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*Research Article*

# Existence of Solutions for Nonhomogeneous $A$ -Harmonic Equations with Variable Growth

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We study the following nonhomogeneous  $A$ -harmonic equations:  $d^*A(x, du(x)) + B(x, u(x)) = 0$ ,  $x \in \Omega$ ,  $u(x) = 0$ ,  $x \in \partial\Omega$ , where  $\Omega \subset \mathbb{R}^n$  is a bounded and convex Lipschitz domain,  $A(x, du(x))$  and  $B(x, u(x))$  satisfy some  $p(x)$ -growth conditions, respectively. We obtain the existence of weak solutions for the above equations in subspace  $\mathfrak{R}_0^{1,p(x)}(\Omega, \Lambda^{l-1})$  of  $W_0^{1,p(x)}(\Omega, \Lambda^{l-1})$ .

## 1. Introduction

Spaces of differential forms have been discussed in great details (see [1, 2] and the references therein). The theory of differential forms is an approach to multivariable calculus that is independent of coordinates and provides a better definition for integrals. Differential forms have played an important role in physical laws of thermodynamics, analytical mechanics, and physical theories, in particular Maxwell's theory, and the Yang-Mills theory, the theory of relativity, see for example [3–6].

In recent years, the study of  $A$ -harmonic equations for differential forms has developed rapidly. Many interesting results concerning  $A$ -harmonic equation have been established recently (see [7–11] and the references therein). In [12], spaces  $L^{p(x)}(\Omega)$  and  $W^{k,p(x)}(\Omega)$  are first introduced, and they used them to study the solutions of nonlinear Dirichlet boundary value problems with  $p(x)$ -growth conditions. In [13], spaces  $L^{p(x)}(\Omega, \Lambda^l, \omega)$  and  $W^{1,p(x)}(\Omega, \Lambda^l, \omega)$  are first introduced and used to study the weak solutions of obstacle problems of  $A$ -harmonic equations with variable growth for differential forms.

Let  $\Omega \subset \mathbb{R}^n$  be a bounded and convex Lipschitz domain. It is our purpose to study the following systems:

$$\begin{aligned} d^* A(x, du(x)) + B(x, u(x)) &= 0, \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where  $u \in \Lambda^{l-1}(\Omega)$ ,  $l = 1, 2, \dots, n$ , and  $A : \Omega \times \Lambda^l(\Omega) \rightarrow \Lambda^l(\Omega)$ ,  $B : \Omega \times \Lambda^{l-1}(\Omega) \rightarrow \Lambda^{l-1}(\Omega)$  satisfy the following conditions.

- (H1)  $A(x, \xi)$  and  $B(x, \zeta)$  are measurable with respect to  $x$  for all  $\xi, \zeta$  and continuous with respect to  $\xi, \zeta$ , respectively, for a.e.  $x \in \Omega$ .
- (H2)  $|A(x, \xi)| + |B(x, \zeta)| \leq C_1 |\xi|^{p(x)-1} + C_2 |\zeta|^{p(x)-1} + G(x)$ , where  $G \in L^{p'(x)}(\Omega)$  and  $C_1, C_2 \geq 0$  are constants.
- (H3)  $\langle A(x, \xi), \xi \rangle \geq a |\xi|^{p(x)} - |h(x)|$ , where  $a > 0$  is a constant and  $h \in L^1(\Omega)$ .
- (H4)  $\langle B(x, \zeta), \zeta \rangle \geq \bar{a} |\zeta|^{p(x)} - |\bar{h}(x)|$ , where  $\bar{a} \geq 0$  is a constant and  $\bar{h} \in L^1(\Omega)$ .
- (H5) For a.e.  $x_0 \in \Omega$ , the mapping  $\xi \rightarrow A(x_0, \xi)$  satisfies

$$\int_D \langle A(x_0, \xi_0 + dv(x)), dv(x) \rangle dx \geq \gamma \int_D |dv(x)|^{p(x)} dx, \tag{1.2}$$

for each  $\xi_0 \in \Lambda^l(\Omega)$ ,  $D \subset \Omega$  and  $v \in C_0^1(\Omega, \Lambda^{l-1})$ , where  $\gamma > 0$  is a constant. Here  $p'$  is the conjugate function of  $p$ . Throughout this paper we suppose (unless declare specially)

$$p \in \mathcal{P}^{\log}(\Omega), \quad 1 < p_* = \operatorname{ess\,inf}_\Omega p(x) \leq p(x) \leq \operatorname{ess\,sup}_\Omega p(x) = p^* < \infty. \tag{1.3}$$

