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Research Article

Some Identities on the Generalized *q*-Bernoulli, *q*-Euler, and *q*-Genocchi Polynomials

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Mahmudov (2012, 2013) introduced and investigated some q-extensions of the q-Bernoulli polynomials $\mathscr{B}_{n,q}^{(\alpha)}(x,y)$ of order α, the q-Euler polynomials $\mathscr{E}_{n,q}^{(\alpha)}(x,y)$ of order α, and the q-Genocchi polynomials $\mathscr{G}_{n,q}^{(\alpha)}(x,y)$ of order α. In this paper, we give some identities for $\mathscr{B}_{n,q}^{(\alpha)}(x,y)$, $\mathscr{G}_{n,q}^{(\alpha)}(x,y)$, and $\mathscr{E}_{n,q}^{(\alpha)}(x,y)$ and the recurrence relations between these polynomials. This is an analogous result to the q-extension of the Srivastava-Pintér addition theorem in Mahmudov (2013).

1. Introduction, Definitions, and Notations

Throughout this paper, we always make use of the following notation: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{C}$ denotes the set of complex numbers. The q-numbers and q-factorial are defined by

$$[a]_q = \frac{1-q^a}{1-q}, \quad q \neq 1, \qquad [n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q, \tag{1}$$

respectively, where $[0]_q! = 1$, $n \in \mathbb{N}$, and $a \in \mathbb{C}$. The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q:q)_n}{(q:q)_{n-k}(q:q)_k}, \tag{2}$$

where $(q:q)_n = (1-q)\cdots(1-q^n)$. The q-analogue of the function $(x+y)_q^n$ is defined by

$$(x+y)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(k(k-1))/2} x^{n-k} y^k. \tag{3}$$

The q-binomial formula is known as

$$(n;q)_a = (1-a)_q^n = \prod_{j=0}^{n-1}\left(1-q^j a\right) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(k(k-1))/2}(-1)^k a^k. \tag{4}$$

The q-exponential functions are given by

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)q^k z)}, \quad 0 < |q| < 1, \quad |z| < \frac{1}{|1-q|},$$

$$E_q(z) = \sum_{n=0}^{\infty} q^{(n(n-1))/2} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \left(1 + (1-q) q^k z\right), \quad 0 < |q| < 1, \ z \in \mathbb{C}. \tag{5}$$

From these forms, we easily see that $e_q(z)E_q(-z) = 1$. Moreover, $D_q e_q(z) = e_q(z)$ and $D_q E_q(z) = E_q(qz)$, where D_q is defined by

$$D_q f(z) = \frac{f(qz) - f(z)}{qz - z}, \quad 0 < |q| < 1, \ 0 \neq z \in \mathbb{C}. \tag{6}$$

The previous q-standard notation can be found in [1, 2]. Carlitz firstly extended the classical Bernoulli numbers and polynomials and Euler numbers and polynomials [3, 4]. There are numerous recent investigations on this subject by many other authors. Among them are Cenkci et al. [5, 6], Choi et al. [1], Cheon [7], Kim [8], Kurt [9], Kurt [10], Luo and Srivastava [11–13], Srivastava et al. [14, 15], Natalini and Bernardini [16], Tremblay et al. [17, 18], Gaboury and Kurt [19], Mahmudov [2, 20, 21], Araci et al. [22], and Kupershmidt [23].

Mahmudov defined and studied the properties of the following generalized q-Bernoulli polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x,y)$ of order α and q-Euler polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x,y)$ of order α as follows [2].

