Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2011, Article ID 549364, 18 pages
doi:10.1155/2011/549364

Research Article

Strong Convergence Theorems of Modified Ishikawa Iterative Method for an Infinite Family of Strict Pseudocontractions in Banach Spaces

Phayap Katchang,[1] Wiyada Kumam,[2] Usa Humphries,[2] and Poom Kumam[2]

[1] *Department of Mathematics and Statistics, Faculty of Science and Agricultural Technology, Rajamangala University of Technology Lanna Tak, Tak 63000, Thailand*
[2] *Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bangmod, Bangkok 10140, Thailand*

Correspondence should be addressed to Poom Kumam, poom.kum@kmutt.ac.th

Received 15 December 2010; Accepted 14 March 2011

Academic Editor: Vittorio Colao

We introduce a new modified Ishikawa iterative process and a new W-mapping for computing fixed points of an infinite family of strict pseudocontractions mapping in the framework of q-uniformly smooth Banach spaces. Then, we establish the strong convergence theorem of the proposed iterative scheme under some mild conditions. The results obtained in this paper extend and improve the recent results of Cai and Hu 2010, Dong et al. 2010, Katchang and Kumam 2011 and many others in the literature.

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$ and C a nonempty closed convex subset of E. Let E^* be the dual space of E, and let $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between E and E^*. For $q > 1$, the *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \left\{ f \in E^* : \langle x, f\rangle = \|x\|^q, \left\|f\right\| = \|x\|^{q-1} \right\}, \tag{1.1}$$

for all $x \in E$. In particular, if $q = 2$, the mapping $J = J_2$ is called the *normalized duality mapping* and $J_q(x) = \|x\|^{q-2} J_2(x)$ for $x \neq 0$. It is well known that if E is smooth, then J_q is single-valued, which is denoted by j_q.

A mapping $T : C \to C$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

We use $F(T)$ to denote the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$.

T is said to be a λ-*strict pseudocontraction* in the terminology of Browder and Petryshyn [1] if there exists a constant $\lambda > 0$ and for some $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Tx - Ty, j_q(x - y)\rangle \leq \|x - y\|^q - \lambda\|(I - T)x - (I - T)y\|^q, \quad \forall x, y \in C. \tag{1.3}$$

T is said to be a *strong pseudocontraction* if there exists $k \in (0, 1)$ such that

$$\langle Tx - Ty, j_q(x - y)\rangle \leq k\|x - y\|^q, \quad \forall x, y \in C. \tag{1.4}$$

Remark 1.1 (see [2]). Let T be a λ-strict pseudocontraction in a Banach space. Let $x \in C$ and $p \in F(T)$. Then,

$$\|Tx - p\| \leq \left(1 + \frac{1}{\lambda^{1/q-1}}\right)\|x - p\|. \tag{1.5}$$

Recall that a self mapping $f : C \to C$ is *contraction* on C if there exists a constant $\alpha \in (0, 1)$ and $x, y \in C$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|. \tag{1.6}$$

We use Π_C to denote the collection of all contractions on C. That is, $\Pi_C = \{f \mid f : C \to C \ a \text{ contraction}\}$. Note that each $f \in \Pi_C$ has a unique fixed point in C.

Very recently, Cai and Hu [3] also proved the strong convergence theorem in Banach spaces. They considered the following iterative algorithm:

$$\begin{gathered} x_1 = x \in C \text{ chosen arbitrarily,} \\ y_n = P_C\left[\beta_n x_n + (1 - \beta_n)\sum_{i=i}^{N} \eta_i^{(n)} T_i x_n\right], \\ x_{n+1} = \alpha_n \gamma f(x_n) + \gamma_n x_n + \left((1 - \gamma_n)I - \alpha_n A\right)y_n, \quad \forall n \geq 1, \end{gathered} \tag{1.7}$$

where T_i is a non-self-λ_i-strictly pseudocontraction, f is a contraction, and A is a strongly positive linear bounded operator.

Dong et al. [2] proved the sequence $\{x_n\}$ converges strongly in Banach spaces under certain appropriate assumptions and used the W_n mapping defined by (1.11). Let the sequences $\{x_n\}$ be generated by

$$\begin{aligned} x_0 &= x \in C \text{ chosen arbitrarily,} \\ y_n &= \delta_n x_n + (1-\delta_n) W_n x_n, \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n y_n, \quad \forall n \geq 0. \end{aligned} \tag{1.8}$$

On the other hand, Katchang and Kumam [4, 5] introduced the following new modified Ishikawa iterative process for computing fixed points of an infinite family nonexpansive mapping in the framework of Banach spaces; let the sequences $\{x_n\}$ be generated by

$$\begin{aligned} x_0 &= x \in C \text{ chosen arbitrarily,} \\ z_n &= \gamma_n x_n + (1-\gamma_n) W_n x_n, \\ y_n &= \beta_n x_n + (1-\beta_n) W_n z_n, \\ x_{n+1} &= \alpha_n \gamma f(x) + (I - \alpha_n A) y_n, \quad \forall n \geq 0, \end{aligned} \tag{1.9}$$

where f is a contraction, A is a strongly positive linear bounded self-adjoint operator, and W_n mapping (see [6, 7]) is defined by

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1-\lambda_n) I, \\ U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1}) I, \\ &\vdots \\ U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1-\lambda_k) I, \\ U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1}) I, \\ &\vdots \\ U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1-\lambda_2) I, \\ W_n = U_{n,1} &= \lambda_1 T_1 U_{n,2} + (1-\lambda_1) I, \end{aligned} \tag{1.10}$$

where $T_1, T_2, \ldots$ is an infinite family of nonexpansive mappings of C into itself and $\lambda_1, \lambda_2, \ldots$ is real numbers such that $0 \leq \lambda_n \leq 1$ for every $n \in \mathbb{N}$. In 2010, Cho [8] considered and proved the strong convergence of the implicit iterative process for an infinite family of strict pseudocontractions in an arbitrary real Banach space.

In this paper, motivated and inspired by Cai and Hu [9], we consider the mapping W_n defined by

$$\begin{aligned}
U_{n,n+1} &= I,\\
U_{n,n} &= t_n T_{n,n} U_{n,n+1} + (1-t_n)I,\\
&\vdots\\
U_{n,k} &= t_k T_{n,k} U_{n,k+1} + (1-t_k)I,\\
U_{n,k-1} &= t_{k-1} T_{n,k-1} U_{n,k} + (1-t_{k-1})I,\\
&\vdots\\
U_{n,2} &= t_2 T_{n,2} U_{n,3} + (1-t_2)I,\\
W_n = U_{n,1} &= t_1 T_{n,1} U_{n,2} + (1-t_1)I,
\end{aligned} \tag{1.11}$$

where $t_1, t_2, \ldots$ are real numbers such that $0 \le t_n \le 1$. $T_{n,k} = \theta_{n,k} S_k + (1-\theta_{n,k})I$, where S_k is a λ_k-strict pseudocontraction of C into itself and $\theta_{n,k} \in (0,\mu]$, $\mu = \min\{1, \{q^{\lambda}/C_q\}^{1/q-1}\}$, where $\lambda = \inf \lambda_k$ for all $k \in \mathbb{N}$. By Lemma 2.3, we know that $T_{n,k}$ is a nonexpansive mapping, and therefore, W_n is a nonexpansive mapping. We note that the W-mapping (1.10) is a special case of a W-mapping (1.11) when $\theta_{n,k} = \theta_k$ is constant for all $n \ge 1$.