## 2. Preliminaries

Let  $e_1, e_2, \dots, e_n$  be the standard orthogonal basis of  $\mathbb{R}^n$ . The space of all  $l$ -forms in  $\mathbb{R}^n$  is denoted by  $\Lambda^l(\mathbb{R}^n)$ . The dual basis to  $e_1, e_2, \dots, e_n$  is denoted by  $e^1, e^2, \dots, e^n$  and referred to as the standard basis for 1-form  $\Lambda^1(\mathbb{R}^n)$ . The Grassman algebra  $\Lambda(\mathbb{R}^n) = \oplus \Lambda^l(\mathbb{R}^n)$  is a graded algebra with respect to the exterior products. The standard ordered basis for  $\Lambda(\mathbb{R}^n)$  consists of the forms

$$1, e^1, e^2, \dots, e^n, e^1 \wedge e^2, \dots, e^{n-1} \wedge e^n, \dots, e^1 \wedge e^2 \dots \wedge e^n. \tag{2.1}$$

For  $\alpha(x) = \sum \alpha_I(x) e^I \in \Lambda^l(\mathbb{R}^n)$  and  $\beta(x) = \sum \beta_I(x) e^I \in \Lambda^l(\mathbb{R}^n)$ , the inner product is obtained by  $\langle \alpha, \beta \rangle = \sum \alpha_I(x) \beta_I(x)$  with summation over all  $l$ -tuples  $I = (i_1, \dots, i_l)$  and all integers  $l = 0, 1, \dots, n$ . The Hodge star operator (see [14])  $\star : \Lambda(\mathbb{R}^n) \rightarrow \Lambda(\mathbb{R}^n)$  is defined by the formulas

$$\star 1 = e^1 \wedge e^2 \dots \wedge e^n, \quad \alpha \wedge \star \beta = \beta \wedge \star \alpha = \langle \alpha, \beta \rangle e^1 \wedge e^2 \dots \wedge e^n. \tag{2.2}$$

Hence, the norm of  $\alpha$  is given by the formula  $|\alpha|^2 = \langle \alpha, \alpha \rangle = \star(\alpha \wedge \star\alpha) = \sum \alpha_I(x)\alpha_I(x) \in \Lambda^0(\mathbb{R}^n) = \mathbb{R}$ . Notice, the Hodge star operator is an isometric isomorphism operator on  $\Lambda(\mathbb{R}^n)$ . Moreover,

$$\star : \Lambda^l(\mathbb{R}^n) \longrightarrow \Lambda^{n-l}(\mathbb{R}^n), \quad \star\star = (-1)^{l(n-l)} : \Lambda^l(\mathbb{R}^n) \longrightarrow \Lambda^l(\mathbb{R}^n), \quad (2.3)$$

where  $I$  is the identity map.

Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain. The coordinate functions  $x_1, x_2, \dots, x_n$  in  $\Omega$  are considered to be differential forms of degree 0. The 1-forms  $dx_1, dx_2, \dots, dx_n$  are constant functions from  $\Omega$  into  $\Lambda^1(\mathbb{R}^n)$ . The value of  $dx_i$  is simply  $e^i, i = 1, 2, \dots, n$ . Therefore, every  $l$ -form  $u : \Omega \rightarrow \Lambda^l(\mathbb{R}^n)$  may be written uniquely as

$$u(x) = \sum_I u_I(x) dx_I = \sum_{1 \leq i_1 < \dots < i_l \leq n} u_{i_1, \dots, i_l}(x) dx_{i_1} \wedge \dots \wedge dx_{i_l}, \quad (2.4)$$

where the coefficients  $u_{i_1, \dots, i_l}(x)$  are distributions from  $\mathfrak{D}'(\Omega)$ , dual to the space of smooth functions with compact support on  $\Omega$ .

We use  $\mathfrak{D}'(\Omega, \Lambda^l)$  to denote the space of all differential  $l$ -forms. For each form  $u(x) \in \mathfrak{D}'(\Omega, \Lambda^l)$ , the exterior differential  $d : \mathfrak{D}'(\Omega, \Lambda^l) \rightarrow \mathfrak{D}'(\Omega, \Lambda^{l+1})$  is expressed by

$$du(x) = \sum_{k=1}^n \sum_{1 \leq i_1 < \dots < i_l \leq n} \frac{\partial u_{i_1, \dots, i_l}(x)}{\partial x_k} dx_k \wedge dx_{i_1} \wedge \dots \wedge dx_{i_l}. \quad (2.5)$$

For  $u \in \mathfrak{D}'(\Omega, \Lambda^l)$ , the vector-valued differential form

$$\nabla u = \left( \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n} \right) \quad (2.6)$$

consists of differential forms  $\partial u / \partial x_i \in \mathfrak{D}'(\Omega, \Lambda^l)$ , where the partial differentiation is applied to the coefficients of  $u$ .

