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Research Article

Subordination Properties for Certain Analytic Functions

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The purpose of the present paper is to derive a subordination result for functions in the class $H_n^*(\alpha,\lambda,b)$ of normalized analytic functions in the open unit disk \mathbb{U} . A number of interesting applications of the subordination result are also considered.

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1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$
 (1.1)

which are analytic in the unit disc $\mathbb{U} = \{z : |z| < 1\}$. We also denote by K the class of functions $f \in A$ that are convex in \mathbb{U} .

Given two functions $f, g \in A$, where f is given by (1.1) and g is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$
 (1.2)

the Hadamard product (or convolution) f*g is defined by

$$(f*g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \quad (z \in \mathbb{U}).$$

$$(1.3)$$

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By using the Hadamard product, Ruscheweyh [1] defined

$$D^{\alpha} f(z) = \frac{z}{(1-z)^{\alpha+1}} * f(z) \quad (\alpha \ge -1).$$
 (1.4)

From the definition of (1.4), we observe that

$$D^{n}f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!},$$
(1.5)

when $n = \alpha \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0,1,2,\dots\}$. The symbol $D^n f(z)$ $(n \in \mathbb{N}_0)$ was called the nth-order Ruscheweyh derivative of f by Al-Amiri [2]. We also note that $D^0 f(z) = f(z)$ and $D^1 f(z) = z f'(z)$.

Definition 1.1. Suppose that $f \in A$. Then the function f is said to be a member of the class $H_n(\alpha, \lambda, b)$ if it satisfies

$$\left| \frac{\lambda(n+1)(D^{n+1}f(z)/z) + [1 - \lambda(n+1)](D^n f(z)/z) - 1}{\lambda(n+1)(D^{n+1}f(z)/z) + [1 - \lambda(n+1)](D^n f(z)/z) + 2b(1-\alpha) - 1} \right| < 1$$

$$(z \in \mathbb{U}; \ 0 < \alpha < 1; \ \lambda > 0; \ b \in \mathbb{C} \setminus \{0\}; \ n \in \mathbb{N}_0).$$
(1.6)

By specializing α , λ , b, and n, one can obtain various subclasses studied by many authors (see, e.g., [3–11]).

Definition 1.2. Let g be analytic and univalent in \mathbb{U} . If f is analytic in \mathbb{U} , f(0) = g(0), and $f(\mathbb{U}) \subset g(\mathbb{U})$, then one says that f is subordinate to g in \mathbb{U} , and we write $f \prec g$ or $f(z) \prec g(z)$. One also says that g is superordinate to f in \mathbb{U} .

Definition 1.3. An infinite sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers will be called a subordinating factor sequence if for every univalent function f in K, one has

$$\sum_{k=1}^{\infty} b_k a_k z^k \prec f(z) \quad (z \in \mathbb{U}; \ a_1 = 1). \tag{1.7}$$

Lemma 1.4 (see [12]). The sequence $\{b_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if

$$\operatorname{Re}\left\{1+2\sum_{k=1}^{\infty}b_{k}z^{k}\right\}>0\quad(z\in\mathbb{U}).\tag{1.8}$$

Now, we prove the following lemma which gives a sufficient condition for functions belonging to the class $H_n(\alpha, \lambda, b)$.

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Lemma 1.5. *If the function* f *which is defined by* (1.1) *satisfies the following condition:*

$$\sum_{k=2}^{\infty} [1 + \lambda(k-1)] C(n,k) |a_k| \le (1-\alpha)|b| \quad (0 \le \alpha < 1; \ \lambda \ge 0; \ b \in \mathbb{C} \setminus \{0\}; \ n \in \mathbb{N}_0), \tag{1.9}$$

where

$$C(n,k) = \prod_{i=2}^{k} \frac{(j+n-1)}{(k-1)!} \quad (k=2,3,...),$$
 (1.10)

then $f \in H_n(\alpha, \lambda, b)$.

