

*Research Article*

# Convergence Theorems for Maximal Monotone Operators, Weak Relatively Nonexpansive Mappings and Equilibrium Problems

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We introduce hybrid-iterative schemes for solving a system of the zero-finding problems of maximal monotone operators, the equilibrium problem, and the fixed point problem of weak relatively nonexpansive mappings. We then prove, in a uniformly smooth and uniformly convex Banach space, strong convergence theorems by using a shrinking projection method. We finally apply the obtained results to a system of convex minimization problems.

## 1. Introduction

Let  $E$  be a real Banach space and  $C$  a nonempty subset of  $E$ . Let  $E^*$  be the dual space of  $E$ . We denote the value of  $x^* \in E^*$  at  $x \in E$  by  $\langle x^*, x \rangle$ . Let  $T : C \rightarrow C$  be a nonlinear mapping. We denote by  $F(T)$  the fixed points set of  $T$ , that is,  $F(T) = \{x \in C : x = Tx\}$ . Let  $A : E \rightarrow 2^{E^*}$  be a set-valued mapping. We denote  $D(A)$  by the domain of  $A$ , that is,  $D(A) = \{x \in E : Ax \neq \emptyset\}$  and also denote  $G(A)$  by the graph of  $A$ , that is,  $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$ . A set-valued mapping  $A$  is said to be *monotone* if  $\langle x^* - y^*, x - y \rangle \geq 0$  whenever  $(x, x^*), (y, y^*) \in G(A)$ . It is said to be *maximal monotone* if its graph is not contained in the graph of any other monotone operators on  $E$ . It is known that if  $A$  is maximal monotone, then the set  $A^{-1}(0^*) = \{z \in E : 0^* \in Az\}$  is closed and convex.

The problem of finding a zero point of maximal monotone operators plays an important role in optimizations. This is because it can be reformulated to a convex minimization

problem and a variational inequality problem. Many authors have studied the convergence of such problems in various spaces (see, e.g., [1–16]). Initiated by Martinet [17], in a real Hilbert space  $H$ , Rockafellar [18] introduced the following iterative scheme:  $x_1 \in H$  and

$$x_{n+1} = J_{\lambda_n} x_n, \quad \forall n \geq 1, \quad (1.1)$$

where  $\{\lambda_n\} \subset (0, \infty)$ ,  $J_\lambda$  is the resolvent of  $A$  defined by  $J_\lambda := J_{\lambda A} = (I + \lambda A)^{-1}$  for all  $\lambda > 0$ , and  $A$  is a maximal monotone operator on  $H$ . Such an algorithm is called the *proximal point algorithm*. It was proved that the sequence  $\{x_n\}$  generated by (1.1) converges weakly to an element in  $A^{-1}(0)$  provided that  $\liminf_{n \rightarrow \infty} \lambda_n > 0$ . Recently, Kamimura and Takahashi [19] introduced the following iteration in a real Hilbert space:  $x_1 \in H$  and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{\lambda_n} x_n, \quad \forall n \geq 1, \quad (1.2)$$

where  $\{\alpha_n\} \subset [0, 1]$  and  $\{\lambda_n\} \subset (0, \infty)$ . The weak convergence theorems are also established in a real Hilbert space under suitable conditions imposed on  $\{\alpha_n\}$  and  $\{\lambda_n\}$ .

In 2004, Kamimura et al. [20] extended the above iteration process to a much more general setting. In fact, they proposed the following algorithm:  $x_1 \in E$  and

$$x_{n+1} = J^{-1}(\alpha_n J(x_n) + (1 - \alpha_n) J(J_{\lambda_n} x_n)), \quad \forall n \geq 1, \quad (1.3)$$

where  $\{\alpha_n\} \subset [0, 1]$ ,  $\{\lambda_n\} \subset (0, \infty)$ , and  $J_\lambda := J_{\lambda A} = (J + \lambda A)^{-1} J$  for all  $\lambda > 0$ . They proved, in a uniformly smooth and uniformly convex Banach space, a weak convergence theorem.

Let  $F : C \times C \rightarrow \mathbb{R}$ , where  $\mathbb{R}$  is the set of real numbers, be a bifunction. The equilibrium problem is to find  $x \in C$  such that

$$F(x, y) \geq 0, \quad \forall y \in C. \quad (1.4)$$

The solutions set of (1.4) is denoted by  $EP(F)$ .

For solving the equilibrium problem, we assume that

- (A1)  $F(x, x) = 0$  for all  $x \in C$ ,
- (A2)  $F$  is monotone, that is  $F(x, y) + F(y, x) \leq 0$  for all  $x, y \in C$ ,
- (A3) for all  $x, y, z \in C$ ,  $\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$ ,
- (A4) for all  $x \in C$ ,  $F(x, \cdot)$  is convex and lower semi-continuous.

Recently, Takahashi and Zembayashi [21] introduced the following iterative scheme for a relatively nonexpansive mapping  $T : C \rightarrow C$  in a uniformly smooth and uniformly convex Banach space:  $x_1 \in C$  and

$$C_1 = C,$$

$$y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T x_n),$$

$$\begin{aligned}
&u_n \in C \text{ such that } F(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0 \quad \forall y \in C, \\
&C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
&x_{n+1} = \Pi_{C_{n+1}}(x_1), \quad \forall n \geq 1,
\end{aligned} \tag{1.5}$$

where  $\{\alpha_n\} \subset [0, 1]$  and  $\{r_n\} \subset (0, \infty)$ . Such an algorithm is called the *shrinking projection method* which was introduced by Takahashi et al. [22]. They proved that the sequence  $\{x_n\}$  converges strongly to an element in  $F(T) \cap EP(F)$  under appropriate conditions. The equilibrium problem has been intensively studied by many authors (see, e.g., [23–31]).

