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Research Article

Persistence and Stability for a Generalized Leslie-Gower Model with Stage Structure and Dispersal

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A generalized version of the Leslie-Gower predator-prey model that incorporates the prey structure and predator dispersal in two-patch environments is introduced. The focus is on the study of the boundedness of solution, permanence, and extinction of the model. Sufficient conditions for global asymptotic stability of the positive equilibrium are derived by constructing a Lyapunov functional. Numerical simulations are also presented to illustrate our main results.

1. Introduction

Lotka-Volterra predator-prey models have been extensively and deeply investigated (see monographs [1–5]). If we let $x(t)$ denote the density of prey and let $y(t)$ be the density of predator, then the classical Lotka-Volterra predator-prey model is given by the following system:

$$\begin{aligned} \frac{dx}{dt} &= (r_1 - c_1 y - b_1 x)x, \\ \frac{dy}{dt} &= (-\varepsilon_2 + \rho_2 x)y. \end{aligned} \tag{1.1}$$

The equations in system (1.1) set no upper limit on the per-capita growth rate of the predator (the second term of model (1.1)) which is unrealistic. For example, for mammals, such a limit will be determined in part by physiological factors (length of the gestation period, the shortest interval between litters, the maximum average number of daughters per-litter, the age at which breeding first starts, and so on [6]). Leslie modeled the effect of such limitations via a predator-prey model where the "carrying capacity" of the predator's environment was assumed to be proportional to the number of prey. Hence, if $x(t)$ and $y(t)$ denote the prey and predator density, respectively, then Leslie's model is given by the following system:

$$\begin{aligned}\frac{dx}{dt} &= (r_1 - c_1 y - b_1 x)x,\\ \frac{dy}{dt} &= \left(r_2 - c_2\frac{y}{x}\right)y,\end{aligned} \tag{1.2}$$

where r_i, c_i, $i = 1,2$, and b_1 are positive constants. The first equation of system (1.2) is standard but the second is not because it contains the "so-called" Leslie-Gower term, namely, $c_2(y/x)$. The rationale behind this term is based on the view that as the prey becomes numerous ($x \to \infty$) then the per-capita growth rate of the predator achieves its maximum ($(1/y)(dy/dt) \to r_2$). Conversely as the prey becomes scarce $x \to 0$, we have that $(1/y)(dy/dt) \to -\infty$. That is, the predator must go extinct. Recently, the use of a Holling-type II functional for the prey has led various researchers [7, 8] to the consideration of the following model (a modification of system (1.2)):

$$\begin{aligned}\frac{dx}{dt} &= \left(r_1 - \frac{c_1 y}{x+k_1} - b_1 x\right)x,\\ \frac{dy}{dt} &= \left(r_2 - \frac{c_2 y}{x+k_2}\right)y,\end{aligned} \tag{1.3}$$

where r_1 is the per-capita growth rate of the prey x; b_1 is a measure of the strength of prey (on prey) interference competition; c_1 is the maximum value of the per-capita reduction rate of x due to y; k_1 measures the extent to which the environment provides protection to prey x (k_2 for y); r_2 gives the maximal per-capita growth rate of y; c_2 has a similar meaning to that of c_1.

In [9], the global stability of the unique coexisting interior equilibrium of system (1.2) is established. In [7], the existence and boundedness of solutions (including that of an attracting set) are established as well as the global stability of the coexisting interior equilibrium for model (1.3). There have been additional extensions, for example, in [10, 11] a Leslie-Gower type model with impulse was introduced and investigated.

The study of the role of dispersal in continuous-time metapopulation models is extensive (see [12–16] and the references cited therein). They show that dispersal can have a stabilizing influence on the system (see [12, 13]) and also can have a destabilizing influence on the system (see [14, 15]).

On the other hand, most prey species have a life history that includes multiple stages (juvenile and adults or immature and mature). In Aiello and Freedman [17], the population dynamics of a single species with two identifiable stages was modeled by the following system:

$$\begin{aligned}x_1'(t) &= \alpha x_2(t) - \gamma x_1(t) - \alpha e^{-\gamma\tau} x_2(t-\tau),\\ x_2'(t) &= \alpha e^{-\gamma\tau} x_2(t-\tau) - \beta x_2^2(t),\end{aligned} \tag{1.4}$$

where $x_1(t)$, $x_2(t)$ denote the immature and mature population densities, respectively. Here, $\alpha > 0$ represents the per-capita birth rate; $\gamma > 0$ is the per-capita immature death rate; $\beta > 0$ is the death rate due to overcrowding, and τ is the "fixed" time to maturity; the term $\alpha e^{-\gamma\tau}x_2(t-\tau)$ models the immature individuals who were born at time $t-\tau$ (i.e., $\alpha x_2(t-\tau)$) and survive and mature at time t. The derivation and analysis of system (1.4) can be found in [17]. More and More researchers (see [16–22] and the references cited therein) have investigated many kinds of predator-prey model under various stage-structure assumptions. In Xu et al. [16], they discussed a Lotka-Volterra-type predator-prey model with stage structure for predator and prey dispersal in two-patch environments. They obtained sufficient conditions of permanence and impermanence and global asymptotic stability of the positive equilibrium; they also discussed the local stability of the positive equilibrium. In [22], they studied a generalized version of the Leslie-Gower predator-prey model that incorporates the prey structure and obtained sufficient conditions of permanence and stability of the nonnegative equilibrium.

Motivated by the above works, in this paper we study the effects of stage structure for prey and predator dispersal on the global dynamics of modified version of the Leslie-Gower and Holling-type II predator-prey system. Following [16, 23], we assumethe following.

