*\right)^2\\
&+\frac{1}{2}\sum_{i=m+1}^{n}\sum_{j=1}^{m} c_i a_{ij}\left(x_i - x_i^*\right)^2\\
&+\frac{1}{2}\sum_{i=m+1}^{n}\sum_{j=1}^{m} c_i a_{ij}\left(x_j - x_j^*\right)^2 + \frac{1}{2}\sum_{i=1}^{n} c_i x_i^* \sigma_i^2\\
= &-\sum_{i=1}^{n} c_i a_{ii}\left(x_i - x_i^*\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j\neq i\\ j=1}}^{n} c_i a_{ij}\left(x_i - x_i^*\right)^2
\end{aligned}$$

$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j\neq i\\ j=1}}^{n}c_j a_{ij}\left(x_i-x_i^*\right)^2+\frac{1}{2}\sum_{i=1}^{n}c_i x_i^*\sigma_i^2$$
$$=-\sum_{i=1}^{n}\left[c_i a_{ii}-\frac{1}{2}\sum_{\substack{j\neq i\\ i=1}}^{n}\left(c_i a_{ij}+c_j a_{ji}\right)\right]\left(x_i-x_i^*\right)^2$$
$$+\frac{1}{2}\sum_{i=1}^{n}c_i x_i^*\sigma_i^2. \tag{12}$$

Note that (10); then the ellipsoid

$$\sum_{i=1}^{n}\left[c_i a_{ii}-\frac{1}{2}\sum_{\substack{j\neq i\\ i=1}}^{n}\left(c_i a_{ij}+c_j a_{ji}\right)\right]\left(x_i-x_i^*\right)^2 = \frac{1}{2}\sum_{i=1}^{n}c_i x_i^*\sigma_i^2 \tag{13}$$

lies entirely in R_+^n. Now we can take U to be a neighborhood of the ellipsoid with $\overline{U}\subseteq E_n=R_+^n$, such that, for $x\in R_+^n\setminus U$, $\mathscr{L}V<0$, which means condition (ii) of Lemma 3 is verified.

Now we begin to verify condition (i) in Lemma 3. Let us define $H(x) = \operatorname{diag}(\sigma_1 x_1,\sigma_2 x_2,\ldots,\sigma_n x_n)$, so the diffusion matrix is $A(x) = H^T(x)H(x)$. It is clear that $\lambda_{\min}\{H^T(x)H(x)\}\geqslant 0$. If $\lambda_{\min}\{H^T(x)H(x)\}=0$ holds, there exists $\xi = (\xi_1,\xi_2,\ldots,\xi_n)^T \in R^n$ such that $|\xi| = 1$ and $\xi^T H^T(x)H(x)\xi = 0$, which implies that $H(x)\xi = 0$. By the definition of σ_i, $i=1,2,\ldots,n$, and $x\in R_+^n\setminus U$, we see $\xi=0$, but it contradicts with $|\xi|=1$. So $\lambda_{\min}\{H^T(x)H(x)\}>0$ for $x\in R_+^n\setminus U$ must hold. That means condition (i) of Lemma 3 is verified. Therefore, we can say that stochastic system (3) has a stationary distribution. □

Remark 6. Theorem 5 shows that system (3) has a unique stationary distribution when the perturbation is small in the sense that

$$\frac{1}{2}\sum_{i=1}^{n}c_i x_i^*\sigma_i^2 < \min_{1\leqslant i\leqslant n}\left\{\left(c_i a_{ii}-\frac{1}{2}\sum_{\substack{j\neq i\\ i=1}}^{n}\left(c_i a_{ij}+c_j a_{ji}\right)\right)\left(x_j^*\right)^2\right\}. \tag{14}$$

4. Extinction

Extinction is one of the most basic questions that can be studied in the population dynamics, which means the population system will die out. Most of the time we need to know the extinction rate of the species for which we have to make a suitable policy in advance and to make useful measures to protect them from becoming extinct.