Motivated by the previous results, we introduce a hybrid-iterative scheme for finding a zero point of maximal monotone operators  $A_i : E \rightarrow 2^{E^*}$  ( $i = 1, 2, \dots, N$ ) which is also a common element in the solutions set of an equilibrium problem for  $F$  and in the fixed points set of weak relatively nonexpansive mappings  $T_i : C \rightarrow C$  ( $i = 1, 2, \dots$ ). Using the projection technique, we also prove that the sequence generated by a constructed algorithm converges strongly to an element in  $[\bigcap_{i=1}^N A_i^{-1}(0^*)] \cap [\bigcap_{i=1}^\infty F(T_i)] \cap EP(F)$  in a uniformly smooth and uniformly convex Banach space. Finally, we apply our results to a system of convex minimization problems.

## 2. Preliminaries and Lemmas

In this section, we give some useful preliminaries and lemmas which will be used in the sequel.

Let  $E$  be a real Banach space and let  $U = \{x \in E : \|x\| = 1\}$  be the unit sphere of  $E$ . A Banach space  $E$  is said to be *strictly convex* if for any  $x, y \in U$ ,

$$x \neq y \text{ implies } \|x + y\| < 2. \tag{2.1}$$

A Banach space  $E$  is said to be *uniformly convex* if, for each  $\varepsilon \in (0, 2]$ , there exists  $\delta > 0$  such that for any  $x, y \in U$ ,

$$\|x - y\| \geq \varepsilon \text{ implies } \|x + y\| < 2(1 - \delta). \tag{2.2}$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. The function  $\delta : [0, 2] \rightarrow [0, 1]$  which is called the *modulus of convexity* of  $E$  is defined as follows:

$$\delta(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon \right\}. \tag{2.3}$$

Then  $E$  is uniformly convex if and only if  $\delta(\varepsilon) > 0$  for all  $\varepsilon \in (0, 2]$ . A Banach space  $E$  is said to be *smooth* if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.4}$$

exists for all  $x, y \in U$ . It is also said to be *uniformly smooth* if the limit (2.4) is attained uniformly for  $x, y \in U$ . The duality mapping  $J : E \rightarrow 2^{E^*}$  is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \quad (2.5)$$

for all  $x \in E$ . It is also known that if  $E$  is uniformly smooth, then  $J$  is uniformly norm-to-norm continuous on bounded subsets of  $E$  (see [32] for more details).

Let  $E$  be a smooth Banach space. The function  $\phi : E \times E \rightarrow \mathbb{R}$  is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad (2.6)$$

for all  $x, y \in E$ . From the definition of  $\phi$ , we see that

$$\begin{aligned} (\|x\| - \|y\|)^2 &\leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \\ \phi(x, y) &= \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle \end{aligned} \quad (2.7)$$

for all  $x, y, z \in E$ .

Let  $C$  be a closed and convex subset of  $E$ , and let  $T$  be a mapping from  $C$  into itself. A point  $p$  in  $C$  is said to be an *asymptotic fixed point* of  $T$  [33] if  $C$  contains a sequence  $\{x_n\}$  which converges weakly to  $p$  such that  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ . The set of asymptotic fixed points of  $T$  will be denoted by  $\widehat{F}(T)$ . A mapping  $T$  is said to be *relatively nonexpansive* [33, 34] if  $\widehat{F}(T) = F(T)$  and  $\phi(p, Tx) \leq \phi(p, x)$  for all  $p \in F(T)$  and  $x \in C$ . A point  $p$  in  $C$  is said to be a *strong asymptotic fixed point* of  $T$  if  $C$  contains a sequence  $\{x_n\}$  which converges strongly to  $p$  such that  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ . The set of strong asymptotic fixed points of  $T$  will be denoted by  $\widetilde{F}(T)$ . A mapping  $T$  is said to be *weak relatively nonexpansive* [35] if  $\widetilde{F}(T) = F(T)$  and  $\phi(p, Tx) \leq \phi(p, x)$  for all  $p \in F(T)$  and  $x \in C$ . It is obvious by definition that the class of weak relatively nonexpansive mappings contains the class of relatively nonexpansive mappings. Indeed, for any mapping  $T : C \rightarrow C$ , we see that  $F(T) \subset \widetilde{F}(T) \subset \widehat{F}(T)$ . Therefore, if  $T$  is a relatively nonexpansive mapping, then  $F(T) = \widetilde{F}(T) = \widehat{F}(T)$ .

Nontrivial examples of weak relatively nonexpansive mappings which are not relatively nonexpansive can be found in [36].

Let  $E$  be a reflexive, strictly convex and smooth Banach space, and let  $C$  be a nonempty, closed, and convex subset of  $E$ . The *generalized projection mapping*, introduced by Alber [37], is a mapping  $\Pi_C : E \rightarrow C$ , that assigns to an arbitrary point  $x \in E$  the minimum point of the function  $\phi(y, x)$ , that is,  $\Pi_C(x) = \bar{x}$ , where  $\bar{x}$  is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min\{\phi(y, x) : y \in C\}. \quad (2.8)$$

In a Hilbert space,  $\Pi_C$  is coincident with the metric projection denoted by  $P_C$ .

