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Research Article

On Homogeneous Production Functions with Proportional Marginal Rate of Substitution

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We completely classify homogeneous production functions with proportional marginal rate of substitution and with constant elasticity of labor and capital, respectively. These classifications generalize some recent results of C. A. Ioan and G. Ioan (2011) concerning the sum production function.

1. Introduction

It is well known that the production function is one of the key concepts of mainstream neoclassical theories, with a lot of applications not only in microeconomics and macroeconomics but also in various fields, like biology [1, 2], educational management [3, 4], and engineering [5–8]. Roughly speaking, the production functions are the mathematical formalization of the relationship between the output of a firm/industry/economy and the inputs that have been used in obtaining it. In fact, a production function is a map f of class C^∞, $f : \mathbb{R}^n_+ \to \mathbb{R}_+$, $f = f(x_1, x_2, \ldots, x_n)$, where f is the quantity of output, n is the number of the inputs, and $x_1, x_2, \ldots, x_n$ are the factor inputs (such as labor, capital, land, and raw materials). In order for these functions to model as well the economic reality, they are required to be homogeneous; that is, there exists a real number p such that $Q(\lambda \cdot x) = \lambda^p Q(x)$, for all $x \in \mathbb{R}^n_+$ and $\lambda \in \mathbb{R}_+$, that means if the inputs are multiplied by same factor, then the output is multiplied by some power of this factor. If $\lambda = 1$, then the function is said to have a constant return to scale, if $\lambda > 1$, then we have an increased return to scale, and if $\lambda < 1$, then we say that the function has a decreased return to scale. Among the family of production functions, the most famous is the Cobb-Douglas (CD) production function, introduced in 1928 by Cobb and Douglas [9], in order to describe the distribution of the national income of the USA. In its most standard form for production of a single good with two factors, the Cobb-Douglas production function is given by

$$f = CK^\alpha L^\beta, \tag{1}$$

where f is the total production, K is the capital input, L is the labor input, and C is a positive constant which signifies the total factor productivity. We note that, in the original definition of Cobb and Douglas, we have $\alpha + \beta = 1$, so the production function had a constant return to scale, but this condition has been later relaxed, and CD production functions were generalized (see [10, 11]). Some very interesting information about other production functions of great interest in economic analysis, like Leontief, Lu-Fletcher, Liu-Hildebrand, Kadiyala, Arrow-Chenery-Minhas-Solow (ACMS), constant elasticity of substitution (CES), and variable elasticity of substitution (VES) production functions, can be found in [12]. Recently, C. A. Ioan and G. Ioan [13] introduced a new class of production functions, called sum production function, as a two-factor production function defined by

$$f(K,L)=\sum_{i=1}^{n}\alpha_i\left(c_{i1}K^{p_{i1}+p_{i2}}+c_{i2}K^{p_{i1}}L^{p_{i2}}+c_{i3}L^{p_{i1}+p_{i2}}\right)^{p_{i3}}, \tag{2}$$

where $n\geq 1$, $\alpha_i\geq 0$, $p_{i3}\in(-\infty,0)\cup[1,\infty)$, $p_{i1}p_{i2}>0$, $p_{i3}(p_{i1}+p_{i2})=1$, $\sum_{i=1}^{n}(c_{i2}+c_{i1}c_{i3})>0$, and $c_{ij}\geq 0$, for all $i\in\{1,2,\ldots,n\}$, for all $j\in\{1,2,3\}$.

It is easy to see that this production function is homogeneous of degree 1 and integrates in an unitary expression various production functions, including CD, CES, and VES. In [13], C. A. Ioan and G. Ioan compute the principal indicators of the sum production function and prove three theorems of characterization for the functions with a proportional marginal rate of substitution, with constant elasticity of labor and for those with constant elasticity of substitution, as follows.

