

## Research Article

# The Ornstein-Uhlenbeck-Type Model with a Hybrid Dividend Strategy

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We consider the Ornstein-Uhlenbeck-type model. We first introduce the model and then find the ordinary differential equations and boundary conditions satisfied by the dividend functions; closed-form solutions for the dividend value functions are given. We also study the distribution of the time value of ruin. Furthermore, the moments and moment-generating functions of total discounted dividends until ruin are discussed.

## 1. Introduction

In recent years, the dividend problem has gained a lot of attention in the actuarial literature. Dividend strategies for insurance risk model were first proposed by de Finetti [1], who considered a discrete time random walk with step size  $\pm 1$  and found that the optimal dividend strategy must be a barrier strategy. From then on, the problem of optimal dividend strategy has been studied in continuous time, for example, Asmussen and Taksar [2], Albrecher et al. [3], Gao and Yin [4], Gerber and Shiu [5, 6], Wan [7] and so on. The optimal dividend problem in a compound Poisson model was studied by Gerber and Shiu [8]. Optimal dividend in an Ornstein-Uhlenbeck-type model with credit and debit interest was considered in Cai et al. [9]. For a class of compound Poisson process perturbed by diffusion with a threshold dividend strategy, the expected discounted penalty function has been studied by Wan [7]. The perturbed Sparre Andersen model with a threshold dividend strategy was settled by Gao and Yin [10].

Recently, the multilayer dividend strategy as an extension of the threshold dividend strategy has drawn many authors attention. For example, the perturbed Sparre Andersen and compound Poisson risk models with multilayer dividend strategy have been studied by Yang and Zhang [11, 12]. The integrodifferential equations for the expected discounted penalty function were derived and solved; when the claims are subexponentially distributed, the asymptotic formula for

ruin probability is obtained. The Ornstein-Uhlenbeck-type model is a very important model in applied probability which has recently gained a lot of attention. See, for example, Cai et al. [9] and Fang and Wu [13]. More general, Wong and Zhao [14] consider the optimal dividends and bankruptcy in an Ornstein-Uhlenbeck process with the surplus-dependent credit/debit interest rate. Motivated by the above work, in this paper, we consider a hybrid dividend strategy which combined a barrier strategy with a threshold strategy in an Ornstein-Uhlenbeck-type model. For simplicity, we consider only one threshold and one barrier.

The remainder of the paper is organized as follows. In Section 2, we describe the model and discuss the dividend functions until ruin, in Section 3, we give the limit of dividends level, and in Section 4 we get the expression by Laplace transform of ruin time. The partial differential equation with boundary conditions satisfied by the moments and moment-generating function is proved in Section 5.

## 2. The Model

Consider the following surplus process:

$$dU_t = (\mu + \rho U_t) dt + \sigma dW_t, \quad t \geq 0, \quad (1)$$

where  $\rho > 0$  is the force of interest,  $\mu > 0$  is the drift coefficient,  $\sigma > 0$  is the diffusion coefficient, and  $\{W_t\}_{t \geq 0}$  is the standard Brownian motion. We will assume that

the company pays dividends according to the following strategy governed by parameters  $b_2 > b_1 > 0$  and  $\alpha > 0$ . Whenever the modified surplus is below the level  $b_1$ , no dividends are paid. However, when the modified surplus is above  $b_1$  and below the  $b_2$ , dividends are paid continuously at a constant rate  $\alpha$ . When the modified surplus is above  $b_2$ , dividends are completely paid. For  $t \geq 0$ , let

$$D(t) = D_1(t) + D_2(t) \tag{2}$$

denote the aggregate dividend paid by time  $t$ , where  $D_1(t)$  and  $D_2(t)$  are caused by the different parts of dividends, respectively. Thus,

$$\widetilde{U}_t = U_t - D(t) \tag{3}$$

is the modified surplus at time  $t$ . Let  $\delta > 0$  be the force of interest for valuation; in this paper, we assume that  $\rho < \delta$ . Let  $I(A)$  be the indicator function of event  $A$  and let  $D$  denote the present value of all dividends until ruin

$$D = \int_0^T e^{-\delta t} dD_1(t) + \int_0^T e^{-\delta t} dD_2(t), \tag{4}$$

where

$$T = \inf \{t \geq 0 : \widetilde{U}_t \leq 0\}, \tag{5}$$

is the time of ruin, and

$$\begin{aligned} D_1(t) &= \alpha \int_0^t I(b_1 \leq \widetilde{U}_s \leq b_2) ds, \\ D_2(t) &= \eta - b_2, \\ \eta &= \max_{0 \leq s \leq t} \{U_s - D_1(s)\}. \end{aligned} \tag{6}$$