**Lemma 2.1** (see [38]). *Let  $E$  be a uniformly convex and smooth Banach space and let  $\{x_n\}, \{y_n\}$  be two sequences in  $E$ . If  $\lim_{n \rightarrow \infty} \phi(x_n, y_n) = 0$  and either  $\{x_n\}$  or  $\{y_n\}$  is bounded, then  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ .*

**Lemma 2.2** (see [37, 38]). *Let  $C$  be a nonempty, closed, and convex subset of a smooth, strictly convex and reflexive Banach space  $E$ , let  $x \in E$  and let  $z \in C$ . Then  $z = \Pi_C(x)$  if and only if  $\langle y - z, Jx - Jz \rangle \leq 0$  for all  $y \in C$ .*

**Lemma 2.3** (see [37, 38]). *Let  $C$  be a nonempty, closed, and convex subset of a smooth, strictly convex, and reflexive Banach space  $E$ . Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, \bar{y}) \leq \phi(x, y) \quad \forall x \in C, y \in E. \quad (2.9)$$

**Lemma 2.4** (see [39]). *Let  $E$  be a smooth and strictly convex Banach space, and let  $C$  be a nonempty, closed, and convex subset of  $E$ . Let  $T$  be a mapping from  $C$  into itself such that  $F(T)$  is nonempty and  $\phi(u, Tx) \leq \phi(u, x)$  for all  $(u, x) \in F(T) \times C$ . Then  $F(T)$  is closed and convex.*

Let  $E$  be a reflexive, strictly convex, and smooth Banach space. It is known that  $A : E \rightarrow 2^{E^*}$  is maximal monotone if and only if  $R(J + \lambda A) = E^*$  for all  $\lambda > 0$ , where  $R(B)$  stands for the range of  $B$ .

Define the *resolvent* of  $A$  by  $J_{\lambda A} = (J + \lambda A)^{-1}J$  for all  $\lambda > 0$ . It is known that  $J_{\lambda A}$  is a single-valued mapping from  $E$  to  $D(A)$  and  $A^{-1}(0^*) = F(J_{\lambda A})$  for all  $\lambda > 0$ . For each  $\lambda > 0$ , the *Yosida approximation* of  $A$  is defined by

$$A_\lambda(x) = \frac{1}{\lambda}(J(x) - JJ_{\lambda A}(x)) \quad (2.10)$$

for all  $x \in E$ . We know that  $A_\lambda(x) \in A(J_{\lambda A}(x))$  for all  $\lambda > 0$  and  $x \in E$ .

**Lemma 2.5** (see [5]). *Let  $E$  be a smooth, strictly convex, and reflexive Banach space, let  $A \subset E \times E^*$  be a maximal monotone operator with  $A^{-1}(0^*) \neq \emptyset$ , and let  $J_{\lambda A} = (J + \lambda A)^{-1}J$  for each  $\lambda > 0$ . Then*

$$\phi(p, J_{\lambda A}(x)) + \phi(J_{\lambda A}(x), x) \leq \phi(p, x) \quad (2.11)$$

for all  $\lambda > 0$ ,  $p \in A^{-1}(0^*)$ , and  $x \in E$ .

**Lemma 2.6** (see [40]). *Let  $C$  be a closed and convex subset of a smooth, strictly convex, and reflexive Banach space  $E$ , let  $F$  be a bifunction from  $C \times C$  to  $\mathbb{R}$  satisfying (A1)–(A4), and let  $r > 0$  and  $x \in E$ . Then, there exists  $z \in C$  such that*

$$F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C. \quad (2.12)$$

**Lemma 2.7** (see [41]). *Let  $C$  be a closed and convex subset of a uniformly smooth, strictly convex, and reflexive Banach space  $E$ , and let  $F$  be a bifunction from  $C \times C$  to  $\mathbb{R}$  satisfying (A1)–(A4). For all  $r > 0$  and  $x \in E$ , define the mapping  $T_r : E \rightarrow C$  as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}. \quad (2.13)$$

Then, the following holds:

- (1)  $T_r$  is single-valued;
- (2)  $T_r$  is a firmly nonexpansive-type mapping [42], that is, for all  $x, y \in E$ ,

$$\langle T_r x - T_r y, J T_r x - J T_r y \rangle \leq \langle T_r x - T_r y, J x - J y \rangle; \quad (2.14)$$

- (3)  $F(T_r) = EP(F)$ ;
- (4)  $EP(F)$  is closed and convex.

**Lemma 2.8** (see [41]). Let  $C$  be a closed and convex subset of a smooth, strictly, and reflexive Banach space  $E$ , let  $F$  be a bifunction from  $C \times C$  to  $\mathbb{R}$  satisfying (A1)–(A4), let  $r > 0$ . Then

$$\phi(p, T_r x) + \phi(T_r x, x) \leq \phi(p, x), \quad (2.15)$$

for all  $x \in E$  and  $p \in F(T_r)$ .

### 3. Strong Convergence Theorems

In this section, we are now ready to prove our main theorem.