($A1$) *The prey population*: the prey only lives in patch 1. For immature prey, α is birth rate, r_1 is death rate, and the term $\alpha e^{-r_1\tau}x_2(t-\tau)$ represents the number of immature prey that was born at time $t-\tau$, which still survive at time t and are transferred from the immature stage to the mature stage at time t. For mature prey, r_2 is death rate, r_3 is the intraspecific competition rate of mature prey, a_1 is the maximum value of the per-capita reduction rate of x_2 due to y_1, and k_1 (resp., k_2) measures the extent to which environment provides protection to prey x_2 (resp., to the predator y_1).

($A2$) *The predator population*: β_i are the birth rate of predator in patch i, $i = 1,2$; D_i is the dispersion rate of predator between two patches; r_4 is death rate of predator in patch 2; a_2 has a similar meaning to a_1. It is assumed that predators in patch 1 do not capture immature prey, then we have the following delayed differential system:

$$\begin{aligned}
\dot{x}_1(t) &= \alpha x_2(t) - r_1 x_1(t) - \alpha e^{-r_1\tau}x_2(t-\tau),\\
\dot{x}_2(t) &= \alpha e^{-r_1\tau}x_2(t-\tau) - r_2 x_2(t) - r_3 x_2^2(t) - \frac{a_1 y_1(t) x_2(t)}{x_2(t)+k_1},\\
\dot{y}_1(t) &= \left(\beta_1 - \frac{a_2 y_1(t)}{x_2(t)+k_2}\right) y_1(t) + D_1\big(y_2(t) - y_1(t)\big),\\
\dot{y}_2(t) &= \big(\beta_2 - r_4 y_2(t)\big) y_2(t) + D_2\big(y_1(t) - y_2(t)\big),
\end{aligned} \tag{1.5}$$

where $x_1(t)$ and $x_2(t)$ represent the densities of immature and mature individual prey in patch 1 at time t, $y_i(t)$ denote the density of predator species in patch i, $i = 1,2$ at time t, all parameters of (1.5) are positive constants.

The initial conditions for system (1.5) take the form of

$$x_i(\theta) = \Phi_i(\theta), \quad y_i(\theta) = \Psi_i(\theta), \quad x_i(0) > 0, \quad y_i(0) > 0, \quad i = 1,2, \tag{1.6}$$

where $(\Phi_1(\theta), \Phi_2(\theta), \Psi_1(\theta), \Psi_2(\theta)) \in C([-\tau,0], R^4_{+0})$, the Banach space of continuous function mapping the interval $[-\tau,0]$ into R^4_{+0}, where $R^4_{+0} = \{(x_1,x_2,x_3,x_4) : x_i \geq 0,\ i = 1,2,3,4\}$.

For continuity of the initial conditions, we further require

$$x_1(0) = \int_{-\tau}^{0} \alpha e^{r_1 s} \Phi_2(s) ds. \tag{1.7}$$

The paper is organized as follows. In Section 2, we will discuss the uniform persistence of system (1.5). In Section 3, we are concerned with the global stability of a positive equilibrium of system (1.5) by constructing Lyapunov functional and also present two numerical simulations to illustrate our main results.

2. Uniform Persistence

In this section, we will discuss the uniform persistence of system (1.5) with initial conditions (1.6) and (1.7).

Definition 2.1. System (1.5) is said to be uniformly persistent if there exists a compact region $D \subset \operatorname{Int} R^4_{+0}$ such that every solution of system (1.5) with initial conditions (1.6) and (1.7) eventually enters and remains in the region D.

Lemma 2.2. *Solutions of system* (1.5) *with initial conditions* (1.6) *and* (1.7) *are positive for all* $t \geq 0$.

Proof. Let $(x_1(t), x_2(t), y_1(t), y_2(t))$ be a solution of system (1.5) with initial conditions (1.6) and (1.7); we first consider $y_1(t)$ and $y_2(t)$ for $t \in [0, \tau]$,

$$\begin{aligned} \dot{y}_1(t)\big|_{y_1=0} &= D_1 y_2(t) > 0 \quad \text{for } y_2 > 0, \\ \dot{y}_2(t)\big|_{y_2=0} &= D_2 y_1(t) > 0 \quad \text{for } y_1 > 0. \end{aligned} \tag{2.1}$$

Thus, it follows that $y_1(t) > 0$, $y_2(t) > 0$ for $t \in [0, \tau]$.

For $t \in [0, \tau]$, it follows from the second equation of system (1.5) that

$$\dot{x}_2(t) \geq \left[-r_2 - r_3 x_2(t) - \frac{a_1 y_1(t)}{x_2(t) + k_1} \right] x_2(t). \tag{2.2}$$

Consider the following auxiliary equation:

$$\begin{aligned} \dot{u}(t) &\geq \left[-r_2 - r_3 u(t) - \frac{a_1 y_1(t)}{u(t) + k_1} \right] u(t), \\ u(t) &= u(0) \exp\left(-\int_0^t \left(r_2 + r_3 u(s) + \frac{a_1 y_1(s)}{u(s) + k_1} \right) d_s \right) > 0. \end{aligned} \tag{2.3}$$

For $t \in [0, \tau]$, $u(0) = x_2(0) > 0$; thus, $x_2(t) \geq u(t) > 0$.

In a similar way, we consider the intervals $[\tau, 2\tau] \cdots [n\tau, (n+1)\tau]$, $n \in N$. Thus, $x_2(t) > 0$, $y_1(t) > 0$, $y_2(t) > 0$ for all $t \geq 0$.

By (1.7) and the first equation of (1.5) we can obtain that

$$x_1(t) = \alpha \int_{t-\tau}^{t} e^{-r_1(t-s)} x_2(s) d_s. \tag{2.4}$$

Therefore the positivity of $x_1(t)$ for $t \geq 0$ follows, this completes the proof. □

In order to discuss the uniform persistence, we need the following result from [24].

