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## Research Article

# Convergence of Path and Approximation of Common Element of Null Spaces of Countably Infinite Family of $m$ -Accretive Mappings in Uniformly Convex Banach Spaces

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We prove path convergence theorems and introduce a new iterative sequence for a countably infinite family of  $m$ -accretive mappings and prove strong convergence of the sequence to a common zero of these operators in uniformly convex real Banach space. Consequently, we obtain strong convergence theorems for a countably infinite family of pseudocontractive mappings. Our theorems extend and improve some important results which are announced recently by various authors.

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## 1. Introduction

Let  $E$  be a real Banach space with dual  $E^*$ . We denote by  $J$  the normalized duality mapping from  $E$  to  $2^{E^*}$  defined by

$$Jx := \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \quad (1.1)$$

where  $\langle \cdot, \cdot \rangle$  denotes the generalized duality pairing between members of  $E$  and  $E^*$ . It is well known that if  $E^*$  is strictly convex, then  $J$  is single-valued (see, e.g., [1, 2]). In the sequel, we will denote the single-valued normalized duality mapping by  $j$ .

A mapping  $A : D(A) \subseteq E \rightarrow E$  is called *accretive* if for all  $x, y \in D(A)$  there exists  $j(x - y) \in J(x - y)$  such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0. \quad (1.2)$$

By the result of Kato [3], (1.2) is equivalent to

$$\|x - y\| \leq \|x - y + s(Ax - Ay)\|, \quad \forall s > 0. \quad (1.3)$$

If  $E$  is a Hilbert space, accretive operators are also called *monotone*. An operator  $A$  is called *m-accretive* if it is accretive and  $R(I + rA)$ , range of  $(I + rA)$ , is  $E$  for all  $r > 0$ ; and  $A$  is said to satisfy the range condition if  $\text{cl}(D(A)) \subseteq R(I + rA)$ , for all  $r > 0$ , where  $\text{cl}(D(A))$  denotes the closure of the domain of  $A$ . It is easy to see that every *m-accretive* operator satisfies the range condition. An operator  $A$  is said to be *maximal accretive* if it is accretive and the inclusion  $G(A) \subseteq G(B)$  implies  $G(A) = G(B)$ , where  $G(A)$  denotes the graph of  $A$  and  $B$  is an accretive operator.

A mapping  $T : D(T) \subseteq E \rightarrow R(T) \subseteq E$  is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in D(T). \quad (1.4)$$

It is not difficult to deduce from (1.3) that a mapping  $A$  is accretive if and only if its resolvent  $J_r := (I + rA)^{-1}$ , for all  $r > 0$ , is nonexpansive and single valued on the range of  $(I + rA)$ . Thus, in particular,  $J_A = J_1 := (I + A)^{-1}$  is nonexpansive and single valued on the range of  $(I + A)$ . Furthermore,  $F(J_A) := A^{-1}(0) := \{x \in D(A) : Ax = 0\}$ . For more details see, for example, [4, 5].

Closely related to the class of accretive operators is the class of pseudocontractive maps. An operator  $T$  with domain  $D(T)$  in  $E$  and range  $R(T)$  in  $E$  is called *pseudocontractive* if  $A := I - T$  is accretive. The importance of these operators in application is well known (see, e.g., [6–9] and the references contained therein).

It is well known that the class of pseudocontractive mappings properly contains the class of nonexpansive mappings (see, e.g., [4]). Construction of fixed points of nonexpansive mappings is an important subject in nonlinear operator theory and its applications, in particular, in image recovery and signal processing (see, e.g., [10]).

Iterative approximation of fixed points and zeros of nonlinear operators have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequality problems (see, e.g., [11–15]). The iterative scheme

$$x_0 \in E, \quad x_{n+1} = J_{r_n} x_n, \quad n \geq 0, \quad (1.5)$$

(where  $J_{r_n}$  is the resolvent of an *m-accretive* operator  $A$ ) for example, has been extensively studied over the past forty years or so for construction of zeros of accretive operators (see, e.g., [16–20]).

Kim and Xu [21] introduced a modification of Mann iterative scheme in a reflexive Banach space having weakly continuous duality mapping for finding a zero of an *m-accretive* operator  $A$  as follows:

$$x_0 = u \in E, \quad x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{r_n} x_n, \quad n \geq 0. \quad (1.6)$$

They proved that the sequence  $\{x_n\}_{n=0}^{\infty}$  generated by (1.6) converges to a zero of  $m$ -accretive operator  $A$  under the following conditions:

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,
- (ii)  $\sum_{n=0}^{\infty} \alpha_n = \infty$  (equivalently,  $\prod_{n=0}^{\infty} (1 - \alpha_n) = 0$ ),
- (iii)  $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ ;  $r_n \geq \epsilon$  for some  $\epsilon > 0$  and for all  $n \geq 0$ ,
- (iv)  $\sum_{n=1}^{\infty} |1 - r_{n-1}/r_n| < \infty$ ,
- (v)  $r_n \geq \epsilon$  for some  $\epsilon > 0$  and for all  $n \geq 0$  and  $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$ .

In 2007, Qin and Su [22] also considered the following iterative scheme in either a uniformly smooth Banach space or a reflexive Banach space having a weakly sequentially continuous duality mapping:

$$\begin{aligned} x_0 &= u \in C, & y_n &= \beta_n x_n + (1 - \beta_n) J_{r_n} x_n, \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n) y_n & n &\geq 0, \end{aligned} \quad (1.7)$$

where  $\{\alpha_n\}_{n=0}^{\infty}$  and  $\{\beta_n\}_{n=0}^{\infty}$  are sequences in  $(0, 1)$ . They proved that the sequence  $\{x_n\}_{n=0}^{\infty}$  generated by (1.7) converges strongly to a zero of  $m$ -accretive operator  $A$  provided that  $\{\alpha_n\}_{n=0}^{\infty}$  and  $\{\beta_n\}_{n=0}^{\infty}$  satisfy conditions (i), (ii), and (iii), and  $\{r_n\}_{n=0}^{\infty}$  satisfies condition (v).

Chen and Zhu [23] considered the following viscosity iterative scheme for resolvent  $J_{r_n}$  of  $m$ -accretive mapping  $A$ :

$$x_0 \in C, \quad x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) J_{r_n} x_n, \quad n \geq 0, \quad (1.8)$$

where  $f$  is a contraction mapping defined on  $C$ . Under the assumption that  $\{r_n\}_{n=0}^{\infty}$  satisfies condition (v), Chen and Zhu [23] proved in a reflexive Banach space having weakly sequentially continuous duality mapping that the sequence  $\{x_n\}_{n=0}^{\infty}$  generated by (1.8) converges strongly to a zero of  $A$ , which solves a certain variational inequality.