Proof. Suppose that the inequality (1.9) holds. Using the identity

$$z(D^n f(z))' = (n+1)D^{n+1} f(z) - nD^n f(z),$$
(1.11)

we have for $z \in \mathbb{U}$,

$$\left| (1 - \lambda) \frac{D^{n} f(z)}{z} + \lambda (D^{n} f(z))' - 1 \right| - \left| 2b(1 - \alpha) + (1 - \lambda) \frac{D^{n} f(z)}{z} + \lambda (D^{n} f(z))' - 1 \right|
= \left| \sum_{k=2}^{\infty} \left[1 + \lambda (k-1) \right] C(n,k) a_{k} z^{k-1} \right| - \left| 2b(1 - \alpha) + \sum_{k=2}^{\infty} \left[1 + \lambda (k-1) \right] C(n,k) a_{k} z^{k-1} \right|
\leq \sum_{k=2}^{\infty} \left[1 + \lambda (k-1) \right] C(n,k) |a_{k}| |z|^{k-1}
- \left\{ 2|b|(1 - \alpha) - \sum_{k=2}^{\infty} \left[1 + \lambda (k-1) \right] C(n,k) |a_{k}| |z|^{k-1} \right\}
\leq 2 \left\{ \sum_{k=2}^{\infty} \left[1 + \lambda (k-1) \right] C(n,k) |a_{k}| - |b|(1 - \alpha) \right\} \leq 0,$$
(1.12)

which shows that f belongs to $H_n(\alpha, \lambda, b)$.

Let $H_n^*(\alpha, \lambda, b)$ denote the class of functions f in A whose Taylor-Maclaurin coefficients a_k satisfy the condition (1.9).

We note that

$$H_n^*(\alpha, \lambda, b) \subseteq H_n(\alpha, \lambda, b). \tag{1.13}$$

Example 1.6. (i) For $0 \le \alpha < 1$, $\lambda > 0$, $b \in \mathbb{C} \setminus \{0\}$, and $n \in \mathbb{N}_0$, the following function defined by:

$$f_0(z) = z + \frac{2b(1-\alpha)}{(n+1)(\lambda+1)} z^2 {}_3F_2\left(1,2,1+\frac{1}{\lambda};2+\frac{1}{\lambda},n+2;z\right) \quad (z \in \mathbb{U}), \tag{1.14}$$

is in the class $H_n(\alpha, \lambda, b)$.

(ii) For $0 \le \alpha < 1$, $\lambda > 0$, $b \in \mathbb{C} \setminus \{0\}$, and $n \in \mathbb{N}_0$, the following functions defined by:

$$f_{1}(z) = z \pm \frac{(1-\alpha)|b|}{(\lambda+1)(n+1)} z^{2} \quad (z \in \mathbb{U}),$$

$$f_{2}(z) = z \pm \frac{(1-\alpha)|b|}{(2\lambda+1)(n+1)(n+2)} z^{3} \quad (z \in \mathbb{U}),$$

$$f_{3}(z) = z \pm \frac{1}{(\lambda+1)(n+1)} z^{2} \pm \frac{2[(1-\alpha)|b|-1]}{(2\lambda+1)(n+1)(n+2)} z^{3} \quad (z \in \mathbb{U})$$

$$(1.15)$$

are in the class $H_n^*(\alpha, \lambda, b)$.

In this paper, we obtain a sharp subordination result associated with the class $H_n^*(\alpha, \lambda, b)$ by using the same techniques as in [13] (see also [14–16]). Some applications of the main result which give important results of analytic functions are also investigated.

2. Main theorem

Theorem 2.1. Let $f \in H_n^*(\alpha, \lambda, b)$. Then

$$\frac{(\lambda+1)(n+1)}{2[(\lambda+1)(n+1)+|b|(1-\alpha)]}(f*g)(z) \prec g(z) \quad (z \in \mathbb{U})$$
 (2.1)

for every function g in K, and

$$\operatorname{Re} f(z) > -\frac{(\lambda + 1)(n+1) + |b|(1-\alpha)}{(\lambda + 1)(n+1)}.$$
 (2.2)

The constant $(\lambda + 1)(n + 1)/2[(\lambda + 1)(n + 1) + |b|(1 - \alpha)]$ cannot be replaced by a larger one.