Theorem 7. *Let condition (4) hold and let $x(t,x_0)$ be the global solution to system (3) with any initial value $x_0\in R_+^n$. Assume that there exists an integer $k\leqslant m$ such that*

$$b_i<\frac{\sigma_i^2}{2},\quad i=1,\ldots,k,$$
$$b_i=\frac{\sigma_i^2}{2},\quad i=k+1,\ldots,m. \tag{15}$$

One then has the following assertions:

(i) *For $i=1,\ldots,k$, the solution $x_i(t,x_0)$ to system (3) has the property that*

$$\lim_{t\to\infty}\frac{\log x_i(t,x_0)}{t}=b_i-\frac{\sigma_i^2}{2}\quad a.s. \tag{16}$$

That is, for each $i = 1,\ldots,k$, the species i will become extinct exponentially with probability one and the exponential extinction rate is $-(\sigma_i/2-b_i)$.

(ii) *For $i=k+1,\ldots,m$, the solution $x_i(t,x_0)$ to system (3) has the property that*

$$\lim_{t\to\infty}x_i(t,x_0)=0,$$
$$\lim_{t\to\infty}\frac{\log x_i(t,x_0)}{t}=0\quad a.s. \tag{17}$$

That is, for each $i = k+1,\ldots,m$, the species i still becomes extinct with zero exponential extinction rate.

(iii) *For $i=m+1,\ldots,n$, the solution $x_i(t,x_0)$ to system (3) has the property that*

$$\operatorname{Lim}_{t\to\infty}\frac{\log x_i(t,x_0)}{t}=-b_i-\frac{\sigma_i^2}{2}\quad a.s. \tag{18}$$

That is, for each $i = m+1,\ldots,n$, the species i will become extinct exponentially with probability one and the exponential extinction rate is $-(\sigma_i/2+b_i)$.

Proof. Let $x_i(t)=x_i(t,x_0)$ for simplicity. To make the proof clear, we are going to divide it into four steps. The first step and the third step are to show the least upper bound of exponential extinction rate for the top k preys and the predators of system (3), respectively. The second step is to show the extinction for the bottom $m-k$ preys of system (3) in the case of $\sigma_i^2=2b_i$, $i=k+1,\ldots,m$. The fourth step is to accomplish the proof of assertions (i)–(iii) based on the proof of Steps 1–3.

Step 1. Applying Itô's formula to $\log x_i(t)$ yields

$$\begin{aligned}\log x_i(t) &= \log x_i(0) + \int_0^t \left(b_i - \frac{\sigma_i^2}{2}\right) ds \\ &\quad - \int_0^t \sum_{j=1}^n a_{ij} x_j(s)\, ds + M_i(t), \\ &\qquad i = 1, \ldots, m,\end{aligned} \tag{19}$$

$$\begin{aligned}\log x_i(t) &= \log x_i(0) \\ &\quad + \int_0^t \left(\sum_{j=1}^m a_{ij} - \sum_{j=m+1}^n a_{ij}\right) x_j(s)\, ds \\ &\quad + \int_0^t \left(-b_i - \frac{\sigma_i^2}{2}\right) ds + M_i(t), \\ &\qquad i = m+1, \ldots, n,\end{aligned} \tag{20}$$

where $M_i(t) = \int_0^t \sigma_i dB_i(s)$, $i = 1, \ldots, n$, is the real-valued continuous local martingale vanishing at $t = 0$, with the quadratic variation $\langle M_i(t), M_i(t)\rangle = \sigma_i^2 t$. Dividing both sides of (19) and (20) by t, we have that