Let $q \in \mathbb{C}$, $\alpha \in \mathbb{N}$, and $0 < |q| < 1$. The q-Bernoulli numbers $\mathcal{B}_{n,q}^{(\alpha)}$ and polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x,y)$ in x and y of order α are defined by means of the generating functions:

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left(\frac{t}{e_q(t) - 1}\right)^{\alpha}, \quad |t| < 2\pi, \tag{7}$$

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{t}{e_q(t) - 1}\right)^{\alpha} e_q(tx) E_q(ty), \quad |t| < 2\pi. \tag{8}$$

The q-Euler numbers $\mathcal{E}_{n,q}^{(\alpha)}$ and polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x,y)$ in x and y of order α are defined by means of the generating functions:

$$\sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left(\frac{2}{e_q(t) + 1}\right)^{\alpha}, \quad |t| < \pi, \tag{9}$$

$$\sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{2}{e_q(t) + 1}\right)^{\alpha} e_q(tx) E_q(ty), \quad |t| < \pi. \tag{10}$$

The q-Genocchi numbers $\mathcal{G}_{n,q}^{(\alpha)}$ and polynomials $\mathcal{G}_{n,q}^{(\alpha)}(x,y)$ in x and y of order α are defined by means of the generating functions:

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,q}^{(\alpha)} \frac{t^n}{[n]_q!} = \left(\frac{2t}{e_q(t) + 1}\right)^{\alpha}, \quad |t| < \pi, \tag{11}$$

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!} = \left(\frac{2t}{e_q(t) + 1}\right)^{\alpha} e_q(tx) E_q(ty), \quad |t| < \pi. \tag{12}$$

The familiar q-Stirling numbers $S_{2,q}(n,k)$ of the second kind are defined by

$$\frac{\left(e_q(t) - 1\right)^k}{[k]_q!} = \sum_{n=0}^{\infty} S_{2,q}(n,k) \frac{t^n}{[n]_q!}. \tag{13}$$

It is obvious that

$$\begin{gathered}
\mathcal{B}_{n,q}^{(1)}(x,y) := \mathcal{B}_{n,q}(x,y), \qquad \mathcal{E}_{n,q}^{(1)}(x,y) := \mathcal{E}_{n,q}(x,y), \\
\mathcal{G}_{n,q}^{(1)}(x,y) := \mathcal{G}_{n,q}(x,y), \qquad \mathcal{B}_{n,q}(0,0) := \mathcal{B}_{n,q}, \\
\mathcal{E}_{n,q}(0,0) := \mathcal{E}_{n,q}, \qquad \mathcal{G}_{n,q}(0,0) := \mathcal{G}_{n,q}, \\
\mathcal{B}_{n,q}^{(\alpha)} = \mathcal{B}_{n,q}^{(\alpha)}(0,0), \\
\lim_{q \to 1^-} \mathcal{B}_{n,q}^{(\alpha)}(x,y) = \mathcal{B}_{n}^{(\alpha)}(x+y), \\
\lim_{q \to 1^-} \mathcal{B}_{n,q}^{(\alpha)} = \mathcal{B}_{n}^{(\alpha)}, \qquad \mathcal{E}_{n,q}^{(\alpha)} = \mathcal{E}_{n,q}^{(\alpha)}(0,0), \\
\lim_{q \to 1^-} \mathcal{E}_{n,q}^{(\alpha)}(x,y) = \mathcal{E}_{n}^{(\alpha)}(x+y), \qquad \lim_{q \to 1^-} \mathcal{E}_{n,q}^{(\alpha)} = \mathcal{E}_{n}^{(\alpha)}, \\
\mathcal{G}_{n,q}^{(\alpha)} = \mathcal{G}_{n,q}^{(\alpha)}(0,0), \qquad \lim_{q \to 1^-} \mathcal{G}_{n,q}^{(\alpha)}(x,y) = \mathcal{G}_{n}^{(\alpha)}(x+y), \\
\lim_{q \to 1^-} \mathcal{G}_{n,q}^{(\alpha)} = \mathcal{G}_{n}^{(\alpha)}.
\end{gathered} \tag{14}$$

