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Research Article

On Open-Open Games of Uncountable Length

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The aim of this paper is to investigate the open-open game of uncountable length. We introduce a cardinal number $\mu(X)$, which says how long the Player I has to play to ensure a victory. It is proved that $c(X) \le \mu(X) \le c(X)^+$. We also introduce the class $\mathcal{C}_\kappa$ of topological spaces that can be represented as the inverse limit of κ-complete system $\{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ with $w(X_\sigma) \le \kappa$ and skeletal bonding maps. It is shown that product of spaces which belong to $\mathcal{C}_\kappa$ also belongs to this class and $\mu(X) \le \kappa$ whenever $X \in \mathcal{C}_\kappa$.

1. Introduction

The following game is due to Daniels et al. [1]: two players take turns playing on a topological space X; a round consists of Player I choosing a nonempty open set $U \subseteq X$ and Player II choosing a nonempty open set $V \subseteq U$; a round is played for each natural number. Player I wins the game if the union of open sets which have been chosen by Player II is dense in X. This game is called the *open-open game*.

In this paper, we consider what happens if one drops restrictions on the length of games. If κ is an infinite cardinal and rounds are played for every ordinal number less than κ, then this modification is called *the open-open game of length* κ. The examination of such games is a continuation of [2–4]. A cardinal number $\mu(X)$ is introduced such that $c(X) \le \mu(X) \le c(X)^+$. Topological spaces, which can be represented as an inverse limit of κ-complete system $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ with $w(X_\sigma) \le \kappa$ and each X_σ is T_0 space and skeletal bonding map π^σ_ϱ, are listed as the class $\mathcal{C}_\kappa$. If $\mu(X) = \omega$, then $X \in \mathcal{C}_\omega$. There exists a space X with $X \notin \mathcal{C}_{\mu(X)}$. The class $\mathcal{C}_\kappa$ is closed under any Cartesian product. In particular, the cellularity number of X^I is equal κ whenever $X \in \mathcal{C}_\kappa$. This implies Theorem of Kurepa that $c(X^I) \le 2^\kappa$, whenever $c(X) \le \kappa$. Undefined notions and symbols are used in accordance with books [5–7]. For example, if κ is a cardinal number, then κ^+ denotes the first cardinal greater than κ.

2. When Games Favor Player I

Let X be a topological space. Denote by $\mathcal{T}$ the family of all nonempty open sets of X. For an ordinal number α, let $\mathcal{T}^\alpha$ denote the set of all sequences of the length α consisting of elements of $\mathcal{T}$. The space X is called κ-*favorable* whenever there exists a function

$$\mathbf{s} : \bigcup\{(\mathcal{T})^\alpha : \alpha < \kappa\} \longrightarrow \mathcal{T}, \tag{2.1}$$

such that for each sequence $\{B_{\alpha+1} : \alpha < \kappa\} \subseteq \mathcal{T}$ with $B_1 \subseteq \mathbf{s}(\emptyset)$ and $B_{\alpha+1} \subseteq \mathbf{s}(\{B_{\gamma+1} : \gamma < \alpha\})$, for each $\alpha < \kappa$, the union $\bigcup\{B_{\alpha+1} : \alpha < \kappa\}$ is dense in X. We may also say that the function $\mathbf{s}$ is witness to κ-favorability of X. In fact, $\mathbf{s}$ is a winning strategy for Player I. For abbreviation we say that $\mathbf{s}$ is κ-winning strategy. Sometimes we do not precisely define a strategy. Just give hints how a player should play. Note that, any winning strategy can be arbitrary on steps for limit ordinals.

A family $\mathcal{B}$ of open non-empty subset is called a π-*base* for X if every non-empty open subset $U \subseteq X$ contains a member of $\mathcal{B}$. The smallest cardinal number $|\mathcal{B}|$, where $\mathcal{B}$ is a π-base for X, is denoted by $\pi(X)$.

Proposition 2.1. *Any topological space X is $\pi(X)$-favorable.*

Proof. Let $\{U_\alpha : \alpha < \pi(X)\}$ be a π-base. Put $\mathbf{s}(f) = U_\alpha$ for any sequence $f \in \mathcal{T}^\alpha$. Each family $\{B_\gamma : B_\gamma \subseteq U_\gamma \text{ and } \gamma < \pi(X)\}$ of open non-empty sets is again a π-base for X. So, its union is dense in X. □

According to [6, p. 86] the cellularity of X is denoted by $c(X)$. Let $\operatorname{sat}(X)$ be the smallest cardinal number κ such that every family of pairwise disjoint open sets of X has cardinality $< \kappa$, compare [8]. Clearly, if $\operatorname{sat}(X)$ is a limit cardinal, then $\operatorname{sat}(X) = c(X)$. In all other cases, $\operatorname{sat}(X) = c(X)^+$. Hence, $c(X) \le \operatorname{sat}(X) \le c(X)^+$. Let

$$\mu(X) = \min\{\kappa : X \text{ is a } \kappa\text{-favorable and } \kappa \text{ is a cardinal number}\}. \tag{2.2}$$

Proposition 2.1 implies $\mu(X) \le \pi(X)$. The next proposition gives two natural strategies and gives more accurate estimation than $c(X) \le \mu(X) \le c^+(X)$.