For  $x \geq 0$ , we use the symbol  $V(x; b_1, b_2)$  to denote the expectation of  $D$ . That is,

$$V(x; b_1, b_2) = E[D | \widetilde{U}_0 = x] = E^x[D]. \tag{7}$$

Define the random times

$$\begin{aligned} \tau_0^- &= \inf(t \geq 0 : U_t \leq 0), \\ \tau_{b_1}^+ &= \inf(t \geq 0 : U_t \geq b_1), \\ \tau_{b_1}^- &= \inf(t \geq 0 : \widetilde{U}_t \leq b_1), \\ \tau_{b_2}^+ &= \inf(t \geq 0 : \widetilde{U}_t \geq b_2), \\ \tau &= \inf(t \geq 0 : \widetilde{U}_t = b_1 \text{ or } \widetilde{U}_t = b_2), \end{aligned} \tag{8}$$

with the convention  $\inf \emptyset = \infty$ .

**Lemma 1.** Assume that  $V(x)$  is twice continuously differentiable on  $(0, b_1) \cup (b_1, b_2) \cup (b_2, \infty)$ . For  $0 < x \leq b_1$ ,  $V(x)$  satisfies the following ordinary differential equation:

$$\frac{1}{2}\sigma^2 V''(x) + (\mu + \rho x) V'(x) - \delta V(x) = 0; \tag{9}$$

for  $b_1 < x \leq b_2$ ,  $V(x)$  satisfies the following ordinary differential equation:

$$\frac{1}{2}\sigma^2 V''(x) + (\mu + \rho x - \alpha) V'(x) - \delta V(x) = 0; \tag{10}$$

for  $x > b_2$ ,  $V(x)$  satisfies the following equation:

$$V(x) = x - b_2 + V(b_2), \tag{11}$$

with boundary conditions

$$V(0; b_1, b_2) = 0, \quad V'(b_2; b_1, b_2) = 1. \tag{12}$$

*Proof.* By virtually the same arguments as in Yin and Wen [15], we can prove (9) and (10). The boundary conditions can be derived the same as in Gerber and Shiu [6] or Cai et al. [9].

The ordinary differential equation (9) has two positive independent solutions  $f_1, f_2$  such that  $f_1$  is strictly decreasing and  $f_2$  is strictly increasing (see e.g., [16]). Let  $f_3, f_4$  be such solution for the ordinary differential equation (10), where  $f_3$  is strictly decreasing and  $f_4$  is strictly increasing. In Cai et al. [9], the authors pointed out that these independent solutions are given by

$$f_1(x) = \exp\left\{-\frac{1}{\rho\sigma^2}(\mu + \rho x)^2\right\} \tag{13}$$

$$\times U\left(\frac{1}{2} + \frac{\delta}{2\rho}, \frac{1}{2}, \frac{1}{\rho\sigma^2}(\mu + \rho x)^2\right),$$

$$f_2(x) = (\mu + \rho x) \exp\left\{-\frac{1}{\rho\sigma^2}(\mu + \rho x)^2\right\} \tag{14}$$

$$\times M\left(1 + \frac{\delta}{2\rho}, \frac{3}{2}, \frac{1}{\rho\sigma^2}(\mu + \rho x)^2\right),$$

$$f_3(x) = \exp\left\{-\frac{1}{\rho\sigma^2}(\mu - \alpha + \rho x)^2\right\} \tag{15}$$

$$\times U\left(\frac{1}{2} + \frac{\delta}{2\rho}, \frac{1}{2}, \frac{1}{\rho\sigma^2}(\mu - \alpha + \rho x)^2\right),$$

$$f_4(x) = (\mu - \alpha + \rho x) \exp\left\{-\frac{1}{\rho\sigma^2}(\mu - \alpha + \rho x)^2\right\} \tag{16}$$

$$\times M\left(1 + \frac{\delta}{2\rho}, \frac{3}{2}, \frac{1}{\rho\sigma^2}(\mu - \alpha + \rho x)^2\right),$$

where  $M$  and  $U$  are called the confluent hypergeometric functions of the first and second kind, respectively.