Throughout this paper, we will assume that $\inf \lambda_i > 0$, $0 < t_n \le b < 1$ for all $n \in \mathbb{N}$ and $\{\theta_{n,k}\}$ satisfies

(H1) $\theta_{n,k} \in (0,\mu]$, $\mu = \min\{1, \inf_i \{q^{\lambda_i}/C_q\}^{1/q-1}\}$ for all $k \in \mathbb{N}$,

(H2) $|\theta_{n+1,k} - \theta_{n,k}| \le a_n$ for all $n \in \mathbb{N}$ and $1 \le k \le n$, where $\{a_n\}$ satisfies $\sum_{n=1}^{\infty} a_n < \infty$.

The hypothesis (H2) secures the existence of $\lim_{n\to\infty} \theta_{n,k}$ for all $k \in \mathbb{N}$. Set $\theta_{1,k} := \lim_{n\to\infty} \theta_{n,k}$ for all $k \in \mathbb{N}$. Furthermore, we assume that

(H3) $\theta_{1,k} > 0$ for all $k \in \mathbb{N}$.

It is obvious that $\theta_{1,k}$ satisfy (H1). Using condition (H3), from $T_{n,k} = \theta_{n,k} S_k + (1-\theta_{n,k})I$, we define mappings $T_{1,k}x := \lim_{n\to\infty} T_{n,k}x = \theta_{1,k} S_k x + (1-\theta_{1,k})x$ for all $x \in C$.

Our results improve and extend the recent ones announced by Cai and Hu [3], Dong et al. [2], Katchang and Kumam [4, 5], and many others.

2. Preliminaries

Recall that $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be *uniformly convex* if, for any $\epsilon \in (0,2]$, there exists $\delta > 0$ such that for any $x, y \in U$, $\|x-y\| \ge \epsilon$ implies $\|(x+y)/2\| \le 1-\delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex (see also [10]). A Banach space E is said to be *smooth* if the limit $\lim_{t\to 0}(\|x+ty\| - \|x\|)/t$ exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U$.

In a smooth Banach space, we define an operator A as strongly positive if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, J(x)\rangle \geq \overline{\gamma}\|x\|^2,\ \|aI - bA\| = \sup_{\|x\|\leq 1}|\langle (aI - bA)x, J(x)\rangle| \quad a \in [0,1],\ b \in [-1,1], \tag{2.1}$$

where I is the identity mapping and J is the normalized duality mapping.

If C and D are nonempty subsets of a Banach space E such that C is a nonempty closed convex and $D \subset C$, then a mapping $Q : C \to D$ is sunny [11, 12] provided that $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \geq 0$ whenever $x + t(x - Q(x)) \in C$. A mapping $Q : C \to C$ is called a *retraction* if $Q^2 = Q$. If a mapping $Q : C \to C$ is a retraction, then $Qz = z$ for all z in the range of Q. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction Q of C onto D. A sunny nonexpansive retraction is a sunny retraction, which is also nonexpansive. Sunny nonexpansive retractions play an important role in our argument. They are characterized as follows [11, 12]: if E is a smooth Banach space, then $Q : C \to D$ is a *sunny nonexpansive retraction* if and only if there holds the inequality

$$\langle x - Qx, J(y - Qx)\rangle \leq 0, \quad \forall x \in C,\ y \in D. \tag{2.2}$$

We need the following lemmas for proving our main results.

Lemma 2.1 (see [13]). *In a Banach space E, the following holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall x, y \in E, \tag{2.3}$$

where $j(x + y) \in J(x + y)$.

Lemma 2.2 (see [14]). *Let E be a real q-uniformly smooth Banach space, then there exists a constant $C_q > 0$ such that*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q x\rangle + C_q\|y\|^q, \quad \forall x, y \in E. \tag{2.4}$$

In particular, if E be a real 2-uniformly smooth Banach space with the best smooth constant K, then the following inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, jx\rangle + 2\|Ky\|^2, \quad \forall x, y \in E. \tag{2.5}$$

The relation between the λ-strict pseudocontraction and the nonexpansive mapping can be obtained from the following lemma.

Lemma 2.3 (see [15]). *Let C be a nonempty convex subset of a real q-uniformly smooth Banach space E and $S : C \to C$ a λ-strict pseudocontraction. For $\alpha \in (0,1)$, one defines $Tx = (1 - \alpha)x + \alpha Sx$. Then, as $\alpha \in (0, \mu]$, $\mu = \min\{1, \{q^\lambda / C_q\}^{1/q-1}\}$, $T : C \to C$ is nonexpansive such that $F(T) = F(S)$, where C_q is the constant in Lemma 2.2.*

Concerning W_n, we have the following lemmas, which are important to prove the main results.

Lemma 2.4 (see [2]). *Let C be a nonempty closed convex subset of a q-uniformly smooth and strictly convex Banach space E. Let $S_i, i = 1, 2, \ldots$, be a λ_i-strict pseudocontraction from C into itself such that $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$, and let $\inf \lambda_i > 0$. Let $t_n, n = 1, 2, \ldots$, be real numbers such that $0 < t_n \leq b < 1$ for any $n \geq 1$. Assume that the sequence $\{\theta_{n,k}\}$ satisfies (H_1) and (H_2). Then, for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 2.4, we define the mappings $U_{1,k}$ and $W : C \to C$ as follows:

$$\begin{aligned} U_{1,k}x &:= \lim_{n\to\infty} U_{n,k}x, \\ Wx &:= \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \end{aligned} \tag{2.6}$$

for all $x \in C$. Such W is called the W-mapping generated by $S_1, S_2, \ldots, t_1, t_2, \ldots$ and $\theta_{n,k}$, for all $n \in \mathbb{N}$ and $1 \leq k \leq n$.