$$\begin{aligned}\frac{1}{t}\log x_i(t) &= \frac{1}{t}\log x_i(0) + \frac{1}{t}\int_0^t \left(b_i - \frac{\sigma_i^2}{2}\right) ds \\ &\quad - \frac{1}{t}\int_0^t \sum_{j=1}^n a_{ij} x_j(s)\, ds + \frac{1}{t} M_i(t), \\ &\qquad i = 1, \ldots, m,\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{1}{t}\log x_i(t) &= \frac{1}{t}\log x_i(0) \\ &\quad + \frac{1}{t}\int_0^t \left(\sum_{j=1}^m a_{ij} - \sum_{j=m+1}^n a_{ij}\right) x_j(s)\, ds \\ &\quad + \frac{1}{t}\int_0^t \left(-b_i - \frac{\sigma_i^2}{2}\right) ds + \frac{1}{t} M_i(t), \\ &\qquad i = m+1, \ldots, n.\end{aligned} \tag{22}$$

Using the strong law of large numbers for martingales [43], we obtain that

$$\lim_{t\to\infty} \frac{M_i(t)}{t} = 0 \quad \text{a.s., } i = 1, \ldots, n. \tag{23}$$

For $i = 1, 2, \ldots, k$, letting $t \to \infty$ in (21) yields that

$$\begin{aligned}\limsup_{t\to\infty} \frac{\log x_i(t)}{t} &\leqslant \lim_{t\to\infty} \frac{1}{t}\int_0^t \left(b_i - \frac{\sigma_i^2}{2}\right) ds = b_i - \frac{\sigma_i^2}{2} \\ &< 0 \quad \text{a.s., } i = 1, \ldots, k.\end{aligned} \tag{24}$$

Step 2. The main aim of this step is to show $\lim_{t\to\infty} x_i(t) = 0$ a.s. for $i = k+1, \ldots, m$. When $\sigma_i^2 = 2b_i$, (21) turns to be the following form:

$$\begin{aligned}\frac{1}{t}\log x_i(t) &= \frac{1}{t}\log x_i(0) - \frac{1}{t}\int_0^t \sum_{j=1}^n a_{ij} x_j(s)\, ds \\ &\quad + \frac{1}{t} M_i(t), \quad i = k+1, \ldots, m.\end{aligned} \tag{25}$$

According to the convergence of $\int_0^\infty x_i(s) ds$, we can decompose the sample space into two exclusive events spaces as follows:

$$\begin{aligned}J_{i,1} &= \left\{\omega : \int_0^\infty x_i(s)\, ds < \infty\right\}, \\ J_{i,2} &= \left\{\omega : \int_0^\infty x_i(s)\, ds = \infty\right\}, \\ &\qquad i = k+1, \ldots, m.\end{aligned} \tag{26}$$

On the other hand we can divide the sample space into three mutually exclusive events as follows:

$$\begin{aligned}E_{i,1} &= \left\{\omega : \lim_{t\to\infty} \sup x_i(t) \geqslant \lim_{t\to\infty} \inf x_i(t) = \gamma_i > 0\right\}, \\ E_{i,2} &= \left\{\omega : \lim_{t\to\infty} \sup x_i(t) > \lim_{t\to\infty} \inf x_i(t) = 0\right\}, \\ E_{i,3} &= \left\{\omega : \lim_{t\to\infty} x_i(t) = 0\right\}.\end{aligned} \tag{27}$$

From the above, the proof of $\lim_{t\to\infty} x_i(t) = 0$ a.s. is equivalent to show $J_{i,1} \subset E_{i,3}$ a.s. and $J_{i,2} \subset E_{i,3}$ a.s. Now we give the process in two parts.

Part 1 of Step 2. Now we show $J_{i,1} \subset E_{i,3}$ a.s. It follows from Lemma 1 that almost every sample path of $x_i(t)$ is locally but uniformly Holder continuous and therefore almost every sample path of $x_i(t, x_0)$ must be uniformly continuous. Considering the definition of $J_{i,1}$ and Lemma 2, we obtain

$$\lim_{t\to\infty} x_i(t) = 0 \quad \text{a.s., } i = k+1, \ldots, m, \tag{28}$$

which means $J_{i,1} \subset E_{i,3}$ a.s.