Denote

$$\begin{aligned} h(x) &= f_1(0) f_2(x) - f_2(0) f_1(x), \\ g(x) &= f_3(b_2) f_4(x) - f_4(b_2) f_3(x), \\ k(x) &= f_3(x) f_4(b_1) - f_3(b_1) f_4(x). \end{aligned} \tag{17}$$

The expressions of the expected discounted dividend payments are given by Theorem 2.  $\square$

**Theorem 2.** (i) For  $0 \leq x \leq b_1$ ,

$$\begin{aligned}
 V(x; b_1, b_2) &= \left( (-\alpha g'(b_1) k'(b_2) + k'(b_1) g(b_1) \delta + \alpha k'(b_1) g'(b_1)) \right. \\
 &\quad \times \left( \delta [h'(b_1) g(b_1) k'(b_2) - g'(b_1) h(b_1) k'(b_2) \right. \\
 &\quad \left. \left. + k'(b_1) g'(b_1) h(b_1)] \right)^{-1} \right) h(x).
 \end{aligned} \tag{18}$$

(ii) For  $b_1 < x \leq b_2$ ,

$$\begin{aligned}
 V(x; b_1, b_2) &= \frac{\alpha}{\delta} + \frac{k(x)}{k'(b_2)} \\
 &\quad + \left( [\delta h(b_1) k'(b_1) - \alpha h'(b_1) k'(b_2)] \right. \\
 &\quad \times \left. [(g(x) k'(b_2) - g'(b_2) k(x))] \right) \\
 &\quad \times \left( k'(b_2) \delta [h'(b_1) g(b_1) k'(b_2) - g'(b_1) h(b_1) k'(b_2) \right. \\
 &\quad \left. + k'(b_1) g'(b_1) h(b_1)] \right)^{-1}.
 \end{aligned} \tag{19}$$

*Proof.* When  $0 < x \leq b_1$ , by the strong Markov property of the process  $U_t$ , we have

$$\begin{aligned}
 E^x [D] &= E^x [DI(\tau_{b_1}^+ < \tau_0^-)] + E^x [DI(\tau_{b_1}^+ > \tau_0^-)] \\
 &= E^x [DI(\tau_{b_1}^+ < \tau_0^-)] \\
 &= E^x [e^{-\delta \tau_{b_1}^+} I(\tau_{b_1}^+ < \tau_0^-)] V(b_1; b_1, b_2).
 \end{aligned} \tag{20}$$

For  $0 < x \leq b_1$ , using Itô's formula or Dynkin's formula as in Li et al. [17], we find that  $f(x) := E^x [e^{-\delta \tau_{b_1}^+} I(\tau_{b_1}^+ < \tau_0^-)]$  satisfies the ordinary differential equation (9), with boundary conditions  $f(b_1) = 1, f(0) = 0$ . Assume that the solution of the equation is  $f(x) = c_1 f_1(x) + c_2 f_2(x)$ ; from the boundary conditions, we obtain

$$\begin{aligned}
 c_1 &= \frac{-f_2(0)}{f_1(0) f_2(b_1) - f_2(0) f_1(b_1)}, \\
 c_2 &= \frac{f_1(0)}{f_1(0) f_2(b_1) - f_2(0) f_1(b_1)}.
 \end{aligned} \tag{21}$$

So we have

$$E^x [e^{-\delta \tau_{b_1}^+} I(\tau_{b_1}^+ < \tau_0^-)] = \frac{h(x)}{h(b_1)}. \tag{22}$$

Similarly, when  $b_1 < x \leq b_2$ ,

$$\begin{aligned}
 E^x [D] &= E^x \left[ \int_0^\tau \alpha e^{-\delta t} dt \right] + E^x [e^{-\delta \tau}] V(\bar{U}_\tau; b_1, b_2) \\
 &= E^x \left[ \left( -\frac{\alpha}{\delta} \right) (e^{-\delta \tau} - 1) \right] \\
 &\quad + E^x [e^{-\delta \tau} I(\tau_{b_1}^- < \tau_{b_2}^+)] V(b_1; b_1, b_2) \\
 &\quad + E^x [e^{-\delta \tau} I(\tau_{b_2}^+ < \tau_{b_1}^-)] V(b_2; b_1, b_2).
 \end{aligned} \tag{23}$$

For  $b_1 < x \leq b_2$ , using Itô's formula or Dynkin's formula, we have that  $f(x) := E^x [e^{-\delta \tau_{b_1}^-} I(\tau_{b_1}^- < \tau_{b_2}^+)]$  satisfies the ordinary differential equation (10) with boundary conditions  $f(b_1) = 1, f(b_2) = 0$ . Assume that the solution of the equation is  $f(x) = c_3 f_3(x) + c_4 f_4(x)$ ; from the boundary conditions, we obtain

$$\begin{aligned}
 c_3 &= \frac{-f_4(b_2)}{f_3(b_2) f_4(b_1) - f_4(b_2) f_3(b_1)}, \\
 c_4 &= \frac{f_3(b_2)}{f_3(b_2) f_4(b_1) - f_4(b_2) f_3(b_1)},
 \end{aligned} \tag{24}$$

so we have

$$E^x [e^{-\delta \tau_{b_1}^-} I(\tau_{b_1}^- < \tau_{b_2}^+)] = \frac{g(x)}{g(b_1)}. \tag{25}$$