Lemma 2.5 (see [2]). *Let $\{x_n\}$ be a bounded sequence in a q-uniformly smooth and strictly convex Banach space E. Under the assumptions of Lemma 2.4, it holds*

$$\lim_{n\to\infty} \|W_n x_n - W x_n\| = 0. \tag{2.7}$$

Lemma 2.6 (see [2]). *Let C be a nonempty closed convex subset of a q-uniformly smooth and strictly convex Banach space E. Let $S_i, i = 1, 2, \ldots$, be a λ_i-strict pseudocontraction from C into itself such that $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$, and let $\inf \lambda_i > 0$. Let $t_n, n = 1, 2, \ldots$, be real numbers such that $0 < t_n \leq b < 1$ for any $n \geq 1$. Assume that the sequence $\{\theta_{n,k}\}$ satisfies (H_1)–(H_3). Then, $F(W) = \cap_{n=1}^{\infty} F(S_n)$.*

Lemma 2.7 (see [16]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \quad n \geq 0, \tag{2.8}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n/\alpha_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.8 (see [17]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X, and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.9 (see [3]). *Assume that A is a strong positive linear bounded operator on a smooth Banach space E with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then, $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

3. Main Results

In this section, we prove a strong convergence theorem.

Theorem 3.1. *Let E be a real q-uniformly smooth and strictly convex Banach space which admits a weakly sequentially continuous duality mapping J from E to E^*. Let C be a nonempty closed and convex subset of E which is also a sunny nonexpansive retraction of E such that $C + C \subset C$. Let A be a strongly positive linear bounded operator on E with coefficient $\overline{\gamma} > 0$ such that $0 < \gamma < \overline{\gamma}/\alpha$, and let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$. Let S_i, $i = 1, 2, \ldots$, be λ_i-strict pseudocontractions from C into itself such that $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and $\inf \lambda_i > 0$. Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ in $(0,1)$ satisfy the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$; *and* $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(iii) $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$,

(iv) $\lim_{n\to\infty} |\delta_{n+1} - \delta_n| = 0$,

(v) $\delta_n(1 + \gamma_n) - 2\gamma_n > a$ *for some* $a \in (0,1)$,

and the sequence $\{\theta_{n,k}\}$ satisfies (H_1)–(H_3). Then, the sequence $\{x_n\}$ generated by

$$\begin{aligned} x_0 &\in C \text{ chosen arbitrarily}, \\ z_n &= \delta_n x_n + (1 - \delta_n) W_n x_n, \\ y_n &= \gamma_n x_n + (1 - \gamma_n) W_n z_n, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + \beta_n x_n + \big((1 - \beta_n) I - \alpha_n A\big) y_n, \quad \forall n \geq 0 \end{aligned} \tag{3.1}$$

converges strongly to $x^* \in \cap_{n=1}^{\infty} F(S_n)$, which solves the following variational inequality:

$$\langle \gamma f(x^*) - Ax^*, J(p - x^*) \rangle \leq 0, \quad \forall f \in \Pi_C,\ p \in \bigcap_{n=1}^{\infty} F(S_n). \tag{3.2}$$

Proof. By (i), we may assume, without loss of generality, that $\alpha_n \leq (1 - \beta_n)\|A\|^{-1}$ for all n. Since A is a strongly positive bounded linear operator on E and by (2.1), we have

$$\|A\| = \sup\{|\langle Ax, J(x)\rangle| : x \in E, \|x\| = 1\}. \tag{3.3}$$

Observe that

$$\begin{aligned} \big\langle \big((1 - \beta_n) I - \alpha_n A\big) x, J(x) \big\rangle &= 1 - \beta_n - \alpha_n \langle Ax, J(x)\rangle \\ &\geq 1 - \beta_n - \alpha_n \|A\| \\ &\geq 0, \quad \forall x \in E. \end{aligned} \tag{3.4}$$

This shows that $(1-\beta_n)I-\alpha_nA$ is positive. It follows that

$$\begin{aligned}\left\|(1-\beta_n)I-\alpha_nA\right\| &= \sup\left\{\left|\left\langle\left((1-\beta_n)I-\alpha_nA\right)x,J(x)\right\rangle\right| : x\in E,\|x\|=1\right\}\\ &= \sup\left\{1-\beta_n-\alpha_n\langle Ax,J(x)\rangle : x\in E,\|x\|=1\right\}\\ &\le 1-\beta_n-\alpha_n\overline{\gamma}.\end{aligned}\tag{3.5}$$

First, we show that $\{x_n\}$ is bounded. Let $p\in\cap_{n=1}^{\infty}F(S_n)$. By the definition of $\{z_n\}$, $\{y_n\}$, and $\{x_n\}$, we have

$$\begin{aligned}\left\|z_n-p\right\| &= \left\|\delta_nx_n+(1-\delta_n)W_nx_n-p\right\|\\ &\le \delta_n\left\|x_n-p\right\|+(1-\delta_n)\left\|W_nx_n-p\right\|\\ &\le \delta_n\left\|x_n-p\right\|+(1-\delta_n)\left\|x_n-p\right\|\\ &= \left\|x_n-p\right\|,\end{aligned}\tag{3.6}$$

and from this, we have

$$\begin{aligned}\left\|y_n-p\right\| &= \left\|\gamma_nx_n+(1-\gamma_n)W_nz_n-p\right\|\\ &\le \gamma_n\left\|x_n-p\right\|+(1-\gamma_n)\left\|W_nz_n-p\right\|\\ &\le \gamma_n\left\|x_n-p\right\|+(1-\gamma_n)\left\|z_n-p\right\|\\ &\le \gamma_n\left\|x_n-p\right\|+(1-\gamma_n)\left\|x_n-p\right\|\\ &= \left\|x_n-p\right\|.\end{aligned}\tag{3.7}$$

It follows that

$$\begin{aligned}\left\|x_{n+1}-p\right\| &= \left\|\alpha_n\gamma f(x_n)+\beta_nx_n+\left((1-\beta_n)I-\alpha_nA\right)y_n-p\right\|\\ &= \left\|\alpha_n\left(\gamma f(x_n)-Ap\right)+\beta_n(x_n-p)+\left((1-\beta_n)I-\alpha_nA\right)(y_n-p)\right\|\\ &\le \alpha_n\left\|\gamma f(x_n)-Ap\right\|+\beta_n\left\|x_n-p\right\|+\left(1-\beta_n-\alpha_n\overline{\gamma}\right)\left\|y_n-p\right\|\\ &\le \alpha_n\left\|\gamma f(x_n)-Ap\right\|+\beta_n\left\|x_n-p\right\|+\left(1-\beta_n-\alpha_n\overline{\gamma}\right)\left\|x_n-p\right\|\\ &\le \alpha_n\left\|\gamma f(x_n)-\gamma f(p)\right\|+\alpha_n\left\|\gamma f(p)-Ap\right\|+\left(1-\alpha_n\overline{\gamma}\right)\left\|x_n-p\right\|\\ &\le \alpha_n\gamma\alpha\left\|x_n-p\right\|+\alpha_n\left\|\gamma f(p)-Ap\right\|+\left(1-\alpha_n\overline{\gamma}\right)\left\|x_n-p\right\|\\ &= \left(1-\left(\overline{\gamma}-\gamma\alpha\right)\alpha_n\right)\left\|x_n-p\right\|+\left(\overline{\gamma}-\gamma\alpha\right)\alpha_n\frac{\left\|\gamma f(p)-Ap\right\|}{\overline{\gamma}-\gamma\alpha}\\ &\le \max\left\{\left\|x_1-p\right\|,\frac{\left\|\gamma f(p)-Ap\right\|}{\overline{\gamma}-\gamma\alpha}\right\}.\end{aligned}\tag{3.8}$$