Part 2 of Step 2. The aim of this part is to prove that $J_{i,2} \subset E_{i,3}$ a.s. It is sufficient to show $P(J_{i,2} \cap E_{i,1}) = 0$ and $P(J_{i,2} \cap E_{i,2}) = 0$.

If this $P(J_{i,2} \cap E_{i,1}) = 0$ is not true, for any $\omega_i \in (J_{i,2} \cap E_{i,1})$ and $\varepsilon_i \in (0, \gamma_i/2)$ there exists $T(\varepsilon_i, \omega_i)$ such that $\forall t > T(\varepsilon_i, \omega_i)$

$$x_i(t) > \gamma_i - \varepsilon_i > \frac{\gamma_i}{2}, \quad i = k+1, \ldots, m \quad \text{a.s.} \tag{29}$$

Simple computations show that

$$\begin{aligned}\frac{1}{t}\int_0^t a_{ij}x_j(s)\,ds &= \frac{1}{t}\int_0^T a_{ij}x_j(s)\,ds \\ &\quad + \frac{1}{t}\int_T^t a_{ij}x_j(s)\,ds \\ &\geqslant \frac{1}{t}\int_0^T a_{ij}x_j(s)\,ds + a_{ij}\frac{t-T}{t}\frac{\gamma_i}{2} \\ &\qquad \text{a.s., } i = k+1,\dots,m.\end{aligned} \tag{30}$$

Letting $t \to \infty$ on both sides of (30) yields

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t a_{ij}x_j(s)\,ds > \frac{1}{2}a_{ij}\gamma_i > 0 \qquad \text{a.s., } i = k+1,\dots,m. \tag{31}$$

This implies that

$$\limsup_{t\to\infty}\frac{\log x_i(t)}{t} \leqslant -\frac{1}{2}\sum_{j=1}^n a_{ij}\gamma_i \quad \text{a.s., } i = k+1,\dots,m, \tag{32}$$

which contradicts with the definition of $J_{i,2}$ and $E_{i,1}$. So $P(J_{i,2}\cap E_{i,1}) = 0$ must hold.

Now we proceed to show $P(J_{i,2}\cap E_{i,2}) > 0$ is false. Now we need more notations such as

$$\begin{aligned} B_t^{\varepsilon} &:= \{0 \leqslant s \leqslant t : x(s) \geqslant \varepsilon\}, \\ h_t^{\varepsilon} &:= \frac{m(B_t^{\varepsilon})}{t}, \\ h^{\varepsilon} &:= \limsup_{t\to\infty} h_t^{\varepsilon}, \\ H^{\varepsilon} &:= \{\omega \in J_{i,2}\cap E_{i,2} : h^{\varepsilon} > 0\}, \end{aligned} \tag{33}$$

where $m(B_t^{\varepsilon})$ means the length of B_t^{ε}. From the definition of H^{ε}, we can easily get that $H^0 = J_{i,2}\cap E_{i,2}$. The following is right for any $\varepsilon_1 < \varepsilon_2$:

$$\begin{aligned} B_t^{\varepsilon_2} &\subset B_t^{\varepsilon_1}, \\ m(B_t^{\varepsilon_2}) &\leqslant m(B_t^{\varepsilon_1}), \\ h_t^{\varepsilon_2} &\leqslant h_t^{\varepsilon_1}, \end{aligned} \tag{34}$$

which yields

$$\begin{aligned} & h_t^{\varepsilon_2} \leqslant h_t^{\varepsilon_1}, \\ & H^{\varepsilon_2} \subset H^{\varepsilon_1}, \\ &\qquad \forall \varepsilon_1 < \varepsilon_2. \end{aligned} \tag{35}$$