From (8) and (10), it is easy to check that

$$\begin{aligned}
\mathcal{B}_{n,q}^{(\alpha)}(x,y) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{B}_{n-k,q}(x,0) \mathcal{B}_{k,q}^{(\alpha-1)}(0,y), \\
\mathcal{E}_{n,q}^{(\alpha)}(x,y) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{E}_{n-k,q}^{(\alpha-1)}(x,0) \mathcal{E}_{k,q}(0,y).
\end{aligned} \tag{15}$$

In this work, we give some identities for the q-Bernoulli polynomials. Also, we give some relations between the q-Bernoulli polynomials and q-Euler polynomials and the q-Genocchi polynomials and q-Bernoulli polynomials. Furthermore, we give a different form of the analogue of the Srivastava-Pintér addition theorem. More precisely, we prove the following theorems.

Theorem 1. *There are the following relations between the q-Bernoulli polynomials and q-Stirling numbers of the second kind:*

$$\mathscr{B}_{n,q}^{(\alpha)}(x,y)=\frac{[k]_q![n]_q!}{[n+k]_q!}\times\sum_{l=0}^{n+k}\begin{bmatrix}n+k\\ l\end{bmatrix}_q\mathscr{B}_{l,q}^{(\alpha+k)}(x,y)\times S_{2,q}(n+k-l,k),\quad(16)$$

$$\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}_q\mathscr{B}_{n-k,q}^{(\alpha)}(x,y)[\alpha]_q!S_{2,q}(k,\alpha)=\sum_{l=0}^{n-\alpha}\begin{bmatrix}n-\alpha\\ l\end{bmatrix}_q\frac{[n]_q!}{[n-\alpha]_q!}x^{n-\alpha-l}y^lq^{\binom{l}{2}},\quad(17)$$

where $q\in\mathbb{C}$, $\alpha,n\in\mathbb{N}$, *and* $0<|q|<1$.

Theorem 2. *The q-Stirling numbers of the second kind satisfy the following relations:*

$$\mathscr{E}_{n,q}^{(\alpha)}(x,y)=\sum_{j=0}^{\infty}\binom{-\alpha}{j}\frac{1}{2^j}[j]_q!\times\sum_{p=0}^{n}\begin{bmatrix}n\\ p\end{bmatrix}_qS_{2,q}(n-p,j)\times\sum_{l=0}^{p}\begin{bmatrix}p\\ l\end{bmatrix}_qx^{p-l}y^lq^{\binom{l}{2}},\quad(18)$$

$$\mathscr{B}_{n,q}^{(\alpha)}=[\alpha]_q!\sum_{j=0}^{\infty}\binom{-\alpha}{j}\times\sum_{k=0}^{j}\binom{j}{k}[k]_q!\frac{S_{2,q}(n+k,k)}{[n+k]_q!}[k]_q!(-1)^{j-k},$$

$$\mathscr{B}_{n,q}^{(-\alpha)}(x,y)=[\alpha]_q!\sum_{m=0}^{n+\alpha}\begin{bmatrix}n+\alpha\\ m\end{bmatrix}_qS_{2,q}(m,\alpha)\times(x+y)_q^{n+\alpha-m}\frac{[n]_q!}{[n+\alpha]_q!},\quad(19)$$

where $q\in\mathbb{C}$, $\alpha,n\in\mathbb{N}$, *and* $0<|q|<1$.

Theorem 3. *The generalized q-Euler polynomials satisfy the following relation:*

$$\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}_q\mathscr{E}_{k,q}(x,y)=2(x+y)_q^n-\mathscr{E}_{n,q}(x,y),\quad(20)$$

where $q\in\mathbb{C}$, α, $n\in\mathbb{N}$, *and* $0<|q|<1$.