Lemma 2.3. *Consider the following equation:*

$$\dot{x}(t) = ax(t-\tau) - bx(t) - cx^2(t), \tag{2.5}$$

where a, b, c, and τ are positive constants, $x(t) > 0$ for $t \in [-\tau, 0]$. We have the following:

(i) *if $a > b$, then $\lim_{t\to+\infty} x(t) = (a-b)/c$;*

(ii) *if $a < b$, then $\lim_{t\to+\infty} x(t) = 0$.*

Lemma 2.4. *Let $(x_1(t), x_2(t), y_1(t), y_2(t))$ be a solution of system* (1.5) *with initial conditions* (1.6) *and* (1.7). *Then there exists a $T_3 > 0$ such that*

$$x_i(t) \leq N, \quad y_i(t) \leq N, \quad (i = 1, 2) \text{ for } t \geq T_3, \tag{2.6}$$

where N is a constant and

$$\begin{aligned}
&N > \max\{N_1, N_2, N^*\},\\
&N_1 = \frac{\alpha N_2}{r_1}\left(1 - e^{-r_1\tau}\right),\\
&N_2 = \frac{\alpha e^{-r_1\tau}}{r_3} + \varepsilon,\\
&N^* = \frac{\alpha^2}{4Ar_3} + \frac{\left(A + D_2 + \beta_1\right)^2 (N_2 + k_2)}{4Aa_2} + \frac{\left(A + D_1 + \beta_2\right)^2}{4Ar_1},\\
&A = \min\{r_1, r_2\}.
\end{aligned} \tag{2.7}$$

Proof. Suppose $X(t) = (x_1(t), x_2(t), y_1(t), y_2(t))$to be any positive solution of system (1.5) with initial conditions (1.6) and (1.7). It follows from the second equation of system (1.5) that for $t \geq \tau$,

$$\dot{x}_2(t) \leq \alpha e^{-r_1\tau} x_2(t-\tau) - r_3 x_2^2(t). \tag{2.8}$$

Consider the following auxiliary equation:

$$\dot{u}(t) = \alpha e^{-r_1\tau} u(t-\tau) - r_3 u^2(t). \tag{2.9}$$

By Lemma 2.3 we obtain that

$$\lim_{t\to+\infty} u(t) = \frac{\alpha e^{-r_1\tau}}{r_3}. \tag{2.10}$$

Using comparison principle, it follows that

$$\lim_{t\to+\infty} \sup x_2(t) \le \frac{\alpha e^{-r_1\tau}}{r_3}. \tag{2.11}$$

Therefore, for sufficiently small $\varepsilon > 0$ there is a $T_1 > \tau$ such that if $t \ge T_1$,

$$x_2(t) \le \frac{\alpha e^{-r_1\tau}}{r_3} + \varepsilon := N_2. \tag{2.12}$$

Setting $T_2 = T_1 + \tau$, it then follows (2.4) and (2.12) that, for $t \ge T_2$,

$$x_1(t) \le \frac{\alpha N_2}{r_1}\left(1 - e^{-r_1\tau}\right) := N_1. \tag{2.13}$$

We define

$$\begin{aligned} &\rho(t) = x_1(t) + x_2(t) + y_1(t) + y_2(t), \\ &\dot{\rho}(t) \le -A\rho(t) + \frac{\alpha^2}{4r_3} + \frac{\left(A + D_2 + \beta_1\right)^2 (N_2 + k_2)}{4a_2} + \frac{\left(A + D_1 + \beta_2\right)^2}{4r_1}, \end{aligned} \tag{2.14}$$

where $A = \min\{r_1, r_2\}$.

It follows from (2.14) that

$$\lim_{t\to+\infty} \sup \rho(t) \le \frac{\alpha^2}{4Ar_3} + \frac{\left(A + D_2 + \beta_1\right)^2 (N_2 + k_2)}{4Aa_2} + \frac{\left(A + D_1 + \beta_2\right)^2}{4Ar_1} := N^*. \tag{2.15}$$

Therefore, there exists a $T_3 = T_2 + \tau$ and

$$N > \max\{N_1, N_2, N^*\}. \tag{2.16}$$

Such that if $t \ge T_3,\ x_i(t) \le N,\ y_i(t) \le N\ (i = 1, 2)$. This completes the proof. □

Theorem 2.5. *System* (1.5) *with initial conditions* (1.6) *and* (1.7) *is uniformly persistent provided that*

$(H1)$ $\alpha e^{-r_1\tau} > r_2 + a_1 N/k_1$, *where* N *is defined by*(2.7).

Proof. Suppose $X(t) = (x_1(t), x_2(t), y_1(t), y_2(t))$to be any positive solution of system (1.5) with initial conditions (1.6) and (1.7). It follows from the second equation of system (1.5) that for $t \geq T_3 + \tau$,

$$\dot{x}_2(t) \geq \alpha e^{-r_1\tau} x_2(t-\tau) - \left(r_2 + \frac{a_1 N}{k_1}\right) x_2(t) - r_3 x_2^2(t). \tag{2.17}$$

Consider the following auxiliary equation:

$$\dot{u}(t) = \alpha e^{-r_1\tau} u(t-\tau) - \left(r_2 + \frac{a_1 N}{k_1}\right) u(t) - r_3 u^2(t). \tag{2.18}$$

By Lemma 2.3, we obtain that

$$\lim_{t\to+\infty} u(t) = \frac{\alpha e^{-r_1\tau} - r_2 - a_1 N/k_1}{r_3}. \tag{2.19}$$

According to comparison principle it follows that

$$\lim_{t\to+\infty} \inf x_2(t) \geq \frac{\alpha e^{-r_1\tau} - r_2 - a_1 N/k_1}{r_3}. \tag{2.20}$$

Therefore, for sufficiently small $\varepsilon > 0$ there is a $T_4 = T_3 + \tau$ such that if $t \geq T_4$,

$$x_2(t) \geq \frac{\alpha e^{-r_1\tau} - r_2 - a_1 N/k_1}{r_3} - \varepsilon := n_2. \tag{2.21}$$