Proposition 2.2. $c(X) \le \mu(X) \le \operatorname{sat}(X)$.

Proof. Suppose $c(X) > \mu(X)$. Fix a family $\{U_\xi : \xi < \mu(X)^+\}$ of pairwise disjoint open sets. If Player II always chooses an open set, which meets at most one U_ξ, then he will not lose the open-open game of the length $\mu(X)$, a contradiction.

Suppose sets $\{B_{\gamma+1} : \gamma < \alpha\}$ are chosen by Player II. If the set

$$X \setminus \operatorname{cl} \bigcup\{B_{\gamma+1} : \gamma < \alpha\} \tag{2.3}$$

is non-empty, then Player I choses it. Player I wins the open-open game of the length $\operatorname{sat}(X)$, when he will use this rule. This gives $\mu(X) \le \operatorname{sat}(X)$. □

Note that, $\omega_0 = c(\{0,1\}^\kappa) = \mu(\{0,1\}^\kappa) \le \mathrm{sat}(\{0,1\}^\kappa) = \omega_1$, where $\{0,1\}^\kappa$ is the Cantor cube of weight κ. There exists a separable space X which is not ω_0-favorable, see Szymański [9] or [1, p.207-208]. Hence we get

$$\omega_0 = c(X) < \mu(X) = \mathrm{sat}(X) = \omega_1. \tag{2.4}$$

3. On Inverse Systems with Skeletal Bonding Maps

Recall that, a continuous surjection is *skeletal* if for any non-empty open sets $U \subseteq X$ the closure of $f[U]$ has non-empty interior. If X is a compact space and Y is a Hausdorff space, then a continuous surjection $f : X \to Y$ is skeletal if and only if $\mathrm{Int}\, f[U] \ne \emptyset$, for every non-empty and open $U \subseteq X$, see Mioduszewski and Rudolf [10].

Lemma 3.1. *A skeletal image of κ-favorable space is a κ-favorable space.*

Proof. A proof follows by the same method as in [11, Theorem 4.1]. In fact, repeat and generalize the proof given in [4, Lemma 1]. □

According to [5], a directed set Σ is said to be κ-*complete* if any chain of length $\le \kappa$ consisting of its elements has the least upper bound in Σ. An inverse system $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ is said to be a κ-*complete*, whenever Σ is κ-complete and for every chain $A \subseteq \Sigma$, where $|A| \le \kappa$, such that $\sigma = \sup A \in \Sigma$ we get

$$X_\sigma = \varprojlim \left\{ X_\alpha, \pi^\beta_\alpha, A \right\}. \tag{3.1}$$

In addition, we assume that bonding maps are surjections.

For ω-favorability, the following lemma is given without proof in [1, Corollary 1.4]. We give a proof to convince the reader that additional assumptions on topology are unnecessary.

Lemma 3.2. *If $Y \subseteq X$ is dense, then X is κ-favorable if and only if Y is κ-favorable.*

Proof. Let a function σ_X be a witness to κ-favorability of X. Put

$$\sigma_Y(\emptyset) = \sigma_X(\emptyset) \cap Y. \tag{3.2}$$

If Player II chooses open set $V_1 \cap Y \subseteq \sigma_Y(\emptyset)$, then put

$$V_1' = V_1 \cap \sigma_X(\emptyset) \subseteq \sigma_X(\emptyset). \tag{3.3}$$

We get $V_1' \cap Y = V_1 \cap Y \subseteq \sigma_Y(\emptyset)$, since $V_1 \cap Y \subset \sigma_X(\emptyset) \cap Y$. Then we put

$$\sigma_Y(V_1 \cap Y) = \sigma_X\left(V_1'\right) \cap Y. \tag{3.4}$$

Suppose we have already defined

$$\sigma_Y\left(\left\{V_{\alpha+1} \cap Y : \alpha < \gamma\right\}\right) = \sigma_X\left(\left\{V'_{\alpha+1} : \alpha < \gamma\right\}\right) \cap Y, \tag{3.5}$$

for $\gamma < \beta < \kappa$. If Player II chooses open set $V_{\beta+1} \cap Y \subseteq \sigma_Y(\{V_{\alpha+1} \cap Y : \alpha < \beta\})$, then put

$$V'_{\beta+1} = V_{\beta+1} \cap \sigma_X(\{V'_{\alpha+1} : \alpha < \beta\}) \subseteq \sigma_X(\{V'_{\alpha+1} : \alpha < \beta\}). \tag{3.6}$$

Finally, put

$$\sigma_Y(\{V_{\alpha+1} \cap Y : \alpha \leq \beta\}) = \sigma_X(\{V'_{\alpha+1} : \alpha \leq \beta\}) \cap Y \tag{3.7}$$

and check that σ_Y is witness to κ-favorability of Y.