Proof. Let $f \in H_n^*(\alpha, \lambda, b)$ and let

$$g(z) = z + \sum_{k=2}^{\infty} c_k z^k$$
 (2.3)

be any function in the class *K*. Then we readily have

$$\frac{(\lambda+1)(n+1)}{2[(\lambda+1)(n+1)+|b|(1-\alpha)]}(f*g)(z) = \frac{(\lambda+1)(n+1)}{2[(\lambda+1)(n+1)+|b|(1-\alpha)]} \left(z + \sum_{k=2}^{\infty} a_k c_k z^k\right). \tag{2.4}$$

Thus, by Definition 1.2, the subordination result (2.1) will hold true if the sequence

$$\left\{ \frac{(\lambda+1)(n+1)a_k}{2[(\lambda+1)(n+1)+|b|(1-\alpha)]} \right\}_{k=1}^{\infty}$$
 (2.5)

is a subordinating factor sequence, with a_1 = 1. In view of Lemma 1.4, this is equivalent to the following inequality:

$$\operatorname{Re}\left\{1 + \sum_{k=1}^{\infty} \frac{(\lambda+1)(n+1)}{\left[(\lambda+1)(n+1) + |b|(1-\alpha)\right]} a_k z^k\right\} > 0 \quad (z \in \mathbb{U}). \tag{2.6}$$

Now, since

$$[1 + \lambda(k-1)]C(n,k) \quad (\lambda \ge 0, \ n \in \mathbb{N}_0)$$

$$(2.7)$$

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is an increasing function of k, we have

$$\operatorname{Re}\left\{1 + \sum_{k=1}^{\infty} \frac{(\lambda+1)(n+1)}{\left[(\lambda+1)(n+1) + |b|(1-\alpha)\right]} a_k z^k\right\}$$

$$= \operatorname{Re}\left\{1 + \frac{(\lambda+1)(n+1)}{\left[(\lambda+1)(n+1) + |b|(1-\alpha)\right]} z + \frac{1}{\left[(\lambda+1)(n+1) + |b|(1-\alpha)\right]} \sum_{k=2}^{\infty} (\lambda+1)(n+1) a_k z^k\right\}$$

$$> 1 - \frac{(\lambda+1)(n+1)}{\left[(\lambda+1)(n+1) + |b|(1-\alpha)\right]} r$$

$$- \frac{1}{\left[(\lambda+1)(n+1) + |b|(1-\alpha)\right]} \sum_{k=2}^{\infty} \left[1 + \lambda(k-1)\right] C(n,k) |a_k| r^k$$

$$> 1 - \frac{(\lambda+1)(n+1)}{\left[(\lambda+1)(n+1) + |b|(1-\alpha)\right]} r - \frac{|b|(1-\alpha)}{\left[(\lambda+1)(n+1) + |b|(1-\alpha)\right]} r > 0 \quad (|z|=r).$$

This proves the inequality (2.6), and hence also the subordination result (2.1) asserted by Theorem 2.1. The inequality (2.2) follows from (2.1) by taking

$$g(z) = \frac{z}{1-z} \in K. \tag{2.9}$$

Next, we consider the function

$$f_1(z) = z - \frac{|b|(1-\alpha)}{(\lambda+1)(n+1)} z^2 \quad (0 \le \alpha < 1; \ \lambda \ge 0; \ b \in \mathbb{C} \setminus \{0\}; \ n \in \mathbb{N}_0)$$
 (2.10)

which is a member of the class $H_n^*(\alpha, \lambda, b)$. Then by using (2.1), we have

$$\frac{(\lambda+1)(n+1)}{2[(\lambda+1)(n+1)+|b|(1-\alpha)]}f_1(z) < \frac{z}{1-z} \quad (z \in \mathbb{U}).$$
 (2.11)

It can be easily verified for the function $f_1(z)$ defined by (2.10) that

$$\inf_{z \in \mathbb{U}} \left\{ \operatorname{Re} \left(\frac{(\lambda + 1)(n+1)}{2[(\lambda + 1)(n+1) + |b|(1-\alpha)]} f_1(z) \right) \right\} = -\frac{1}{2} \quad (z \in \mathbb{U})$$
 (2.12)

which completes the proof of Theorem 2.1.