By induction on n, we obtain $\|x_n - p\| \leq \max\{\|x_0 - p\|, \|\gamma f(p) - Ap\|/\overline{\gamma} - \gamma\alpha\|\}$ for every $n \geq 0$ and $x_0 \in C$, then $\{x_n\}$ is bounded. So, $\{y_n\}, \{z_n\}, \{Ay_n\}, \{W_n x_n\}$, $\{W_n z_n\}$, and $\{f(x_n)\}$ are also bounded.

Next, we claim that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Let $x \in C$ and $p \in \cap_{n=1}^{\infty} F(S_n)$. Fix $k \in \mathbb{N}$ for any $n \in \mathbb{N}$ with $n \geq k$, and since $T_{n,k}$ and $U_{n,k}$ are nonexpansive, we have $\|T_{n,k}x - p\| \leq \|x - p\|$ and $\|U_{n,k}x - p\| \leq \|x - p\|$, respectively. From (1.5), it follows that $\|S_k x - p\| \leq (1 + (1/\lambda_k^{1/q-1}))\sup_n \|x - p\|$. We can set

$$
\begin{aligned}
M_1 &= \inf_i \left(2 + \frac{1}{\lambda_i^{1/q-1}}\right) \sup_n \|x_n - p\| < \infty, \\
M_2 &= \inf_i \left(2 + \frac{1}{\lambda_i^{1/q-1}}\right) \sup_n \|z_n - p\| < \infty.
\end{aligned}
\tag{3.9}
$$

From (1.11), we have

$$
\begin{aligned}
\|W_{n+1}x_n - W_n x_n\| &= \|U_{n+1,1}x_n - U_{n,1}x_n\| \\
&= \|t_1 T_{n+1,1}U_{n+1,2}x_n + (1 - t_1)x_n - t_1 T_{n,1}U_{n,2}x_n - (1 - t_1)x_n\| \\
&= t_1\|T_{n+1,1}U_{n+1,2}x_n - T_{n,1}U_{n,2}x_n\| \\
&= t_1\|(\theta_{n+1,1}S_1 + (1 - \theta_{n+1,1}))U_{n+1,2}x_n - T_{n,1}U_{n,2}x_n\| \\
&= t_1\|(\theta_{n,1}S_1 + (1 - \theta_{n,1}))U_{n+1,2}x_n \\
&\qquad -T_{n,1}U_{n,2}x_n + (\theta_{n+1,1} - \theta_{n,1})(S_1 U_{n+1,2}x_n - U_{n+1,2}x_n)\| \\
&\leq t_1\|T_{n,1}U_{n+1,2}x_n - T_{n,1}U_{n,2}x_n\| + t_1|\theta_{n+1,1} - \theta_{n,1}|\|S_1 U_{n+1,2}x_n - U_{n+1,2}x_n\| \\
&\leq t_1\|U_{n+1,2}x_n - U_{n,2}x_n\| + t_1|\theta_{n+1,1} - \theta_{n,1}|M_1 \\
&\leq t_1\|U_{n+1,2}x_n - U_{n,2}x_n\| + t_1 a_n M_1 \\
&\vdots \\
&\leq \prod_{i=1}^{n} t_i\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| + a_n M_1 \sum_{j=1}^{n}\prod_{i=1}^{j} t_i \\
&\leq \prod_{i=1}^{n} t_i\|t_{n+1}T_{n+1,n+1}x_n + (1 - t_{n+1})x_n - x_n\| + a_n M_1 \frac{b}{1-b} \\
&\leq \prod_{i=1}^{n+1} t_i\|T_{n+1,n+1}x_n - x_n\| + a_n M_1 \frac{b}{1-b} \\
&\leq \left(b^{n+1} + a_n \frac{b}{1-b}\right) M_1,
\end{aligned}
\tag{3.10}
$$

for all $n \geq 0$. Similarly, we also have $\|W_{n+1}z_n - W_n z_n\| \leq (b^{n+1} + a_n(b/(1-b)))M_2$ for all $n \geq 0$. We compute that

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \|(\delta_{n+1}x_{n+1} + (1-\delta_{n+1})W_{n+1}x_{n+1}) - (\delta_n x_n + (1-\delta_n)W_n x_n)\| \\
&\leq (1-\delta_{n+1})\|W_{n+1}x_{n+1} - W_{n+1}x_n\| + |\delta_{n+1} - \delta_n|\|x_n - W_{n+1}x_n\| + \delta_{n+1}\|x_{n+1} - x_n\| \\
&\quad + (1-\delta_n)\|W_{n+1}x_n - W_n x_n\| \\
&\leq (1-\delta_{n+1})\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|\|W_{n+1}x_n - x_n\| + \delta_{n+1}\|x_{n+1} - x_n\| \\
&\quad + (1-\delta_n)\|W_{n+1}x_n - W_n x_n\| \\
&= \|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|\|W_{n+1}x_n - x_n\| + (1-\delta_n)\|W_{n+1}x_n - W_n x_n\| \\
&\leq \|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|\|W_{n+1}x_n - x_n\| + \|W_{n+1}x_n - W_n x_n\| \\
&\leq \|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|\|W_{n+1}x_n - x_n\| + \left(b^{n+1} + a_n\frac{b}{1-b}\right)M_1,
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|(\gamma_{n+1}x_{n+1} + (1-\gamma_{n+1})W_{n+1}z_{n+1}) - (\gamma_n x_n + (1-\gamma_n)W_n z_n)\| \\
&= \|\gamma_{n+1}x_{n+1} + (1-\gamma_{n+1})W_{n+1}z_{n+1} - (1-\gamma_{n+1})W_{n+1}z_n + (1-\gamma_{n+1})W_{n+1}z_n \\
&\quad -\gamma_n x_n - (1-\gamma_n)W_n z_n - (1-\gamma_n)W_{n+1}z_n + (1-\gamma_n)W_{n+1}z_n - \gamma_{n+1}x_n + \gamma_{n+1}x_n\| \\
&= \|(1-\gamma_{n+1})(W_{n+1}z_{n+1} - W_{n+1}z_n) + (\gamma_n - \gamma_{n+1})W_{n+1}z_n + \gamma_{n+1}(x_{n+1} - x_n) \\
&\quad +(\gamma_{n+1} - \gamma_n)x_n + (1-\gamma_n)(W_{n+1}z_n - W_n z_n)\| \\
&\leq (1-\gamma_{n+1})\|W_{n+1}z_{n+1} - W_{n+1}z_n\| + |\gamma_{n+1} - \gamma_n|\|x_n - W_{n+1}z_n\| + \gamma_{n+1}\|x_{n+1} - x_n\| \\
&\quad + (1-\gamma_n)\|W_{n+1}z_n - W_n z_n\| \\
&\leq (1-\gamma_{n+1})\|z_{n+1} - z_n\| + |\gamma_{n+1} - \gamma_n|\|W_{n+1}z_n - x_n\| + \gamma_{n+1}\|x_{n+1} - x_n\| \\
&\quad + \|W_{n+1}z_n - W_n z_n\| \\
&\leq (1-\gamma_{n+1})\left(\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|\|W_{n+1}x_n - x_n\| + \left(b^{n+1} + a_n\frac{b}{1-b}\right)M_1\right) \\
&\quad + |\gamma_{n+1} - \gamma_n|\|W_{n+1}z_n - x_n\| + \gamma_{n+1}\|x_{n+1} - x_n\| + \left(b^{n+1} + a_n\frac{b}{1-b}\right)M_2 \\
&\leq \|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|\|W_{n+1}x_n - x_n\| + |\gamma_{n+1} - \gamma_n|\|W_{n+1}z_n - x_n\| \\
&\quad + 2\left(b^{n+1} + a_n\frac{b}{1-b}\right)M,
\end{aligned} \tag{3.12}$$