Theorem 4. *The polynomials* $B_{n,q}(x,y)$ *and* $\mathscr{G}_{n,q}(x,y)$ *satisfy the following difference relationships:*

$$\mathscr{B}_{n,q}(x,y)=\sum_{\substack{l=0\\ l\neq n}}^{n+1}\begin{bmatrix}n+1\\ l\end{bmatrix}_q\frac{1}{[n+1]_q}\mathscr{G}_{l,q}(x,y)\mathscr{B}_{n+1-l,q},\quad(21)$$

$$\mathscr{G}_{n,q}(x,y)=-2\sum_{\substack{l=0\\ l\neq n}}^{n}\begin{bmatrix}n\\ l\end{bmatrix}_q\frac{1}{[l+1]_q}\mathscr{G}_{l+1,q}\mathscr{B}_{n-l,q}(x,y),\quad(22)$$

where $q\in\mathbb{C}$, $\alpha,n\in\mathbb{N}$, *and* $0<|q|<1$.

Theorem 5. *There is the following relation between the generalized q-Euler polynomials and generalized q-Bernoulli polynomials:*

$$\mathscr{E}_{n,q}^{(\alpha)}(x,y)=\left\{\sum_{s=0}^{n+1}\begin{bmatrix}n+1\\ s\end{bmatrix}_q\sum_{l=0}^{s}\begin{bmatrix}s\\ l\end{bmatrix}_q\mathscr{B}_{s-l,q}(mx,0)-\sum_{l=0}^{n+1}\begin{bmatrix}n+1\\ l\end{bmatrix}_q\mathscr{B}_{n+1-l,q}(mx,0)\right\}\times\frac{m}{[n+1]_q!}\mathscr{E}_{l,q}^{(\alpha)}(0,y)m^{l-n-1},\quad(23)$$

where $q\in\mathbb{C}$, $\alpha,n\in\mathbb{N}$, *and* $0<|q|<1$.

2. Proof of the Theorems

Lemma 6. *The generalized q-Bernoulli polynomials, q-Euler polynomials, and q-Genocchi polynomials satisfy the following relations:*

$$\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}_q\mathscr{B}_{k,q}^{(\alpha)}(x,y)\mathscr{B}_{n-k,q}^{(-\alpha)}=(x+y)_q^n,$$

$$\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}_q\mathscr{B}_{k,q}^{(\alpha)}(0,y)\mathscr{B}_{n-k,q}^{(-\alpha)}=q^{(n(n-1))/2}y^n,$$

$$\mathscr{B}_{n,q}^{(\alpha)}(x,y)=\sum_{l=0}^{n}\begin{bmatrix}n\\ l\end{bmatrix}_q\mathscr{B}_{n-l,q}^{(\alpha)}(0,y)\times\sum_{k=0}^{l}\begin{bmatrix}l\\ k\end{bmatrix}_q\mathscr{E}_{k,q}^{(\alpha)}(x,0)\mathscr{E}_{l-k,q}^{(-\alpha)},$$

$$\mathscr{E}_{n,q}^{(\alpha)}(x,y) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathscr{E}_{n-l,q}^{(\alpha)}(0,y) \times \sum_{k=0}^{l} \begin{bmatrix} l \\ k \end{bmatrix}_q \mathscr{B}_{k,q}^{(\alpha)}(x,0)\, \mathscr{B}_{l-k,q}^{(-\alpha)},$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{G}_{k,q}(x,y) + \mathscr{G}_{n,q}(x,y) = 2[n]_q (x+y)_q^{n-1},$$

$$\mathscr{G}_{n,q}^{(\alpha-\beta)}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathscr{G}_{k,q}^{(\alpha)}(x,0)\, \mathscr{G}_{n-k,q}^{(-\beta)}(0,y). \tag{24}$$