Theorem 1 (see [13]). *The sum production function has a proportional marginal rate of substitution if and only if it reduces to the Cobb-Douglas function.*

Theorem 2 (see [13]). *The sum production function has a constant elasticity of labor if and only if it reduces to the Cobb-Douglas function.*

Theorem 3 (see [13]). *If $n=1$, then the sum production function has constant elasticity of substitution if and only if it reduces to the Cobb-Douglas or CES function.*

We recall that, for a production function f with two factors (K-capital and L-labor), the marginal rate of substitution (between capital and labor) is given by

$$\text{MRS}=\frac{\partial f/\partial L}{\partial f/\partial K}, \tag{3}$$

where the elasticities of L and K are defined as

$$E_L=\frac{\partial f/\partial L}{f/L},\qquad E_K=\frac{\partial f/\partial K}{f/K}, \tag{4}$$

while the elasticity of substitution is given by

$$\begin{aligned}\sigma=&\left((1/(K(\partial f/\partial K)))+(1/(L(\partial f/\partial L)))\right)\\&\times\Big(-\left(\left(\partial^2 f/\partial K^2\right)/(\partial f/\partial K)^2\right)\\&\quad+\left(\left(2\left(\partial^2 f/\partial K\partial L\right)\right)/((\partial f/\partial K)(\partial f/\partial L))\right)\\&\quad-\left(\left(\partial^2 f/\partial L^2\right)/(\partial f/\partial L)^2\right)\Big)^{-1}.\end{aligned} \tag{5}$$

It is easy to verify that, in the case of constant return to scale, Euler's theorem implies the following more simple expression for the elasticity of substitution:

$$\sigma=\frac{(\partial f/\partial L)(\partial f/\partial K)}{f\left(\partial^2 f/\partial K\partial L\right)}. \tag{6}$$

We note that it was proved by Losonczi [14] that twice differentiable two-input homogeneous production functions with constant elasticity of substitution (CES) property are Cobb-Douglas and ACMS production functions, which is obviously a more general result than Theorem 3. This result was recently generalized by Chen for an arbitrary number of inputs [15]. In the next section, we prove the following result which is a generalization of Theorems 1 and 2.

Theorem 4. *Let f be a twice differentiable, homogeneous of degree r, nonconstant, real valued production function with two inputs (K-capital and L-labor). Then, one has the following.*

(i) *f has a constant elasticity of labor k if and only if it is a Cobb-Douglas production function given by*

$$f(K,L)=CK^{r-k}L^{k}, \tag{7}$$

where C is a positive constant.

(ii) *f has a constant elasticity of capital k if and only if it is a Cobb-Douglas production function given by*

$$f(K,L)=CK^{k}L^{r-k}, \tag{8}$$

where C is a positive constant.

(iii) *f satisfies the proportional rate of substitution property between capital and labor (i.e., MRS $=k(K/L)$, where k is a positive constant) if and only if it is a Cobb-Douglas production function given by*

$$f(K,L)=CK^{r/(k+1)}L^{rk/(k+1)}, \tag{9}$$

where C is a positive constant.

In the last section of the paper, we generalize the above theorem for an arbitrary number of inputs $n\geq 3$. We note that other classification results concerning production functions were proved recently in [16–20].

2. Proof of Theorem 4

Proof. Consider the following.

(i) We first suppose that f has a constant elasticity of labor k. Then, we have

$$\frac{\partial f}{\partial L}=k\frac{f}{L}. \tag{10}$$

But with f being homogeneous of degree r, it follows that it can be written in the form

$$f(K,L)=K^{r}h(u) \tag{11}$$

or

$$f(K,L)=L^{r}h(u), \tag{12}$$

where $u = L/K$ (with $K \neq 0$), respectively, $u = K/L$ (with $L \neq 0$), and h is a real valued function of u, of class C^2 on its domain of definition. We can suppose, without loss of generality, that the first situation occurs, so $f(K, L) = K^r h(u)$, with $u = L/K$. Then, we have