Recently, Jung [24] introduced the following viscosity iterative method:

$$\begin{aligned} x_0 &\in C, & y_n &= \beta_n x_n + (1 - \beta_n) J_{r_n} x_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) y_n & n &\geq 0. \end{aligned} \quad (1.9)$$

Under certain appropriate conditions on the parameters  $\{\alpha_n\}_{n=0}^{\infty}$ ,  $\{\beta_n\}_{n=0}^{\infty}$ ,  $\{r_n\}_{n=0}^{\infty}$  and the sequence  $\{x_n\}_{n=0}^{\infty}$ ; Jung [24] established strong convergence of the sequence  $\{x_n\}_{n=0}^{\infty}$  generated by (1.9) to a zero of  $A$ , which is a unique solution of a certain variational inequality problem, in either a reflexive Banach space having a weakly sequentially continuous duality mapping or a reflexive Banach space having a uniformly Gâteaux differentiable norm such that every weakly compact convex subset of  $E$  has the fixed point property for nonexpansive mappings.

In [5], Zegeye and Shahzad proved the following theorem.

**Theorem ZS.** *Let  $E$  be a strictly convex reflexive real Banach space which has uniformly Gâteaux differentiable norm and let  $K$  be a nonempty closed convex subset of  $E$ . Assume that every nonempty closed convex and bounded subset of  $E$  has the fixed point property for nonexpansive mappings.*

Let  $A_i : K \rightarrow E, i = 1, 2, \dots, r$  be a finite family of  $m$ -accretive mappings with  $\bigcap_{i=1}^r A_i^{-1}(0) \neq \emptyset$ . For given  $u, x_1 \in K$ , let  $\{x_n\}_{n \geq 1}$  be generated by the algorithm

$$x_{n+1} = \theta_n u + (1 - \theta_n) S_r x_n, \quad \forall n \geq 1, \quad (1.10)$$

where  $S_r = a_0 I + a_1 J_{A_1} + \dots + a_r J_{A_r}$ , with  $J_{A_i} = (I + A_i)^{-1}, 0 < a_i < 1, i = 1, \dots, r, \sum_{i=1}^r a_i = 1$ , and  $\{\theta_n\}_{n \geq 1}$  is a sequence in  $(0, 1)$  satisfying the following conditions:

- (i)  $\lim_{n \rightarrow \infty} \theta_n = 0$ ;
- (ii)  $\sum_{n=1}^{\infty} \theta_n = \infty$ ;
- (iii)  $\sum_{n=1}^{\infty} |\theta_n - \theta_{n-1}| < \infty$  or  $\lim_{n \rightarrow \infty} (|\theta_n - \theta_{n-1}| / \theta_n) = 0$ .

Then,  $\{x_n\}_{n \geq 1}$  converges strongly to a common solution of the equation  $A_i x = 0$  for  $i = 1, 2, \dots, r$ .

Motivated by the results of the authors mentioned above, it is our purpose in this paper to prove new path convergence theorems and introduce a new iteration process for a countably infinite family of  $m$ -accretive mappings and prove strong convergence of the sequence to a common zero of these operators in uniformly convex real Banach spaces. As a result, we obtain strong convergence theorems for a countably infinite family of pseudocontractive mappings. Our theorems extend and improve some important results which are announced recently by various authors.

## 2. Preliminaries

Let  $E$  be a real normed linear space. Let  $S := \{x \in E : \|x\| = 1\}$ .  $E$  is said to have a *Gâteaux differentiable* norm (and  $E$  is called *smooth*) if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.1)$$

exists for each  $x, y \in S$ ;  $E$  is said to have a *uniformly Gâteaux differentiable* norm if for each  $y \in S$  the limit is attained uniformly for  $x \in S$ . Furthermore,  $E$  is said to be *uniformly smooth* if the limit exists uniformly for  $(x, y) \in S \times S$ .

Let  $E$  be a real normed linear space. The modulus of convexity of  $E$  is the function  $\delta_E : [0, 2] \rightarrow [0, 1]$  defined by

$$\delta_E(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \epsilon = \|x - y\| \right\}. \quad (2.2)$$

The space  $E$  is said to be *uniformly convex* if and only if  $\delta_E(\epsilon) > 0$  for all  $\epsilon \in (0, 2]$ .  $E$  is said to be *strictly convex* if for all  $x, y \in E$  such that  $\|x\| = \|y\| = 1$  and for all  $\lambda \in (0, 1)$  we have  $\|\lambda x + (1 - \lambda)y\| < 1$ . It is well known that every uniformly convex Banach space is strictly convex.

A mapping  $T$  with domain  $D(T)$  and range  $R(T)$  in  $E$  is said to be *demiclosed* at  $p$  if whenever  $\{x_n\}$  is a sequence in  $D(T)$  such that  $x_n \rightarrow x \in D(T)$  and  $Tx_n \rightarrow p$ , then  $Tx = p$ .

A mapping  $T : D(T) \subseteq E \rightarrow E$  is said to be *demicompact* at  $h$  if for any bounded sequence  $\{x_n\}$  in  $D(T)$  such that  $(x_n - Tx_n) \rightarrow h$  as  $n \rightarrow \infty$ , there exists a subsequence say  $\{x_{n_j}\}$  of  $\{x_n\}$  and  $x^* \in D(T)$  such that  $\{x_{n_j}\}$  converges strongly to  $x^*$  and  $x^* - Tx^* = h$ .

We need the following lemmas in the sequel.

**Lemma 2.1.** *Let  $E$  be a real normed space, then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad (2.3)$$

for all  $x, y \in E$  and for all  $j(x + y) \in J(x + y)$ .

**Lemma 2.2** (Lemma 3 of Bruck [25]). *Let  $K$  be a nonempty closed convex subset of a strictly convex real Banach space  $E$ . Let  $\{T_i\}_{i=1}^\infty$  be a sequence of nonexpansive mappings from  $K$  to  $E$  such that  $\bigcap_{i=1}^\infty F(T_i) \neq \emptyset$ . Let  $\{\lambda_i\}_{i=1}^\infty$  be a sequence of positive numbers such that  $\sum_{i=1}^\infty \lambda_i = 1$ , then a mapping  $G$  on  $K$  defined by  $Gx := \sum_{i=1}^\infty \lambda_i T_i x$  for all  $x \in K$  is well defined, nonexpansive, and  $F(G) = \bigcap_{i=1}^\infty F(T_i)$ .*

**Lemma 2.3** (Xu [26]). *Let  $\{a_n\}$  be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \sigma_n, \quad n \geq 1, \quad (2.4)$$

where  $\{\alpha_n\}_{n=1}^\infty \subset [0, 1]$  and  $\{\sigma_n\}_{n=1}^\infty$  is a sequence in  $\mathbb{R}$  satisfying (i)  $\sum \alpha_n = \infty$ ; (ii)  $\limsup \sigma_n \leq 0$ . Then,  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ .

**Lemma 2.4** (Suzuki [27]). *Let  $\{x_n\}$  and  $\{y_n\}$  be bounded sequences in a Banach space  $X$  and let  $\{\beta_n\}$  be a sequence in  $[0, 1]$  with  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ . Suppose that  $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$  for all integers  $n \geq 1$  and  $\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$ . Then,  $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$ .*

**Lemma 2.5** (Cioranescu [28]). *Let  $A$  be a continuous accretive operator defined on a real Banach space  $E$  with  $D(A) = E$ . Then,  $A$  is  $m$ -accretive.*

**Lemma 2.6** (C. E. Chidume and C. O. Chidume [29]). *Let  $K$  be a nonempty closed convex subset of a real Banach space  $E$ . For arbitrary  $r > 0$ , let  $B_r(0) := \{x \in E : \|x\| \leq r\}$ . Then, there exists a continuous strictly increasing function  $g : [0, \infty) \rightarrow [0, \infty)$ ,  $g(0) = 0$  such that for every  $x, y \in B_r(0)$  and for  $p \in (1, \infty)$ , the following inequality holds:*

$$4.2^p g\left(\frac{1}{2}\|x + y\|\right) \leq (p.2^p - 4)\|x\|^p + p.2^p \langle y, j_p(x) \rangle + 4\|y\|^p. \quad (2.5)$$

### 3. Path Convergence Theorems

We begin with the following lemma.

**Lemma 3.1.** *Let  $K$  be a nonempty closed convex subset of a reflexive strictly convex Banach space  $E$ . Let  $T : K \rightarrow K$  be a nonexpansive mapping. Let  $\{x_n\}_{n=1}^\infty$ , a bounded sequence in  $K$ , be an approximate fixed point sequence of  $T$ , that is  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ . Let  $\varphi(x) = \mu_n \|x_n - x\|^2$ , for all  $x \in K$  and*

let  $\Gamma = \{x \in K \cap B : \varphi(x) = \min_{z \in K} \varphi(z)\}$ , where  $B$  is any bounded closed convex nonempty subset of  $E$  such that  $x_n \in B$  for all  $n \in \mathbb{N}$ . Then  $T$  has a fixed point in  $\Gamma$ , provided that  $F(T) \neq \emptyset$ .

*Proof.* Since  $E$  is a reflexive Banach space, then  $\Gamma$  is a bounded closed convex nonempty subset of  $E$ . Since  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ , we have that for all  $x \in \Gamma$ ,

$$\begin{aligned} \varphi(Tx) &= \mu_n \|x_n - Tx\|^2 \leq \mu_n (\|x_n - Tx_n\| + \|Tx_n - Tx\|)^2 \\ &\leq \mu_n \|x_n - x\|^2 = \varphi(x). \end{aligned} \quad (3.1)$$

Hence,  $Tx \in \Gamma$ , for all  $x \in \Gamma$ , that is,  $\Gamma$  is invariant under  $T$ . Let  $x^* \in F(T)$ . Then since every closed convex nonempty subset of a reflexive strictly convex Banach space is a Chebyshev set (see, e.g., [30, Corollary 5.1.19]), there exists a unique  $u^* \in \Gamma$  such that

$$\|x^* - u^*\| = \inf_{z \in \Gamma} \|x^* - z\|, \quad (3.2)$$

but  $x^* = Tx^*$  and  $Tu^* \in \Gamma$ . Thus,

$$\|x^* - Tu^*\| = \|Tx^* - Tu^*\| \leq \|x^* - u^*\|. \quad (3.3)$$

So,  $Tu^* = u^*$ . Hence,  $F(T) \cap \Gamma \neq \emptyset$ . This completes the proof.  $\square$

**Proposition 3.2.** Let  $K$  be a nonempty closed convex subset of a real Banach space  $E$ . Let  $A_i : K \rightarrow E$ ,  $i = 1, 2, \dots$ , be a countably infinite family of  $m$ -accretive mappings and define  $J_{A_i} := (I + A_i)^{-1}$ ,  $i = 1, 2, \dots$ . Let  $\{\alpha_n\}_{n=1}^\infty$ ,  $\{\sigma_{i,n}\}_{n=1}^\infty$ ,  $i = 1, 2, \dots$  be sequences in  $(0, 1)$  such that  $\sum_{i=1}^\infty \sigma_{i,n} = (1 - \alpha_n)$ . Fix  $\delta \in [\gamma_1, \gamma_2]$ , for some  $\gamma_1, \gamma_2 \in (0, 1)$ . For arbitrary fixed  $u \in K$ , define a map  $T_n : K \rightarrow K$  by

$$T_n x = \alpha_n u + \sum_{i=1}^\infty \sigma_{i,n} ((1 - \delta)x + \delta J_{A_i} x), \quad \forall x \in K. \quad (3.4)$$

Then,  $T_n$  is a strict contraction on  $K$ .

*Proof.* Let  $x, y \in K$ , then

$$\begin{aligned} \|T_n x - T_n y\| &= \left\| \sum_{i=1}^\infty \sigma_{i,n} ((1 - \delta)(x - y) + \delta(J_{A_i} x - J_{A_i} y)) \right\| \\ &\leq \sum_{i=1}^\infty \sigma_{i,n} ((1 - \delta)\|x - y\| + \delta\|J_{A_i} x - J_{A_i} y\|) \\ &\leq (1 - \alpha_n)\|x - y\|. \end{aligned} \quad (3.5) \quad \square$$

Thus, for each  $n \in \mathbb{N}$ , there is a unique  $z_n \in K$  satisfying

$$z_n = \alpha_n u + \sum_{i=1}^\infty \sigma_{i,n} ((1 - \delta)z_n + \delta J_{A_i} z_n). \quad (3.6)$$

**Lemma 3.3.** *Let  $K$  be a nonempty closed convex subset of a real Banach space  $E$ . For each  $i \geq 1$ , let  $A_i : K \rightarrow E$  be a countably infinite family of  $m$ -accretive mappings. For  $n \in \mathbb{N}$ , let  $\{z_n\}$  be a sequence satisfying (3.6) and assume  $\bigcap_{i=1}^{\infty} A_i^{-1}(0) \neq \emptyset$ . Then,  $\{z_n\}$  is bounded.*

*Proof.* Let  $x^* \in \bigcap_{i=1}^{\infty} A_i^{-1}(0) = \bigcap_{i=1}^{\infty} F(J_{A_i})$ . Then, using (3.6), we obtain

$$\begin{aligned} \|z_n - x^*\|^2 &= \left\langle \alpha_n(u - x^*) + \sum_{i=1}^{\infty} \sigma_{i,n}((1 - \delta)z_n + \delta J_{A_i} z_n - x^*), j(z_n - x^*) \right\rangle \\ &\leq \alpha_n \langle u - x^*, j(z_n - x^*) \rangle + \sum_{i=1}^{\infty} \sigma_{i,n} \|z_n - x^*\|^2 \\ &= \alpha_n \langle u - x^*, j(z_n - x^*) \rangle + (1 - \alpha_n) \|z_n - x^*\|^2, \end{aligned} \tag{3.7}$$

which implies that  $\|z_n - x^*\| \leq \|u - x^*\|$ . Thus,  $\{z_n\}$  is bounded. □