Proof. The proof of this lemma can be found from (7)–(12). □

Proof of Theorem 1. By (8) and (13) we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathscr{B}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!}
&= \left(\frac{t}{e_q(t)-1} \right)^{\alpha} e_q(tx) E_q(ty) \\
&\quad \times \frac{[k]_q!}{\left(e_q(t)-1\right)^k} \frac{\left(e_q(t)-1\right)^k}{[k]_q!} \\
&= [k]_q! \frac{t^{\alpha}}{\left(e_q(t)-1\right)^{\alpha+k}} e_q(tx) E_q(ty) \\
&\quad \times \sum_{m=0}^{\infty} S_{2,q}(m,k) \frac{t^m}{[m]_q!} \\
&= [k]_q! t^{-k} \sum_{n=0}^{\infty} \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathscr{B}_{l,q}^{(\alpha+k)} \\
&\quad \times (x,y) S_{2,q}(n-l,k) \frac{t^n}{[n]_q!} \\
&= [k]_q! \sum_{n=0}^{\infty} \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \mathscr{B}_{l,q}^{(\alpha+k)} \\
&\quad \times (x,y) S_{2,q}(n-l,k) \frac{t^{n-k}}{[n]_q!} \\
&= [k]_q! \sum_{n=-k}^{\infty} \sum_{l=0}^{n+k} \begin{bmatrix} n+k \\ l \end{bmatrix}_q \mathscr{B}_{l,q}^{(\alpha+k)} \\
&\quad \times (x,y) S_{2,q}(n+k-l,k) \frac{t^{n-k}}{[n]_q!}.
\end{aligned} \tag{25}$$

Equating the coefficients of $(t^n/[n]_q!)$, we obtain (16).
Similarly, we have (17). □

Proof of Theorem 2. Combining (10) and (13), we obtain

$$\begin{aligned}
\left(\frac{2}{e_q(t)+1} \right)^{\alpha} &= \left(1 + \frac{e_q(t)-1}{2} \right)^{(-\alpha)} \\
&= \sum_{j=0}^{\infty} \binom{-\alpha}{j} \left(\frac{e_q(t)-1}{2} \right)^{(j)}, \\
\sum_{n=0}^{\infty} \mathscr{E}_{n,q}^{(\alpha)}(x,y) \frac{t^n}{[n]_q!}
&= \sum_{j=0}^{\infty} \binom{-\alpha}{j} \left(\frac{e_q(t)-1}{2} \right)^{(j)} e_q(tx) E_q(ty) \\
&= \sum_{j=0}^{\infty} \binom{-\alpha}{j} \frac{1}{2^j} [j]_q! \sum_{n=0}^{\infty} S_{2,q}(n,j) \frac{t^n}{[n]_q!} \\
&\quad \times \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} y^k q^{\binom{k}{2}} \frac{t^n}{[n]_q!} \\
&= \sum_{j=0}^{\infty} \binom{-\alpha}{j} \frac{1}{2^j} \\
&\quad \times \sum_{n=0}^{\infty} \sum_{p=0}^{n} \begin{bmatrix} n \\ p \end{bmatrix}_q [j]_q! S_{2,q}(n-p,j) \\
&\quad \times \sum_{l=0}^{p} \begin{bmatrix} p \\ l \end{bmatrix}_q x^{p-l} y^l q^{\binom{l}{2}} \frac{t^n}{[n]_q!}.
\end{aligned} \tag{26}$$

Comparing the coefficients of $(t^n/[n]_q!)$, we find (18). Similarly, we have (19). □

Proof of Theorem 3. It is obvious that

$$\frac{-2}{\left(e_q(t)+1\right) e_q(t)} = \frac{2}{\left(e_q(t)+1\right)} - \frac{2}{e_q(t)}. \tag{27}$$

