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Research Article

On Generalized Flett's Mean Value Theorem

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We present a new proof of generalized Flett's mean value theorem due to Pawlikowska (from 1999) using only the original Flett's mean value theorem. Also, a Trahan-type condition is established in general case.

1. Introduction

Mean value theorems play an essential role in analysis. The simplest form of the mean value theorem due to Rolle is well known.

Theorem 1.1 (Rolle's mean value theorem). *If $f : \langle a,b\rangle \to \mathbb{R}$ is continuous on $\langle a,b\rangle$ and differentiable on (a,b) and $f(a) = f(b)$, then there exist a number $\eta \in (a,b)$ such that $f'(\eta) = 0$.*

A geometric interpretation of Theorem 1.1 states that if the curve $y = f(x)$ has a tangent at each point in (a,b) and $f(a) = f(b)$, then there exists a point $\eta \in (a,b)$ such that the tangent at $(\eta, f(\eta))$ is parallel to the x-axis. One may ask a natural question: *What if we remove the boundary condition $f(a) = f(b)$?* The answer is well known as the Lagrange mean value theorem. For the sake of brevity put

$$ {}_a^b\mathcal{K}\left(f^{(n)}, g^{(n)}\right) = \frac{f^{(n)}(b) - f^{(n)}(a)}{g^{(n)}(b) - g^{(n)}(a)}, \quad n \in \mathbb{N} \cup \{0\}, \tag{1.1} $$

for functions f, g defined on $\langle a,b\rangle$ (for which the expression has a sense). If $g^{(n)}(b) - g^{(n)}(a) = b - a$, we simply write ${}_a^b\mathcal{K}(f^{(n)})$.

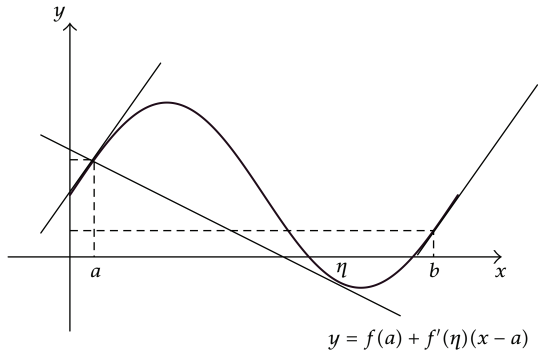

Figure 1: Geometric interpretation of Flett's mean value theorem.

Theorem 1.2 (Lagrange's mean value theorem). *If $f : \langle a,b\rangle \to \mathbb{R}$ is continuous on $\langle a,b\rangle$ and differentiable on (a,b), then there exist a number $\eta \in (a,b)$ such that $f'(\eta) = {}_a^b\mathcal{K}(f)$.*

Clearly, Theorem 1.2 reduces to Theorem 1.1 if $f(a) = f(b)$. Geometrically, Theorem 1.2 states that given a line ℓ joining two points on the graph of a differentiable function f, namely, $(a, f(a))$ and $(b, f(b))$, then there exists a point $\eta \in (a,b)$ such that the tangent at $(\eta, f(\eta))$ is parallel to the given line ℓ.

In connection with Theorem 1.1, the following question may arise: *Are there changes if in Theorem 1.1 the hypothesis $f(a) = f(b)$ refers to higher-order derivatives?* Flett, see [1], first proved in 1958 the following answer to this question for $n = 1$ which gives a variant of Lagrange's mean value theorem with the Rolle-type condition.

Theorem 1.3 (Flett's mean value theorem). *If $f : \langle a,b\rangle \to \mathbb{R}$ is a differentiable function on $\langle a,b\rangle$ and $f'(a) = f'(b)$, then there exists a number $\eta \in (a,b)$ such that*

$$f'(\eta) = {}_a^\eta\mathcal{K}(f). \tag{1.2}$$

Flett's original proof, see [1], uses Theorem 1.1. A slightly different proof which uses Fermat's theorem instead of Rolle's can be found in [2]. There is a nice geometric interpretation of Theorem 1.3: if the curve $y = f(x)$ has a tangent at each point in $\langle a,b\rangle$ and if the tangents at $(a, f(a))$ and $(b, f(b))$ are parallel, then there exists a point $\eta \in (a,b)$ such that the tangent at $(\eta, f(\eta))$ passes through the point $(a, f(a))$; see Figure 1.