**Theorem 3.1.** Let  $E$  be a uniformly smooth and uniformly convex Banach space, and let  $C$  be a nonempty, closed and convex subset of  $E$ . Let  $A_i : E \rightarrow 2^{E^*}$  ( $i = 1, 2, \dots, N$ ) be maximal monotone operators, let  $F : C \times C \rightarrow \mathbb{R}$  be a bifunction, and let  $T_i : C \rightarrow C$  ( $i = 1, 2, \dots$ ) be weak relatively nonexpansive mappings such that  $\mathcal{F} := [\bigcap_{i=1}^N A_i^{-1}(0^*)] \cap [\bigcap_{i=1}^{\infty} F(T_i)] \cap EP(F) \neq \emptyset$ . Let  $\{e_n\}_{n=1}^{\infty} \subset E$  be the sequence such that  $\lim_{n \rightarrow \infty} e_n = 0$ . Define the sequence  $\{x_n\}_{n=1}^{\infty}$  in  $C$  as follows:

$$\begin{aligned} x_1 &\in C_1 = C, \\ y_n &= J_{\lambda_n^N A_N} \circ J_{\lambda_n^{N-1} A_{N-1}} \circ \cdots \circ J_{\lambda_n^1 A_1}(x_n + e_n), \\ u_n &= T_{r_n} y_n, \\ C_{n+1} &= \left\{ z \in C_n : \sup_{i \geq 1} \phi(z, T_i u_n) \leq \phi(z, x_n + e_n) \right\}, \\ x_{n+1} &= \Pi_{C_{n+1}}(x_1), \quad \forall n \geq 1. \end{aligned} \quad (3.1)$$

If  $\liminf_{n \rightarrow \infty} \lambda_n^i > 0$  for each  $i = 1, 2, \dots, N$  and  $\liminf_{n \rightarrow \infty} r_n > 0$ , then the sequence  $\{x_n\}$  converges strongly to  $q = \Pi_{\mathcal{F}}(x_1)$ .

*Proof.* We split the proof into several steps as follows.

*Step 1.*  $\mathcal{F} \subset C_n$  for all  $n \geq 1$ .

From Lemma 2.4, we know that  $\bigcap_{i=1}^{\infty} F(T_i)$  is closed and convex. From Lemma 2.7(4), we also know that  $EP(F)$  is closed and convex. On the other hand, since  $A_i$  ( $i = 1, 2, \dots, N$ ) are maximal monotone,  $A_i^{-1}(0^*)$  are closed and convex for each  $i = 1, 2, \dots, N$ ; consequently,  $\bigcap_{i=1}^N A_i^{-1}(0^*)$  is closed and convex. Hence  $\mathcal{F}$  is a nonempty, closed, and convex subset of  $C$ .

We next show that  $C_n$  is closed and convex for all  $n \geq 1$ . Obviously,  $C_1 = C$  is closed and convex. Now suppose that  $C_k$  is closed and convex for some  $k \in \mathbb{N}$ . Then, for each  $z \in C_k$  and  $i \geq 1$ , we see that  $\phi(z, T_i u_k) \leq \phi(z, x_k)$  is equivalent to

$$2\langle z, Jx_k \rangle - 2\langle z, JT_i u_k \rangle \leq \|x_k\|^2 - \|T_i u_k\|^2. \quad (3.2)$$

By the construction of the set  $C_{k+1}$ , we see that

$$\begin{aligned} C_{k+1} &= \left\{ z \in C_k : \sup_{i \geq 1} \phi(z, T_i u_k) \leq \phi(z, x_k) \right\} \\ &= \bigcap_{i=1}^{\infty} \{ z \in C_k : \phi(z, T_i u_k) \leq \phi(z, x_k) \}. \end{aligned} \quad (3.3)$$

Hence,  $C_{k+1}$  is closed and convex. This shows, by induction, that  $C_n$  is closed and convex for all  $n \geq 1$ . It is obvious that  $\mathcal{F} \subset C_1 = C$ . Now, suppose that  $\mathcal{F} \subset C_k$  for some  $k \in \mathbb{N}$ . For any  $p \in \mathcal{F}$ , by Lemmas 2.5 and 2.8, we have

$$\begin{aligned} \phi(p, T_i u_k) &\leq \phi(p, u_k) = \phi(p, T_{r_k} y_k) \\ &\leq \phi(p, y_k) \\ &= \phi\left(p, J_{\lambda_k^N A_N} \circ J_{\lambda_k^{N-1} A_{N-1}} \circ \cdots \circ J_{\lambda_k^1 A_1}(x_k + e_k)\right) \\ &\leq \phi\left(p, J_{\lambda_k^{N-1} A_{N-1}} \circ J_{\lambda_k^{N-2} A_{N-2}} \circ \cdots \circ J_{\lambda_k^1 A_1}(x_k + e_k)\right) \\ &\quad \vdots \\ &\leq \phi\left(p, J_{\lambda_k^2 A_2} \circ J_{\lambda_k^1 A_1}(x_k + e_k)\right) \\ &\leq \phi\left(p, J_{\lambda_k^1 A_1}(x_k + e_k)\right) \\ &\leq \phi(p, x_k + e_k). \end{aligned} \quad (3.4)$$

This shows that  $\mathcal{F} \subset C_{k+1}$ . By induction, we can conclude that  $\mathcal{F} \subset C_n$  for all  $n \geq 1$ .

*Step 2.*  $\lim_{n \rightarrow \infty} \phi(x_n, x_1)$  exists.

From  $x_n = \Pi_{C_n}(x_1)$  and  $x_{n+1} = \Pi_{C_{n+1}}(x_1) \in C_{n+1} \subset C_n$ , we have

$$\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1), \quad \forall n \geq 1. \quad (3.5)$$

From Lemma 2.3, for any  $p \in \mathcal{F} \subset C_n$ , we have

$$\phi(x_n, x_1) = \phi(\Pi_{C_n}(x_1), x_1) \leq \phi(p, x_1) - \phi(p, x_n) \leq \phi(p, x_1). \quad (3.6)$$

Combining (3.5) and (3.6), we conclude that  $\lim_{n \rightarrow \infty} \phi(x_n, x_1)$  exists.