Assume that σ_Y is a witness to κ-favorability of Y. If $\sigma_Y(\emptyset) = U_0 \cap Y$ and $U_0 \subseteq X$ is open, then put $\sigma_X(\emptyset) = U_0$. If Player II chooses open set $V_1 \subseteq \sigma_X(\emptyset)$, then $V_1 \cap Y \subseteq \sigma_Y(\emptyset)$. Put $\sigma_X(V_1) = U_1$, where $\sigma_Y(V_1 \cap Y) = U_1 \cap Y$ and $U_1 \subseteq X$ is open. Suppose

$$\sigma_Y(\{V_{\alpha+1} \cap Y : \alpha < \gamma\}) = U_\gamma \cap Y, \qquad \sigma_X(\{V_{\alpha+1} : \alpha < \gamma\}) = U_\gamma \tag{3.8}$$

have been already defined for $\gamma < \beta < \kappa$. If II Player chooses open set $V_{\beta+1} \subseteq \sigma_X(\{V_{\alpha+1} : \alpha < \beta\})$, then put $\sigma_X(\{V_{\alpha+1} : \alpha < \beta+1\}) = U_{\beta+1}$, where open set $U_{\beta+1} \subseteq X$ X is determined by $\sigma_Y(\{V_{\alpha+1} \cap Y : \alpha < \beta+1\}) = U_{\beta+1} \cap Y$. □

The next theorem is similar to [12, Theorem 2]. We replace a continuous inverse system with indexing set being a cardinal, by κ-complete inverse system, and also $c(X)$ is replaced by $\mu(X)$. Let κ be a fixed cardinal number.

Theorem 3.3. *Let X be a dense subset of the inverse limit of the κ-complete system $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$, where $\kappa = \sup\{\mu(X_\sigma) : \sigma \in \Sigma\}$. If all bonding maps are skeletal, then $\mu(X) = \kappa$.*

Proof. By Lemma 3.2, one can assume that $X = \lim_{\leftarrow}\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$. Fix functions $\mathbf{s}_\sigma : \mathcal{T}_\sigma^{<\kappa} \to \mathrm{T}_\sigma$, each one is a witness to $\mu(X_\sigma)$-favorability of X_σ. This does not reduce the generality, because $\mu(X_\sigma) \leq \kappa$ for every $\sigma \in \Sigma$. In order to explain the induction, fix a bijection $f : \kappa \to \kappa \times \kappa$ such that

(1) if $f(\alpha) = (\beta, \zeta)$, then $\beta, \zeta \leq \alpha$;

(2) $f^{-1}(\beta, \gamma) < f^{-1}(\beta, \zeta)$ if and only if $\gamma < \zeta$;

(3) $f^{-1}(\gamma, \beta) < f^{-1}(\zeta, \beta)$ if and only if $\gamma < \zeta$.

One can take as f an isomorphism between κ and $\kappa \times \kappa$, with canonical well-ordering, see [7]. The function f will indicate the strategy and sets that we have taken in the following induction.

We construct a function $\mathbf{s} : \mathcal{T}^{<\kappa} \to \mathcal{T}$ which will provide κ-favorability of X. The first step is defined for $f(0) = (0, 0)$. Take an arbitrary $\sigma_1 \in \Sigma$ and put

$$\mathbf{s}(\emptyset) = \pi_{\sigma_1}^{-1}(\mathbf{s}_{\sigma_1}(\emptyset)). \tag{3.9}$$

Assume that Player II chooses non-empty open set $B_1 = \pi_{\sigma_2}^{-1}(V_1) \subseteq \mathbf{s}(\emptyset)$, where $V_1 \subseteq X_{\sigma_2}$ is open. Let

$$\mathbf{s}(\{B_1\}) = \pi_{\sigma_1}^{-1}(\mathbf{s}_{\sigma_1}(\{\operatorname{Int} \operatorname{cl} \pi_{\sigma_1}(B_1) \cap \mathbf{s}_{\sigma_1}(\emptyset)\})) \tag{3.10}$$

and denote $D_0^0 = \text{Int cl}\,\pi_{\sigma_1}(B_1) \cap \mathbf{s}_{\sigma_1}(\emptyset)$. So, after the first round and the next respond of Player I, we know: indexes σ_1 and σ_2, the open set $B_1 \subseteq X$ and the open set $D_0^0 \subseteq X_{\sigma_1}$.

Suppose that sequences of open sets $\{B_{\alpha+1} \subseteq X : \alpha < \gamma\}$, indexes $\{\sigma_{\alpha+1} : \alpha < \gamma\}$, and sets $\{D_\zeta^\varphi : f^{-1}(\varphi,\zeta) < \gamma\}$ have been already defined such that.