By the third and forth equation of system (1.5), we have

$$\begin{aligned}
\dot{y}_1(t) &\geq \left(\beta_1 - \frac{a_2 y_1(t)}{n_2 + k_2}\right) y_1(t) + D_1\big(y_2(t) - y_1(t)\big), \\
\dot{y}_2(t) &= \big(\beta_2 - r_4 y_2(t)\big) y_2(t) + D_2\big(y_1(t) - y_2(t)\big), \quad t \geq T_4 + \tau.
\end{aligned} \tag{2.22}$$

Consider the following auxiliary equation:

$$\begin{aligned}
\dot{u}_1(t) &= \left(\beta_1 - \frac{a_2 u_1(t)}{n_2 + k_2}\right) u_1(t) + D_1(u_2(t) - u_1(t)), \\
\dot{u}_2(t) &= \big(\beta_2 - r_4 u_2(t)\big) u_2(t) + D_2(u_1(t) - u_2(t)).
\end{aligned} \tag{2.23}$$

Define

$$V_{11}(t) = \min\{u_1(t), u_2(t)\}. \tag{2.24}$$

Using a similar argument in the proof of [25, Lemma 2.1] we obtain

$$\lim_{t\to+\infty}\inf V_{11}(t) \geq \min\left\{\frac{\beta_1(n_2+k_2)}{a_2}, \frac{\beta_2}{r_4}\right\} := n_3^*. \tag{2.25}$$

Therefore, for sufficiently small $\varepsilon > 0$ there is a $T_5 = T_4 + \tau$ such that if $t \geq T_5$,

$$y_1(t) \geq n_3^* - \varepsilon := n_3, \qquad y_2(t) \geq n_3^* - \varepsilon := n_3. \tag{2.26}$$

Setting $T_6 = T_5 + \tau$, then by (2.4), we have

$$x_1(t) \geq \frac{\alpha n_2(1-e^{-r_1\tau})}{r_1} := n_1, \quad t \geq T_6. \tag{2.27}$$

This completes the proof. □

We now state a result on the extinction of the mature and immature prey.

Theorem 2.6. *The mature and immature prey population will go to extinction if* $(H2)$ *holds*

$(H2)$ $\alpha e^{-r_1\tau} < r_2$.

Remark 2.7. From the $(H2)$, we know that if the death rate of mature prey r_2 is more than the product of birth rate of immature prey α and the surviving probability of each immature prey becomes mature $e^{-r_1\tau}$, then the mature and immature prey population will go to extinction.

Proof. Suppose $X(t) = (x_1(t), x_2(t), y_1(t), y_2(t))$to be any positive solution of system (1.5) with initial conditions (1.6) and (1.7). It follows from the second equation of system (1.5) that there is a $T_{11} > 0$,

$$\dot{x}_2(t) \leq \alpha e^{-r_1\tau}x_2(t-\tau) - r_2x_2(t) - r_3x_2^2(t). \tag{2.28}$$

Consider the following auxiliary equation:

$$\dot{u}(t) = \alpha e^{-r_1\tau}u(t-\tau) - r_2u(t) - r_3u^2(t). \tag{2.29}$$

By Lemma 2.3, we derived from (2.29) and $(H2)$ that

$$\lim_{t\to+\infty} u(t) = 0. \tag{2.30}$$

A standard comparison argument shows that

$$\lim_{t\to+\infty} x_2(t) = 0. \tag{2.31}$$

Therefore, $\forall \varepsilon > 0$, there is a $T_7 > T_6$ such that if $t \geq T_7$, $0 < x_2(t) < r_1\varepsilon/2\alpha(1-e^{-r_1\tau})$. Thus, we derive from (2.4) that for $t \geq T_7 + \tau$,

$$x_1(t) \leq \alpha \int_{t-\tau}^{t} e^{-r_1(t-s)} \frac{r_1\varepsilon}{2\alpha(1-e^{-r_1\tau})} d_s < \varepsilon. \tag{2.32}$$

We therefore obtain that

$$\lim_{t\to+\infty} x_1(t) = 0. \tag{2.33}$$

This completes the proof. □

3. Global Stability

In this section, we study the global asymptotic stability of a positive equilibrium of system (1.5). By Theorem 2.5 we see that if $(H1)$ satisfies, system (1.5) is uniformly persistent, which implies that system (1.5) must have at least one positive equilibrium. So in the following we assume that a positive equilibrium exists and denote it by $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$.

Theorem 3.1. *Let $(H1)$ hold. Assume further that*

$(H3)$ $\overline{A}_i > 0$, $i = 1, 3$, *where*

$$\begin{aligned} \overline{A}_1 &= \frac{a_2}{N+k_2} + \frac{a_1x_2^*}{4(x_2^*+k_1)} - \frac{a_2y_1^*}{4(n_2+k_2)(x_2^*+k_2)}, \\ \overline{A}_3 &= r_3n_2 + \frac{a_1k_1n_3}{(N+k_1)(x_2^*+k_1)} + \frac{a_1(x_2^*-y_1^*)}{x_2^*+k_1} - \frac{a_2y_1^*}{(n_2+k_2)(x_2^*+k_2)}, \end{aligned} \tag{3.1}$$

where $n_2 = ((\alpha e^{-r_1\tau} - r_2) - a_1N/k_1)/r_3 - \varepsilon$, $n_3 = \min\{\beta_1(n_2+k_2)/a_2, \beta_2/r_4\} - \varepsilon$, $\varepsilon > 0$ is a sufficient small constant, and N is defined by (2.7).

Then the positive equilibrium $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$ of system (1.5) *is globally asymptotically stable.*

Remark 3.2. Theorem 3.1 shows that if the time delay due to maturity is sufficiently small, the positive equilibrium of system (1.5) is globally asymptotically stable.