where $M = \inf_i(2 + (1/\lambda_i^{1/q-1}))\sup_n(\|x_n - p\| + \|z_n - p\|) < \infty$. Observe that we put $l_n = (x_{n+1} - \beta_n x_n)/1 - \beta_n$, then

$$x_{n+1} = (1-\beta_n)l_n + \beta_n x_n, \quad \forall n \geq 0. \tag{3.13}$$

Now, we have

$$\begin{aligned}
\|l_{n+1} - l_n\| &= \left\| \frac{\alpha_{n+1}\gamma f(x_{n+1}) + ((1-\beta_{n+1})I - \alpha_{n+1}A)y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n \gamma f(x_n) + ((1-\beta_n)I - \alpha_n A)y_n}{1-\beta_n} \right\| \\
&= \left\| \frac{\alpha_{n+1}\gamma f(x_{n+1})}{1-\beta_{n+1}} + \frac{(1-\beta_{n+1})y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_{n+1}Ay_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n \gamma f(x_n)}{1-\beta_n} - \frac{(1-\beta_n)y_n}{1-\beta_n} + \frac{\alpha_n A y_n}{1-\beta_n} \right\| \\
&= \left\| \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1}) - Ay_{n+1}) + \frac{\alpha_n}{1-\beta_n}(Ay_n - \gamma f(x_n)) + y_{n+1} - y_n \right\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|Ay_n - \gamma f(x_n)\| + \|y_{n+1} - y_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|Ay_n - \gamma f(x_n)\| \\
&\quad + \|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|\|W_{n+1}x_n - x_n\| + |\gamma_{n+1} - \gamma_n|\|W_{n+1}z_n - x_n\| \\
&\quad + 2\left(b^{n+1} + a_n \frac{b}{1-b}\right)M.
\end{aligned} \tag{3.14}$$

Therefore, we have

$$\begin{aligned}
\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|Ay_n - \gamma f(x_n)\| \\
&\quad + |\delta_{n+1} - \delta_n|\|W_{n+1}x_n - x_n\| + |\gamma_{n+1} - \gamma_n|\|W_{n+1}z_n - x_n\| \\
&\quad + 2\left(b^{n+1} + a_n \frac{b}{1-b}\right)M.
\end{aligned} \tag{3.15}$$

From the conditions (i)–(iv), (H2), $0 < b < 1$ and the boundedness of $\{x_n\}$, $\{f(x_n)\}$, $\{Ay_n\}$, $\{W_n x_n\}$, and $\{W_n z_n\}$, we obtain

$$\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{3.16}$$

It follows from Lemma 2.8 that $\lim_{n\to\infty}\|l_n - x_n\| = 0$. Noting (3.13), we see that

$$\|x_{n+1} - x_n\| = (1-\beta_n)\|l_n - x_n\| \longrightarrow 0, \tag{3.17}$$

as $n \to \infty$. Therefore, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.18}$$

We also have $\|y_{n+1} - y_n\| \to 0$ and $\|z_{n+1} - z_n\| \to 0$ as $n \to \infty$. Observing that

$$\begin{aligned} \|x_n - y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n \|\gamma f(x_n) - A y_n\| + \beta_n \|x_n - y_n\|, \end{aligned} \tag{3.19}$$

it follows that

$$(1 - \beta_n)\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \alpha_n \|\gamma f(x_n) - A y_n\|. \tag{3.20}$$

By the conditions (i), (ii), (3.18), and the boundedness of $\{x_n\}$, $\{f(x_n)\}$, and $\{Ay_n\}$, we obtain

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.21}$$

Consider

$$\begin{aligned} \|y_n - W_n z_n\| &= \|\gamma_n x_n + (1 - \gamma_n) W_n z_n - W_n z_n\| = \gamma_n \|x_n - W_n z_n\|, \\ \|z_n - x_n\| &= \|\delta_n x_n + (1 - \delta_n) W_n x_n - x_n\| = (1 - \delta_n)\|W_n x_n - x_n\|. \end{aligned} \tag{3.22}$$

It follows that

$$\begin{aligned} \|x_n - W_n x_n\| &\le \|x_n - y_n\| + \|y_n - W_n z_n\| + \|W_n z_n - W_n x_n\| \\ &\le \|x_n - y_n\| + \gamma_n \|x_n - W_n z_n\| + \|z_n - x_n\| \\ &\le \|x_n - y_n\| + \gamma_n \|x_n - W_n x_n\| + \gamma_n \|W_n x_n - W_n z_n\| + \|z_n - x_n\| \\ &\le \|x_n - y_n\| + \gamma_n \|x_n - W_n x_n\| + (1 + \gamma_n)\|z_n - x_n\| \\ &= \|x_n - y_n\| + \gamma_n \|x_n - W_n x_n\| + (1 + \gamma_n)(1 - \delta_n)\|W_n x_n - x_n\|. \end{aligned} \tag{3.23}$$

This implies that

$$(\delta_n(1 + \gamma_n) - 2\gamma_n)\|W_n x_n - x_n\| \le \|x_n - y_n\|. \tag{3.24}$$