**Lemma 3.4.** *Let  $K$  be a nonempty closed convex subset of a uniformly convex real Banach space  $E$ . For each  $i \geq 1$ , let  $A_i : K \rightarrow E$  be a countably infinite family of  $m$ -accretive mappings such that  $\bigcap_{i=1}^{\infty} A_i^{-1}(0) \neq \emptyset$ . Let  $\{\alpha_n\}$  be a sequence in  $(0, 1)$  such that  $\lim_{n \rightarrow \infty} (\alpha_n / \sigma_{i,n}) = 0$ , for all  $i \geq 1$ ,  $\sum_{i=1}^{\infty} \sigma_{i,n} = (1 - \alpha_n)$ . Let  $\{z_n\}$  be a sequence satisfying (3.6). Then,  $\lim_{n \rightarrow \infty} \|z_n - J_{A_i} z_n\| = 0$ , for all  $i \geq 1$ . Furthermore, if  $\{\lambda_i\}_{i=1}^{\infty}$  is a sequence in  $(0, 1)$  such that  $\sum_{i=1}^{\infty} \lambda_i = 1$ ;  $\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} |\sigma_{i,n} - \lambda_i| = 0$  and define  $G := (1 - \delta)I + \delta T$ , where  $T := \sum_{i=1}^{\infty} \lambda_i J_{A_i}$ , then  $\lim_{n \rightarrow \infty} \|z_n - G z_n\| = 0$ .*

*Proof.* We start by showing that  $\lim_{n \rightarrow \infty} \|z_n - J_{A_i} z_n\| = 0$ , for all  $i \geq 1$ . For this, let  $S_i := (1 - \delta)I + \delta J_{A_i}$ , where  $I$  is the identity operator on  $K$ . Since  $\{z_n\}_{n=1}^{\infty}$  is bounded, then for each  $i \geq 1$  and  $x^* \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$ , we have the following using (2.5):

$$\begin{aligned} 4.2^p g\left(\frac{1}{2} \|S_i z_n - z_n\|\right) &= 4.2^p g\left(\frac{1}{2} \|S_i z_n - x^* + x^* - z_n\|\right) \\ &\leq (p.2^p - 4) \|x^* - z_n\|^p + p.2^p \langle S_i z_n - x^*, j_p(x^* - z_n) \rangle + 4 \|S_i z_n - x^*\|^p \\ &\leq (p.2^p - 4) \|x^* - z_n\|^p + p.2^p \langle S_i z_n - z_n + z_n - x^*, j_p(x^* - z_n) \rangle \\ &\quad + 4 \|S_i z_n - x^*\|^p \\ &= (p.2^p - 4) \|x^* - z_n\|^p + p.2^p \langle S_i z_n - z_n, j_p(x^* - z_n) \rangle \\ &\quad - p.2^p \langle x^* - z_n, j_p(x^* - z_n) \rangle + 4 \|S_i z_n - x^*\|^p \\ &\leq p.2^p \langle z_n - S_i z_n, j_p(z_n - x^*) \rangle. \end{aligned} \tag{3.8}$$

Hence,

$$\frac{4}{p} g\left(\frac{1}{2} \|S_i z_n - z_n\|\right) \leq \langle z_n - S_i z_n, j_p(z_n - x^*) \rangle, \tag{3.9}$$

and so,

$$\frac{4}{p} \sum_{i=1}^{\infty} \sigma_{i,n} g\left(\frac{1}{2} \|S_i z_n - z_n\|\right) \leq \frac{4}{p} \sum_{i=1}^{\infty} \sigma_{i,n} \langle z_n - S_i z_n, j_p(z_n - x^*) \rangle. \quad (3.10)$$

Using (3.6), we have

$$\begin{aligned} \langle z_n - x^*, j_p(z_n - x^*) \rangle &= \alpha_n \langle u - x^*, j_p(z_n - x^*) \rangle \\ &\quad + \sum_{i=1}^{\infty} \sigma_{i,n} \langle S_i z_n - z_n + z_n - x^*, j_p(z_n - x^*) \rangle \\ &= \alpha_n \langle u - x^*, j_p(z_n - x^*) \rangle + \sum_{i=1}^{\infty} \sigma_{i,n} \langle S_i z_n - z_n, j_p(z_n - x^*) \rangle \\ &\quad + (1 - \alpha_n) \langle z_n - x^*, j_p(z_n - x^*) \rangle, \end{aligned} \quad (3.11)$$

which implies

$$\sum_{i=1}^{\infty} \sigma_{i,n} \langle z_n - S_i z_n, j_p(z_n - x^*) \rangle = \alpha_n \langle u - z_n, j_p(z_n - x^*) \rangle. \quad (3.12)$$

Using this and (3.10), we get

$$\frac{4}{p} \sum_{i=1}^{\infty} \sigma_{i,n} g\left(\frac{1}{2} \|S_i z_n - z_n\|\right) \leq \alpha_n \langle u - z_n, j_p(z_n - x^*) \rangle. \quad (3.13)$$

Since  $\{z_n\}$  is bounded, we have that

$$\sum_{i=1}^{\infty} \sigma_{i,n} g\left(\frac{1}{2} \|S_i z_n - z_n\|\right) \leq \alpha_n M, \quad (3.14)$$

for some constant  $M > 0$ . This yields

$$g\left(\frac{1}{2} \|S_i z_n - z_n\|\right) \leq \frac{\alpha_n}{\sigma_{i,n}} M. \quad (3.15)$$

Thus, since  $g$  is continuous, strictly increasing,  $g(0) = 0$ , and  $\lim_{n \rightarrow \infty} (\alpha_n / \sigma_{i,n}) = 0$ , for all  $i \geq 1$ , we have

$$2g\left(\frac{1}{2} \lim_{n \rightarrow \infty} \|S_i z_n - z_n\|\right) = 0. \quad (3.16)$$



So,  $\lim_{n \rightarrow \infty} \|S_i z_n - z_n\| = 0$ , for all  $i \geq 1$ , but

$$\begin{aligned} \|S_i z_n - z_n\| &= \|(1 - \delta)z_n + \delta J_{A_i} z_n - z_n\| \\ &= \|\delta(J_{A_i} z_n - z_n)\| \\ &= \delta \|J_{A_i} z_n - z_n\|. \end{aligned} \tag{3.17}$$

Thus,

$$\lim_{n \rightarrow \infty} \|J_{A_i} z_n - z_n\| = 0, \quad \forall i \geq 1. \tag{3.18}$$

Next, we show that  $\lim_{n \rightarrow \infty} \|z_n - Gz_n\| = 0$ . Observe that

$$z_n - Gz_n = \alpha_n u + \sum_{i=1}^{\infty} (\sigma_{i,n} - \lambda_i) [(1 - \delta)z_n + \delta J_{A_i} z_n]. \tag{3.19}$$