Similarly as in the case of Rolle's theorem, we may ask about possibility to remove the boundary assumption $f'(a) = f'(b)$ in Theorem 1.3. As far as we know, the first result of that kind is given in the book [3].

Theorem 1.4 (Riedel-Sahoo). *If $f : \langle a,b\rangle \to \mathbb{R}$ is a differentiable function on $\langle a,b\rangle$, then there exist a number $\eta \in (a,b)$ such that*

$$f'(\eta) = {}_a^\eta\mathcal{K}(f) + {}_a^b\mathcal{K}(f') \cdot \frac{\eta - a}{2}. \tag{1.3}$$

We point out that there are also other sufficient conditions guaranteeing the existence of a point $\eta \in (a,b)$ satisfying (1.2). First such a condition was published in Trahan's work [4].

An interesting idea is presented in paper [5] where the discrete and integral arithmetic mean is used. We suppose that this idea may be further generalized for the case of means studied, for example, in [6–8].

In recent years there has been renewed interest in Flett's mean value theorem; see a survey in [9]. Among the many other extensions and generalizations of Theorem 1.3, see, for example, [10–13], we focus on that of Iwona Pawlikowska [14] solving the question of Zsolt Pales raised at the 35th International Symposium on Functional Equations held in Graz in 1997.

Theorem 1.5 (Pawlikowska). *Let f be n-times differentiable on $\langle a,b\rangle$ and $f^{(n)}(a) = f^{(n)}(b)$. Then there exists $\eta \in (a,b)$ such that*

$$f(\eta) - f(a) = \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i!} (\eta - a)^i f^{(i)}(\eta). \tag{1.4}$$

Observe that Pawlikowska's theorem has a close relationship with the nth Taylor polynomial of f. Indeed, for

$$T_n(f, x_0)(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n, \tag{1.5}$$

Pawlikowska's theorem has the following very easy form $f(a) = T_n(f,\eta)(a)$.

Pawlikowska's proof follows up the original idea of Flett; see [1], considering the auxiliary function

$$G_f(x) = \begin{cases} g^{(n-1)}(x), & x \in (a,b\rangle, \\ \dfrac{1}{n} f^{(n)}(a), & x = a, \end{cases} \tag{1.6}$$

where $g(x) = {}_a^x\mathcal{K}(f)$ for $x \in (a,b\rangle$ and using Theorem 1.1. In what follows we provide a different proof of Theorem 1.5 which uses only iterations of an appropriate auxiliary function and Theorem 1.3. In Section 3 we give a general version of Trahan condition, compare [4] under which Pawlikowska's theorem holds.

2. New Proof of Pawlikowska's Theorem

The key tool in our proof consists in using the auxiliary function

$$\varphi_k(x) = x f^{(n-k+1)}(a) + \sum_{i=0}^{k} \frac{(-1)^{i+1}}{i!} (k - i)(x - a)^i f^{(n-k+i)}(x), \quad k = 1, 2, \ldots, n. \tag{2.1}$$

Running through all indices $k = 1, 2, \ldots, n$, we show that its derivative fulfills assumptions of Flett's mean value theorem and it implies the validity of Flett's mean value theorem for the lth derivative of f, where $l = n-1, n-2, \ldots, 1$.