From the continuity of probability, we can obviously get

$$P(H^{\varepsilon}) \longrightarrow P(H^0) = P(J_{i,2}\cap E_{i,2}), \quad \text{as } \varepsilon \longrightarrow 0. \tag{36}$$

Based on the hypothesis $P(J_{i,2}\cap E_{i,2}) > 0$, we can claim that there exists $\theta > 0$ such that $P(H^{\theta}) > 0$. It is therefore to see that, for any $\omega \in H^{\theta}$,

$$\begin{aligned} \frac{1}{t}\sum_{j=1}^n\int_0^t a_{ij}x_j(s)\,ds &= \frac{1}{t}\sum_{j=1}^n\int_{H_t^{\theta}} a_{ij}x_j(s)\,ds \\ &\quad + \frac{1}{t}\sum_{j=1}^n\int_{[0,t]\setminus H_t^{\theta}} a_{ij}x_j(s)\,ds \\ &\geqslant \frac{1}{t}\sum_{j=1}^n\int_{H_t^{\theta}} a_{ij}x_j(s)\,ds \\ &\geqslant \sum_{j=1}^n a_{ij}\theta\frac{m(B_t^{\theta})}{t} \\ &\qquad \text{a.s., } i = k+1,\dots,m. \end{aligned} \tag{37}$$

Letting $t \to \infty$ on both sides of (37) yields

$$\frac{1}{t}\sum_{j=1}^n\int_0^t a_{ij}x_j(s)\,ds \geqslant \sum_{j=1}^n a_{ij}\theta h^{\theta} \quad \text{a.s., } i = k+1,\dots,m, \tag{38}$$

which means

$$\limsup_{t\to\infty}\frac{1}{t}\log x_i(t) \leqslant -\sum_{j=1}^n a_{ij}\theta h^{\theta} \qquad \text{a.s., } i = k+1,\dots,m. \tag{39}$$

This also contradicts with the definition of $J_{i,2}$ and $E_{i,2}$. It immediately yields that the assertion $P(J_{i,2}\cap E_{i,2}) = 0$ must hold. Now we can claim that $P(J_{i,2}\cap E_{i,1}) = 0$ and $P(J_{i,2}\cap E_{i,2}) = 0$, which means $J_{i,2} \subset E_{i,3}$. Combining with the fact $J_{i,1} \subset E_{i,3}$ and $J_{i,2} \subset E_{i,3}$, we have

$$\lim_{t\to\infty} x_i(t) = 0 \quad \text{a.s., } i = k+1,\dots,m. \tag{40}$$

Step 3. It follows from (24) and (40) that

$$\lim_{t\to\infty} x_i(t) = 0 \quad \text{a.s., } i = 1,\dots,m. \tag{41}$$

This implies

$$\lim_{t\to\infty}\frac{1}{t}\sum_{j=1}^m\int_0^t a_{ij}x_j(s)\,ds = 0 \quad \text{a.s.} \tag{42}$$

Now letting $t \to \infty$ on both sides of (22) yields

$$\limsup_{t\to\infty}\frac{\log x_i(t)}{t} \leqslant -b_i - \frac{\sigma_i^2}{2} < 0 \qquad \text{a.s., } i = m+1,\dots,n. \tag{43}$$

Step 4. It is immediate from (40) and (43) that

$$\lim_{t\to\infty} x_i(t) = 0 \quad \text{a.s., } i = 1,\dots,n. \tag{44}$$

This implies

$$\lim_{t\to\infty}\frac{1}{t}\sum_{j=1}^{n}\int_0^t a_{ij}x_j(s)\,ds = 0 \quad \text{a.s., } i = 1,\ldots,n. \tag{45}$$

By taking limit on both sides of (21) and (22), we have

$$\begin{aligned}\lim_{t\to\infty}\frac{\log x_i(t)}{t} &= b_i - \frac{\sigma_i^2}{2} \quad \text{a.s., } i = 1,\ldots,k,\\ \lim_{t\to\infty}\frac{\log x_i(t)}{t} &= -b_i - \frac{\sigma_i^2}{2} \quad \text{a.s., } i = m+1,\ldots,n.\end{aligned} \tag{46}$$