$$\frac{\partial f}{\partial L} = K^{r-1} h'(u). \tag{13}$$

From (10) and (13), we obtain

$$K^{r-1} h'(u) = k \frac{K^r h(u)}{L}, \tag{14}$$

and therefore we deduce that the constant elasticity of labor property implies the following differential equation:

$$h'(u) = k \frac{h(u)}{u}. \tag{15}$$

Solving the above separable differential equation, we obtain

$$h(u) = C u^k, \tag{16}$$

where C is a positive constant. Finally, from (11) and (16), we derive that f is a Cobb-Douglas production function given by

$$f(K, L) = C \cdot K^{r-k} L^k. \tag{17}$$

The converse is easy to verify.

(ii) The proof follows similarly as in (i).

(iii) Since the production function satisfies the proportional rate of substitution property, it follows that

$$\frac{\partial f}{\partial L} = k \frac{K}{L} \frac{\partial f}{\partial K}. \tag{18}$$

On the other hand, from Euler's homogeneous function theorem, we have

$$K \frac{\partial f}{\partial K} + L \frac{\partial f}{\partial L} = r f(K, L). \tag{19}$$

Combining now (18) and (19), we obtain

$$\frac{\partial f}{\partial K} = \frac{r}{k+1} \frac{f}{K}. \tag{20}$$

From (20), we deduce that

$$f(K, L) = C K^{r/(k+1)} u(L), \tag{21}$$

where C is a real constant. But with f being a homogeneous function of degree r, it follows from (21) that

$$u(L) = L^{rk/(k+1)}. \tag{22}$$

Therefore, from (21) and (22), we derive that

$$f(K, L) = C K^{r/(k+1)} L^{rk/(k+1)}, \tag{23}$$

where C is a real constant. Finally, since f is a nonconstant production function, it follows that $f > 0$, and therefore we deduce that C is in fact a positive constant. So, f is a Cobb-Douglas production function.

The converse is easy to check, and the proof is now complete. □

3. Generalization to an Arbitrary Number of Inputs

Let f be a homogeneous production function with n inputs $x_1, x_2, \ldots, x_n$, $n > 2$. Then, the elasticity of production with respect to a certain factor of production x_i is defined as

$$E_{x_i} = \frac{\partial f / \partial x_i}{f / x_i}, \tag{24}$$

while the marginal rate of technical substitution of input j for input i is given by

$$\mathrm{MRS}_{ij} = \frac{\partial f / \partial x_j}{\partial f / \partial x_i}. \tag{25}$$

A production function is said to satisfy the proportional marginal rate of substitution property if and only if $\mathrm{MRS}_{ij} = x_i / x_j$, for all $1 \leq i \neq j \leq n$. Now, we are able to prove the following result, which generalizes Theorem 4 for an arbitrary number of inputs.

Theorem 5. *Let f be a twice differentiable, homogeneous of degree r, nonconstant, real valued function of n variables $(x_1, x_2, \ldots, x_n)$ defined on $D = \mathbb{R}^n_+$, where $n > 2$. Then, one has the following.*

(i) The elasticity of production is a constant k_i with respect to a certain factor of production x_i if and only if

$$f(x_1, x_2, \ldots, x_n) = x_i^{k_i} x_j^{r-k_i} F(u_1, \ldots, u_{n-2}), \tag{26}$$

where j is any element settled from the set $\{1, \ldots, n\} \setminus \{i\}$ and F is a twice differentiable real valued function of $n-2$ variables

$$\{u_1, \ldots, u_{n-2}\} = \left\{ \frac{x_k}{x_j} \mid k \in \{1, \ldots, n\} \setminus \{i, j\} \right\}. \tag{27}$$

(ii) *The elasticity of production is a constant k_i with respect to all factors of production x_i, $i \in \{1, 2, \ldots, n\}$, if and only if*

$$k_1 + k_2 + \cdots + k_n = r, \tag{28}$$

and f reduces to the Cobb-Douglas production function given by

$$f(x_1, x_2, \ldots, x_n) = Cx_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}, \quad (29)$$

where C is a positive constant.