Similarly,  $f(x) := E^x [e^{-\delta \tau_{b_2}^+} I(\tau_{b_2}^+ < \tau_{b_1}^-)]$  satisfies (10) with boundary conditions  $f(b_1) = 0, f(b_2) = 1$ , so we get

$$\begin{aligned}
 E^x [e^{-\delta \tau_{b_2}^+} I(\tau_{b_2}^+ < \tau_{b_1}^-)] &= \frac{f_3(x) f_4(b_1) - f_3(b_1) f_4(x)}{g(b_1)} \\
 &= \frac{k(x)}{g(b_1)},
 \end{aligned} \tag{26}$$

where

$$g(b_1) = k(b_2). \tag{27}$$

For  $0 < x \leq b_1$ , dividends will be payable if the surplus process without ruin and reaches  $b_1$ , so that

$$\begin{aligned}
 V(x; b_1, b_2) &= E^x [e^{-\delta \tau_{b_1}^+} I(\tau_{b_1}^+ < \tau_0^-)] V(b_1; b_1, b_2) \\
 &= \frac{h(x)}{h(b_1)} V(b_1; b_1, b_2).
 \end{aligned} \tag{28}$$

For  $b_1 < x \leq b_2$ , dividends are paid continuously at a constant rate  $\alpha$ , so we obtain

$$\begin{aligned} V(x; b_1, b_2) &= \frac{\alpha}{\delta} + E^x \left[ e^{-\delta\tau} I(\tau_{b_1}^- < \tau_{b_2}^+) \right] \left( V(b_1; b_1, b_2) - \frac{\alpha}{\delta} \right) \\ &\quad + E^x \left[ e^{-\delta\tau} I(\tau_{b_2}^+ < \tau_{b_1}^-) \right] \left( V(b_2; b_1, b_2) - \frac{\alpha}{\delta} \right) \\ &= \frac{\alpha}{\delta} + \frac{g(x)}{g(b_1)} \left( V(b_1; b_1, b_2) - \frac{\alpha}{\delta} \right) \\ &\quad + \frac{k(x)}{g(b_1)} \left( V(b_2; b_1, b_2) - \frac{\alpha}{\delta} \right). \end{aligned} \tag{29}$$

Using  $V'(b_2; b_1, b_2) = 1$  and  $V'(b_1+; b_1, b_2) = V'(b_1-; b_1, b_2)$ , we get

$$\begin{aligned} \frac{g'(b_2)}{g(b_1)} \left( V(b_1; b_1, b_2) - \frac{\alpha}{\delta} \right) + \frac{k'(b_2)}{g(b_1)} \left( V(b_2; b_1, b_2) - \frac{\alpha}{\delta} \right) &= 1, \\ \frac{h'(b_1)}{h(b_1)} V(b_1; b_1, b_2) &= \frac{g'(b_1)}{g(b_1)} \left( V(b_1; b_1, b_2) - \frac{\alpha}{\delta} \right) \\ &\quad + \frac{k'(b_1)}{g(b_1)} \left( V(b_2; b_1, b_2) - \frac{\alpha}{\delta} \right). \end{aligned} \tag{30}$$

With some careful calculations, we obtain  $V(b_1; b_1, b_2)$  and  $V(b_2; b_1, b_2)$ , so we get the results (18) and (19).  $\square$

### 3. The Special Dividends Strategy

In this section, we consider the limit of dividends level. Let  $b_2 \rightarrow \infty$ ; by the expressions of  $k$  and  $g$ , we have

$$\begin{aligned} g(b_1) &\sim -f_3(b_1) f_4(b_2), \\ g'(b_1) &\sim -f'(b_1) f_4(b_2), \\ k'(b_2) &\sim -f_3(b_1) f_4'(b_2), \end{aligned} \tag{31}$$