We write it as

$$\begin{aligned}
\frac{-2}{e_q(t)+1} \frac{e_q(tx) E_q(ty)}{e_q(t)} &= \frac{2}{e_q(t)+1} e_q(tx) E_q(ty) \\
&\quad - \frac{2}{e_q(t)} e_q(tx) E_q(ty), \\
\frac{-2}{e_q(t)+1} e_q(tx) E_q(ty) &= \frac{2}{e_q(t)+1} e_q(tx) E_q(ty) \\
&\quad - 2 e_q(tx) E_q(ty)
\end{aligned}$$

$$-\sum_{n=0}^{\infty}\mathscr{E}_{n,q}(x,y)\frac{t^n}{[n]_q!}$$
$$=\sum_{n=0}^{\infty}\mathscr{E}_{n,q}(x,y)\frac{t^n}{[n]_q!}$$
$$\times\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}-2\sum_{n=0}^{\infty}(x+y)_q^n\frac{t^n}{[n]_q!}. \tag{28}$$

Using the Cauchy product and comparing the coefficients of $(t^n/[n]_q!)$, we have

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q\mathscr{E}_{k,q}(x,y)=2(x+y)_q^n-\mathscr{E}_{k,q}(x,y). \tag{29}$$

□

Finally, we consider the interesting relationships between the q-Bernoulli polynomials and q-Genocchi polynomials and the q-Euler polynomials and q-Bernoulli polynomials. These relations are q-analogues to the Srivastava-Pintér addition theorems.

Proof of Theorem 4. It follows immediately that

$$\sum_{n=0}^{\infty}\mathscr{B}_{n,q}(x,y)\frac{t^n}{[n]_q!}$$
$$=\frac{1}{2}\frac{2te_q(tx)E_q(ty)}{e_q(t)+1}$$
$$+\frac{1}{t}\left(\frac{t}{e_q(t)-1}\right)\frac{2t}{e_q(t)+1}e_q(tx)E_q(ty)$$
$$=\frac{1}{2}\sum_{n=0}^{\infty}\mathscr{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}+\frac{1}{t}$$
$$\times\sum_{n=0}^{\infty}\mathscr{B}_{n,q}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathscr{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}$$
$$=\frac{1}{2}\sum_{n=0}^{\infty}\mathscr{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}$$
$$+\sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{bmatrix} n \\ l\end{bmatrix}_q\frac{1}{[n]_q}\mathscr{G}_{l,q}(x,y)$$
$$\times\mathscr{B}_{n-l,q}\frac{t^{n-1}}{[n-1]_q!}$$
$$=\frac{1}{2}\sum_{n=0}^{\infty}\mathscr{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}$$
$$+\sum_{n=0}^{\infty}\left(-\frac{1}{2}\mathscr{G}_{n,q}(x,y)\right.$$
$$+\sum_{l=0}^{n+1}\begin{bmatrix} n+1 \\ l\end{bmatrix}_q\frac{1}{[n+1]_q}$$
$$\left.\times\mathscr{G}_{l,q}(x,y)\mathscr{B}_{n+1-l,q}\right)\frac{t^n}{[n]_q!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{\substack{l=0\\ l\neq n}}^{n+1}\begin{bmatrix} n+1 \\ l\end{bmatrix}_q\frac{1}{[n+1]_q}\right.$$
$$\left.\times\mathscr{G}_{l,q}(x,y)\mathscr{B}_{n+1-l,q}\right)\frac{t^n}{[n]_q!}. \tag{30}$$

Equating the coefficients of $(t^n/[n]_q!)$, we have (21).