Proof. We first consider the following subsystem:

$$\begin{aligned} \dot{x}_2(t) &= \alpha e^{-r_1\tau}x_2(t-\tau) - r_2x_2(t) - r_3x_2^2(t) - \frac{a_1y_1(t)x_2(t)}{x_2(t)+k_1}, \\ \dot{y}_1(t) &= \left(\beta_1 - \frac{a_2y_1(t)}{x_2(t)+k_2}\right)y_1(t) + D_1(y_2(t)-y_1(t)), \\ \dot{y}_2(t) &= (\beta_2 - r_4y_2(t))y_2(t) + D_2(y_1(t)-y_2(t)). \end{aligned} \tag{3.2}$$

Noting that $E^*(x_2^*, y_1^*, y_2^*)$ is a positive equilibrium of system (3.2), we can rewrite system (3.2) as

$$\begin{aligned}
\dot{x}_2(t) &= \alpha e^{-r_1\tau}\left(x_2(t-\tau)-x_2^*\right)-r_2\left(x_2(t)-x_2^*\right)-r_3\left(x_2(t)+x_2^*\right)\left(x_2(t)-x_2^*\right) \\
&\quad -\frac{a_1k_1y_1(t)\left(x_2(t)-x_2^*\right)}{\left(x_2(t)+k_1\right)\left(x_2^*+k_1\right)}-\frac{a_1x_2^*\left(y_1(t)-y_1^*\right)}{x_2^*+k_1}, \\
\dot{y}_1(t) &= \left(-\frac{a_2\left(y_1(t)-y_1^*\right)}{x_2(t)+k_2}+\frac{a_2\left(y_1(t)-y_1^*\right)}{\left(x_2(t)+k_2\right)\left(x_2^*+k_2\right)}\right)y_1(t) \\
&\quad -\frac{D_1}{y_1^*}y_2(t)\left(y_1(t)-y_1^*\right)+\frac{D_1}{y_1^*}y_1(t)\left(y_2(t)-y_2^*\right), \\
\dot{y}_2(t) &= \left(-r_4\left(y_2(t)-y_2^*\right)\right)y_2(t)-\frac{D_2}{y_2^*}y_1(t)\left(y_2(t)-y_2^*\right)+\frac{D_2}{y_2^*}y_2(t)\left(y_1(t)-y_1^*\right).
\end{aligned} \tag{3.3}$$

Define

$$V_1(t)=\sum_{i=1}^{2}c_i\left(y_i(t)-y_i^*-y_i^*\ln\frac{y_i(t)}{y_i^*}\right)+\frac{1}{2}c_3\left(x_2(t)-x_2^*\right)^2. \tag{3.4}$$

Calculating the derivative of $V_1(t)$ along solution of system (1.5), we have

$$\begin{aligned}
\frac{dV_1(t)}{d_t} &= \sum_{i=1}^{2}c_i\left(y_i(t)-y_i^*\right)\frac{\dot{y}_i(t)}{y_i(t)}+c_3\left(x_2(t)-x_2^*\right)\dot{x}_2(t) \\
&= -c_1\frac{a_2\left(y_1(t)-y_1^*\right)^2}{x_2(t)+k_2}+c_1\frac{a_2y_1^*\left(x_2(t)-x_2^*\right)\left(y_1(t)-y_1^*\right)}{\left(x_2(t)+k_2\right)\left(x_2^*+k_2\right)}-\frac{c_1D_1}{y_1^*y_1(t)}y_2(t)\left(y_1(t)-y_1^*\right)^2 \\
&\quad +\frac{c_1D_1}{y_1^*}\left(y_1(t)-y_1^*\right)\left(y_2(t)-y_2^*\right)-c_2r_4\left(y_2(t)-y_2^*\right)^2-\frac{c_2D_2}{y_2^*y_2(t)}y_1(t)\left(y_2(t)-y_2^*\right)^2 \\
&\quad +\frac{c_2D_2}{y_2^*}\left(y_1(t)-y_1^*\right)\left(y_2(t)-y_2^*\right)+c_3\alpha e^{-r_1\tau}\left(x_2(t-\tau)-x_2^*\right)\left(x_2(t)-x_2^*\right) \\
&\quad -c_3r_2\left(x_2(t)-x_2^*\right)^2-c_3r_3\left(x_2(t)+x_2^*\right)\left(x_2(t)-x_2^*\right)^2 \\
&\quad -\frac{c_3a_1k_1y_1(t)\left(x_2(t)-x_2^*\right)^2}{\left(x_2(t)+k_1\right)\left(x_2^*+k_1\right)}-\frac{c_3a_1x_2^*\left(x_2(t)-x_2^*\right)\left(y_1(t)-y_1^*\right)}{x_2^*+k_1}.
\end{aligned} \tag{3.5}$$

Setting $c_1 = 1$, $c_2 = D_1 y_2^*/D_2 y_1^*$. By (3.5) we obtain

$$\begin{aligned}
\frac{dV_1(t)}{d_t} = &-\frac{a_2\left(y_1(t)-y_1^*\right)^2}{x_2(t)+k_2} - \frac{D_1 y_2^*}{D_2 y_1^*} r_4\left(y_2(t)-y_2^*\right)^2 \\
&- \frac{D_1}{y_1^*}\left[\sqrt{\frac{y_2(t)}{y_1(t)}}\left(y_1(t)-y_1^*\right) - \sqrt{\frac{y_1(t)}{y_2(t)}}\left(y_2(t)-y_2^*\right)\right]^2 \\
&+ \frac{a_2 y_1^*\left(x_2(t)-x_2^*\right)\left(y_1(t)-y_1^*\right)}{\left(x_2(t)+k_2\right)\left(x_2^*+k_2\right)} - \frac{c_3 a_1 x_2^*\left(x_2(t)-x_2^*\right)\left(y_1(t)-y_1^*\right)}{x_2^*+k_1} \\
&+ c_3 \alpha e^{-r_1\tau}\left(x_2(t-\tau)-x_2^*\right)\left(x_2(t)-x_2^*\right) - c_3 r_2\left(x_2(t)-x_2^*\right)^2 \\
&- c_3 r_3\left(x_2(t)+x_2^*\right)\left(x_2(t)-x_2^*\right)^2 - \frac{c_3 a_1 k_1 y_1(t)\left(x_2(t)-x_2^*\right)^2}{\left(x_2(t)+k_1\right)\left(x_2^*+k_1\right)}.
\end{aligned} \tag{3.6}$$