If $\alpha < \gamma$ and $f(\alpha) = (\varphi,\eta)$, then

$$B_{\alpha+1} = \pi^{-1}_{\sigma_{\alpha+2}}(V_{\alpha+1}) \subseteq \mathbf{s}\left(\left\{B_{\xi+1} : \xi < \alpha\right\}\right) = \pi^{-1}_{\sigma_{\varphi+1}}\left(\mathbf{s}_{\sigma_{\varphi+1}}\left(\left\{D_\nu^\varphi : \nu < \eta\right\}\right)\right), \tag{3.11}$$

where $D_\nu^\varphi = \text{Int cl}\,\pi_{\sigma_{\varphi+1}}(B_{f^{-1}((\varphi,\nu))+1}) \cap \mathbf{s}_{\sigma_{\varphi+1}}(\{D_\zeta^\varphi : \zeta < \nu\})$ and $V_{\alpha+1} \subseteq X_{\sigma_{\alpha+2}}$ are open.

If $f(\gamma) = (\theta,\lambda)$ and $\beta < \lambda$, then take

$$D_\beta^\theta = \text{Int cl}\,\pi_{\sigma_{\theta+1}}(B_{f^{-1}((\theta,\beta))+1}) \cap \mathbf{s}_{\sigma_{\theta+1}}\left(\left\{D_\zeta^\theta : \zeta < \beta\right\}\right) \tag{3.12}$$

and put

$$\mathbf{s}\left(\left\{B_{\alpha+1} : \alpha < \gamma\right\}\right) = \pi^{-1}_{\sigma_{\theta+1}}\left(\mathbf{s}_{\sigma_{\theta+1}}\left(\left\{D_\alpha^\theta : \alpha < \lambda\right\}\right)\right). \tag{3.13}$$

Since Σ is κ-complete, one can assume that the sequence $\{\sigma_{\alpha+1} : \alpha < \kappa\}$ is increasing and $\sigma = \sup\{\sigma_{\xi+1} : \xi < \kappa\} \in \Sigma$.

We will prove that $\bigcup_{\alpha<\kappa} B_{\alpha+1}$ is dense in X. Since $\pi_\sigma^{-1}(\pi_\sigma(B_{\alpha+1})) = B_{\alpha+1}$ for each $\alpha < \kappa$ and π_σ is skeletal map, it is sufficient to show that $\bigcup_{\alpha<\kappa} \pi_\sigma(B_{\alpha+1})$ is dense in X_σ. Fix arbitrary open set $(\pi^\sigma_{\sigma_{\xi+1}})^{-1}(W)$ where W is an open set of $X_{\xi+1}$. Since $\mathbf{s}_{\sigma_{\xi+1}}$ is winning strategy on $X_{\sigma_{\xi+1}}$, there exists D_α^ξ such that $D_\alpha^\xi \cap W \neq \emptyset$, and $D_\alpha^\xi \subseteq \text{Int cl}\,\pi_{\sigma_{\xi+1}}(B_{f^{-1}((\xi,\alpha))+1})$. Therefore we get

$$\left(\pi^\sigma_{\sigma_{\xi+1}}\right)^{-1}(W) \cap \pi_\sigma(B_{\delta+1}) \neq \emptyset, \tag{3.14}$$

where $\delta = f^{-1}((\xi,\alpha))$. Indeed, suppose that $(\pi^\sigma_{\sigma_{\xi+1}})^{-1}(W) \cap \pi_\sigma(B_{\delta+1}) = \emptyset$. Then

$$\emptyset = \pi^\sigma_{\sigma_{\xi+1}}\left[\left(\pi^\sigma_{\sigma_{\xi+1}}\right)^{-1}(W) \cap \pi_\sigma(B_{\delta+1})\right] = W \cap \pi^\sigma_{\sigma_{\xi+1}}[\pi_\sigma(B_{\delta+1})] = W \cap \pi_{\sigma_{\xi+1}}(B_{\delta+1}). \tag{3.15}$$

Hence we have $W \cap \text{Int cl}\,\pi_{\sigma_{\xi+1}}(B_{\delta+1}) = \emptyset$, a contradiction. □

Corollary 3.4. *If X is dense subset of an inverse limit of $\mu(X)$-complete system $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$, where all bonding map are skeletal, then*

$$c(X) = \sup\{c(X_\sigma) : \sigma \in \Sigma\}. \tag{3.16}$$

Proof. Let $X = \lim_{\leftarrow}\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$. Since $c(X) \geq c(X_\sigma)$, for every $\sigma \in \Sigma$, we will show that

$$c(X) \leq \sup\{c(X_\sigma) : \sigma \in \Sigma\}. \tag{3.17}$$

Suppose that $\sup\{c(X_\sigma) : \sigma \in \Sigma\} = \tau < c(X)$. Using Proposition 2.2 and Theorem 3.3, check that

$$\mu(X) = \sup\{\mu(X_\sigma) : \sigma \in \Sigma\} \leq \sup\{c(X_\sigma)^+ : \sigma \in \Sigma\} \leq \tau^+ \leq c(X). \tag{3.18}$$

So, we get $\mu(X) = c(X) = \tau^+$. Therefore, there exists a family $\mathcal{R}$, of size τ^+, which consists of pairwise disjoint open subset of X. We can assume that

$$\mathcal{R} \subseteq \left\{\pi_\sigma^{-1}(U) : U \text{ is an open subset of } X_\sigma,\ \sigma \in \Sigma\right\}. \tag{3.19}$$

Since $\{X_\sigma, \pi_\varrho^\sigma, \Sigma\}$ is $\mu(X)$-complete inverse system and $|\mathcal{R}| = \mu(X)$, there exists $\beta \in \Sigma$ such that

$$\mathcal{R} \subseteq \left\{\pi_\beta^{-1}(U) : U \text{ is an open subset of } X_\beta\right\}, \tag{3.20}$$

a contradiction with $c(X_\beta) < \tau^+$. □

The above corollary is similar to [12, Theorem 1], but we replaced a continuous inverse system, whose indexing set is a cardinal number by κ-complete inverse system.