Step 3.  $\lim_{n \rightarrow \infty} \|J(T_i y_n) - J(x_n + e_n)\| = 0$ .

Since  $x_m = \Pi_{C_m}(x_1) \in C_m \subset C_n$  for  $m > n \geq 1$ , by Lemma 2.3, it follows that

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n}(x_1)) \leq \phi(x_m, x_1) - \phi(\Pi_{C_n}(x_1), x_1) \\ &= \phi(x_m, x_1) - \phi(x_n, x_1). \end{aligned} \quad (3.7)$$

Letting  $m, n \rightarrow \infty$ , we have  $\phi(x_m, x_n) \rightarrow 0$ . By Lemma 2.1, it follows that  $\|x_m - x_n\| \rightarrow 0$  as  $m, n \rightarrow \infty$ . Therefore,  $\{x_n\}$  is a Cauchy sequence. By the completeness of the space  $E$  and the closedness of  $C$ , we can assume that  $x_n \rightarrow q \in C$  as  $n \rightarrow \infty$ . In particular, we obtain that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (3.8)$$

Since  $e_n \rightarrow 0$ , we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - (x_n + e_n)\| = 0. \quad (3.9)$$

Since  $x_{n+1} = \Pi_{C_{n+1}}(x_1) \in C_{n+1}$ , for each  $i \geq 1$ ,

$$\begin{aligned} \phi(x_{n+1}, T_i u_n) &\leq \phi(x_{n+1}, x_n + e_n) \\ &= \langle x_{n+1}, J(x_{n+1}) - J(x_n + e_n) \rangle + \langle x_{n+1} - (x_n + e_n), J(x_{n+1}) \rangle. \end{aligned} \quad (3.10)$$

Since  $E$  is uniformly smooth,  $J$  is uniformly norm-to-norm continuous on bounded sets. It follows from (3.9) and by the boundedness of  $\{x_n\}$  that

$$\lim_{n \rightarrow \infty} \phi(x_{n+1}, T_i u_n) = 0 \quad (3.11)$$

for all  $i = 1, 2, \dots$ . So from Lemma 2.1, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_{n+1} - T_i u_n\| &= 0, \\ \lim_{n \rightarrow \infty} \|T_i u_n - x_n\| &= 0, \end{aligned} \quad (3.12)$$

and, since  $e_n \rightarrow 0$ , therefore

$$\lim_{n \rightarrow \infty} \|T_i u_n - (x_n + e_n)\| = 0, \quad (3.13)$$

for all  $i = 1, 2, \dots$ . Since  $J$  is uniformly norm-to-norm continuous on bounded subsets of  $E$ ,

$$\lim_{n \rightarrow \infty} \|J(T_i u_n) - J(x_n + e_n)\| = 0 \quad (3.14)$$

for all  $i = 1, 2, \dots$



Step 4.  $\lim_{n \rightarrow \infty} \|T_i u_n - u_n\| = 0$  for all  $i = 1, 2, \dots$

Denote that  $\Theta_n^i = J_{\lambda_n A_i} \circ J_{\lambda_n^{i-1} A_{i-1}} \circ \dots \circ J_{\lambda_n A_1}$  for each  $i \in \{1, 2, \dots, N\}$  and  $\Theta_n^0 = I$  for each  $n \geq 1$ . We note that  $y_n = \Theta_n^N(x_n + e_n)$  for each  $n \geq 1$ .

To this end, we will show that

$$\lim_{n \rightarrow \infty} \left\| J\left(\Theta_n^i(x_n + e_n)\right) - J\left(\Theta_n^{i-1}(x_n + e_n)\right) \right\| = 0 \quad (3.15)$$

for all  $i = 1, 2, \dots, N$ .

For any  $p \in \mathcal{F}$ , by (3.4), we see that

$$\begin{aligned} \phi\left(p, \Theta_n^{N-1}(x_n + e_n)\right) &\leq \phi\left(p, \Theta_n^{N-2}(x_n + e_n)\right) \\ &\leq \phi\left(p, \Theta_n^{N-3}(x_n + e_n)\right) \\ &\vdots \\ &\leq \phi\left(p, (x_n + e_n)\right). \end{aligned} \quad (3.16)$$

Since  $p \in \mathcal{F}$ , by Lemma 2.5 and (3.16), it follows that

$$\begin{aligned} \phi\left(y_n, \Theta_n^{N-1}(x_n + e_n)\right) &\leq \phi\left(p, \Theta_n^{N-1}(x_n + e_n)\right) - \phi\left(p, y_n\right) \\ &\leq \phi\left(p, (x_n + e_n)\right) - \phi\left(p, y_n\right) \\ &\leq \phi\left(p, (x_n + e_n)\right) - \phi\left(p, u_n\right) \\ &\leq \phi\left(p, (x_n + e_n)\right) - \phi\left(p, T_i u_n\right) \\ &= \|x_n + e_n\|^2 - \|T_i u_n\|^2 - 2\langle p, J(x_n + e_n) - J(T_i u_n) \rangle. \end{aligned} \quad (3.17)$$

From (3.13) and (3.14), we get that  $\lim_{n \rightarrow \infty} \phi\left(y_n, \Theta_n^{N-1}(x_n + e_n)\right) = 0$ . So we obtain that

$$\lim_{n \rightarrow \infty} \left\| y_n - \Theta_n^{N-1}(x_n + e_n) \right\| = 0. \quad (3.18)$$