So assertions (i)–(iii) of Theorem 7 must hold. □

Remark 8. Note that, for $m = n$, system (3) becomes the following classic stochastic Lotka-Volterra competitive systems, which have received much attention (see [12, 36, 38]):

$$dx_i = x_i\left(b_i - \sum_{j=1}^{n}a_{ij}x_j\right)dt + \sigma_i x_i dB_i(t), \quad i = 1,\ldots,n. \tag{47}$$

And condition (4) becomes the following form:

$$\begin{aligned}&b_i > 0, \quad i = 1,\ldots,n,\\ &a_{ii} > 0, \quad i = 1,\ldots,n,\\ &a_{ij} \geqslant 0 \quad (i \neq j).\end{aligned} \tag{48}$$

Thus, by Theorem 7, we have the sufficient conditions on extinction for system (45) as Corollary 9.

Corollary 9. *Let condition (48) hold and let $x(t, x_0)$ be the global solution to system (47) with any initial value x_0. Assume that there exists an integer $k \leqslant n$ such that*

$$\begin{aligned}b_i &< \frac{\sigma_i^2}{2}, \quad i = 1,\ldots,k,\\ b_i &= \frac{\sigma_i^2}{2}, \quad i = k+1,\ldots,n.\end{aligned} \tag{49}$$

One then has the following assertions:

(i) *For $i = 1,\ldots,k$, the solution $x_i(t, x_0)$ to system (47) has the property that*

$$\lim_{t\to\infty}\frac{\log x_i(t)}{t} = b_i - \frac{\sigma_i^2}{2} \quad \text{a.s.} \tag{50}$$

That is, for each $i = 1,\ldots,k$, the species i will become extinct exponentially with probability one and the exponential extinction rate is $-(\sigma_i/2 - b_i)$.

(ii) *For $i = k+1,\ldots,n$, the solution $x_i(t, x_0)$ to system (47) has the property that*

$$\begin{aligned}&\lim_{t\to\infty} x_i(t) = 0,\\ &\lim_{t\to\infty}\frac{\log x_i(t)}{t} = 0 \quad \text{a.s.}\end{aligned} \tag{51}$$

That is, for each $i = k+1,\ldots,n$, the species i still becomes extinct with zero exponential extinction rate.

By using some novel stochastic analysis techniques, we point out that species i is still extinct when $\sigma^2 = 2b_i$. In comparison with Theorem 4.1 in [12], the conditions imposed on the extinction are weaker.

5. Example and Simulations

In this section, we present a numerical example to illustrate the usefulness and flexibility of the theorem developed in the previous section.

Example 1. Consider a 3-dimensional stochastic Lotka-Volterra predator-prey system as follows:

$$\begin{aligned}dx_1 &= x_1(0.9 - 0.8x_1 - 0.2x_2 - 0.4x_3)\,dt\\ &\quad + \sigma_1 x_1 dB_1(t),\\ dx_2 &= x_2(0.8 - 0.3x_1 - 0.9x_2 - 0.5x_3)\,dt\\ &\quad + \sigma_2 x_2 dB_2(t),\\ dx_2 &= x_3(-0.1 + 0.3x_2 + 0.1x_2 - 0.6x_3)\,dt\\ &\quad + \sigma_3 x_3 dB_3(t).\end{aligned} \tag{52}$$