4. Classes $\mathcal{C}_\kappa$

Let κ be an infinite cardinal number. Consider inverse limits of κ-complete system $\{X_\sigma, \pi_\varrho^\sigma, \Sigma\}$ with $w(X_\sigma) \leq \kappa$. Let $\mathcal{C}_\kappa$ be a class of such inverse limits with skeletal bonding maps and X_σ being T_0-space. Now, we show that the class $\mathcal{C}_\kappa$ is stable under Cartesian products.

Theorem 4.1. *The Cartesian product of spaces from $\mathcal{C}_\kappa$ belongs to $\mathcal{C}_\kappa$.*

Proof. Let $X = \prod\{X_s : s \in S\}$ where each $X_s \in \mathcal{C}_\kappa$. For each $s \in S$, let $X_s = \lim_{\leftarrow}\{X_\sigma, s_\rho^\sigma, \Sigma_s\}$ be a κ-complete inverse system with skeletal bonding map such that each T_0-space X_σ has the weight $\leq \kappa$. Consider the union

$$\Gamma = \bigcup\left\{\prod_{s\in A}\Sigma_s : A \in [S]^\kappa\right\}. \tag{4.1}$$

Introduce a partial order on Γ as follows:

$$f \preceq g \Longleftrightarrow \operatorname{dom}(f) \subseteq \operatorname{dom}(g), \quad \forall_{a\in\operatorname{dom}(f)} f(a) \leq_a g(a), \tag{4.2}$$

where $\leq_a$ is the partial order on Σ_a. The set Γ with the relation $\preceq$ is upward directed and κ-complete.

If $f \in \Gamma$, then Y_f denotes the Cartesian product

$$\prod\{X_{f(a)} : a \in \operatorname{dom}(f)\}. \tag{4.3}$$

If $f \preceq g$, then put

$$p_f^g = \left(\prod_{a \in \mathrm{dom}(f)} a_{f(a)}^{g(a)} \right) \circ \pi_{\mathrm{dom}(f)}^{\mathrm{dom}(g)}, \tag{4.4}$$

where $\pi_{\mathrm{dom}(f)}^{\mathrm{dom}(g)}$ is the projection of $\prod\{X_{g(a)} : a \in \mathrm{dom}(g)\}$ onto $\prod\{X_{g(a)} : a \in \mathrm{dom}(f)\}$ and $\prod_{a \in \mathrm{dom}(f)} a_{f(a)}^{g(a)}$ is the Cartesian product of the bonding maps $a_{f(a)}^{g(a)} : X_{g(a)} \to X_{f(a)}$. We get the inverse system$\{Y_f, p_f^g, \Gamma\}$ which is κ-complete, bonding maps are skeletal and $w(Y_f) \le \kappa$. So, we can take $Y = \lim_{\leftarrow}\{Y_f, p_f^g, \Gamma\}$.

Now, define a map $h : X \to Y$ by the following formula:

$$h(\{x_s\}_{s \in S}) = \{x_f\}_{f \in \Gamma}, \tag{4.5}$$

where $x_f = \{x_{f(a)}\}_{a \in \mathrm{dom}(f)} \in Y_f$ and $f \in \prod\{\Sigma_a : a \in \mathrm{dom}(f)\}$ and $\mathrm{dom}(f) \in [S]^\kappa$. By the property

$$\{x_s\}_{s \in S} = \{t_s\}_{s \in S} \Longleftrightarrow \forall_{s \in S} \forall_{\sigma \in \Sigma_s}, \quad x_\sigma = t_\sigma \Longleftrightarrow \forall_{f \in \Gamma}, \quad x_f = t_f, \tag{4.6}$$

the map h is well defined and it is injection.

The map h is surjection. Indeed, let $\{b_f\}_{f \in \Gamma} \in Y$. For each $s \in S$ and each $\sigma \in \Sigma_s$ we fix $f_\sigma^s \in \Gamma$ such that $s \in \mathrm{dom}(f_\sigma^s)$ and $f_\sigma^s(s) = \sigma$. Let $\pi_{f(s)} : Y_f \to X_{f(s)}$ be a projection for each $f \in \Gamma$.