From the condition (v) and (3.21), we get

$$\lim_{n\to\infty} \|W_n x_n - x_n\| = 0. \tag{3.25}$$

On the other hand,

$$\|W x_n - x_n\| \le \|W x_n - W_n x_n\| + \|W_n x_n - x_n\|. \tag{3.26}$$

From the boundedness of $\{x_n\}$ and using (2.7), we have $\|Wx_n - W_n x_n\| \to 0$ as $n \to \infty$. It follows that

$$\lim_{n\to\infty} \|Wx_n - x_n\| = 0. \tag{3.27}$$

Next, we prove that

$$\limsup_{n\to\infty} \langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle \le 0, \tag{3.28}$$

where $x^* = \lim_{t\to 0} x_t$ with x_t being the fixed point of contraction $x \mapsto t\gamma f(x) + (1 - tA)Wx$. Noticing that x_t solves the fixed point equation $x_t = t\gamma f(x_t) + (1 - tA)Wx_t$, it follows that

$$\|x_t - x_n\| = \left\|(I - tA)(Wx_t - x_n) + t\left(\gamma f(x_t) - Ax_n\right)\right\|. \tag{3.29}$$

It follows from Lemma 2.1 that

$$\begin{aligned}
\|x_t - x_n\|^2 &= \left\|(I - tA)(Wx_t - x_n) + t\left(\gamma f(x_t) - Ax_n\right)\right\|^2 \\
&\le \left(1 - \overline{\gamma}t\right)^2 \|Wx_t - x_n\|^2 + 2t\langle \gamma f(x_t) - Ax_n, J(x_t - x_n)\rangle \\
&= \left(1 - \overline{\gamma}t\right)^2 \|Wx_t - Wx_n + Wx_n - x_n\|^2 + 2t\langle \gamma f(x_t) - Ax_n, J(x_t - x_n)\rangle \\
&\le \left(1 - \overline{\gamma}t\right)^2 \left[\|Wx_t - Wx_n\|^2 + 2\langle Wx_n - x_n, J(Wx_t - x_n)\rangle\right] \\
&\quad + 2t\langle \gamma f(x_t) - Ax_n, J(x_t - x_n)\rangle \\
&\le \left(1 - \overline{\gamma}t\right)^2 \left[\|x_t - x_n\|^2 + 2\|Wx_n - x_n\|\|Wx_t - x_n\|\right] + 2t\langle \gamma f(x_t) - Ax_n, J(x_t - x_n)\rangle \\
&\le \left(1 - \overline{\gamma}t\right)^2 \left[\|x_t - x_n\|^2 + 2\|Wx_n - x_n\|(\|x_t - x_n\| + \|Wx_n - x_n\|)\right] \\
&\quad + 2t\langle \gamma f(x_t) - Ax_n, J(x_t - x_n)\rangle \\
&= \left(1 - 2\overline{\gamma}t + \left(\overline{\gamma}t\right)^2\right)\|x_t - x_n\|^2 + 2\left(1 - \overline{\gamma}t\right)^2 \|Wx_n - x_n\|(\|x_t - x_n\| + \|Wx_n - x_n\|) \\
&\quad + 2t\langle \gamma f(x_t) - Ax_t, J(x_t - x_n)\rangle + 2t\langle Ax_t - Ax_n, J(x_t - x_n)\rangle,
\end{aligned} \tag{3.30}$$

where

$$f_n(t) = 2\left(1 - \overline{\gamma}t\right)^2 \|Wx_n - x_n\|(\|x_t - x_n\| + \|Wx_n - x_n\|) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.31}$$

Since A is linearly strong and positive and using (2.1), we have

$$\langle Ax_t - Ax_n, J(x_t - x_n)\rangle = \langle A(x_t - x_n), J(x_t - x_n)\rangle \ge \overline{\gamma}\|x_t - x_n\|^2. \tag{3.32}$$

Substituting (3.32) in (3.30), we have

$$\begin{aligned}\langle Ax_t - \gamma f(x_t), J(x_t - x_n)\rangle &\le \left(\frac{\overline{\gamma}^2 \mathrm{t}}{2} - \overline{\gamma}\right)\|x_t - x_n\|^2 + \frac{1}{2t}f_n(t) + \langle Ax_t - Ax_n, J(x_t - x_n)\rangle \\ &\le \left(\frac{\overline{\gamma}t}{2} - 1\right)\langle Ax_t - Ax_n, J(x_t - x_n)\rangle \\ &\quad + \frac{1}{2t}f_n(t) + \langle Ax_t - Ax_n, J(x_t - x_n)\rangle \\ &= \frac{\overline{\gamma}t}{2}\langle Ax_t - Ax_n, J(x_t - x_n)\rangle + \frac{1}{2t}f_n(t).\end{aligned} \tag{3.33}$$

Letting $n \to \infty$ in (3.33) and noting (3.31) yield that

$$\limsup_{n\to\infty}\langle Ax_t - \gamma f(x_t),\ J(x_t - x_n)\rangle \le \frac{t}{2}M_3, \tag{3.34}$$

where $M_3 > 0$ is a constant such that $M_3 \ge \overline{\gamma}\langle Ax_t - Ax_n, J(x_t - x_n)\rangle$ for all $t \in (0,1)$ and $n \ge 0$. Taking $t \to 0$ from (3.34), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle Ax_t - \gamma f(x_t), J(x_t - x_n)\rangle \le 0. \tag{3.35}$$

On the other hand, we have

$$\begin{aligned}\langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle &= \langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle \\ &\quad - \langle \gamma f(x^*) - Ax^*, J(x_n - x_t)\rangle + \langle \gamma f(x^*) - Ax^*, J(x_n - x_t)\rangle \\ &\quad - \langle \gamma f(x^*) - Ax_t, J(x_n - x_t)\rangle + \langle \gamma f(x^*) - Ax_t, J(x_n - x_t)\rangle \\ &\quad - \langle \gamma f(x_t) - Ax_t, J(x_n - x_t)\rangle + \langle \gamma f(x_t) - Ax_t, J(x_n - x_t)\rangle \\ &= \langle \gamma f(x^*) - Ax^*, J(x_n - x^*) - J(x_n - x_t)\rangle \\ &\quad + \langle Ax_t - Ax^*, J(x_n - x_t)\rangle \\ &\quad + \langle \gamma f(x^*) - \gamma f(x_t), J(x_n - x_t)\rangle + \langle \gamma f(x_t) - Ax_t, J(x_n - x_t)\rangle,\end{aligned} \tag{3.36}$$

which implies that

$$\begin{aligned}\limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle &\le \limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J(x_n - x^*) - J(x_n - x_t)\rangle \\ &\quad + \|A\|\|x_t - x^*\|\limsup_{n\to\infty}\|x_n - x_t\| \\ &\quad + \gamma\alpha\|x^* - x_t\|\limsup_{n\to\infty}\|x_n - x_t\| \\ &\quad + \limsup_{n\to\infty}\langle \gamma f(x_t) - Ax_t, J(x_n - x_t)\rangle.\end{aligned} \tag{3.37}$$

Noticing that J is norm-to-norm uniformly continuous on bounded subsets of C, it follows from (3.35) that

$$\limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle = \limsup_{t\to 0}\limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle \le 0. \tag{3.38}$$

Therefore, we obtain that (3.28) holds.