The formal adjoint operator, called the Hodge codifferential, is given by

$$d^* = (-1)^{nl-1} \star d \star : \mathfrak{D}'(\Omega, \Lambda^{l+1}) \longrightarrow \mathfrak{D}'(\Omega, \Lambda^l). \quad (2.7)$$

By  $C^\infty(\Omega, \Lambda^l)$  denote the space of infinitely differentiable  $l$ -forms on  $\Omega$  and by  $C_0^\infty(\Omega, \Lambda^l)$  denote the subspace of  $C^\infty(\Omega, \Lambda^l)$  with compact support on  $\Omega$ .

Let  $\mathcal{P}(\Omega)$  be the set of all Lebesgue measurable functions  $p : \Omega \rightarrow (1, \infty)$ . For  $p \in \mathcal{P}(\Omega)$ , we put  $p_* = \text{essinf}_\Omega p(x)$  and  $p^* = \text{esssup}_\Omega p(x)$ . Given  $p \in \mathcal{P}(\Omega)$  we define the conjugate function  $p' \in \mathcal{P}(\Omega)$  by

$$p'(x) = \frac{p(x)}{p(x) - 1}, \quad \forall x \in \Omega. \quad (2.8)$$

*Definition 2.1* (see [15]). A Lebesgue measurable function  $p : \Omega \rightarrow \mathbb{R}$  is called globally log-Hölder continuous in  $\Omega$  if there exist  $p_\infty \in \mathbb{R}$  and a constant  $C > 0$  such that

$$|p(x) - p(y)| \leq \frac{C}{\log(e + 1/|x - y|)}, \quad |p(x) - p_\infty| \leq \frac{C}{\log(e + |x|)} \quad (2.9)$$

hold for all  $x, y \in \Omega$ .  $\mathcal{P}^{\log}(\Omega)$  is defined by

$$\mathcal{P}^{\log}(\Omega) = \left\{ p \in \mathcal{P}(\Omega) : \frac{1}{p} \text{ is globally log-Hölder continuous} \right\}. \quad (2.10)$$

For a differential  $l$ -form  $u(x)$  on  $\Omega$ ,  $l = 0, 1, \dots, n$ , define the functional  $\rho_{p(x)}$  by

$$\rho_{p(x), \Lambda^l}(u) = \int_{\Omega} |u(x)|^{p(x)} dx. \quad (2.11)$$

The space  $L^{p(x)}(\Omega, \Lambda^l) = \{u \in \Lambda^l(\Omega) : \exists \lambda > 0, \rho_{p(x), \Lambda^l}(\lambda u) < \infty\}$  is a reflexive Banach space endowed with the norm

$$\|u\|_{L^{p(x)}(\Omega, \Lambda^l)} = \inf \left\{ \lambda > 0 : \rho_{p(x), \Lambda^l} \left( \frac{u}{\lambda} \right) \leq 1 \right\}. \quad (2.12)$$

The space  $W^{1,p(x)}(\Omega, \Lambda^l) = \{u \in \Lambda^l(\Omega) : u \in L^{p(x)}(\Omega, \Lambda^l) \text{ and } du \in L^{p(x)}(\Omega, \Lambda^{l+1})\}$  is a reflexive Banach space endowed with the norm

$$\|u\|_{W^{1,p(x)}(\Omega, \Lambda^l)} = \|u\|_{L^{p(x)}(\Omega, \Lambda^l)} + \|du\|_{L^{p(x)}(\Omega, \Lambda^{l+1})}. \quad (2.13)$$

Note that  $L^{p(m)}(\Omega, \Lambda^0)$  and  $W^{1,p(m)}(\Omega, \Lambda^0)$  are spaces of functions on  $\Omega$ . In this paper, we denote them by  $L^{p(m)}(\Omega)$  and  $W^{1,p(m)}(\Omega)$ .

Iwaniec and Lutoborski proved the following results in [2].