(iii) *The production function satisfies the proportional marginal rate of substitution property if and only if it reduces to the Cobb-Douglas production function given by*

$$f(x_1, x_2, \ldots, x_n) = Cx_1^{r/n} x_2^{r/n} \cdots x_n^{r/n}, \quad (30)$$

where C is a positive constant.

Proof. Consider the following.

(i) The if part of the statement is easy to verify. Next, we prove the only if part. Since the elasticity of production with respect to a certain factor of production x_i is a constant k_i, we have

$$\frac{\partial f}{\partial x_i} = k_i \frac{f}{x_i}. \quad (31)$$

On the other hand, since f is a homogeneous of degree r, it follows that it can be expressed in the form

$$f(x_1, \ldots, x_n) = x_j^r h(u_1, \ldots, u_{n-1}), \quad (32)$$

where j can be settled in the set $\{1, \ldots, n\}$ and

$$u_k = \begin{cases} \dfrac{x_k}{x_j}, & 1 \leq k \leq j-1, \\ \dfrac{x_{k+1}}{x_j}, & j \leq k \leq n-1. \end{cases} \quad (33)$$

If we settle j such that $j \neq i$, then we derive from (32)

$$\frac{\partial f}{\partial x_i} = \begin{cases} x_j^{r-1} \dfrac{\partial h}{\partial u_i}, & \text{if } i < j, \\ x_j^{r-1} \dfrac{\partial h}{\partial u_{i-1}}, & \text{if } i > j. \end{cases} \quad (34)$$

Replacing now (34) in (31), we obtain

$$k_i h = \begin{cases} u_i \dfrac{\partial h}{\partial u_i}, & \text{if } i < j, \\ u_{i-1} \dfrac{\partial h}{\partial u_{i-1}}, & \text{if } i > j, \end{cases} \quad (35)$$

and solving the partial differential equations in (35), we derive

$$h(u_1, \ldots, u_{n-1}) = \begin{cases} Cu_i^{k_i} F(u_1, \ldots, \widehat{u_i}, \ldots, u_{n-1}), & \text{if } i < j, \\ Cu_{i-1}^{k_i} F(u_1, \ldots, \widehat{u_{i-1}}, \ldots, u_{n-1}), & \text{if } i > j, \end{cases} \quad (36)$$

where C is a positive constant, F is a twice differentiable real valued function of $n-2$ variables and the symbol "$\hat{\ }$" means that the corresponding term is omitted.

The conclusion follows now easily from (32) and (36), taking into account (33).

(ii) This assertion follows immediately from (i).

(iii) It is easy to show that if f is a Cobb-Douglas production function given by

$$f(x_1, x_2, \ldots, x_n) = Cx_1^{r/n} x_2^{r/n} \cdots x_n^{r/n}, \quad (37)$$

then f satisfies the proportional marginal rate of substitution property. We prove now the converse. Since f satisfies the proportional marginal rate of substitution property, it follows that

$$x_1 \frac{\partial f}{\partial x_1} = x_2 \frac{\partial f}{\partial x_2} = \cdots = x_n \frac{\partial f}{\partial x_n}. \quad (38)$$

On the other hand, since f is a homogeneous of degree r, the Euler homogeneous function theorem implies that

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \cdots + x_n \frac{\partial f}{\partial x_n} = rf. \quad (39)$$

From (38) and (39), we obtain

$$x_i \frac{\partial f}{\partial x_i} = \frac{r}{n} f, \quad i \in \{1, 2, \ldots, n\}. \quad (40)$$

Finally, from the above system of partial differential equations, we obtain the solution

$$f(x_1, x_2, \ldots, x_n) = Cx_1^{r/n} x_2^{r/n} \cdots x_n^{r/n}, \quad (41)$$

where C is a positive constant and the conclusion follows.

□

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