For each $t \in S$ let define $b_t = \{b_\sigma\}_{\sigma \in \Sigma_t}$, where $b_\sigma = \pi_{f_\sigma^t(t)}(b_{f_\sigma^t})$. We will prove that an element b_t is a thread of the space X_t. Indeed, if $\sigma \ge \rho$ and $\sigma, \rho \in \Sigma_t$, then take functions f_σ^t and g_ρ^t. For abbreviation, denote $f = f_\sigma^t$ and $g = g_\rho^t$. Define a function $h : \mathrm{dom}(f) \cup \mathrm{dom}(g) \to \bigcup\{\Sigma_t : t \in \mathrm{dom}(f) \cup \mathrm{dom}(g)\}$ in the following way:

$$h(s) = \begin{cases} g(s), & \text{if } s \in \mathrm{dom}(g) \setminus \mathrm{dom}(f), \\ f(s), & \text{if } s \in \mathrm{dom}(f). \end{cases} \tag{4.7}$$

The function h is element of Γ and $f, g \preceq h$. Note that $h \mid \mathrm{dom}(f) = f$ and $h \mid \mathrm{dom}(g) \setminus \{t\} = g \mid \mathrm{dom}(g) \setminus \{t\}$. Since

$$\begin{aligned} \{b_{g(s)}\}_{s \in \mathrm{dom}(g)} = b_g = p_g^h(b_h) &= \left(\prod_{s \in \mathrm{dom}(g)} s_{g(s)}^{h(s)} \right) \left(\pi_{\mathrm{dom}(g)}^{\mathrm{dom}(h)}(b_h) \right) \\ &= \left(\prod_{s \in \mathrm{dom}(g)} s_{g(s)}^{h(s)} \right) \left(\{b_{h(s)}\}_{s \in \mathrm{dom}(g)} \right) = \left\{ s_{g(s)}^{h(s)}(b_{h(s)}) \right\}_{s \in \mathrm{dom}(g)}, \end{aligned} \tag{4.8}$$

we get

$$b_\rho = b_{g(t)} = s_{g(t)}^{h(t)}\left(b_{h(t)}\right) = s_{g(t)}^{f(t)}\left(b_{f(t)}\right) = s_\rho^\sigma(b_\sigma). \tag{4.9}$$

It is clear that $h(\{a_t\}_{t\in S}) = \{b_f\}_{f\in\Gamma}$.

We shall prove that the map h is continuous. Take an open subset $U = \prod_{s\in \mathrm{dom}(f)} A_{f(s)} \subseteq Y_f$ such that

$$A_{f(s)} = \begin{cases} V, & \text{if } s = s_0, \\ X_{f(s)}, & \text{otherwise}, \end{cases} \tag{4.10}$$

where $V \subseteq X_{f(s_0)}$ is open subset. A map p_f is projection from the inverse limit Y to Y_f. It is sufficient to show that

$$h^{-1}\left((p_f)^{-1}(U)\right) = \prod_{s\in S} B_s, \tag{4.11}$$

where

$$B_s = \begin{cases} W, & \text{if } s = s_0, \\ X_s, & \text{otherwise}, \end{cases} \tag{4.12}$$

and $W = \pi_{f(s_0)}^{-1}(V)$ and $\pi_{f(s_0)} : Y_f \to X_{\sigma_0}$ is the projection and $f(s_0) = \sigma_0$. We have

$$\begin{aligned} \{x_s\}_{s\in S} \in h^{-1}\left((p_f)^{-1}(U)\right) &\Longleftrightarrow p_f(h(\{x_s\}_{s\in S})) \in U \\ &\Longleftrightarrow p_f\left(\{x_f\}_{f\in\Gamma}\right) = x_f \in U \Longleftrightarrow x_{f(s_0)} \in V \\ &\Longleftrightarrow x_{s_0} \in W \Longleftrightarrow x \in \prod_{s\in S} B_s \subseteq \prod_{s\in S} X_s = X. \end{aligned} \tag{4.13}$$

Since the map h is bijection and

$$(p_f)^{-1}(U) = h\left(h^{-1}\left((p_f)^{-1}(U)\right)\right) = h\left(\prod_{s\in S} B_s\right) \tag{4.14}$$

for any subbase subset $\prod_{s\in S} B_s \subseteq X$, the map h is open. □

In the case $\kappa = \omega$ we have well-known results that product of I-favorable space is I-favorable space (see [1] or [2]).

Corollary 4.2. *Every I-favorable space is stable under any product.*

If D is a set and κ is cardinal number then we denote $\bigcup_{\alpha<\kappa} D^\alpha$ by $D^{<\kappa}$.

The following result probably is known but we give a proof for the sake of completeness.