Let  $\Omega \subset \mathbb{R}^n$  be a bounded and convex domain. If  $u(x) \in \Lambda^l(\mathbb{R}^n)$  is defined for some  $x \in \Omega$ , then the value of  $u(x)$  at the vectors  $\xi_1, \dots, \xi_l \in \mathbb{R}^n$  is denoted by  $u(x)(\xi_1, \dots, \xi_l)$ . Then to each  $y \in \Omega$ , there corresponds a linear operator  $K_y : L_{\text{loc}}^1(\Omega, \Lambda^l) \rightarrow L_{\text{loc}}^1(\Omega, \Lambda^{l-1})$  defined by

$$K_y u(x)(\xi_1, \xi_2, \dots, \xi_{l-1}) = \int_0^1 t^{l-1} u(tx + y - ty)(x - y, \xi_1, \xi_2, \dots, \xi_{l-1}) dt. \quad (2.14)$$

The homotopy operator  $T : L_{\text{loc}}^1(\Omega, \Lambda^l) \rightarrow L_{\text{loc}}^1(\Omega, \Lambda^{l-1})$  is defined by averaging  $K_y$  over all points  $y \in \Omega$

$$Tu(x) = \int_{\Omega} \varphi(y) K_y u(x) dy, \quad (2.15)$$

where  $\varphi \in C_0^\infty(\Omega)$  is normalized so that  $\int_\Omega \varphi(y) dy = 1$ . Then we have a pointwise estimate

$$|Tu(x)| \leq 2^n \mu(\Omega) \int_\Omega \frac{|u(y)|}{|x-y|^{n-1}} dy, \quad \forall x \in \Omega, \quad (2.16)$$

where

$$\mu(\Omega) = (\text{diam } \Omega)^{n+1} \inf \left\{ \frac{\|\nabla \varphi\|_{L^\infty(\Omega)}}{\|\varphi\|_{L^1(\Omega)}} : \varphi \in C_0^\infty(\Omega) \right\}, \quad (2.17)$$

further infimum is attained at  $\varphi(x) = \text{diam}(x, \partial\Omega)$ , and the decomposition

$$u = dTu + Tdu \quad (2.18)$$

holds for  $u \in L_{\text{loc}}^1(\Omega, \Lambda^l)$ .

*Definition 2.2.* For  $u \in L_{\text{loc}}^1(\Omega, \Lambda^l)$ , define the  $l$ -form  $u_\Omega \in \mathfrak{D}'(\Omega, \Lambda^l)$  by

$$u_\Omega = \begin{cases} \frac{1}{\text{meas}(\Omega)} \int_\Omega u(x) dx, & \text{for } l = 0, \\ dTu, & \text{for } l = 1, 2, \dots, n, \end{cases} \quad (2.19)$$

and the Maximal operator is defined by

$$(Mu)(x) = \sup_{r>0} \frac{1}{\text{meas}(B_r(x))} \int_{B_r(x)} |u(y)| dy, \quad (2.20)$$

where  $B_r(x) = \{y \in \mathbb{R}^n : |y-x| < r\}$ .

**Lemma 2.3** (see [15]). *Let  $p(x)$  satisfies (1.3). Then the inequality*

$$\|(Mu)(x)\|_{L^{p(x)}(\mathbb{R}^n)} \leq C(n, p) \|u(x)\|_{L^{p(x)}(\mathbb{R}^n)} \quad (2.21)$$

*holds for every  $u \in L^{p(x)}(\mathbb{R}^n)$ .*

**Lemma 2.4** (see [15]). *Let  $\Omega \subset \mathbb{R}^n$  be a bounded convex domain,  $x \in \Omega$  and  $u \in L_{\text{loc}}^1(\mathbb{R}^n)$ . Then*

$$\int_\Omega \frac{|u(y)|}{|x-y|^{n-1}} dy \leq C(n) (\text{diam } \Omega) (Mu)(x). \quad (2.22)$$

**Lemma 2.5** (see [15]). *Let  $\Psi$  be a Calderón-Zygmund operator with Calderón-Zygmund kernel  $K$  on  $\mathbb{R}^n \times \mathbb{R}^n$ . Then  $\Psi$  is bounded on  $L^{p(x)}(\mathbb{R}^n)$ . Further there exists a constant  $C = C(n, p)$  such that*

$$\|\Psi u(x)\|_{L^{p(x)}(\mathbb{R}^n)} \leq C(n, p) \|u(x)\|_{L^{p(x)}(\mathbb{R}^n)} \quad (2.23)$$

holds for every  $u \in L^{p(x)}(\mathbb{R}^n)$ .