3. Some applications

Taking n = 0 in Theorem 2.1, we obtain the following.

Corollary 3.1. *If the function* f *defined by* (1.1) *satisfies*

$$\sum_{k=2}^{\infty} \left[1 + \lambda(k-1) \right] \left| a_k \right| \le m \quad \left(\lambda \ge 0, \ m > 0 \right)$$
 (3.1)

then for every function g in K, one has

$$\frac{(\lambda+1)}{2(\lambda+m+1)}(f*g)(z) < g(z), \quad (z \in \mathbb{U}),$$

$$\operatorname{Re} f(z) > -\left(1 + \frac{m}{\lambda+1}\right). \tag{3.2}$$

The constant $(\lambda + 1)/2(\lambda + m + 1)$ cannot be replaced by larger one.

Putting $\lambda = 0$ in Theorem 2.1, we have the following corollary.

Corollary 3.2. *If the function* f *defined by* (1.1) *satisfies*

$$\sum_{k=2}^{\infty} C(n,k) \left| a_k \right| \le m, \quad m > 0, \ n \in \mathbb{N}_0, \tag{3.3}$$

where C(n,k) is defined by (1.10), then for every function g in K, one has

$$\frac{(n+1)}{2(n+m+1)}(f*g)(z) < g(z) \quad (z \in \mathbb{U}),$$

$$\operatorname{Re} f(z) > -\left(1 + \frac{m}{n+1}\right). \tag{3.4}$$

The constant (n+1)/2(n+m+1) cannot be replaced by larger one.

Next, letting $\lambda = 1$ and n = 0, in Theorem 2.1, we obtain the following corollary.

Corollary 3.3. *If the function f satisfies*

$$\sum_{k=0}^{\infty} k \left| a_k \right| \le m \quad (m > 0), \tag{3.5}$$

then for every function g in K, one has

$$\frac{1}{(m+2)}(f*g)(z) < g(z) \quad (z \in \mathbb{U}),$$

$$\operatorname{Re} f(z) > -\left(1 + \frac{m}{2}\right). \tag{3.6}$$

The constant 1/(m+2) cannot be replaced by a larger one.

Remark 3.4. Putting $\lambda = 1$, m = 1, and n = 0, in Theorem 2.1, we obtain the result due to Singh [17].

Also, by taking $\lambda = 0$ and n = 0, in Theorem 2.1, we have the following.

Corollary 3.5. *If the function f satisfies*

$$\sum_{k=2}^{\infty} |a_k| \le m \quad (m>0), \tag{3.7}$$

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then for every function g in K, one has

$$\frac{1}{2(m+1)}(f*g)(z) \prec g(z) \quad (z \in \mathbb{U}),$$

$$\operatorname{Re} f(z) > -(1+m).$$
(3.8)

The constant 1/2(m+1) cannot be replaced by a larger one.

It is clearly from the proof of Theorem 2.1 that the function $f(z) = z - mz^2$ (m > 0, $z \in \mathbb{U}$) is the extremal function of Corollary 3.5. Also, the following example gives a nonpolynomial extremal function for the same corollary.

Example 3.6. Let the function h be defined by

$$h(z) = \frac{(m+1)z}{(m+1) + mz} \quad (m > 0, \ z \in \mathbb{U}), \tag{3.9}$$

the above function is analytic in \mathbb{U} and it is equivalent to

$$h(z) = z + \sum_{k=2}^{\infty} \left(\frac{-m}{m+1}\right)^{k-1} z^k.$$
 (3.10)

Then we have

$$\sum_{k=2}^{\infty} \left| \left(\frac{-m}{m+1} \right)^{k-1} \right| = m. \tag{3.11}$$

Therefore, the Taylor-Maclaurin coefficients of the function h satisfy the condition in Corollary 3.5. Moreover, it can be easily verified that

$$\inf_{z \in \mathbb{U}} \operatorname{Re} h(z) = h(-1) = -(m+1). \tag{3.12}$$

Then, the constant -(m+1) cannot be replaced by a larger one. Therefore, the function h is the extremal function of Corollary 3.5.