where

$$f_4(b_2) \rightarrow \infty, \quad f_4'(b_2) \rightarrow \infty. \tag{32}$$

Substituting the above expressions into (18) and (19), and setting

$$\begin{aligned} A &= -\alpha f_3'(b_1) f_4(b_2) f_3(b_1) f_4'(b_2), \\ B &= \delta k'(b_1) f_3(b_1) f_4(b_2), \\ C &= \alpha k'(b_1) f_3'(b_1) f_4(b_2), \\ D &= h'(b_1) f_3(b_1) f_4'(b_2) f_3(b_1) f_4(b_2), \\ E &= f_3'(b_1) f_4(b_2) f_3(b_1) f_4'(b_2), \\ F &= f_3'(b_1) f_4(b_2) k'(b_1) h(b_1), \end{aligned} \tag{33}$$

we obtain, for  $0 < x \leq b_1$ ,

$$V(x; b_1, b_2) = \frac{A - B - C}{\delta [D - E - F]} h(x). \tag{34}$$

Let

$$\begin{aligned} A' &= \delta h(b_1) k'(b_1) + \alpha h'(b_1) f_3(b_1) f_4'(b_2), \\ B' &= -(g(x) f_3(b_1) f_4'(b_2) - g'(b_2) k(x)), \\ C' &= h'(b_1) f_3^2(b_1) f_4(b_2) f_4'(b_2), \\ D' &= k'(b_1) f_4(b_2) h(b_1) f_3(b_1) f_4'(b_2), \\ E' &= k'^2(b_1) f_4(b_2) h(b_1); \end{aligned} \tag{35}$$

we obtain, for  $b_1 < x \leq \infty$ ,

$$\begin{aligned} V(x; b_1, b_2) &= \frac{\alpha}{\delta} + \frac{k(x)}{-f_3(b_1) f_4'(b_2)} \\ &\quad - \frac{A' B'}{f_3(b_1) f_4'(b_2) \delta [C' - D' - E']}. \end{aligned} \tag{36}$$

Then dividing numerator and denominator of (34) and (36) by  $f_4(b_2) f_4'(b_2)$ , we get the expected discounted dividend payments for the threshold strategy

$$\begin{aligned} V(x; b) &= \begin{cases} \frac{\alpha}{\delta} \frac{f_3'(b) h(x)}{f_3'(b) h(b) - f_3(b) h'(b)}, & \text{if } 0 < x \leq b_1, \\ \frac{\alpha}{\delta} + \frac{\alpha}{\delta} \frac{f_3(x) h'(b)}{f_3'(b) h(b) - f_3(b) h'(b)}, & \text{if } x > b_2, \end{cases} \end{aligned} \tag{37}$$

which is (15) in Fang and Wu [13].

*Remark 3.* Similarly, when  $b_1 = b_2 = b$ , we get the expected discounted dividend payments for the barrier strategy

$$V(x; b) = \begin{cases} \frac{h(x)}{h'(b)}, & \text{if } 0 < x \leq b, \\ x - b + \frac{h(b)}{h'(b)}, & \text{if } x > b, \end{cases} \tag{38}$$

which is (9) and (10) in Cai et al. [9].

### 4. The Time Value of Ruin under a Hybrid Dividend Strategy

In this section, we focus on the Laplace transform of the time value of ruin. We assume that dividends are paid according to threshold strategy with parameters  $b_1$ ,  $\alpha$  and barrier strategy with parameter  $b_2$ . Let  $L(x; b_1, b_2)$  denote the Laplace transform of the time value of ruin; for  $x \geq 0$ ,

$$L(x; b_1, b_2) = E \left[ e^{-\delta T} \mid U_0 = x \right]. \tag{39}$$

Let

$$\rho(x) = f_1(x) f_2(b_1) - f_2(x) f_1(b_1). \tag{40}$$

**Theorem 4.** For  $0 < x \leq b_1$ , one has

$$L(x; b_1, b_2) = \frac{h(x)}{h(b_1)}A + \frac{\rho(x)}{h(b_1)}, \tag{41}$$

and, for  $b_1 < x \leq b_2$ , one has

$$L(x; b_1, b_2) = \frac{g(x)}{g(b_1)}A + \frac{k(x)}{g(b_1)}B, \tag{42}$$

where

$$\begin{aligned} A &= (k'(b_1)h(b_1)g(b_1) - \rho'(b_1)k'(b_2)g(b_1)) \\ &\quad \times (g'(b_2)k'(b_1)h(b_1) + g(b_1)h'(b_1)k'(b_2) \\ &\quad - g'(b_1)h(b_1)k'(b_2))^{-1}, \\ B &= (g^2(b_1)h'(b_1) - g(b_1)g'(b_1)h(b_1)) \\ &\quad + g(b_1)g'(b_2)\rho'(b_1) \\ &\quad \times (g'(b_2)k'(b_1)h(b_1) + g(b_1)h'(b_1)k'(b_2) \\ &\quad - g'(b_1)h(b_1)k'(b_2))^{-1}. \end{aligned} \tag{43}$$