**Lemma 2.6.** *If  $u \in L^{p(x)}(\Omega, \Lambda^l)$ , then*

$$\|Tu\|_{L^{p(x)}(\Omega, \Lambda^{l-1})} \leq C(n, p) \mu(\Omega) (\text{diam } \Omega) \|u\|_{L^{p(x)}(\Omega, \Lambda^l)}. \quad (2.24)$$

Moreover, if  $u \in W^{1, p(x)}(\Omega, \Lambda^l)$ , then

$$\|u_\Omega\|_{L^{p(x)}(\Omega, \Lambda^l)} \leq C(p) \|u\|_{L^{p(x)}(\Omega, \Lambda^l)} + C(n, p) \mu(\Omega) (\text{diam } \Omega) \|du\|_{L^{p(x)}(\Omega, \Lambda^{l+1})}. \quad (2.25)$$

*Proof.* First define  $u(x) = 0$  if  $x \in \mathbb{R}^n \setminus \Omega$ . From pointwise estimate (2.16) and Lemma 2.4,

$$|Tu(x)| \leq C(n) \mu(\Omega) (\text{diam } \Omega) M(|u|)(x), \quad \forall x \in \Omega. \quad (2.26)$$

In view of Lemma 2.3, we have

$$\|Tu\|_{L^{p(x)}(\Omega)} \leq C(n, p) \mu(\Omega) (\text{diam } \Omega) \|u\|_{L^{p(x)}(\Omega)}, \quad (2.27)$$

that is to say, (2.24) holds.

From the definition of  $u_\Omega$  and (2.18), we have  $u_\Omega = u - Tdu$ . Therefore,

$$\|u_\Omega\|_{L^{p(x)}(\Omega, \Lambda^l)} \leq C(p) \|u\|_{L^{p(x)}(\Omega, \Lambda^l)} + C(n, p) \|Tdu\|_{L^{p(x)}(\Omega, \Lambda^l)}. \quad (2.28)$$

Now in (2.24) replace  $u$  with  $du$ , we obtain (2.25). □

**Lemma 2.7.** *Let  $p(x)$  satisfies (1.3).*

- (1)  $C_0^\infty(\Omega, \Lambda^l)$  is dense in  $L^{p(x)}(\Omega, \Lambda^l)$ ,
- (2)  $L^{p(x)}(\Omega, \Lambda^l)$  is separable.

*Proof.* (1) For any  $u(x) = \sum_I u_I(x) dx_I \in L^{p(x)}(\Omega, \Lambda^l)$ , since  $C_0^\infty(\Omega)$  is dense in  $L^{p(x)}(\Omega)$  and  $u_I(x) \in L^{p(x)}(\Omega)$  for all  $I$ , we can find a sequence  $\{u_{Ik}\}_{k=1}^\infty \subset C_0^\infty(\Omega)$  which converges to  $u_I(x)$

in  $L^{p(x)}(\Omega)$  for each  $I$ . Now let  $u_k(x) = \sum_I u_{Ik} dx_I$ , then the sequence  $\{u_k(x)\} \subset C_0^\infty(\Omega, \Lambda^l)$  converges to  $u(x)$  in  $L^{p(x)}(\Omega, \Lambda^l)$ , since

$$\begin{aligned} \int_{\Omega} |u(x) - u_k(x)|^{p(x)} dx &= \int_{\Omega} \left( \left( \sum_I |u_I(x) - u_{Ik}(x)|^2 \right)^{1/2} \right)^{p(x)} dx \\ &\leq \int_{\Omega} \left( \sum_I |u_I(x) - u_{Ik}(x)| \right)^{p(x)} dx \\ &\leq 2^{p^*} \sum_I \int_{\Omega} |u_I(x) - u_{Ik}(x)|^{p(x)} dx. \end{aligned} \tag{2.29}$$

That is to say,  $C_0^\infty(\Omega, \Lambda^l)$  is dense in  $L^{p(x)}(\Omega, \Lambda^l)$ .