So,

$$\|z_n - Gz_n\| \leq \alpha_n \|u\| + M \sum_{i=1}^{\infty} |\sigma_{i,n} - \lambda_i| \tag{3.20}$$

for some  $M > 0$ . Hence,

$$\lim_{n \rightarrow \infty} \|z_n - Gz_n\| = 0. \tag{3.21}$$

This completes the proof. □

**Theorem 3.5.** *Let  $K$  be a nonempty closed convex subset of a uniformly convex real Banach space  $E$  with uniformly Gâteaux differentiable norm. Let  $A_i : K \rightarrow E, i = 1, 2, \dots$ , be a countably infinite family of  $m$ -accretive mappings such that  $\bigcap_{i=1}^{\infty} A_i^{-1}(0) \neq \emptyset$ . Let  $\{z_n\}$  be a sequence satisfying (3.6). Let  $\{\lambda_i\}_{i=1}^{\infty}$  be a sequence in  $(0, 1)$  such that  $\sum_{i=1}^{\infty} \lambda_i = 1$  and  $\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} |\sigma_{i,n} - \lambda_i| = 0$ . Let  $G := (1 - \delta)I + \delta T$ , where  $T := \sum_{i=1}^{\infty} \lambda_i J_{A_i}$ . Then,  $\{z_n\}$  converges strongly to an element in  $\bigcap_{i=1}^{\infty} A_i^{-1}(0)$ .*

*Proof.* Observe that by Lemma 2.2,  $T := \sum_{i=1}^{\infty} \lambda_i J_{A_i}$  is well defined, nonexpansive, and  $F(T) = \bigcap_{i=1}^{\infty} F(J_{A_i}) = \bigcap_{i=1}^{\infty} A_i^{-1}(0)$ . Furthermore, it is easy to see that  $G$  is nonexpansive and that  $F(G) = F(T) = \bigcap_{i=1}^{\infty} A_i^{-1}(0)$ . Now, since  $\{z_n\}$  is bounded and  $\lim_{n \rightarrow \infty} \|Gz_n - z_n\| = 0$ , we have by Lemma 3.1 that there exists a unique  $z^*$  in the set  $\Omega^* := \{x \in K \cap B^* : \mu_n \|z_n - x\|^2 = \min_{y \in K} \|z_n - y\|\}$  such that  $Gz^* = z^*$ , where  $B^*$  is a bounded closed convex nonempty subset of  $E$  such that  $u, z_n \in B^*$  for all  $n \in \mathbb{N}$ . Thus,  $z^* \in F(G) = \bigcap_{i=1}^{\infty} A_i^{-1}(0)$ . Let  $t \in (0, 1)$ , then by convexity of  $K \cap B^*$ , we have that  $(1 - t)z^* + tu \in K \cap B^*$ . Thus,  $\mu_n \|z_n - z^*\|^2 \leq \mu_n \|z_n - ((1 - t)z^* + tu)\|^2 = \mu_n \|z_n - z^* - t(u - z^*)\|^2$ . Moreover, we have, by Lemma 2.1 that

$$\|z_n - z^* - t(u - z^*)\|^2 \leq \|z_n - z^*\|^2 - 2t \langle u - z^*, j(z_n - z^* - t(u - z^*)) \rangle. \tag{3.22}$$

This implies that  $\mu_n \langle u - z^*, j(z_n - z^* - t(u - z^*)) \rangle \leq 0$ . Furthermore, since  $E$  has uniformly Gâteaux differentiable norm, we obtain that

$$\lim_{t \rightarrow 0} (\langle u - z^*, j(z_n - z^*) \rangle - \langle u - z^*, j(z_n - z^* - t(u - z^*)) \rangle) = 0. \quad (3.23)$$

Thus, given  $\epsilon > 0$ , there exists  $\delta_\epsilon > 0$  such that for all  $t \in (0, \delta_\epsilon)$  and for all  $n \in \mathbb{N}$ ,

$$\langle u - z^*, j(z_n - z^*) \rangle < \epsilon + \langle u - z^*, j(z_n - z^* - t(u - z^*)) \rangle. \quad (3.24)$$

Taking Banach limit on both sides of this inequality, we obtain

$$\mu_n \langle u - z^*, j(z_n - z^*) \rangle \leq \epsilon; \quad (3.25)$$

and since  $\epsilon > 0$  is arbitrary, we have that

$$\mu_n \langle u - z^*, j(z_n - z^*) \rangle \leq 0. \quad (3.26)$$

Now, using (3.6), we have that

$$\begin{aligned} \|z_n - z^*\|^2 &= \left\langle \alpha_n(u - z^*) + \sum_{i=1}^{\infty} \sigma_{i,n}(((1 - \delta)z_n + \delta J_{A_i} z_n) - z^*), j(z_n - z^*) \right\rangle \\ &\leq \alpha_n \langle u - z^*, j(z_n - z^*) \rangle + (1 - \alpha_n) \|z_n - z^*\|^2. \end{aligned} \quad (3.27)$$

So,

$$\|z_n - z^*\|^2 \leq \langle u - z^*, j(z_n - z^*) \rangle. \quad (3.28)$$

Again, taking Banach limit, we obtain

$$\mu_n \|z_n - z^*\|^2 \leq \mu_n \langle u - z^*, j(z_n - z^*) \rangle \leq 0, \quad (3.29)$$

so that  $\mu_n \|z_n - z^*\|^2 = 0$ . Hence, there exists a subsequence  $\{z_{n_l}\}_{l=1}^{\infty}$  of  $\{z_n\}_{n=1}^{\infty}$  such that  $\lim_{l \rightarrow \infty} z_{n_l} = z^*$ . We now show that  $\{z_n\}_{n=1}^{\infty}$  actually converges to  $z^*$ . Suppose there is another subsequence  $\{z_{n_k}\}_{k=1}^{\infty}$  of  $\{z_n\}_{n=1}^{\infty}$  such that  $\lim_{k \rightarrow \infty} z_{n_k} = u^*$ . Then, since  $\lim_{n \rightarrow \infty} \|J_{A_i} z_n - z_n\| = 0$  and  $J_{A_i}$  is continuous for all  $i \in \mathbb{N}$ , we have that  $u^* \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$ .