Using the inequality $ab \le (1/2)ka^2 + (1/2k)b^2$, it follows from (3.6) that

$$\begin{aligned}
\frac{dV_1(t)}{d_t} \le &-\frac{a_2}{x_2(t)+k_2}\left(y_1(t)-y_1^*\right)^2 - \frac{D_1 y_2^*}{D_2 y_1^*} r_4\left(y_2(t)-y_2^*\right)^2 \\
&+ \left(\frac{a_2 y_1^*}{\left(x_2(t)+k_2\right)\left(x_2^*+k_2\right)} - \frac{c_3 a_1 x_2^*}{x_2^*+k_1}\right)\left(\frac{A\left(x_2(t)-x_2^*\right)^2}{2} + \frac{\left(y_1(t)-y_1^*\right)^2}{2A}\right) \\
&+ c_3 \alpha e^{-r_1\tau}\left(\frac{B\left(x_2(t)-x_2^*\right)^2}{2} + \frac{\left(x_2(t-\tau)-x_2^*\right)^2}{2B}\right) - c_3 r_2\left(x_2(t)-x_2^*\right)^2 \\
&- c_3 r_3\left(x_2(t)+x_2^*\right)\left(x_2(t)-x_2^*\right)^2 - \frac{c_3 a_1 k_1 y_1(t)}{\left(x_2(t)+k_1\right)\left(x_2^*+k_1\right)}\left(x_2(t)-x_2^*\right)^2,
\end{aligned} \tag{3.7}$$

where parameters A, B are positive constants to be determined.

Define

$$V(t) = V_1(t) + \frac{1}{2B} c_3 \alpha e^{-r_1\tau}\int_{t-\tau}^{t}\left(x_2(s)-x_2^*\right)^2 d_s. \tag{3.8}$$

Setting $A = 2$, $B = 1$, $c_3 = 1$, then it follows from (3.7) and (3.8) that

$$\begin{aligned}
\frac{dV_1(t)}{d_t} \le& -\left\{\frac{a_2}{x_2(t)+k_2}+\frac{a_1x_2^*}{4(x_2^*+k_1)}-\frac{a_2y_1^*}{4(x_2(t)+k_2)(x_2^*+k_2)}\right\}(y_1(t)-y_1^*)^2-\frac{D_1y_2^*}{D_2y_1^*}r_4(y_2(t)-y_2^*)^2\\
&-\left\{r_3x_2(t)+\frac{a_1k_1y_1(t)}{(x_2(t)+k_1)(x_2^*+k_1)}+\frac{a_1(x_2^*-y_1^*)}{x_2^*+k_1}-\frac{a_2y_1^*}{(x_2(t)+k_2)(x_2^*+k_2)}\right\}(x_2(t)-x_2^*)^2\\
\le& -\left\{\frac{a_2}{N+k_2}+\frac{a_1x_2^*}{4(x_2^*+k_1)}-\frac{a_2y_1^*}{4(n_2+k_2)(x_2^*+k_2)}\right\}(y_1(t)-y_1^*)^2-\frac{D_1y_2^*}{D_2y_1^*}r_4(y_2(t)-y_2^*)^2\\
&-\left\{r_3n_2+\frac{a_1k_1n_3}{(N+k_1)(x_2^*+k_1)}+\frac{a_1(x_2^*-y_1^*)}{x_2^*+k_1}-\frac{a_2y_1^*}{(n_2+k_2)(x_2^*+k_2)}\right\}(x_2(t)-x_2^*)^2\\
:=& -\overline{A}_1(y_1(t)-y_1^*)^2-\overline{A}_2(y_2(t)-y_2^*)^2-\overline{A}_3(x_2(t)-x_2^*)^2,
\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}
\overline{A}_1 &= \frac{a_2}{N+k_2}+\frac{a_1x_2^*}{4(x_2^*+k_1)}-\frac{a_2y_1^*}{4(n_2+k_2)(x_2^*+k_2)},\\
\overline{A}_2 &= \frac{D_1y_2^*}{D_2y_1^*}r_4,\\
\overline{A}_3 &= r_3n_2+\frac{a_1k_1n_3}{(N+k_1)(x_2^*+k_1)}+\frac{a_1(x_2^*-y_1^*)}{x_2^*+k_1}-\frac{a_2y_1^*}{(n_2+k_2)(x_2^*+k_2)}.
\end{aligned} \tag{3.10}$$

N, n_2, and n_3 are defined in (2.16), (2.21), and (2.26), respectively.

If $(H1)$ and $(H3)$ hold and $\varepsilon > 0$ is sufficiently small, we have $\overline{A}_i > 0$, $i = 1,3$. In view of Lyapunov theorem [26], we conclude that the positive equilibrium $E^*(x_2^*, y_1^*, y_2^*)$ of system (3.2) is globally asymptotically stable. Thus, we have

$$\lim_{t\to+\infty} x_2(t) = x_2^*, \qquad \lim_{t\to+\infty} y_1(t) = y_1^*, \qquad \lim_{t\to+\infty} y_2(t) = y_2^*. \tag{3.11}$$

Using L'Hospital's rule, it follows from (2.4) and (3.11) that

$$\begin{aligned}
\lim_{t\to+\infty} x_1(t) &= \lim_{t\to+\infty} \alpha\int_{t-\tau}^{t} e^{-r_1(t-s)}x_2(s)d_s\\
&= \lim_{t\to+\infty}\frac{\alpha}{r_1}x_2(t)-e^{-r_1\tau}x_2(t-\tau)\\
&= \frac{\alpha x_2^*}{r_1}\left(1-e^{-r_1\tau}\right) = x_1^*.
\end{aligned} \tag{3.12}$$

This completes the proof. □

It is interesting to discuss the local stability of the positive equilibrium $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$ of system (1.5).