Theorem 4.3. *Let κ be an infinite cardinal and let T be a set such that $|T| \geq \kappa^{\kappa}$. If $A \in [T]^{\kappa}$ and $f_{\delta} : T^{<\kappa} \to T$ for all $\delta < \kappa^{<\kappa}$ then there exists a set $B \subseteq T$ such that $|B| \leq \tau$ and $A \subseteq B$ and $f_{\delta}(C) \in B$ for every $C \in B^{<\kappa}$ and every $\delta < \kappa^{<k}$, where*

$$\tau = \begin{cases} \kappa^{<\kappa}, & \text{for regular } \kappa, \\ \kappa^{\kappa}, & \text{otherwise}. \end{cases} \tag{4.15}$$

Proof. Assume that κ is regular cardinal. Let $A \in [T]^{\kappa}$ and let $f_{\delta} : \bigcup_{\alpha<\kappa} T^{\alpha} \to T$ for $\delta < \kappa^{<\kappa}$. Let $A_0 = A$. Assume that we have defined A_{α} for $\alpha < \beta$ such that $|A_{\alpha}| \leq \kappa^{|\alpha|}$. Put

$$A_{\beta} = \left(\bigcup_{\alpha<\beta} A_{\alpha}\right) \cup \left\{ f_{\delta}(C) : C \in \left(\bigcup_{\alpha<\beta} A_{\alpha}\right)^{<\beta},\ \delta < \kappa^{|\beta|} \right\}. \tag{4.16}$$

Calculate the size of the set A_{β}:

$$|A_{\beta}| \leq \left|\left(\bigcup_{\alpha<\beta} A_{\alpha}\right)\right| \left|\kappa^{|\beta|}\right| \left|\left(\bigcup_{\alpha<\beta} A_{\alpha}\right)^{<\beta}\right| \leq \kappa^{|\beta|} \left|\left(\kappa^{|\beta|}\right)^{|\beta|}\right| \leq \kappa^{|\beta|}. \tag{4.17}$$

Let $B = \bigcup_{\beta<\kappa} A_{\beta}$, so we get $|B| \leq \kappa^{<\kappa}$. Fix a sequence $\langle b_{\alpha} : \alpha < \beta \rangle \subseteq B$ and f_{γ}. Since $\mathrm{cf}(\kappa) = \kappa$ there exists $\delta < \kappa$ such that $C = \{b_{\alpha} : \alpha < \beta\} \subseteq A_{\delta}$ and $f_{\gamma}(C) \in A_{\sigma+1}$ for some $\sigma < \kappa$.

In the second case $\mathrm{cf}(\kappa) < \kappa$, we proceed the above induction up to $\beta = \kappa$. Let $B = A_{\kappa}$, so we get $|B| \leq \kappa^{\kappa}$ and $B = \bigcup_{\beta<\kappa^{+}} A_{\beta}$. Similarly to the first case we get that B is closed under all function f_{δ}, $\delta < \kappa^{<\kappa}$. □

Theorem 4.4. *If X belongs to the class $\mathcal{C}_{\kappa}$ then $c(X) \leq \kappa$.*

Proof. If $X \in \mathcal{C}_{\kappa}$ then by Theorems 3.3 and Proposition 2.2 we get $c(X) \leq \mu(X) \leq \kappa$. □

We apply some facts from the paper [3]. Let $\mathcal{P}$ be a family of open subset of topological space X and $x, y \in X$. We say that $x \sim_{\mathcal{P}} y$ if and only if $x \in V \Leftrightarrow y \in V$ for every $V \in \mathcal{P}$. The family of all sets $[x]_{\mathcal{P}} = \{y : y \sim_{\mathcal{P}} x\}$ we denote by $X/\mathcal{P}$. Define a map $q : X \to X/\mathcal{P}$ as follows $q[x] = [x]_{\mathcal{P}}$. The set $X/\mathcal{P}$ is equipped with topology $\mathcal{T}_{\mathcal{P}}$ generated by all images $q[V]$ where $V \in \mathcal{P}$.

Recall Lemma 1 from paper [3]: if $\mathcal{P}$ is a family of open set of X and $\mathcal{P}$ is closed under finite intersection then the mapping $q : X \to X/\mathcal{P}$ is continuous. Moreover if $X = \bigcup \mathcal{P}$ then the family $\{q[V] : V \in \mathcal{P}\}$ is a base for the topology $\mathcal{T}_{\mathcal{P}}$.

Notice that if $\mathcal{P}$ has a property

$$\begin{gathered} \forall (W \in \mathcal{P}) \exists (\{V_n : n < \omega\} \subseteq \mathcal{P}) \exists (\{U_n : n < \omega\} \subseteq \mathcal{P}), \\ W = \bigcup_{n<\omega} U_n, \quad \forall (n < \omega) U_n \subseteq X \setminus V_n \subseteq U_{n+1}, \end{gathered} \tag{seq}$$

then $\bigcup \mathcal{P} = X$ and by [3, Lemma 3] the topology $\mathcal{T}_{\mathcal{P}}$ is Hausdorff. Moreover if $\mathcal{P}$ is closed under finite intersection then by [3, Lemma 4] the topology $\mathcal{T}_{\mathcal{P}}$ is regular. Theorem 5 and Lemma 9 [3] yeild.

Theorem 4.5. *If $\mathcal{P}$ is a set of open subset of topological space X such that*

(1) *is closed under κ-winning strategy, finite union and intersection,*

(2) *has property (seq),*

then $X/\mathcal{P}$ with topology $\mathcal{T}_{\mathcal{P}}$ is completely regular space and $q : X \to X/\mathcal{P}$ is skeletal.