Indeed, for $k = 1$ we have $\varphi_1(x) = -f^{(n-1)}(x) + xf^{(n)}(a)$ and $\varphi_1'(x) = -f^{(n)}(x) + f^{(n)}(a)$. Clearly, $\varphi_1'(a) = 0 = \varphi_1'(b)$, so applying Flett's mean value theorem for φ_1 on $\langle a, b\rangle$ there exists $u_1 \in (a, b)$, such that $\varphi_1'(u_1)(u_1 - a) = \varphi_1(u_1) - \varphi_1(a)$; that is,

$$f^{(n-1)}(u_1) - f^{(n-1)}(a) = (u_1 - a)f^{(n)}(u_1). \tag{2.2}$$

Then for $\varphi_2(x) = -2f^{(n-2)}(x) + (x - a)f^{(n-1)}(x) + xf^{(n-1)}(a)$, we get

$$\varphi_2'(x) = -f^{(n-1)}(x) + (x - a)f^{(n)}(x) + f^{(n-1)}(a) \tag{2.3}$$

and $\varphi_2'(a) = 0 = \varphi_2'(u_1)$ by (2.2). So, by Flett's mean value theorem for φ_2 on $\langle a, u_1\rangle$, there exists $u_2 \in (a, u_1) \subset (a, b)$ such that $\varphi_2'(u_2)(u_2 - a) = \varphi_2(u_2) - \varphi_2(a)$, which is equivalent to

$$f^{(n-2)}(u_2) - f^{(n-2)}(a) = (u_2 - a)f^{(n-1)}(u_2) - \frac{1}{2}(u_2 - a)^2 f^{(n)}(u_2). \tag{2.4}$$

Continuing this way after $n - 1$ steps, $n \geq 2$, there exists $u_{n-1} \in (a, b)$ such that

$$f'(u_{n-1}) - f'(a) = \sum_{i=1}^{n-1} \frac{(-1)^{i+1}}{i!}(u_{n-1} - a)^i f^{(i+1)}(u_{n-1}). \tag{2.5}$$

Considering the function φ_n, we get

$$\begin{aligned}\varphi_n'(x) &= -f'(x) + f'(a) + \sum_{i=1}^{n-1} \frac{(-1)^{i+1}}{i!}(x - a)^i f^{(i)}(x) \\ &= f'(a) + \sum_{i=0}^{n-1} \frac{(-1)^{i+1}}{i!}(x - a)^i f^{(i+1)}(x).\end{aligned} \tag{2.6}$$

Clearly, $\varphi_n'(a) = 0 = \varphi_n'(u_{n-1})$ by (2.5). Then by Flett's mean value theorem for φ_n on $\langle a, u_{n-1}\rangle$, there exists $\eta \in (a, u_{n-1}) \subset (a, b)$ such that

$$\varphi_n'(\eta)(\eta - a) = \varphi_n(\eta) - \varphi_n(a). \tag{2.7}$$

Since

$$\begin{aligned}\varphi_n'(\eta)(\eta - a) &= (\eta - a)f'(a) + \sum_{i=1}^{n} \frac{(-1)^i}{(i-1)!}(\eta - a)^i f^{(i)}(\eta), \\ \varphi_n(\eta) - \varphi_n(a) &= (\eta - a)f'(a) - n(f(\eta) - f(a)) + \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i!}(n - i)(\eta - a)^i f^{(i)}(\eta),\end{aligned} \tag{2.8}$$

the equality (2.7) yields

$$-n(f(\eta)-f(a)) = \sum_{i=1}^{n} \frac{(-1)^i}{(i-1)!}(\eta-a)^i f^{(i)}(\eta)\left(1+\frac{n-i}{i}\right) = n\sum_{i=1}^{n} \frac{(-1)^i}{i!}(\eta-a)^i f^{(i)}(\eta), \tag{2.9}$$

which corresponds to (1.4).

It is also possible to state the result which no longer requires any endpoint conditions. If we consider the auxiliary function

$$\psi_k(x) = \varphi_k(x) + \frac{(-1)^{k+1}}{(k+1)!}(x-a)^{k+1} \cdot {}_a^b\mathcal{K}\left(f^{(n)}\right), \tag{2.10}$$

then the analogous way as in the proof of Theorem 1.5 yields the following result also given in [14] including Riedel-Sahoo's Theorem 1.4 as a special case ($n = 1$).