Again, since  $p \in \mathcal{F}$ ,

$$\begin{aligned} \phi\left(\Theta_n^{N-1}(x_n + e_n), \Theta_n^{N-2}(x_n + e_n)\right) &\leq \phi\left(p, \Theta_n^{N-2}(x_n + e_n)\right) - \phi\left(p, \Theta_n^{N-1}(x_n + e_n)\right) \\ &\leq \phi\left(p, (x_n + e_n)\right) - \phi\left(p, \Theta_n^{N-1}(x_n + e_n)\right) \\ &\leq \phi\left(p, (x_n + e_n)\right) - \phi\left(p, T_i u_n\right). \end{aligned} \quad (3.19)$$

From (3.13) and (3.14), we get that

$$\lim_{n \rightarrow \infty} \phi\left(\Theta_n^{N-1}(x_n + e_n), \Theta_n^{N-2}(x_n + e_n)\right) = 0. \quad (3.20)$$

It also follows that

$$\lim_{n \rightarrow \infty} \left\| \Theta_n^{N-1}(x_n + e_n) - \Theta_n^{N-2}(x_n + e_n) \right\| = 0. \quad (3.21)$$

Continuing in this process, we can show that

$$\lim_{n \rightarrow \infty} \left\| \Theta_n^{N-2}(x_n + e_n) - \Theta_n^{N-3}(x_n + e_n) \right\| = \dots = \lim_{n \rightarrow \infty} \left\| \Theta_n^1(x_n + e_n) - (x_n + e_n) \right\| = 0. \quad (3.22)$$

So, we now conclude that

$$\lim_{n \rightarrow \infty} \left\| \Theta_n^i(x_n + e_n) - \Theta_n^{i-1}(x_n + e_n) \right\| = 0 \quad (3.23)$$

for each  $i = 1, 2, \dots, N$ . By the uniform norm-to-norm continuity of  $J$ , we also have

$$\lim_{n \rightarrow \infty} \left\| J\left(\Theta_n^i(x_n + e_n)\right) - J\left(\Theta_n^{i-1}(x_n + e_n)\right) \right\| = 0 \quad (3.24)$$

for each  $i = 1, 2, \dots, N$ . Using (3.23), it is easily seen that

$$\lim_{n \rightarrow \infty} \left\| y_n - (x_n + e_n) \right\| = 0. \quad (3.25)$$

From  $u_n = T_{r_n}y_n$ , by Lemma 2.8, it follows that

$$\begin{aligned} \phi(u_n, y_n) &= \phi(T_{r_n}y_n, y_n) \\ &\leq \phi(p, y_n) - \phi(p, T_{r_n}y_n) \\ &\leq \phi(p, x_n + e_n) - \phi(p, u_n) \\ &\leq \phi(p, x_n + e_n) - \phi(p, T_i u_n). \end{aligned} \quad (3.26)$$

This implies that  $\lim_{n \rightarrow \infty} \phi(u_n, y_n) = 0$  and hence

$$\lim_{n \rightarrow \infty} \left\| u_n - y_n \right\| = 0. \quad (3.27)$$

Combining (3.13), (3.25), and (3.27), we obtain that

$$\lim_{n \rightarrow \infty} \left\| T_i u_n - u_n \right\| = 0 \quad (3.28)$$

for all  $i \geq 1$ .

*Step 5.*  $q \in \bigcap_{i=1}^{\infty} F(T_i)$ .

Since  $x_n \rightarrow q$  and  $e_n \rightarrow 0$ ,  $x_n + e_n \rightarrow q$ . So from (3.25) and (3.27), we have  $u_n \rightarrow q$ . Note that  $T_i$  ( $i = 1, 2, \dots$ ) are weak relatively nonexpansive. Using (3.28), we can conclude that  $q \in \tilde{F}(T_i) = F(T_i)$  for all  $i \geq 1$ . Hence  $q \in \bigcap_{i=1}^{\infty} F(T_i)$ .

Step 6.  $q \in \bigcap_{i=1}^N A_i^{-1}(0^*)$ .

Noting that  $\Theta_n^i(x_n + e_n) = J_{\lambda_n^i A_i} \Theta_n^{i-1}(x_n + e_n)$  for each  $i = 1, 2, \dots, N$ , we obtain that

$$\|A_{\lambda_n^i} \Theta_n^{i-1}(x_n + e_n)\| = \frac{1}{\lambda_n^i} \|J(\Theta_n^{i-1}(x_n + e_n)) - J(\Theta_n^i(x_n + e_n))\|. \quad (3.29)$$

From (3.24) and  $\liminf_{n \rightarrow \infty} \lambda_n^i > 0$ , we have

$$\lim_{n \rightarrow \infty} \|A_{\lambda_n^i} \Theta_n^{i-1}(x_n + e_n)\| = 0. \quad (3.30)$$

We note that  $(\Theta_n^i(x_n + e_n), A_{\lambda_n^i} \Theta_n^{i-1}(x_n + e_n)) \in G(A_i)$  for each  $i = 1, 2, \dots, N$ . If  $(w, w^*) \in G(A_i)$  for each  $i = 1, 2, \dots, N$ , then it follows from the monotonicity of  $A_i$  that

$$\langle w^* - A_{\lambda_n^i} \Theta_n^{i-1}(x_n + e_n), w - \Theta_n^i(x_n + e_n) \rangle \geq 0. \quad (3.31)$$

We see that  $\Theta_n^i(x_n + e_n) \rightarrow q$  for each  $i = 1, 2, \dots, N$ . Thus, from (3.30) and (3.31), we have

$$\langle w^*, w - q \rangle \geq 0. \quad (3.32)$$

By the maximality of  $A_i$ , it follows that  $q \in A_i^{-1}(0^*)$  for each  $i = 1, 2, \dots, N$ . Therefore,  $q \in \bigcap_{i=1}^N A_i^{-1}(0^*)$ .