If a topological space X has the cardinal number $\mu(X) = \omega$ then $X \in \mathcal{C}_{\omega}$, but for $\mu(X)$ equals for instance ω_1 we get only $X \in \mathcal{C}_{\omega_1{}^{\omega}}$.

Theorem 4.6. *Each Tichonov space X with $\mu(X) = \kappa$ can be dense embedded into inverse limit of a system $\{X_\sigma, \pi_\varrho^\sigma, \Sigma\}$, where all bonding map are skeletal, indexing set Σ is τ-complete each X_σ is Tichonov space with $w(X_\sigma) \le \tau$ and*

$$\tau = \begin{cases} \kappa^{<\kappa}, & \text{for regular } \kappa, \\ \kappa^{\kappa}, & \text{otherwise.} \end{cases} \tag{4.18}$$

Proof. Let $\mathcal{B}$ be a π-base for topological space X consisting of cozero sets and $\sigma : \bigcup\{\mathcal{B}^\alpha : \alpha < \kappa\} \to \mathcal{B}$ be a κ-winning strategy. We can define a function of finite intersection property and finite union property as follows: $g(\{B_0, B_1, \ldots, B_n\}) = B_0 \cap B_1 \cap \cdots \cap B_n$ and $h(\{B_0, B_1, \ldots, B_n\}) = B_0 \cup B_1 \cup \cdots \cup B_n$. For each cozero set $V \in \mathcal{B}$ fix a continuous function $f_V : X \to [0,1]$ such that $V = f_V^{-1}((0,1])$. Put $\sigma_{2n}(V) = f_V^{-1}((1/n, 1])$ and $\sigma_{2n+1}(V) = f_V^{-1}([0, 1/n))$. By Theorem 4.3 for each $\mathcal{R} \in [\mathcal{B}]^\kappa$ and all functions h, g, σ_n, σ there is subset $\mathcal{P} \subseteq \mathcal{B}$ such that

(1) $|\mathcal{P}| \le \tau$, where

$$\tau = \begin{cases} \kappa^{<\kappa}, & \text{for regular } \kappa, \\ \kappa^{\kappa}, & \text{otherwise,} \end{cases} \tag{4.19}$$

(2) $\mathcal{R} \subseteq \mathcal{P}$,

(3) $\mathcal{P}$ is closed under κ-winning strategy σ, function of finite intersection property and finite union property,

(4) $\mathcal{P}$ is closed under σ_n, $n < \omega$, hence $\mathcal{P}$ holds property (seq).

Therefore by Theorem 4.5 we get skeletal mapping $q_{\mathcal{P}} : X \to X/\mathcal{P}$. Let $\Sigma \subseteq [\mathcal{B}]^{\le\tau}$ be a set of families which satisfies above condition (1), (2), (3) and the (4). If Σ is directed by inclusion. It is easy to check that Σ is τ-complete. Similar to [3, Theorem 11] we define a function $f : X \to Y$ as follows $f(x) = \{f_{\mathcal{P}}(x)\}$, where $f(x)_{\mathcal{P}} = q_{\mathcal{P}}(x)$ and $Y = \lim_{\leftarrow}\{X/\mathcal{R}, q_{\mathcal{P}}^{\mathcal{R}}, \mathcal{C}\}$. If $\mathcal{R}, \mathcal{P} \in \mathcal{C}$ and $\mathcal{P} \subseteq \mathcal{R}$, then $q_{\mathcal{P}}^{\mathcal{R}}(f(x)_{\mathcal{R}}) = f(x)_{\mathcal{P}}$. Thus $f(x)$ is a thread, that is, $f(x) \in Y$. It easy to see that f is homeomorphism onto its image and $f[X]$ is dense in Y, compare [3, proof of Theorem 11]. □

Theorem 4.6 suggests question.

Does each space X belong to $\mathcal{C}_{\mu(X)}$?

Fleissner [13] proved that there exists a space Y such that $c(Y) = \aleph_0$ and $c(Y^3) = \aleph_2$. Hence, we get $\mu(Y) = \aleph_1$, by Theorem 3.3 and Corollary 4.2. Suppose that $Y \in \mathcal{C}_{\mu(X)}$ then $c(Y^3) \leq \aleph_1$, by Theorem 4.4, a contradiction.

Corollary 4.7. *If X is topological space with $\mu(X) = \kappa$ then $c(X^I) \leq \tau$ and*

$$\tau = \begin{cases} \kappa^{<\kappa}, & \text{for regular } \kappa, \\ \kappa^{\kappa}, & \text{otherwise.} \end{cases} \tag{4.20}$$

Proof. By Theorem 4.3 we get $X^I \in \mathcal{C}_\tau$. Hence by Theorems 4.4 and 4.1 we have $c(X^I) \leq \tau$. □

By above Corollary we get the following.

Corollary 4.8 (see [14, Kurepa]). *If $\{X_s : s \in S\}$ is a family of topological spaces and $c(X_s) \leq \kappa$ for each $s \in S$, then $c(\prod\{X_s : s \in S\}) \leq 2^\kappa$.*

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