System (52) is exactly system (3) with $n = 3$, $m = 2$, $a_{11} = 0.8 > 0$, $a_{12} = 0.2 > 0$, $a_{13} = 0.4 > 0$, $a_{21} = 0.3 > 0$, $a_{22} = 0.9 > 0$, $a_{23} = 0.5 > 0$, $a_{31} = 0.3 > 0$, $a_{32} = 0.1 > 0$, $a_{33} = 0.6 > 0$, $b_1 = 0.9 > 0$, $b_2 = 0.8 > 0$, and $b_3 = 0.1 > 0$. We compute that the equilibrium $(x_1^*, x_2^*, x_3^*)' = (0.9107, 0.4808, 0.1882)'$. The existence and uniqueness of the solution follow from Lemma 1. On the condition of the suitable parameters, we can get the simulation figures with initial value $(x_1(0), x_2(0), x_3(0)) = (0.6, 0.4, 0.4)$ by MATLAB. In Figures 1–3, the blue line represents the population of $x_1(t)$,

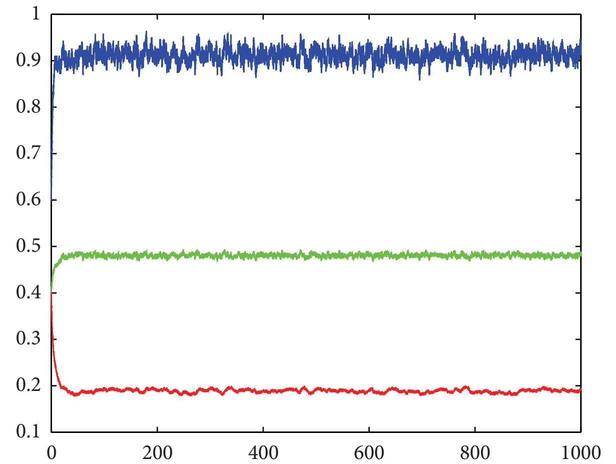

Figure 1: Computer simulation of $x_1(t)$, $x_2(t)$, and $x_3(t)$ generated by the Heun scheme for time step $\Delta = 10^{-3}$ for system (52) on $[0, 1000]$, respectively.

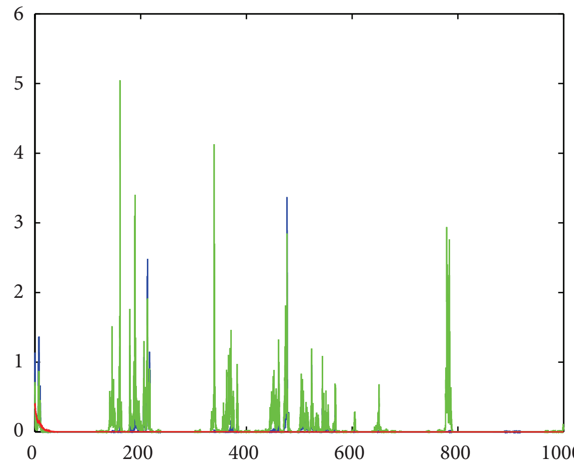

FIGURE 2: Computer simulation of a single path of $x_i(t)$, $i = 1, 2, 3$, generated by the Heun scheme for time step $\Delta = 10^{-3}$ for system (52) on $[0, 1000]$, respectively.

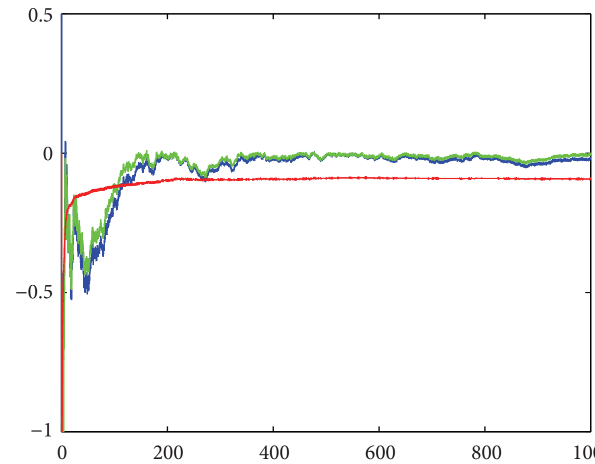

FIGURE 3: Computer simulation of $(\log x_i(t))/t$, $i = 1, 2, 3$, generated by the Heun scheme for time step $\Delta = 10^{-3}$ for system (52) on $[0, 1000]$, respectively.

the green line represents the population of $x_2(t)$, and the red line represents the population of $x_3(t)$.