*Claim 1* ( $u^* = z^*$ ). Suppose for contradiction that  $u^* \neq z^*$ , then  $\|u^* - z^*\| > 0$ , but using (3.6), we have that

$$\begin{aligned} \|z_{n_i} - u^*\|^2 &= \left\langle \alpha_{n_i}(u - u^*) + \sum_{i=1}^{\infty} \sigma_{i,n_i}(((1 - \delta)z_{n_i} + \delta J_{A_i} z_{n_i}) - u^*), j(z_{n_i} - u^*) \right\rangle \\ &= \alpha_{n_i} \langle u - z^*, j(z_{n_i} - u^*) \rangle + \alpha_{n_i} \langle z^* - z_{n_i}, j(z_{n_i} - u^*) \rangle \\ &\quad + \alpha_{n_i} \|z_{n_i} - u^*\|^2 + \sum_{i=1}^{\infty} \sigma_{i,n_i} \langle (1 - \delta)z_{n_i} + \delta J_{A_i} z_{n_i} - u^*, j(z_{n_i} - u^*) \rangle \\ &\leq \alpha_{n_i} \langle u - z^*, j(z_{n_i} - u^*) \rangle + \alpha_{n_i} \langle z^* - z_{n_i}, j(z_{n_i} - u^*) \rangle \\ &\quad + \alpha_{n_i} \|z_{n_i} - u^*\|^2 + (1 - \delta)(1 - \alpha_{n_i}) \|z_{n_i} - u^*\|^2 + \delta(1 - \alpha_{n_i}) \|z_{n_i} - u^*\|^2 \\ &= \alpha_{n_i} \langle u - z^*, j(z_{n_i} - u^*) \rangle + \alpha_{n_i} \langle z^* - z_{n_i}, j(z_{n_i} - u^*) \rangle + \|z_{n_i} - u^*\|^2. \end{aligned} \tag{3.30}$$

Thus,

$$\langle u - z^*, j(u^* - z_{n_i}) \rangle \leq \|z_{n_i} - u^*\| \|z_{n_i} - z^*\|. \tag{3.31}$$

Using the fact that  $\{z_n\}_{n=1}^{\infty}$  is bounded and that  $E$  has a uniformly Gâteaux differentiable norm, we obtain from (3.31) that

$$\langle u - z^*, j(u^* - z^*) \rangle \leq 0. \tag{3.32}$$

Similarly, we also obtain that  $\langle u - u^*, j(z^* - u^*) \rangle \leq 0$  or rather

$$\langle u^* - u, j(u^* - z^*) \rangle \leq 0. \tag{3.33}$$

Adding (3.32) and (3.33), we have that  $\|z^* - u^*\| \leq 0$ , a contradiction. Thus,  $z^* = u^*$ . Hence,  $\{z_n\}_{n=1}^{\infty}$  converges strongly to  $z^* \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$ . This completes the proof.  $\square$

**Theorem 3.6.** *Let  $K$  be a nonempty closed convex subset of a uniformly convex real Banach space  $E$  with uniformly Gâteaux differentiable norm. Let  $A_i : K \rightarrow E, i = 1, 2, \dots$ , be a countably infinite family of  $m$ -accretive mappings such that  $\bigcap_{i=1}^{\infty} A_i^{-1}(0) \neq \emptyset$ . Let  $\{z_n\}$  be a sequence satisfying (3.6). If at least one of the mappings  $J_{A_i}$  is demicompact, then  $\{z_n\}$  converges strongly to an element of  $\bigcap_{i=1}^{\infty} A_i^{-1}(0)$ .*

*Proof.* For fixed  $s \in \mathbb{N}$ , let  $J_{A_s}$  be demicompact. Since  $\lim_{n \rightarrow \infty} \|z_n - J_{A_s} z_n\| = 0$ , there exists a subsequence say  $\{z_{n_k}\}$  of  $\{z_n\}$  that converges strongly to some point  $z^* \in K$ . By continuity of  $J_{A_i}$  and the fact that  $\lim_{k \rightarrow \infty} \|z_{n_k} - J_{A_i} z_{n_k}\| = 0, i = 1, 2, \dots$ , we have that  $z^* \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$ . Assuming that there is another subsequence  $\{z_{n_j}\}_{j=1}^{\infty}$  of  $\{z_n\}$  that converges strongly to a point say,  $q^*$ , then following the argument of the last part of proof of Theorem 3.5, we obtain that  $\{z_n\}_{n=1}^{\infty}$  converges strongly to  $z^* \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$ . This completes the proof.  $\square$

**Theorem 3.7.** *Let  $K$  be a nonempty closed convex subset of a uniformly convex real Banach space  $E$  with uniformly Gâteaux differentiable norm and also admits weakly sequential continuous duality*

mapping. Let  $A_i : K \rightarrow E, i = 1, 2, \dots$ , be a countably infinite family of  $m$ -accretive mappings such that  $\bigcap_{i=1}^{\infty} A_i^{-1}(0) \neq \emptyset$ . Let  $\{z_n\}$  be a sequence satisfying (3.6). Then,  $\{z_n\}$  converges strongly to an element of  $\bigcap_{i=1}^{\infty} A_i^{-1}(0)$ .

*Proof.* Since  $\{z_n\}$  is bounded, there exists a subsequence say  $\{z_{n_k}\}$  of  $\{z_n\}$  that converges weakly to some point  $z^* \in K$ . Using the demiclosedness property of  $(I - J_{A_i})$  at 0 for each  $i \geq 1$  (since  $J_{A_i}$  is nonexpansive for each  $i \in \mathbb{N}$ , see, e.g., [31]) and the fact that  $\lim_{k \rightarrow \infty} \|z_{n_k} - J_{A_i} z_{n_k}\| = 0$ , we get that  $z^* \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$ . We also observe from (3.6) that

$$\begin{aligned} \|z_{n_k} - z^*\|^2 &= \left\langle \alpha_{n_k} u + \sum_{i=1}^{\infty} \sigma_{i,n_k} ((1 - \delta) z_{n_k} + \delta J_{A_i} z_{n_k}) - z^*, j(z_{n_k} - z^*) \right\rangle \\ &\leq \alpha_{n_k} \langle u - z^*, j(z_{n_k} - z^*) \rangle + \sum_{i=1}^{\infty} \sigma_{i,n_k} \|z_{n_k} - z^*\|^2 \\ &= \alpha_{n_k} \langle u - z^*, j(z_{n_k} - z^*) \rangle + (1 - \alpha_{n_k}) \|z_{n_k} - z^*\|^2. \end{aligned} \quad (3.34)$$

This implies that  $\|z_{n_k} - z^*\|^2 \leq \langle u - z^*, j(z_{n_k} - z^*) \rangle$ . Using the fact that  $j$  is weakly sequential continuous, then we have from the last inequality that  $\{z_{n_k}\}$  converges strongly to  $z^*$ . Then following the argument of the last part of proof of Theorem 3.5, we obtain that  $\{z_n\}_{n=1}^{\infty}$  converges strongly to  $z^* \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$ . This completes the proof.  $\square$

#### 4. Convergence Theorems for Countably Infinite Family of $m$ -Accretive Mappings

For the rest of this paper,  $\{\alpha_n\}_{n=1}^{\infty}$  and  $\{\sigma_{i,n}\}_{n=1}^{\infty}$  are in  $(0, 1)$  satisfying the following additional conditions:

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,
- (ii)  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,
- (iii)  $\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} |\sigma_{i,n+1} - \sigma_{i,n}| = 0$ .