References

- [1] S. Ruscheweyh, "New criteria for univalent functions," *Proceedings of the American Mathematical Society*, vol. 49, no. 1, pp. 109–115, 1975.
- [2] H. S. Al-Amiri, "On Ruscheweyh derivatives," *Annales Polonici Mathematici*, vol. 38, no. 1, pp. 88–94, 1980.
- [3] M. P. Chen, "On functions satisfying Re $\{f(z)/z\} > \alpha$," Tamkang Journal of Mathematics, vol. 5, pp. 231–234, 1974.
- [4] M. P. Chen, "On the regular functions satisfying Re $\{f(z)/z\} > \alpha$," Bulletin of the Institute of Mathematics. Academia Sinica, vol. 3, no. 1, pp. 65–70, 1975.
- [5] K. K. Dixit and S. K. Pal, "On a class of univalent functions related to complex order," *Indian Journal of Pure and Applied Mathematics*, vol. 26, no. 9, pp. 889–896, 1995.
- [6] T. G. Èzrohi, "Certain estimates in special classes of univalent functions regular in the circle |z| < 1," Dopovidi Akademiji Nauk Ukrajins'koji RSR, vol. 1965, pp. 984–988, 1965.

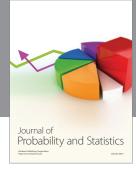
- [7] R. M. Goel, "The radius of convexity and starlikeness for certain classes of analytic functions with fixed second coefficients," Annales Universitatis Mariae Curie-Skłodowska. Sectio A, vol. 25, pp. 33–39, 1971.
- [8] Y. C. Kim and F. Rønning, "Integral transforms of certain subclasses of analytic functions," *Journal of Mathematical Analysis and Applications*, vol. 258, no. 2, pp. 466–489, 2001.
- [9] T. H. MacGregor, "Functions whose derivative has a positive real part," *Transactions of the American Mathematical Society*, vol. 104, no. 3, pp. 532–537, 1962.
- [10] A. Swaminathan, "Certain sufficiency conditions on Gaussian hypergeometric functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 5, no. 4, article 83, pp. 1–10, 2004.
- [11] K. Yamaguchi, "On functions satisfying Re $\{f(z)/z\}$ < 0," Proceedings of the American Mathematical Society, vol. 17, no. 3, pp. 588–591, 1966.
- [12] H. S. Wilf, "Subordinating factor sequences for convex maps of the unit circle," *Proceedings of the American Mathematical Society*, vol. 12, no. 5, pp. 689–693, 1961.
- [13] H. M. Srivastava and A. A. Attiya, "Some subordination results associated with certain subclasses of analytic functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 5, no. 4, pp. 1–6, 2004, article 82.
- [14] R. M. Ali, V. Ravichandran, and N. Seenivasagan, "Subordination by convex functions," *International Journal of Mathematics and Mathematical Sciences*, vol. 2006, Article ID 62548, 6 pages, 2006.
- [15] A. A. Attiya, "On some applications of a subordination theorem," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 489–494, 2005.
- [16] B. A. Frasin, "Subordination results for a class of analytic functions defined by a linear operator," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 7, no. 4, article 134, pp. 1–7, 2006.
- [17] S. Singh, "A subordination theorem for spirallike functions," *International Journal of Mathematics and Mathematical Sciences*, vol. 24, no. 7, pp. 433–435, 2000.



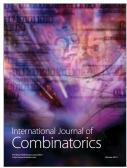








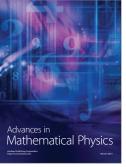


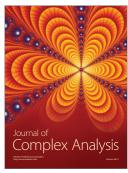




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