Theorem 2.1. *If $f : \langle a,b\rangle \to \mathbb{R}$ is n-times differentiable on $\langle a,b\rangle$, then there exists $\eta \in (a,b)$ such that*

$$f(a) = T_n(f,\eta)(a) + \frac{(a-\eta)^{n+1}}{(n+1)!} \cdot {}_a^b\mathcal{K}\left(f^{(n)}\right). \tag{2.11}$$

Note that the case $n = 1$ is used to extend Flett's mean value theorem for holomorphic functions; see [10]. An easy generalization of Pawlikowska's theorem involving two functions is the following one.

Theorem 2.2. *Let f, g be n-times differentiable on $\langle a,b\rangle$ and $g^{(n)}(a) \neq g^{(n)}(b)$. Then there exists $\eta \in \langle a,b\rangle$ such that*

$$f(a) - T_n(f,\eta)(a) = {}_a^b\mathcal{K}\left(f^{(n)}, g^{(n)}\right) \cdot [g(a) - T_n(g,\eta)(a)]. \tag{2.12}$$

This may be verified applying Pawlikowska's theorem to the auxiliary function

$$h(x) = f(x) - {}_a^b\mathcal{K}\left(f^{(n)}, g^{(n)}\right) \cdot ,g(x), \qquad x \in \langle a,b\rangle. \tag{2.13}$$

A different proof will be presented in the following section.

3. A Trahan-Type Condition

In [4] Trahan gave a sufficient condition for the existence of a point $\eta \in (a, b\rangle$ satisfying (1.2) under the assumptions of differentiability of f on $\langle a, b\rangle$ and inequality

$$\left(f'(b) - {}_a^b\mathcal{K}(f)\right) \cdot \left(f'(a) - {}_a^b\mathcal{K}(f)\right) \geq 0. \tag{3.1}$$

Modifying Trahan's original proof using Pawlikowska's auxiliary function G_f, we are able to state the following condition for validity (1.4).

Theorem 3.1. *Let f be n-times differentiable on $\langle a, b\rangle$ and*

$$\left(\frac{f^{(n)}(a)(a-b)^n}{n!} + M_f\right)\left(\frac{f^{(n)}(b)(a-b)^n}{n!} + M_f\right) \geq 0, \tag{3.2}$$

where $M_f = T_{n-1}(f, b)(a) - f(a)$. Then there exists $\eta \in (a, b\rangle$ satisfying (1.4).

Proof. Since G_f is continuous on $\langle a, b\rangle$ and differentiable on $(a, b\rangle$ with

$$G_f'(x) = g^{(n)}(x) = \frac{(-1)^n n!}{(x-a)^{n+1}}\left(f(x) - f(a) + \sum_{i=1}^{n} \frac{(-1)^i}{i!}(x-a)^i f^{(i)}(x)\right), \tag{3.3}$$

for $x \in (a, b\rangle$, [14, page 282], then

$$\begin{aligned}(G_f(b) - G_f(a))G_f'(b) &= \left(g^{(n-1)}(b) - \frac{1}{n}f^{(n)}(a)\right)g^{(n)}(b)\\ &= -\frac{n!(n-1)!}{(b-a)^{2n+1}}\left(\frac{f^{(n)}(a)(a-b)^n}{n!} + T_{n-1}(f, b)(a) - f(a)\right)\\ &\quad\cdot\left(\frac{f^{(n)}(b)(a-b)^n}{n!} + T_{n-1}(f, b)(a) - f(a)\right) \leq 0.\end{aligned} \tag{3.4}$$

According to [4, Lemma 1], there exists $\eta \in (a, b\rangle$ such that $G_f'(\eta) = 0$ which corresponds to (1.4). □

Now we provide an alternative proof of Theorem 2.2 which does not use, original Pawlikowska's theorem.