*Proof.* For  $0 < x \leq b_1$ , applying the strong Markov property, we obtain

$$\begin{aligned} L(x; b_1, b_2) &= E^x [e^{-\delta T}] \\ &= E^x [e^{-\delta T} I(\tau_{b_1}^+ < \tau_0^-)] + E^x [e^{-\delta T} I(\tau_{b_1}^+ > \tau_0^-)] \\ &= E^x [e^{-\delta \tau_{b_1}^+} I(\tau_{b_1}^+ < \tau_0^-)] L(b_1; b_1, b_2) \\ &\quad + E^x [e^{-\delta \tau_0^-} I(\tau_{b_1}^+ > \tau_0^-)] \\ &= \frac{h(x)}{h(b_1)}L(b_1; b_1, b_2) + \frac{\rho(x)}{h(b_1)}. \end{aligned} \tag{44}$$

Similarly, for  $b_1 < x \leq b_2$ ,

$$\begin{aligned} L(x; b_1, b_2) &= E^x [e^{-\delta T}] \\ &= E^x [e^{-\delta \tau} L(x(\tau); b_1, b_2)] \\ &= E^x [e^{-\delta \tau} I(\tau_{b_1}^- < \tau_{b_2}^+)] L(b_1; b_1, b_2) \\ &\quad + E^x [e^{-\delta \tau} I(\tau_{b_1}^- > \tau_{b_2}^+)] L(b_2; b_1, b_2) \\ &= \frac{g(x)}{g(b_1)}L(b_1; b_1, b_2) + \frac{k(x)}{g(b_1)}L(b_2; b_1, b_2). \end{aligned} \tag{45}$$

It can be verified that  $L'(b_2; b_1, b_2) = 1$  and  $L'(b_1+; b_1, b_2) = L'(b_1-; b_1, b_2)$ , so we get

$$\begin{aligned} \frac{g'(b_2)}{g(b_1)}L(b_1; b_1, b_2) + \frac{k'(b_2)}{g(b_1)}L(b_2; b_1, b_2) &= 1, \\ \frac{h'(b_1)}{h(b_1)}L(b_1; b_1, b_2) + \frac{\rho'(x)}{h(b_1)} &= \frac{g'(b_1)}{g(b_1)}L(b_1; b_1, b_2) \\ &\quad + \frac{k'(b_1)}{g(b_1)}L(b_2; b_1, b_2). \end{aligned} \tag{46}$$

From (46) we obtain  $L(b_1; b_1, b_2)$  and  $L(b_2; b_1, b_2)$ , so we get the results (41) and (42).  $\square$

*Remark 5.* (1) Let  $\phi(x; b_1, b_2)$  denote the expected discounted penalty at ruin; in this model, the penalty at ruin is a constant  $M > 0$ , so we get

$$\phi(x; b_1, b_2) = ML(x; b_1, b_2). \tag{47}$$

Substitution of (41) and (42) into (47) yields an expression for  $\phi(x; b_1, b_2)$ .

(2) Let  $\psi(x; b_1, b_2) = P(T < \infty \mid U_0 = x)$  be the probability of ruin. Note that

$$\begin{aligned} E[e^{-\delta T} \mid U_0 = x] &= E^x [e^{-\delta T} I(T < \infty)] \\ &\quad + E^x [e^{-\delta T} I(T = \infty)] \\ &= E^x [e^{-\delta T} I(T < \infty)]; \end{aligned} \tag{48}$$

then

$$\psi(x; b_1, b_2) = \lim_{\delta \rightarrow 0} L(x; b_1, b_2) = P(T < \infty). \tag{49}$$

From (14)–(18), we have

$$\begin{aligned} \lim_{\delta \rightarrow 0} f_1(x) &= \exp \left\{ -\frac{1}{\rho\sigma^2}(\mu + \rho x)^2 \right\} \\ &\quad \times U \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{\rho\sigma^2}(\mu + \rho x)^2 \right), \\ \lim_{\delta \rightarrow 0} f_2(x) &= (\mu + \rho x) \exp \left\{ -\frac{1}{\rho\sigma^2}(\mu + \rho x)^2 \right\} \\ &\quad \times M \left( 1, \frac{3}{2}, \frac{1}{\rho\sigma^2}(\mu + \rho x)^2 \right), \end{aligned} \tag{50}$$

$$\begin{aligned} \lim_{\delta \rightarrow 0} f_3(x) &= \exp \left\{ -\frac{1}{\rho\sigma^2}(\mu - \alpha + \rho x)^2 \right\} \\ &\quad \times U \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{\rho\sigma^2}(\mu - \alpha + \rho x)^2 \right), \\ \lim_{\delta \rightarrow 0} f_4(x) &= (\mu - \alpha + \rho x) \exp \left\{ -\frac{1}{\rho\sigma^2}(\mu - \alpha + \rho x)^2 \right\} \\ &\quad \times M \left( 1, \frac{3}{2}, \frac{1}{\rho\sigma^2}(\mu - \alpha + \rho x)^2 \right). \end{aligned}$$