Finally, we prove that $x_n \to x^*$ as $n \to \infty$. Now, from Lemma 2.1, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\|\alpha_n\gamma f(x_n) + \beta_n x_n + \left[(1-\beta_n)I - \alpha_n A\right]y_n - x^*\right\|^2 \\
&= \left\|\left[(1-\beta_n)I - \alpha_n A\right](y_n - x^*) + \alpha_n(\gamma f(x_n) - Ax^*) + \beta_n(x_n - x^*)\right\|^2 \\
&\le \left(1-\beta_n-\alpha_n\overline{\gamma}\right)^2\|y_n - x^*\|^2 + 2\langle \alpha_n(\gamma f(x_n) - Ax^*) + \beta_n(x_n - x^*), J(x_{n+1} - x^*)\rangle \\
&= \left(1-\beta_n-\alpha_n\overline{\gamma}\right)^2\|y_n - x^*\|^2 + 2\langle \alpha_n(\gamma f(x_n) - Ax^*), J(x_{n+1} - x^*)\rangle \\
&\quad + 2\beta_n\langle x_n - x^*, J(x_{n+1} - x^*)\rangle \\
&= \left(1-\beta_n-\alpha_n\overline{\gamma}\right)^2\|y_n - x^*\|^2 + 2\alpha_n\gamma\langle f(x_n) - f(x^*), J(x_{n+1} - x^*)\rangle \\
&\quad + 2\alpha_n\langle \gamma f(x^*) - Ax^*, J(x_{n+1} - x^*)\rangle + 2\beta_n\langle x_n - x^*, J(x_{n+1} - x^*)\rangle \\
&\le \left(1-\beta_n-\alpha_n\overline{\gamma}\right)^2\|x_n - x^*\|^2 + \alpha_n\gamma\alpha\left(\|x_{n+1} - x^*\|^2 + \|x_n - x^*\|^2\right) \\
&\quad + 2\alpha_n\langle \gamma f(x^*) - Ax^*, J(x_{n+1} - x^*)\rangle + \beta_n\left(\|x_{n+1} - x^*\|^2 + \|x_n - x^*\|^2\right) \\
&= \left[\left(1-\beta_n-\alpha_n\overline{\gamma}\right)^2 + \alpha_n\gamma\alpha + \beta_n\right]\|x_n - x^*\|^2 + \left(\alpha_n\gamma\alpha + \beta_n\right)\|x_{n+1} - x^*\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(x^*) - Ax^*, J(x_{n+1} - x^*)\rangle,
\end{aligned} \tag{3.39}$$

and consequently,

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \frac{\left(1-\beta_n-\alpha_n\overline{\gamma}\right)^2 + \alpha_n\gamma\alpha + \beta_n}{1-\alpha_n\gamma\alpha-\beta_n}\|x_n - x^*\|^2 \\
&\quad + \frac{2\alpha_n}{1-\alpha_n\gamma\alpha-\beta_n}\langle \gamma f(x^*) - Ax^*, J(x_{n+1} - x^*)\rangle \\
&= \left[1 - \frac{2\alpha_n(\overline{\gamma}-\gamma\alpha)}{1-\alpha_n\gamma\alpha-\beta_n}\right]\|x_n - x^*\|^2 + \frac{\beta_n^2 + 2\beta_n\alpha_n\overline{\gamma} + \alpha_n^2\overline{\gamma}^2}{1-\alpha_n\gamma\alpha-\beta_n}\|x_n - x^*\|^2 \\
&\quad + \frac{2\alpha_n}{1-\alpha_n\gamma\alpha-\beta_n}\langle \gamma f(x^*) - Ax^*, J(x_{n+1} - x^*)\rangle \\
&= \left[1 - \frac{2\alpha_n(\overline{\gamma}-\gamma\alpha)}{1-\alpha_n\gamma\alpha-\beta_n}\right]\|x_n - x^*\|^2 + \frac{2\alpha_n(\overline{\gamma}-\gamma\alpha)}{1-\alpha_n\gamma\alpha-\beta_n} \\
&\quad \times \left[\frac{\beta_n^2 + 2\beta_n\alpha_n\overline{\gamma} + \alpha_n^2\overline{\gamma}^2}{2\alpha_n(\overline{\gamma}-\gamma\alpha)}M_4 + \frac{1}{\overline{\gamma}-\gamma\alpha}\langle \gamma f(x^*) - Ax^*, J(x_{n+1} - x^*)\rangle\right],
\end{aligned} \tag{3.40}$$

where M_4 is an appropriate constant such that $M_4 \geq \sup_{n\geq 0}\|x_n - x^*\|^2$. Setting $c_n = 2\alpha_n(\overline{\gamma} - \gamma\alpha)/(1 - \alpha_n\gamma\alpha - \beta_n)$ and $b_n = (\beta_n^2 + 2\beta_n\alpha_n\overline{\gamma} + \alpha_n^2\overline{\gamma}^2)/(2\alpha_n(\overline{\gamma} - \gamma\alpha))M_4 + (1/(\overline{\gamma} - \gamma\alpha))\langle \gamma f(x^*) - Ax^*, J(x_{n+1} - x^*)\rangle$, then we have

$$\|x_{n+1} - x^*\|^2 \leq (1 - c_n)\|x_n - x^*\|^2 + c_n b_n. \tag{3.41}$$

By (3.28), (i) and applying Lemma 2.7 to (3.41), we have $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

Corollary 3.2. *Let E be a real q-uniformly smooth and strictly convex Banach space which admits a weakly sequentially continuous duality mapping J from E to E^*. Let C be a nonempty closed and convex subset of E which is also a sunny nonexpansive retraction of E such that $C + C \subset C$. Let A be a strongly positive linear bounded operator on E with coefficient $\overline{\gamma} > 0$ such that $0 < \gamma < \overline{\gamma}/\alpha$, and let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$. Let S_i, $i = 1, 2, \ldots$, be λ_i-strict pseudocontractions from C into itself such that $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and $\inf \lambda_i > 0$. Assume that the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, and $\{\delta_n\}$ in $(0,1)$ satisfy the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$; *and* $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(iii) $\lim_{n\to\infty}|\gamma_{n+1} - \gamma_n| = 0$,