In a similar fashion, (12) yields

$$\sum_{n=0}^{\infty}\mathscr{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}$$
$$=\frac{1}{t}\left(\frac{2t}{e_q(t)+1}\left(e_q(t)-1\right)\right)\left(\frac{te_q(tx)E_q(ty)}{e_q(t)-1}\right)$$
$$=\frac{1}{t}\left(2t-2\frac{2t}{e_q(t)+1}\right)\left(\frac{t}{e_q(t)-1}e_q(tx)E_q(ty)\right)$$
$$=\frac{1}{t}\left(2t-2\sum_{n=0}^{\infty}\mathscr{G}_{n,q}\frac{t^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty}\mathscr{B}_{n,q}(x,y)\frac{t^n}{[n]_q!}\right)$$
$$=\frac{1}{t}\left(-2\sum_{l=0}^{\infty}\frac{1}{[l+1]_q!}\mathscr{G}_{l+1,q}\frac{t^{l+1}}{[l]_q!}\right)\left(\sum_{n=0}^{\infty}\mathscr{B}_{n,q}(x,y)\frac{t^n}{[n]_q!}\right)$$
$$=\sum_{n=1}^{\infty}\left(-2\sum_{\substack{l=0\\ l\neq n}}^{n}\begin{bmatrix} n \\ l\end{bmatrix}_q\frac{\mathscr{G}_{l+1,q}}{[l+1]_q}\mathscr{B}_{n-l,q}(x,y)\right)\frac{t^n}{[n]_q!}. \tag{31}$$

Comparing the coefficients of $(t^n/[n]_q!)$, we have (22). □

Proof of Theorem 5. By (10), we write

$$\sum_{n=0}^{\infty}\mathscr{E}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}$$
$$=\left(\frac{2}{e_q(t)+1}\right)^{\alpha}$$
$$\times E_q(ty)\frac{e_q(t/m)-1}{(t/m)}\frac{(t/m)}{e_q(t/m)\ -1}e_q((t/m)mx)$$

$$= \frac{m}{t}\left\{\sum_{n=0}^{\infty}\mathscr{E}_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}\right.$$
$$\times\sum_{n=0}^{\infty}\mathscr{B}_{n,q}(mx,0)\frac{t^n}{m^n[n]_q!}$$
$$\times\sum_{n=0}^{\infty}\frac{t^n}{m^n[n]_q!}-\sum_{n=0}^{\infty}\mathscr{E}_{n,q}^{(\alpha)}(0,y)\frac{t^n}{[n]_q!}$$
$$\left.\times\sum_{n=0}^{\infty}\mathscr{B}_{n,q}(mx,0)\frac{t^n}{m^n[n]_q!}\right\}$$
$$= m\sum_{n=-1}^{\infty}\frac{1}{[n+1]_q}$$
$$\times\left\{\sum_{s=0}^{n+1}\begin{bmatrix}n+1\\ s\end{bmatrix}_q\sum_{l=0}^{s}\begin{bmatrix}s\\ l\end{bmatrix}_q\mathscr{B}_{s-l,q}(mx,0)\right.$$
$$\left.-\sum_{l=0}^{n+1}\begin{bmatrix}n+1\\ l\end{bmatrix}_q\mathscr{B}_{n+1-l,q}(mx,0)\right\}$$
$$\times\frac{m}{[n+1]_q!}\mathscr{E}_{l,q}^{(\alpha)}(0,y)m^{l-n-1}\frac{t^n}{[n]_q!}. \tag{32}$$

By equating the coefficients of $(t^n/[n]_q!)$, we get the theorem. □

Remark 7. There are many different relationships which are analogues to the Srivastava-Pintér addition theorems at these polynomials.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] J. Choi, P. J. Anderson, and H. M. Srivastava, "Some q-extensions of the Apostol-Bernoulli and the Apostol-Euler polynomials of order n, and the multiple Hurwitz zeta function," *Applied Mathematics and Computation*, vol. 199, no. 2, pp. 723–737, 2008.

[2] N. I. Mahmudov, "On a class of q-Bernoulli and q-Euler polynomials," *Advances in Difference Equations*, vol. 2013, article 108, 2013.

[3] L. Carlitz, "q-Bernoulli numbers and polynomials," *Duke Mathematical Journal*, vol. 15, pp. 987–1050, 1948.

[4] L. Carlitz, "Expansions of q-Bernoulli numbers," *Duke Mathematical Journal*, vol. 25, pp. 355–364, 1958.

[5] M. Cenkci, M. Can, and V. Kurt, "q-extensions of Genocchi numbers," *Journal of the Korean Mathematical Society*, vol. 43, no. 1, pp. 183–198, 2006.