(i) $(\sigma_1, \sigma_2, \sigma_3)' = (0.02, 0.01, 0.005)'$: choosing $c_1 = 1$, $c_2 = 1$, $c_3 = 1.5$, we further compute that

$$\begin{aligned}
& c_1 a_{11} - \frac{1}{2}\left(c_1 a_{12} + c_2 a_{21} + c_1 a_{13} + c_3 a_{31}\right) = 0.05 > 0, \\
& c_2 a_{22} - \frac{1}{2}\left(c_2 a_{21} + c_1 a_{12} + c_2 a_{23} + c_3 a_{32}\right) = 0.325 > 0, \\
& c_3 a_{33} - \frac{1}{2}\left(c_3 a_{31} + c_1 a_{13} + c_3 a_{32} + c_2 a_{23}\right) = 0.20 > 0, \\
& c_1 a_{11} \\
& \quad - \frac{1}{2}\left[\left(c_1 a_{12} + c_2 a_{21}\right)\left(x_2^*\right)^2 + \left(c_1 a_{13} + c_3 a_{31}\right)\left(x_3^*\right)^2\right] \\
& \quad = 0.72715466, \\
& c_2 a_{22} \\
& \quad - \frac{1}{2}\left[\left(c_2 a_{21} + c_1 a_{12}\right)\left(x_1^*\right)^2 + \left(c_2 a_{23} + c_3 a_{32}\right)\left(x_3^*\right)^2\right] \\
& \quad = 0.68114512, \\
& c_3 a_{33} \\
& \quad - \frac{1}{2}\left[\left(c_3 a_{31} + c_1 a_{13}\right)\left(x_1^*\right)^2 + \left(c_3 a_{32} + c_2 a_{23}\right)\left(x_2^*\right)^2\right] \\
& \quad = 0.47238604, \\
& \frac{1}{2}\sum_{i=1}^{3} c_i x_i^* \sigma_i^2 = 0.000209709 \\
& \quad < \min_{1\leqslant i\leqslant 3}\{0.72715466, 0.68114512, 0.47238604\}.
\end{aligned} \tag{53}$$

By virtue of Theorem 5, system (52) has a unique stationary distribution.

(ii) $(\sigma_1, \sigma_2, \sigma_3)' = (1.38, \sqrt{1.6}, 0.05)'$: note that $\sigma_1^2 = 1.9044 > 2b_1 = 1.8$, $\sigma_2^2 = 1.6 = 2b_2 = 1.6$; by virtue of Theorem 7, system (52) is extinctive. From Figure 2, we can see that the predator population will die out though it suffers the small white noise when the prey population becomes extinct.

But we cannot see the value of the extinction rate of the three populations clearly. So we give Figure 3 to show $(\log x_i(t))/t$ for $i = 1, 2, 3$. According to Theorem 7, we can compute that, for $i = 1$, the growth rate is −0.05220 (it is said that the extinction rate is 0.05220). By the same method, we can know that the extinction rate is 0.00000 for $i = 2$ and the extinction rate is 0.10125 for $i = 3$.

6. Conclusion

This paper is devoted to the existence of stationary distribution and extinction for multispecies stochastic Lotka-Volterra predator-prey system. Firstly, by applying Lyapunov methods, sufficient conditions for ensuring the existence of stationary distribution of the system are obtained. Secondly, some novel techniques have been developed to derive weaker sufficient conditions under which the system is extinctive. Finally, numerical experiment is provided to illustrate the effectiveness of our results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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