The characteristic equation of the positive equilibrium E^* of system (1.5) is of the form

$$(\lambda + r_1)\left[P(\lambda) + Q(\lambda)e^{-\lambda\tau}\right] = 0, \tag{3.13}$$

where

$$\begin{gathered} P(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0, \\ Q(\lambda) = b_2\lambda^2 + b_1\lambda + b_0, \end{gathered} \tag{3.14}$$

here

$$\begin{aligned}
a_0 &= \left(r_2 + 2r_3 + \frac{a_1k_1y_1^*}{(x_2^*+k_1)^2}\right)\left[\left(\frac{a_2y_1^*}{x_2^*+k_2} + D_1\frac{y_2^*}{y_1^*}\right)\left(r_4y_2^* + D_2\frac{y_1^*}{y_2^*}\right) - D_1D_2\right] \\
&\quad + \frac{a_1a_2x_2^*y_1^{*2}}{(x_2^*+k_1)(x_2^*+k_2)^2}\left(r_4y_2^* + D_2\frac{y_1^*}{y_2^*}\right), \\
a_1 &= \left(r_2 + 2r_3 + \frac{a_1k_1y_1^*}{(x_2^*+k_1)^2}\right)\left(\frac{a_2y_1^*}{x_2^*+k_2} + D_1\frac{y_2^*}{y_1^*} + r_4y_2^* + D_2\frac{y_1^*}{y_2^*}\right) \\
&\quad + \left[\left(\frac{a_2y_1^*}{x_2^*+k_2} + D_1\frac{y_2^*}{y_1^*}\right)\left(r_4y_2^* + D_2\frac{y_1^*}{y_2^*}\right) - D_1D_2\right] + \frac{a_1a_2x_2^*y_1^{*2}}{(x_2^*+k_1)(x_2^*+k_2)^2}, \\
a_2 &= r_2 + 2r_3 + \frac{a_1k_1y_1^*}{(x_2^*+k_1)^2} + \frac{a_2y_1^*}{x_2^*+k_2} + D_1\frac{y_2^*}{y_1^*} + r_4y_2^* + D_2\frac{y_1^*}{y_2^*}, \\
b_0 &= -\alpha e^{-r_1\tau}\left[\left(\frac{a_2y_1^*}{x_2^*+k_2} + D_1\frac{y_2^*}{y_1^*}\right)\left(r_4y_2^* + D_2\frac{y_1^*}{y_2^*}\right) - D_1D_2\right], \\
b_1 &= -\alpha e^{-r_1\tau}\left(\frac{a_2y_1^*}{x_2^*+k_2} + D_1\frac{y_2^*}{y_1^*} + r_4y_2^* + D_2\frac{y_1^*}{y_2^*}\right), \\
b_2 &= -\alpha e^{-r_1\tau}.
\end{aligned} \tag{3.15}$$

Clearly, $\lambda = -r_1$ is a negative eigenvalue. If $r_3x_2^* - a_1x_2^*y_1^*/(x_2^*+k_1)^2 > 0$, which implies that $a_i + b_i > 0$ $(i = 1,2,3)$, and $(a_1+b_1)(a_2+b_2) - (a_0+b_0) > 0$, then by Routh-Hurwitz Theorem the positive equilibrium E^* of system (1.5) is locally asymptotically stable when $\tau = 0$.

Let

$$F(y) = |P(iy)|^2 - |Q(iy)|^2 = y^6 + ly^4 + my^2 + n = 0, \tag{3.16}$$

where

$$\begin{aligned} l &= a_2^2 - 2a_1 - b_2^2, \\ m &= a_1^2 - 2a_0a_2 + 2b_0b_2 - b_1^2, \\ n &= a_0^2 - b_0^2. \end{aligned} \tag{3.17}$$

Let $z = y^2$, and then (3.16) becomes

$$z^3 + lz^2 + mz + nz = 0. \tag{3.18}$$

By applying the results on the distribution of roots of (3.16) and (3.18) in [27] and [26, Theorem 4.1, page 83], we therefore derive the following results on the stability of the positive equilibrium E^*.

Theorem 3.3. *Suppose that system* (1.5) *admits a positive equilibrium* $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$ *and* $r_3x_2^* - a_1x_2^*y_1^*/(x_2^* + k_1)^2 > 0$.

(1) *If* $\Delta = l^2 - 3m \leq 0$, *then the positive equilibrium* E^* *of system* (1.5) *is locally asymptotically stable.*

(2) *If* $\Delta = l^2 - 3m > 0$ *and* $h(z_1^*) \leq 0$, *then there exists a positive number* τ_0 *such that the positive equilibrium* E^* *of system* (1.5) *is locally asymptotically stable if* $0 < \tau < \tau_0$ *and is locally unstable if* $\tau > \tau_0$; *further, as* τ *increases through* τ_0, E^* *bifurcates into small amplitude periodic solutions, here,* $z_1^* = (-l + \sqrt{\Delta})/3$, $h(z) = z^3 + lz^2 + mz + nz$.

4. Two Examples

In this section, we give two examples to illustrate our main results.

Example 4.1. Consider the following system:

$$\begin{aligned} \dot{x}_1(t) &= 5x_2(t) - x_1(t) - 5e^{-\tau}x_2(t-\tau), \\ \dot{x}_2(t) &= 5e^{-\tau}x_2(t-\tau) - 1.5x_2(t) - 3x_2^2(t) - \frac{0.8y_1(t)x_2(t)}{x_2(t)+8}, \\ \dot{y}_1(t) &= \left(0.2 - \frac{1.5y_1(t)}{x_2(t)+1.5}\right)y_1(t) + 0.5\big(y_2(t) - y_1(t)\big), \\ \dot{y}_2(t) &= \big(1.5 - y_2(t)\big)y_2(t) + 0.5\big(y_1(t) - y_2(t)\big), \end{aligned} \tag{4.1}$$

where the parameter τ is a positive constant.