**Theorem 4.1.** Let  $K$  be a nonempty closed convex subset of a uniformly convex real Banach space  $E$  with uniformly Gâteaux differentiable norm. Let  $A_i : K \rightarrow E, i = 1, 2, \dots$ , be a countably infinite family of  $m$ -accretive mappings such that  $\bigcap_{i=1}^{\infty} A_i^{-1}(0) \neq \emptyset$ . For fixed  $\delta \in [\gamma_1, \gamma_2]$ , for some  $\gamma_1, \gamma_2 \in (0, 1)$ ,  $u \in K$ , let  $\{x_n\}_{n=1}^{\infty}$  be generated by

$$x_1 \in K, \quad x_{n+1} = \alpha_n u + \sum_{i=1}^{\infty} \sigma_{i,n} ((1 - \delta) x_n + \delta J_{A_i} x_n), \quad n \geq 1, \quad (4.1)$$

then  $\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} \sigma_{i,n} ((1 - \delta) x_n + \delta J_{A_i} x_n - x_n) = 0$ . Furthermore, if  $\{\alpha_n\}_{n \geq 1}$  is such that

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^{\infty} \sigma_{i,n} ((1 - \delta) x_n + \delta J_{A_i} x_n - x_n)}{\alpha_n} = 0, \quad (4.2)$$

then  $\{x_n\}_{n=1}^{\infty}$  converges strongly to a common zero of  $\{A_i\}_{i=1}^{\infty}$ .

*Proof.* Using mathematical induction, it is easy to see that for  $x^* \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$  fixed

$$\|x_n - x^*\| \leq \max\{\|u - x^*\|, \|x_1 - x^*\|\}, \quad \forall n \geq 1. \tag{4.3}$$

Hence,  $\{x_n\}_{n=1}^{\infty}$  is bounded and so  $\{J_{A_i}x_n\}_{n=1}^{\infty}$  is also bounded.

Now, define the sequences  $\{\beta_n\}_{n=1}^{\infty}$  and  $\{y_n\}_{n=1}^{\infty}$  by  $\beta_n := (1 - \delta)\alpha_n + \delta$  and  $y_n := (x_{n+1} - x_n + \beta_n x_n) / \beta_n$ . Then,

$$y_n = \frac{\alpha_n u + \delta \sum_{i \geq 1} \sigma_{i,n} ((1 - \delta)x_n + \delta J_{A_i} x_n)}{\beta_n}. \tag{4.4}$$

Observe that  $\{y_n\}_{n=1}^{\infty}$  is bounded and that

$$\begin{aligned} \|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| &\leq \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| \|u\| \\ &\quad + \left| \frac{\delta(1 - \alpha_{n+1})}{\beta_{n+1}} - 1 \right| \|x_{n+1} - x_n\| \\ &\quad + \frac{\delta M}{\beta_{n+1}\beta_n} \sum_{i=1}^{\infty} |\sigma_{i,n+1} - \sigma_{i,n}| + \frac{\delta M}{\beta_{n+1}\beta_n} |\beta_{n+1} - \beta_n|, \end{aligned} \tag{4.5}$$

for some  $M > 0$ . Thus,

$$\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{4.6}$$

Hence, by Lemma 2.4, we have  $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$ . Consequently, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} \beta_n \|y_n - x_n\| = 0. \tag{4.7}$$

From (4.1), we have that

$$x_{n+1} - x_n = \alpha_n (u - x_n) + \sum_{i=1}^{\infty} \sigma_{i,n} (((1 - \delta)x_n + \delta J_{A_i} x_n) - x_n), \tag{4.8}$$

which implies that  $\|\sum_{i=1}^{\infty} \sigma_{i,n} ((1 - \delta)x_n + \delta J_{A_i} x_n - x_n)\| \leq \|x_{n+1} - x_n\| + \alpha_n \|u - x_n\|$  and thus

$$\lim_{n \rightarrow \infty} \left\| \sum_{i=1}^{\infty} \sigma_{i,n} (((1 - \delta)x_n + \delta J_{A_i} x_n) - x_n) \right\| = 0. \tag{4.9}$$

Let  $\{z_n\}_{n=1}^{\infty}$  be a sequence satisfying (3.6). Then, by Theorem 3.5,  $z_n \rightarrow z^* \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$ . Using Lemma 2.1, we have that

$$\begin{aligned} \|z_n - x_n\|^2 &\leq \left\| \sum_{i=1}^{\infty} \sigma_{i,n} (((1-\delta)z_n + \delta J_{A_i} z_n) - ((1-\delta)x_n + \delta J_{A_i} x_n) + ((1-\delta)x_n + \delta J_{A_i} x_n) - x_n) \right\|^2 \\ &\quad + 2\alpha_n \langle u - x_n, j(z_n - x_n) \rangle \\ &\leq \left( (1-\alpha_n) \|z_n - x_n\| + \left\| \sum_{i=1}^{\infty} \sigma_{i,n} (((1-\delta)x_n + \delta J_{A_i} x_n) - x_n) \right\| \right)^2 \\ &\quad + 2\alpha_n \langle u - z_n, j(z_n - x_n) \rangle. \end{aligned} \tag{4.10}$$

This implies that

$$\begin{aligned} &\langle u - z_n, j(x_n - z_n) \rangle \\ &\leq \frac{\alpha_n}{2} \|z_n - x_n\|^2 + \frac{(1-\alpha_n) \|z_n - x_n\| \cdot \left\| \sum_{i=1}^{\infty} \sigma_{i,n} (((1-\delta)x_n + \delta J_{A_i} x_n) - x_n) \right\|}{\alpha_n} \\ &\quad + \frac{\left\| \sum_{i=1}^{\infty} \sigma_{i,n} (((1-\delta)x_n + \delta J_{A_i} x_n) - x_n) \right\|^2}{2\alpha_n} \end{aligned} \tag{4.11}$$

and hence,

$$\limsup_{n \rightarrow \infty} \langle u - z_n, j(x_n - z_n) \rangle \leq 0. \tag{4.12}$$

Moreover, we have that

$$\begin{aligned} \langle u - z_n, j(x_n - z_n) \rangle &= \langle u - z^*, j(x_n - z^*) \rangle + \langle u - z^*, j(x_n - z_n) - j(x_n - z^*) \rangle \\ &\quad + \langle z^* - z_n, j(x_n - z_n) \rangle. \end{aligned} \tag{4.13}$$

Using the boundedness of  $\{x_n\}_{n=1}^{\infty}$ , we have

$$\langle z^* - z_n, j(x_n - z_n) \rangle \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{4.14}$$

Also, since  $j$  is norm-to-weak\* uniformly continuous on bounded subsets of  $E$ , we have that

$$\langle u - z^*, j(x_n - z_n) - j(x_n - z^*) \rangle \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{4.15}$$

From (4.12) and (4.13), we obtain that

$$\limsup_{n \rightarrow \infty} \langle u - z^*, j(x_n - z^*) \rangle \leq 0. \quad (4.16)$$

Finally, using Lemma 2.1 on (4.1), we get

$$\begin{aligned} \|x_{n+1} - z^*\|^2 &\leq \left\| \sum_{i=1}^{\infty} \sigma_{i,n+1} ((1-\delta)x_n + \delta J_{A_i} x_n) - z^* \right\|^2 \\ &\quad + 2\alpha_n \langle u - z^*, j(x_{n+1} - z^*) \rangle \\ &\leq (1 - \alpha_n) \|x_n - z^*\|^2 + 2\alpha_n \langle u - z, j(x_{n+1} - z^*) \rangle. \end{aligned} \quad (4.17)$$

Using (4.16) and Lemma 2.3 in (4.17), we get that  $\{x_n\}_{n=1}^{\infty}$  converges strongly to common zero of the family  $\{A_i\}_{i=1}^{\infty}$  of  $m$ -accretive operators.  $\square$

*Remark 4.2.* If  $K$  is replaced with  $E$  in Theorems 3.5, 3.6, 3.7, and 4.1, then by Lemma 2.5, the assumption that  $A_i$  is  $m$ -accretive for each  $i \geq 1$  could be replaced with  $A_i$  is continuous for each  $i \geq 1$ .