Taking limit in (41) and (42) yields  $\psi(x; b_1, b_2)$ .

### 5. The Moment-Generating Function of $D$

In this section, the moment-generating function of the hybrid dividend payments is discussed. We adopt a similar approach to that of Gao and Yin in Section 3 [10].

Let

$$M(x, y; b_1, b_2) = E[e^{yD} | U_0 = x] \tag{51}$$

denote the moment-generating function of  $D$ , and let

$$V_m(x; b_1, b_2) = E[D^m U_0 = x], \quad m = 1, 2, 3, \dots, \tag{52}$$

denote the  $m$ th moment function. The following theorem provides differential equations for the function  $M(x, y; b_1, b_2)$ .

**Theorem 6.** *The moment-generating function  $M(x, y; b_1, b_2)$  satisfies the partial differential equations*

$$\begin{aligned} \frac{\sigma^2}{2} \frac{\partial^2 M}{\partial x^2} + \frac{\partial M}{\partial x} (\rho x + \mu) - \frac{\partial M}{\partial y} y \delta &= 0, \quad 0 < x \leq b_1, \\ \frac{\sigma^2}{2} \frac{\partial^2 M}{\partial x^2} + \frac{\partial M}{\partial x} (\rho x + \mu - \alpha) - \frac{\partial M}{\partial y} y \delta &= 0, \quad b_1 < x \leq b_2, \end{aligned} \tag{53}$$

with boundary conditions

$$M(0, y; b_1, b_2) = 1, \quad \lim_{x \rightarrow \infty} M(x, y; b_1, b_2) = e^{y\alpha/\delta}. \tag{54}$$

*Proof.* (i) We first provide the solution of  $U_t$ . Consider the SDE

$$dU_t = (\mu + \rho U_t) dt + \sigma dW_t; \tag{55}$$

note that, in the case  $\sigma = 0$ , the solution to the ODE is  $(1/\rho)(\rho U_0 + \mu)e^{\rho t} - \mu/\rho$ . To solve the SDE, consider the process

$$Y(t) = \frac{1}{\rho} (\rho U_0 + \mu) e^{-\rho t} - \frac{\mu}{\rho}. \tag{56}$$

Let  $P(t) = U_t e^{-\rho t}$  and  $Q(t) = (\mu/\rho)(e^{-\rho t} - 1)$ , so we have  $dY(t) = dP(t) + dQ(t)$ ,

$$\begin{aligned} dP(t) &= dU_t e^{-\rho t} \\ &= U_t (-\rho) e^{-\rho t} dt + e^{-\rho t} dU_t + d[U_t, e^{-\rho t}] \\ &= U_t (-\rho) e^{-\rho t} dt + e^{-\rho t} dU_t \\ &= U_t (-\rho) e^{-\rho t} dt + e^{-\rho t} (\mu + \rho U_t) dt + \sigma dW_t \\ &= e^{-\rho t} \mu dt + e^{-\rho t} \sigma dW_t, \\ dQ(t) &= -e^{-\rho t} \mu dt; \end{aligned} \tag{57}$$

we get  $dY(t) = e^{-\rho t} \sigma dW_t$ , so we obtain

$$Y(t) = Y(0) + \int_0^t e^{-\rho s} \sigma dW_s. \tag{58}$$

Now the solution for  $U_t$  is

$$U_t = e^{\rho t} \left\{ U_0 - \frac{\mu}{\rho} (e^{-\rho t} - 1) + \int_0^t e^{-\rho s} \sigma dW_s \right\}. \tag{59}$$

(ii) Now we derive the integrodifferential equations for  $M(x, y; b_1, b_2)$ .