System (4.1) has a unique positive equilibrium $E^*(0.9589, 0.4874, 0.7466, 1.2895)$. It is easy to show that if $\tau < 0.8973$, then $(H1)$ and $(H3)$ hold for system (4.1). By Theorem 2.5 we see that system (4.1) is uniformly persistent when $\tau < 0.8973$. By Theorem 3.1 we see that the positive equilibrium of system (4.1) is globally asymptotically stable when $\tau = 0.5$. Numerical

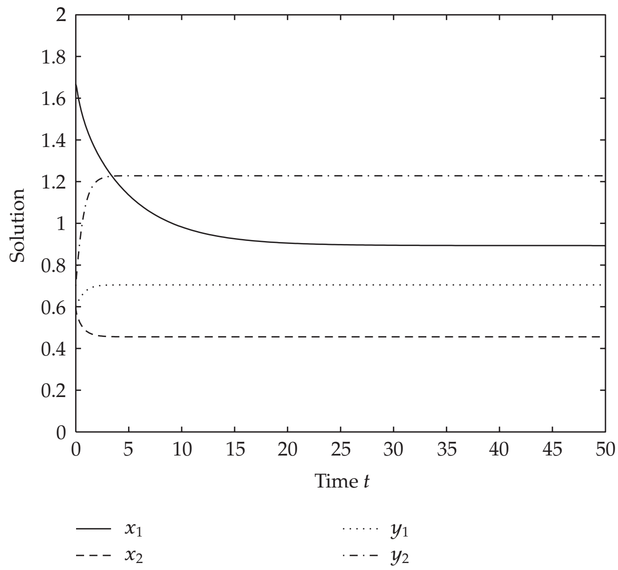

Figure 1: The temporal solution found by numerical integration of system (4.1) with $\tau = 0.5$ and $(\Phi_1(\theta), \Phi_2(\theta), \Psi_1(\theta), \Psi_2(\theta)) = (5(1 - e^{-0.5}), 1, 0.6, 0.6)$.

integration can be carried out using standard MATLAB algorithm. Numerical simulation also confirms the fact (see Figure 1).

Example 4.2. Consider the following system:

$$\begin{aligned}
\dot{x}_1(t) &= 5x_2(t) - x_1(t) - 5e^{-1}x_2(t-1), \\
\dot{x}_2(t) &= 5e^{-1}x_2(t-1) - 2x_2(t) - 3x_2^2(t) - \frac{2y_1(t)x_2(t)}{x_2(t)+8}, \\
\dot{y}_1(t) &= \left(1 - \frac{2y_1(t)}{x_2(t)+2}\right)y_1(t) + y_2(t) - y_1(t), \\
\dot{y}_2(t) &= (1 - y_2(t))y_2(t) + y_1(t) - y_2(t).
\end{aligned} \tag{4.2}$$

System (4.2) has a unique boundary equilibrium $E^*(0,0,1,1)$. It is easy to show that $(H2)$holds for system (4.2). By Theorem 2.6 we see that mature and immature prey population goes to extinction. Numerical integration can be carried out using standard MATLAB algorithm. Numerical simulation also confirms the fact (see Figure 2).

5. Discussion

In this paper, we discussed a generalized Leslie-Gower-type predator-prey model with stage structure for prey and predator dispersal in two-patch environments. By using comparison arguments we established sufficient conditions for system (1.5) to be permanent. By constructing Lyapunov functionals, sufficient conditions are derived for the global

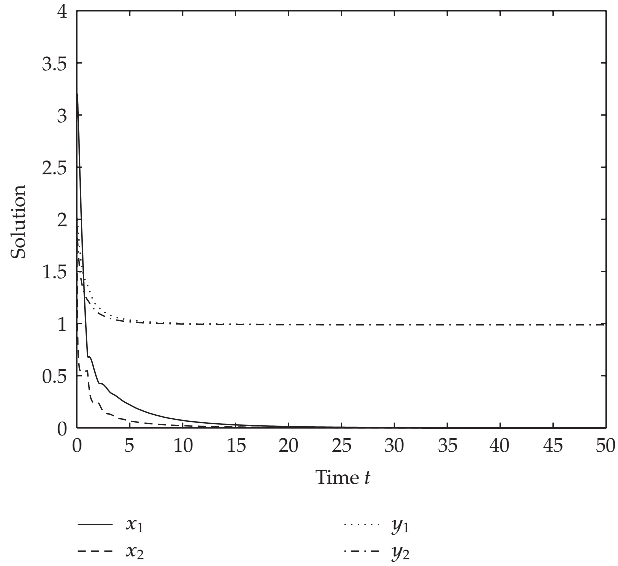

Figure 2: The temporal solution found by numerical integration of system (4.2) with $(\Phi_1(\theta), \Phi_2(\theta), \Psi_1(\theta), \Psi_2(\theta)) = (5(1 - e^{-1}), 2, 2, 2)$.

asymptotic stability of the positive equilibrium of system (1.5). By Theorem 3.1 we see that if the birth rate of immature prey and the extent to which environment provides protection to mature prey and predator in patch 1, respectively, are high and the maximum value of the per-capita reduction rate of mature prey due to predator in patch 1 is low satisfying $(H1)$ and $(H3)$, the positive equilibrium of system (1.5) is globally asymptotically stable. By Theorem 2.6 we see that if the death rate of mature prey is more than the transformation rate of immatures to matures satisfying $(H2)$, the immature and mature prey population will go to extinction.

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