Hence, we have the following theorem.

**Theorem 4.3.** *Let  $E$  be a uniformly convex real Banach space with uniformly Gâteaux differentiable norm. Let  $A_i : E \rightarrow E$ ,  $i = 1, 2, \dots$ , be a countably infinite family of continuous accretive operators such that  $\bigcap_{i=1}^{\infty} A_i^{-1}(0) \neq \emptyset$ . For arbitrary but fixed  $\delta \in (0, 1)$ ,  $u \in K$ , let  $\{x_n\}_{n=1}^{\infty}$  be generated by  $x_1 \in K$ ,*

$$x_{n+1} = \alpha_n u + \sum_{i=1}^{\infty} \sigma_{i,n} ((1-\delta)x_n + \delta J_{A_i} x_n), \quad n \geq 1, \quad (4.18)$$

*then,  $\{x_n\}_{n=1}^{\infty}$  converges strongly to a common zero of  $\{A_i\}_{i=1}^{\infty}$ .*

*Proof.* By Lemma 2.5, we have that  $A_i$  is  $m$ -accretive for each  $i \geq 1$ . Then, the rest of the proof follows from Theorem 4.1.  $\square$

We also have the following theorems.

**Theorem 4.4.** *Let  $K$ ,  $E$ ,  $A_i$  and  $\{x_n\}_{n \geq 1}$  be as an Theorem 4.1. Suppose that at least one of  $J_{A_i}$  is demicompact, then  $\{x_n\}_{n \geq 1}$  converges strongly to a common zero of  $A_i$ ,  $i = 1, 2, \dots$*

*Proof.* The proof follows as in the proof of Theorem 4.1 but using Theorem 3.6.  $\square$

**Theorem 4.5.** *Let  $K$ ,  $E$ ,  $A_i$  and  $\{x_n\}_{n \geq 1}$  be as an Theorem 4.1. Suppose that, in addition,  $E$  admits weakly sequential continuous duality mapping, then  $\{x_n\}_{n \geq 1}$  converges strongly to a common zero of  $A_i$ ,  $i = 1, 2, \dots$*

*Proof.* The proof follows as in the proof of Theorem 4.1 but using Theorem 3.7.  $\square$

## 5. Convergence Theorems for Countably Infinite Family of Pseudocontractive Mappings

**Theorem 5.1.** Let  $K$  be a nonempty closed convex subset of a uniformly convex real Banach space  $E$  with uniformly Gâteaux differentiable norm. Let  $T_i : K \rightarrow E$ ,  $i = 1, 2, \dots$ , be a countably infinite family of pseudocontractive mappings such that for each  $i \geq 1$ ,  $(I - T_i)$  is  $m$ -accretive and  $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ . Let  $J_{T_i} = (2I - T_i)^{-1}$ ,  $i \geq 1$ . For fixed  $\delta \in [\gamma_1, \gamma_2]$ , for some  $\gamma_1, \gamma_2 \in (0, 1)$  and  $u \in K$ , let  $\{x_n\}_{n=1}^{\infty}$  be generated by  $x_1 \in K$ :

$$x_{n+1} = \alpha_n u + \sum_{i=1}^{\infty} \sigma_{i,n} ((1 - \delta)x_n + \delta J_{T_i} x_n), \quad n \geq 1, \quad (5.1)$$

then  $\{x_n\}_{n=1}^{\infty}$  converges strongly to a common fixed point of  $\{T_i\}_{i=1}^{\infty}$ .

*Proof.* Put  $A_i := (I - T_i)$ ,  $i \geq 1$ . It is then obvious that  $A_i^{-1}(0) = F(T_i)$ , for all  $i \in \mathbb{N}$  and hence  $\bigcap_{i=1}^{\infty} A_i^{-1}(0) = \bigcap_{i=1}^{\infty} F(T_i)$ . Furthermore,  $A_i$  is  $m$ -accretive for each  $i \geq 1$ . Thus, we obtain the conclusion from Theorem 4.1 with  $J_{A_i}$  in the definition of  $z_n$  replaced with  $J_{T_i}$ .  $\square$

**Theorem 5.2.** Let  $E$  be a uniformly convex real Banach space with uniformly Gâteaux differentiable norm. For each  $i \geq 1$ , let  $T_i : E \rightarrow E$  be a countably infinite family of continuous pseudocontractive mappings such that for each  $i \geq 1$  and  $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ . Let  $J_{T_i} = (2I - T_i)^{-1}$ ,  $i \geq 1$ . For arbitrary but fixed  $\delta \in [\gamma_1, \gamma_2]$ , for some  $\gamma_1, \gamma_2 \in (0, 1)$  and  $u \in K$ , let  $\{x_n\}_{n=1}^{\infty}$  be generated by  $x_1 \in K$ :

$$x_{n+1} = \alpha_n u + \sum_{i=1}^{\infty} \sigma_{i,n} ((1 - \delta)x_n + \delta J_{T_i} x_n), \quad n \geq 1, \quad (5.2)$$

then  $\{x_n\}_{n=1}^{\infty}$  converges strongly to a common fixed point of  $\{T_i\}_{i=1}^{\infty}$ .

*Remark 5.3.* Theorems similar to Theorems 4.4 and 4.5 could also be obtained for countably infinite family of pseudocontractive mappings.

*Remark 5.4.* Prototypes for our iteration parameters are

$$\alpha_n = \frac{1}{n+1}, \quad \sigma_{i,n} = \frac{n}{2^i(n+1)}, \quad \lambda_i = \frac{1}{2^i}. \quad (5.3)$$

*Remark 5.5.* The addition of bounded error terms in any of our recursion formulas leads to no further generalization.

*Remark 5.6.* If  $f : K \rightarrow K$  is a contraction map and we replace  $u$  by  $f(x_n)$  in the recursion formulas of our theorems, we obtain what some authors now call *viscosity* iteration process. We observe that all our theorems in this paper carry over trivially to the so-called viscosity process. One simply replaces  $u$  by  $f(x_n)$ , repeats the argument of this paper, using the fact that  $f$  is a contraction map.



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