(iv) $\lim_{n\to\infty}|\delta_{n+1} - \delta_n| = 0$,

(v) $\delta_n(1 + \gamma_n) - 2\gamma_n > a$ *for some* $a \in (0,1)$,

and the sequence $\{\theta_n\}$ satisfies (H_1). Then, the sequence $\{x_n\}$ generated by

$$\begin{aligned} x_0 &\in C \text{ chosen arbitrarily}, \\ z_n &= \delta_n x_n + (1 - \delta_n)W_n x_n, \\ y_n &= \gamma_n x_n + (1 - \gamma_n)W_n z_n, \\ x_{n+1} &= \alpha_n\gamma f(x_n) + \beta_n x_n + \big((1 - \beta_n)I - \alpha_n A\big)y_n, \quad \forall n \geq 0 \end{aligned} \tag{3.42}$$

converges strongly to $x^* \in \cap_{n=1}^{\infty} F(S_n)$, which solves the following variational inequality:

$$\langle \gamma f(x^*) - Ax^*, J(p - x^*)\rangle \leq 0, \quad \forall f \in \Pi_C,\ p \in \bigcap_{n=1}^{\infty} F(S_n). \tag{3.43}$$

Corollary 3.3. *Let E be a real q-uniformly smooth and strictly convex Banach space which admits a weakly sequentially continuous duality mapping J from E to E^*. Let C be a nonempty closed and convex subset of E which is also a sunny nonexpansive retraction of E such that $C + C \subset C$. Let A be a strongly positive linear bounded operator on E with coefficient $\overline{\gamma} > 0$ such that $0 < \gamma < \overline{\gamma}/\alpha$, and let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$. Let S_i, $i = 1, 2, \ldots$, be a nonexpansive mapping from C into itself such that $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and $\inf \lambda_i > 0$. Assume that the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, and $\{\delta_n\}$ in $(0,1)$ satisfy the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$; *and* $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,

(iii) $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$,

(iv) $\lim_{n\to\infty} |\delta_{n+1} - \delta_n| = 0$,

(v) $\delta_n(1+\gamma_n) - 2\gamma_n > a$ *for some* $a \in (0,1)$.

Then, the sequence $\{x_n\}$ *generated by*

$$\begin{aligned} x_0 &\in C \text{ chosen arbitrarily}, \\ z_n &= \delta_n x_n + (1-\delta_n) W_n x_n, \\ y_n &= \gamma_n x_n + (1-\gamma_n) W_n z_n, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + \beta_n x_n + \left((1-\beta_n) I - \alpha_n A\right) y_n, \quad \forall n \ge 0 \end{aligned} \tag{3.44}$$

converges strongly to $x^* \in \cap_{n=1}^{\infty} F(S_n)$, *which solves the following variational inequality:*

$$\left\langle \gamma f(x^*) - Ax^*, J(p - x^*) \right\rangle \le 0, \quad \forall f \in \Pi_C,\ p \in \bigcap_{n=1}^{\infty} F(S_n). \tag{3.45}$$

Remark 3.4. Theorem 3.1, Corollaries 3.2, and 3.3, improve and extend the corresponding results of Cai and Hu [3], Dong et al. [2], and Katchang and Kumam [4, 5] in the following senses.

(i) For the mappings, we extend the mappings from an infinite family of nonexpansive mappings to an infinite family of strict pseudocontraction mappings.

(ii) For the algorithms, we propose new modified Ishikawa iterative algorithms, which are different from the ones given in [2–5] and others.

Acknowledgments

The authors would like to thank The National Research Council of Thailand (NRCT) and the Faculty of Science KMUTT for financial support. Furthermore, they also would like to thank the National Research University Project of Thailand's Office of the Higher Education Commission for financial support (NRU-CSEC Project no. 54000267). Finally, they are grateful for the reviewers for the careful reading of the paper and for the suggestions which improved the quality of this work.

References

[1] F. E. Browder and W. V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.

[2] Q.-L. Dong, S. He, and F. Su, "Strong convergence of an iterative algorithm for an infinite family of strict pseudo-contractions in Banach spaces," *Applied Mathematics and Computation*, vol. 216, no. 3, pp. 959–969, 2010.

[3] G. Cai and C. S. Hu, "Strong convergence theorems of a general iterative process for a finite family of λ_i-strict pseudo-contractions in q-uniformly smooth Banach spaces," *Computers & Mathematics with Applications*, vol. 59, no. 1, pp. 149–160, 2010.

[4] P. Katchang and P. Kumam, "Strong convergence of the modified Ishikawa iterative method for infinitely many nonexpansive mappings in Banach spaces," *Computers & Mathematics with Applications*, vol. 59, no. 4, pp. 1473–1483, 2010.
[5] P. Katchang and P. Kumam, "Corrigendum to "Strong convergence of the modified Ishikawa iterative method for infinitely many nonexpansive mappings in Banach spaces" [Comput. Math. Appl. 59 (2010) 1473-1483]," *Computers & Mathematics with Applications*, vol. 61, no. 1, p. 148, 2011.
[6] K. Shimoji and W. Takahashi, "Strong convergence to common fixed points of infinite nonexpansive mappings and applications," *Taiwanese Journal of Mathematics*, vol. 5, no. 2, pp. 387–404, 2001.
[7] W. Takahashi, "Weak and strong convergence theorems for families of nonexpansive mappings and their applications," *Annales Universitatis Mariae Curie-Skłodowska. Sectio A*, vol. 51, no. 2, pp. 277–292, 1997.
[8] Y. J. Cho, S. M. Kang, and X. Qin, "Strong convergence of an implicit iterative process for an infinite family of strict pseudocontractions," *Bulletin of the Korean Mathematical Society*, vol. 47, no. 6, pp. 1259–1268, 2010.
[9] G. Cai and C. S. Hu, "Strong convergence theorems of modified Ishikawa iterative process with errors for an infinite family of strict pseudo-contractions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 12, pp. 6044–6053, 2009.
[10] W. Takahashi, *Nonlinear Functional Analysis: Fixed Point Theory and Its Applications*, Yokohama Publishers, Yokohama, Japan, 2000.
[11] R. E. Bruck Jr., "Nonexpansive projections on subsets of Banach spaces," *Pacific Journal of Mathematics*, vol. 47, pp. 341–355, 1973.
[12] S. Reich, "Asymptotic behavior of contractions in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 44, pp. 57–70, 1973.
[13] S.-S. Chang, "On Chidume's open questions and approximate solutions of multivalued strongly accretive mapping equations in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 216, no. 1, pp. 94–111, 1997.
[14] H. K. Xu, "Inequalities in Banach spaces with applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 16, no. 12, pp. 1127–1138, 1991.
[15] H. Zhang and Y. Su, "Strong convergence theorems for strict pseudo-contractions in q-uniformly smooth Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 9, pp. 3236–3242, 2009.
[16] H. K. Xu, "An iterative approach to quadratic optimization," *Journal of Optimization Theory and Applications*, vol. 116, no. 3, pp. 659–678, 2003.
[17] T. Suzuki, "Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.