[6] M. Cenkci, V. Kurt, S. H. Rim, and Y. Simsek, "On (i, q) Bernoulli and Euler numbers," *Applied Mathematics Letters*, vol. 21, no. 7, pp. 706–711, 2008.

[7] G.-S. Cheon, "A note on the Bernoulli and Euler polynomials," *Applied Mathematics Letters*, vol. 16, no. 3, pp. 365–368, 2003.

[8] T. Kim, "Some formulae for the q-Bernoulli and Euler polynomials of higher order," *Journal of Mathematical Analysis and Applications*, vol. 273, no. 1, pp. 236–242, 2002.

[9] B. Kurt, "A further generalization of the Bernoulli polynomials and on the 2D-Bernoulli polynomials $B_{n,q}^{(\alpha)}$," *Applied Mathematical Sciences*, vol. 4, no. 47, pp. 2315–2322, 2010.

[10] V. Kurt, "A new class of generalized q-Bernoulli and q-Euler polynomials," in *Proceedings of the International Western Balkans Conference of Mathematical Sciences*, Elbasan, Albania, May 2013.

[11] Q.-M. Luo, "Some results for the q-Bernoulli and q-Euler polynomials," *Journal of Mathematical Analysis and Applications*, vol. 363, no. 1, pp. 7–18, 2010.

[12] Q.-M. Luo and H. M. Srivastava, "Some relationships between the Apostol-Bernoulli and Apostol-Euler polynomials," *Computers & Mathematics with Applications*, vol. 51, no. 3-4, pp. 631–642, 2006.

[13] Q.-M. Luo and H. M. Srivastava, "q-extensions of some relationships between the Bernoulli and Euler polynomials," *Taiwanese Journal of Mathematics*, vol. 15, no. 1, pp. 241–257, 2011.

[14] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic, London, UK, 2001.

[15] H. M. Srivastava and A. Pintér, "Remarks on some relationships between the Bernoulli and Euler polynomials," *Applied Mathematics Letters*, vol. 17, no. 4, pp. 375–380, 2004.

[16] P. Natalini and A. Bernardini, "A generalization of the Bernoulli polynomials," *Journal of Applied Mathematics*, no. 3, pp. 155–163, 2003.

[17] R. Tremblay, S. Gaboury, and B.-J. Fugère, "A new class of generalized Apostol-Bernoulli polynomials and some analogues of the Srivastava-Pintér addition theorem," *Applied Mathematics Letters*, vol. 24, no. 11, pp. 1888–1893, 2011.

[18] R. Tremblay, S. Gaboury, and B. J. Fegure, "Some new classes of generalized Apostol Bernoulli and Apostol-Genocchi polynomials," *International Journal of Mathematics and Mathematical Sciences*, vol. 2012, Article ID 182785, 14 pages, 2012.

[19] S. Gaboury and B. Kurt, "Some relations involving Hermite-based Apostol-Genocchi polynomials," *Applied Mathematical Sciences*, vol. 6, no. 81–84, pp. 4091–4102, 2012.

[20] N. I. Mahmudov, "q-analogues of the Bernoulli and Genocchi polynomials and the Srivastava-Pintér addition theorems," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 169348, 8 pages, 2012.

[21] N. I. Mahmudov and M. E. Keleshteri, "On a class of generalized q-Bernoulli and q-Euler polynomials," *Advances in Difference Equations*, vol. 2013, article 115, 2013.

[22] S. Araci, J. J. Seo, and M. Acikgoz, "A new family of q-analogue of Genocchi polynomials of higher order," *Kyungpook Mathematical Journal*. In press.

[23] B. A. Kupershmidt, "Reflection symmetries of q-Bernoulli polynomials," *Journal of Nonlinear Mathematical Physics*, vol. 12, no. 1, pp. 412–422, 2005.