Step 7.  $q \in \text{EP}(F)$ .

From  $u_n = T_{r_n} y_n$ , we have

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C. \quad (3.33)$$

By (A2), we have

$$\begin{aligned} \|y - u_n\| \frac{\|Ju_n - Jy_n\|}{r_n} &\geq \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \\ &\geq -F(u_n, y) \geq F(y, u_n), \quad \forall y \in C. \end{aligned} \quad (3.34)$$

Note that  $\|Ju_n - Jy_n\|/r_n \rightarrow 0$  since  $\liminf_{n \rightarrow \infty} r_n > 0$ . From (A4) and  $u_n \rightarrow q$ , we get  $F(y, q) \leq 0$  for all  $y \in C$ . For  $0 < t < 1$  and  $y \in C$ , define that  $y_t = ty + (1-t)q$ . Then  $y_t \in C$ , which implies that  $F(y_t, q) \leq 0$ . From (A1), we obtain that  $0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, q) \leq tF(y_t, y)$ . Thus,  $F(y_t, y) \geq 0$ . From (A3), we have  $F(q, y) \geq 0$  for all  $y \in C$ . Hence,  $q \in \text{EP}(F)$ . From Steps 5, 6, and 7, we now can conclude that  $q \in \mathcal{F}$ .

Step 8.  $q = \Pi_{\mathcal{F}}(x_1)$ .

From  $x_n = \Pi_{C_n}(x_1)$ , we have

$$\langle J(x_1) - J(x_n), x_n - z \rangle \geq 0, \quad \forall z \in C_n. \quad (3.35)$$

Since  $\mathcal{F} \subset C_n$ , we also have

$$\langle J(x_1) - J(x_n), x_n - z \rangle \geq 0, \quad \forall z \in \mathcal{F}. \quad (3.36)$$

Letting  $n \rightarrow \infty$  in (3.36), we obtain that

$$\langle J(x_1) - J(q), q - z \rangle \geq 0, \quad \forall z \in \mathcal{F}. \quad (3.37)$$

This shows that  $q = \Pi_{\mathcal{F}}(x_1)$  by Lemma 2.2. We thus complete the proof.  $\square$

As a direct consequence of Theorem 3.1, we can also apply to a system of convex minimization problems.

**Theorem 3.2.** *Let  $E$  be a uniformly smooth and uniformly convex Banach space, and let  $C$  be a nonempty, closed, and convex subset of  $E$ . Let  $f_i : E \rightarrow (-\infty, \infty]$  ( $i = 1, 2, \dots, N$ ) be proper lower semicontinuous convex functions, let  $F : C \times C \rightarrow \mathbb{R}$  be a bifunction, and let  $T_i : C \rightarrow C$  ( $i = 1, 2, \dots$ ) be weak relatively nonexpansive mappings such that  $\mathcal{F} := [\bigcap_{i=1}^N (\partial f_i^{-1})(0^*)] \cap [\bigcap_{i=1}^{\infty} F(T_i)] \cap EP(F) \neq \emptyset$ . Let  $\{e_n\}_{n=1}^{\infty} \subset E$  be the sequence such that  $\lim_{n \rightarrow \infty} e_n = 0$ . Define the sequence  $\{x_n\}_{n=1}^{\infty}$  in  $C$  as follows:*

$$\begin{aligned} x_1 &\in C_1 = C, \\ z_n^1 &= \arg \min_{y \in E} \left\{ f_1(y) + \frac{1}{2\lambda_n^1} \|y\|^2 + \frac{1}{\lambda_n^1} \langle y, J(x_n + e_n) \rangle \right\}, \\ &\vdots \\ z_n^{N-1} &= \arg \min_{y \in E} \left\{ f_{N-1}(y) + \frac{1}{2\lambda_n^{N-1}} \|y\|^2 + \frac{1}{\lambda_n^{N-1}} \langle y, J(z_n^{N-2}) \rangle \right\}, \\ y_n &= \arg \min_{y \in E} \left\{ f_N(y) + \frac{1}{2\lambda_n^N} \|y\|^2 + \frac{1}{\lambda_n^N} \langle y, J(z_n^{N-1}) \rangle \right\}, \\ u_n &= T_{r_n} y_n, \\ C_{n+1} &= \left\{ z \in C_n : \sup_{i \geq 1} \phi(z, T_i u_n) \leq \phi(z, x_n + e_n) \right\}, \\ x_{n+1} &= \Pi_{C_{n+1}}(x_1), \quad \forall n \geq 1. \end{aligned} \quad (3.38)$$

If  $\liminf_{n \rightarrow \infty} \lambda_n^i > 0$  for each  $i = 1, 2, \dots, N$  and  $\liminf_{n \rightarrow \infty} r_n > 0$ , then the sequence  $\{x_n\}$  converges strongly to  $q = \Pi_{\mathcal{F}}(x_1)$ .