Proof of Theorem 2.2. For $x \in (a, b\rangle$ put $\varphi(x) = {}_a^x\mathcal{K}(f)$ and $\psi(x) = {}_a^x\mathcal{K}(g)$. Define the auxiliary function F as follows:

$$F(x) = \begin{cases} \varphi^{(n-1)}(x) - {}_a^b\mathcal{K}\left(f^{(n)}, g^{(n)}\right) \cdot \psi^{(n-1)}(x), & x \in (a, b\rangle, \\ \dfrac{1}{n}\left[f^{(n)}(a) - {}_a^b\mathcal{K}\left(f^{(n)}, g^{(n)}\right) \cdot g^{(n)}(a)\right], & x = a. \end{cases} \tag{3.5}$$

Clearly, F is continuous on $\langle a, b\rangle$, differentiable on $(a, b\rangle$, and for $x \in (a, b\rangle$ there holds

$$F'(x) = \varphi^{(n)}(x) - {}_a^b\mathcal{K}\left(f^{(n)}, g^{(n)}\right) \cdot \psi^{(n)}(x). \tag{3.6}$$

Then

$$F'(b)[F(b) - F(a)] = -\frac{n}{(b-a)}(F(b) - F(a))^2 \leq 0, \tag{3.7}$$

and by [4, Lemma 1] there exists $\eta \in (a, b\rangle$ such that $F'(\eta) = 0$; that is,

$$f(a) - T_n(f, \eta)(a) = {}_a^b\mathcal{K}\left(f^{(n)}, g^{(n)}\right) \cdot (g(a) - T_n(g, \eta)(a)), \tag{3.8}$$

which completes the proof. □

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References

[1] T. M. Flett, "A mean value theorem," *The Mathematical Gazette*, vol. 42, pp. 38–39, 1958.
[2] T.-L. Rădulescu, V. D. Rădulescu, and T. Andreescu, *Problems in Real Analysis: Advanced Calculus on the Real Axis*, Springer, New York, NY, USA, 2009.
[3] P. K. Sahoo and T. Riedel, *Mean Value Theorems and Functional Equations*, World Scientific, River Edge, NJ, USA, 1998.
[4] D. H. Trahan, "A new type of mean value theorem," *Mathematics Magazine*, vol. 39, pp. 264–268, 1966.
[5] J. Tong, "On Flett's mean value theorem," *International Journal of Mathematical Education in Science & Technology*, vol. 35, pp. 936–941, 2004.
[6] J. Haluška and O. Hutník, "Some inequalities involving integral means," *Tatra Mountains Mathematical Publications*, vol. 35, pp. 131–146, 2007.
[7] O. Hutník, "On Hadamard type inequalities for generalized weighted quasi-arithmetic means," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 7, no. 3, article 96, 2006.
[8] O. Hutník, "Some integral inequalities of Hölder and Minkowski type," *Colloquium Mathematicum*, vol. 108, no. 2, pp. 247–261, 2007.
[9] O. Hutník and J. Molnárová, "On Flett's mean value theorem," In press.
[10] R. M. Davitt, R. C. Powers, T. Riedel, and P. K. Sahoo, "Flett's mean value theorem for holomorphic functions," *Mathematics Magazine*, vol. 72, no. 4, pp. 304–307, 1999.
[11] M. Das, T. Riedel, and P. K. Sahoo, "Flett's mean value theorem for approximately differentiable functions," *Radovi Matematički*, vol. 10, no. 2, pp. 157–164, 2001.
[12] I. Jędrzejewska and B. Szkopińska, "On generalizations of Flett's theorem," *Real Analysis Exchange*, vol. 30, no. 1, pp. 75–86, 2004.
[13] R. C. Powers, T. Riedel, and P. K. Sahoo, "Flett's mean value theorem in topological vector spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 27, no. 11, pp. 689–694, 2001.
[14] I. Pawlikowska, "An extension of a theorem of Flett," *Demonstratio Mathematica*, vol. 32, no. 2, pp. 281–286, 1999.


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