For  $0 < x \leq b_1$ , we consider a small time interval  $[0, t]$ , the time  $t > 0$  such that the surplus will not reach  $b_1$  before  $t$ ; in view of the strong Markov property of the surplus process  $\{U_t, t \geq 0\}$ , we have

$$M(x, y; b_1, b_2) = E[M(U_t, ye^{-\delta t}; b_1, b_2)] + o(t). \tag{60}$$

By Taylor expansion, we have

$$\begin{aligned} E[M(U_t, ye^{-\delta t}; b_1, b_2)] \\ &= E[M(x, y; b_1, b_2)] + \frac{\partial M}{\partial x} (U_t - x) + \frac{\partial M}{\partial y} (ye^{-\delta t} - y) \\ &\quad + \frac{1}{2} \left( \frac{\partial^2 M}{\partial x^2} (U_t - x)^2 + \frac{\partial^2 M}{\partial y^2} (ye^{-\delta t} - y)^2 \right) \\ &\quad + \frac{\partial M}{\partial x \partial y} (U_t - x)(ye^{-\delta t} - y) + o(t). \end{aligned} \tag{61}$$

Subtracting  $M(x, y; b_1, b_2)$  from each side of the above equation, dividing by  $t$ , and then letting  $t \rightarrow 0$ , we achieve

$$\frac{\sigma^2}{2} \frac{\partial^2 M}{\partial x^2} + \frac{\partial M}{\partial x} \rho \left( x + \frac{\mu}{\rho} \right) - \frac{\partial M}{\partial y} y \delta = 0. \tag{62}$$

Similarly, for  $b_1 < x \leq b_2$ , we consider a small time interval  $[0, t]$ , the time  $t > 0$  such that the surplus will not reach  $b_2$ ; we have

$$M(x, y; b_1, b_2) = E[M(U_t - \alpha t, ye^{-\delta t}; b_1, b_2)] + o(t). \tag{63}$$

By Taylor expansion, we get

$$\begin{aligned} E[M(U_t - \alpha t, ye^{-\delta t}; b_1, b_2)] \\ &= E[M(x, y; b_1, b_2)] + \frac{\partial M}{\partial x} (U_t - x - \alpha t) \\ &\quad + \frac{\partial M}{\partial y} (ye^{-\delta t} - y) \\ &\quad + \frac{1}{2} \left( \frac{\partial^2 M}{\partial x^2} (U_t - x - \alpha t)^2 + \frac{\partial^2 M}{\partial y^2} (ye^{-\delta t} - y)^2 \right) \\ &\quad + \frac{\partial M}{\partial x \partial y} (U_t - x - \alpha t)(ye^{-\delta t} - y) + o(t). \end{aligned} \tag{64}$$

Subtracting  $M(x, y; b_1, b_2)$  from each side of (63), dividing by  $t$ , and then letting  $t \rightarrow 0$ , we achieve

$$\frac{\sigma^2}{2} \frac{\partial^2 M}{\partial x^2} + \frac{\partial M}{\partial x} (\rho x + \mu - \alpha) - \frac{\partial M}{\partial y} y \delta = 0. \tag{65}$$

The proof of boundary conditions is routine. This ends the proof of Theorem 6.  $\square$

**Theorem 7.** For  $0 < x \leq b_1$ ,  $V_m(x; b_1, b_2)$  satisfies

$$\begin{aligned} \frac{\sigma^2}{2} V_m''(x; b_1, b_2) + (\rho x + \rho) V_m'(x; b_1, b_2) \\ - \delta m V_m(x; b_1, b_2) = 0, \end{aligned} \quad (66)$$

and, for  $b_1 < x \leq b_2$ ,  $V_m(x; b_1, b_2)$  satisfies

$$\begin{aligned} \frac{\sigma^2}{2} V_m''(x; b_1, b_2) + (\rho x + \rho - \alpha) V_m'(x; b_1, b_2) \\ - \delta m V_m(x; b_1, b_2) = 0, \end{aligned} \quad (67)$$

with the boundary conditions

$$V_m(0; b_1, b_2) = 0, V_m'(x; b_1, b_2)|_{x=b_2} = m V_{m-1}(b_2; b_1, b_2). \quad (68)$$

*Proof.* Recall that  $M(x, y; b_1, b_2) = E[e^{yD}]$ , and  $V_m(x; b_1, b_2) = E[D^m]$ ; using the representation

$$M(x, y; b_1, b_2) = 1 + \sum_{m=1}^{\infty} \frac{y^m}{m!} V_m(x; b_1, b_2) \quad (69)$$

and equating the coefficients of  $y^m$  in (53) yield the ordinary differential equations (66) and (67).  $\square$

*Remark 8.* When  $m = 1$ , we get  $V_1(x; b_1, b_2) = V(x; b_1, b_2)$ .

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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