*Proof.* By Rockafellar's theorem [43, 44],  $\partial f_i$  are maximal monotone operators for each  $i = 1, 2, \dots, N$ . Let  $\lambda^i > 0$  for each  $i = 1, 2, \dots, N$ . Then,  $z^i = J_{\lambda^i \partial f_i}(x)$  if and only if

$$\begin{aligned} 0 &\in \partial f_i(z^i) + \frac{1}{\lambda^i} (J(z^i) - J(x)) \\ &= \partial \left( f_i + \frac{1}{\lambda^i} \left( \frac{\|\cdot\|^2}{2} - J(x) \right) \right) (z^i), \end{aligned} \tag{3.39}$$

which is equivalent to

$$z^i = \arg \min_{y \in E} \left\{ f_i(y) + \frac{1}{\lambda^i} \left( \frac{\|y\|^2}{2} - \langle y, J(x) \rangle \right) \right\}. \tag{3.40}$$

Using Theorem 3.1, we thus complete the proof. □

If  $E = H$  is a real Hilbert space, we then obtain the following results.

**Corollary 3.3.** *Let  $C$  be a nonempty, closed and convex subset of a real Hilbert space  $H$ . Let  $A_i : H \rightarrow 2^H$  ( $i = 1, 2, \dots, N$ ) be maximal monotone operators, let  $F : C \times C \rightarrow \mathbb{R}$  be a bifunction, and let  $T_i : C \rightarrow C$  ( $i = 1, 2, \dots$ ) be weak relatively nonexpansive mappings such that  $\mathcal{F} := [\bigcap_{i=1}^N A_i^{-1}(0)] \cap [\bigcap_{i=1}^\infty F(T_i)] \cap EP(F) \neq \emptyset$ . Let  $\{e_n\}_{n=1}^\infty \subset H$  be the sequence such that  $\lim_{n \rightarrow \infty} e_n = 0$ . Define the sequence  $\{x_n\}_{n=1}^\infty$  in  $C$  as follows:*

$$\begin{aligned} x_1 &\in C_1 = C, \\ y_n &= J_{\lambda_n^N A_N} \circ J_{\lambda_n^{N-1} A_{N-1}} \circ \dots \circ J_{\lambda_n^1 A_1} (x_n + e_n), \\ u_n &= T_{r_n} y_n, \\ C_{n+1} &= \left\{ z \in C_n : \sup_{i \geq 1} \|z - T_i u_n\| \leq \|z - (x_n + e_n)\| \right\}, \\ x_{n+1} &= P_{C_{n+1}}(x_1), \quad \forall n \geq 1. \end{aligned} \tag{3.41}$$

*If  $\liminf_{n \rightarrow \infty} \lambda_n^i > 0$  for each  $i = 1, 2, \dots, N$  and  $\liminf_{n \rightarrow \infty} r_n > 0$ , then the sequence  $\{x_n\}$  converges strongly to  $q = P_{\mathcal{F}}(x_1)$ .*

**Corollary 3.4.** *Let  $C$  be a nonempty, closed, and convex subset of a real Hilbert space  $H$ . Let  $f_i : H \rightarrow (-\infty, \infty]$  ( $i = 1, 2, \dots, N$ ) be proper lower semi-continuous convex functions, let  $F : C \times C \rightarrow \mathbb{R}$  be a bifunction, and let  $T_i : C \rightarrow C$  ( $i = 1, 2, \dots$ ) be weak relatively nonexpansive mappings such*

that  $\mathcal{F} := [\bigcap_{i=1}^N (\partial f_i^{-1})(0)] \cap [\bigcap_{i=1}^{\infty} F(T_i)] \cap EP(F) \neq \emptyset$ . Let  $\{e_n\}_{n=1}^{\infty} \subset H$  be the sequence such that  $\lim_{n \rightarrow \infty} e_n = 0$ . Define the sequence  $\{x_n\}_{n=1}^{\infty}$  in  $C$  as follows:

$$\begin{aligned} x_1 &\in C_1 = C, \\ z_n^1 &= \arg \min_{y \in H} \left\{ f_1(y) + \frac{1}{2\lambda_n^1} \|y\|^2 + \frac{1}{\lambda_n^1} \langle y, x_n + e_n \rangle \right\}, \\ &\vdots \\ z_n^{N-1} &= \arg \min_{y \in H} \left\{ f_{N-1}(y) + \frac{1}{2\lambda_n^{N-1}} \|y\|^2 + \frac{1}{\lambda_n^{N-1}} \langle y, z_n^{N-2} \rangle \right\}, \\ y_n &= \arg \min_{y \in H} \left\{ f_N(y) + \frac{1}{2\lambda_n^N} \|y\|^2 + \frac{1}{\lambda_n^N} \langle y, z_n^{N-1} \rangle \right\}, \\ u_n &= T_{r_n} y_n, \\ C_{n+1} &= \left\{ z \in C_n : \sup_{i \geq 1} \|z - T_i u_n\| \leq \|z - (x_n + e_n)\| \right\}, \\ x_{n+1} &= P_{C_{n+1}}(x_1), \quad \forall n \geq 1. \end{aligned} \tag{3.42}$$

If  $\liminf_{n \rightarrow \infty} \lambda_n^i > 0$  for each  $i = 1, 2, \dots, N$  and  $\liminf_{n \rightarrow \infty} r_n > 0$ , then the sequence  $\{x_n\}$  converges strongly to  $q = P_{\mathcal{F}}(x_1)$ .

*Remark 3.5.* Using the shrinking projection method, we can construct a hybrid-proximal point algorithm for solving a system of the zero-finding problems, the equilibrium problems, and the fixed point problems of weak relatively nonexpansive mappings.

*Remark 3.6.* Since every relatively nonexpansive mapping is weak relatively nonexpansive, our results also hold if  $T_i : C \rightarrow C$  ( $i = 1, 2, \dots$ ) are relatively nonexpansive mappings.

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