(2) Let  $u(x) = \sum_I u_I(x) dx_I \in L^{p(x)}(\Omega, \Lambda^l)$ . Since  $L^{p(x)}(\Omega)$  is separable, there exists a countable dense subset  $K$  of  $L^{p(x)}(\Omega)$ . Then for any  $u_I(x)$  above we can extract a sequence  $\{u_{Ik}(x)\}$  in  $K$  which converges to  $u_I(x)$  in  $L^{p(x)}(\Omega)$ . Similar to (1), the sequence  $\{u_k : u_k(x) = \sum_I u_{Ik}(x) dx_I\}$  converges to  $u(x)$  in  $L^{p(x)}(\Omega, \Lambda^l)$ . That is to say,  $L^{p(x)}(\Omega, \Lambda^l)$  is separable.  $\square$

Let  $\mathfrak{R}^{1,p(x)}(\Omega, \Lambda^l) = \{u(x) = \mathfrak{D}(x) - \mathfrak{D}_\Omega(x) : \mathfrak{D} \in W^{1,p(x)}(\Omega, \Lambda^l)\}$ . Note that  $u \in \mathfrak{R}^{1,p(x)}(\Omega, \Lambda^l)$  if and only if  $u_\Omega = 0$ .

**Lemma 2.8.** *Let  $p(x)$  satisfies (1.3). Then  $\mathfrak{R}^{1,p(x)}(\Omega, \Lambda^l)$  is a closed subspace of  $W^{1,p(x)}(\Omega, \Lambda^l)$ . In particular, it is a reflexive Banach space.*

*Proof.* Set a sequence  $\{u_k(x)\} \subset \mathfrak{R}^{1,p(x)}(\Omega, \Lambda^l)$  convergent to  $u(x)$  in  $W^{1,p(x)}(\Omega, \Lambda^l)$ , then  $(u_k)_\Omega = 0$ . By Lemma 2.6, the operator  $T$  is continuous on  $\mathfrak{R}^{1,p(x)}(\Omega, \Lambda^l)$ . Therefore,  $u_\Omega = 0$ , we have  $u(x) \in \mathfrak{R}^{1,p(x)}(\Omega, \Lambda^l)$ . That is to say,  $\mathfrak{R}^{1,p(x)}(\Omega, \Lambda^l)$  is a closed subspace of  $W^{1,p(x)}(\Omega, \Lambda^l)$ .  $\square$

In [2], Iwaniec and Lutoborski obtained

$$\frac{\partial}{\partial x_i}(Tu) = A_i u + S_i u, \tag{2.30}$$

where

$$|A_i u(x)| \leq \frac{2^n \mu(\Omega)}{\text{diam}(\Omega)} \int_{\Omega} \frac{|u(z)|}{|x-z|^{n-1}} dz, \tag{2.31}$$

$$S_i u(x)(\xi) = \int_{\Omega} u(z)(K_i(z, x-z), \xi) dz, \tag{2.32}$$

where  $\xi = (\xi_1, \xi_2, \dots, \xi_{l-1})$  and

$$\begin{aligned} K_i(z, x-z) &= \frac{e_i}{|x-z|^n} \int_0^\infty s^{n-1} \varphi\left(z - s \frac{x-z}{|x-z|}\right) ds \\ &\quad - \frac{x-z}{|x-z|^{n+1}} \int_0^\infty s^n \varphi_i\left(z - s \frac{x-z}{|x-z|}\right) ds. \end{aligned} \quad (2.33)$$

Further for each  $z \in \Omega$  and  $h \in \mathbb{R}^n - \{0\}$ ,  $K_i(z, h)$  satisfies the following properties:

- (i)  $K_i(z, h) \leq \mu(\Omega)|h|^{-n}$ ,
- (ii)  $K_i(z, sh) = s^{-n}K_i(z, h)$ ,  $s > 0$ ,
- (iii)  $\int_{|h|=1} K_i(z, h) = 0$  for all  $z \in \Omega$ .

Let  $K_i(z, h) = (K_{i1}, K_{i2}, \dots, K_{in})$ . Then  $K_{i\alpha}$  satisfies the conditions of Calderón-Zygmund kernel on  $\mathbb{R}^n \times \mathbb{R}^n$  for each  $\alpha = 1, 2, \dots, n$ .

**Lemma 2.9.** *Let  $u \in L^{p(x)}(\Omega, \Lambda^l)$ . Then*

$$\|\nabla T u\|_{L^{p(x)}(\Omega)} \leq C(n, p, \Omega) \|u\|_{L^{p(x)}(\Omega, \Lambda^l)}. \quad (2.34)$$

*Proof.* By Lemmas 2.3 and 2.4, and (2.31),

$$\|A_i u\|_{L^{p(x)}(\Omega, \Lambda^l)} \leq C(n, p) \mu(\Omega) \|u\|_{L^{p(x)}(\Omega, \Lambda^l)}. \quad (2.35)$$

Let

$$S_i u(x) = \sum_{1 \leq j_1 < j_2 < \dots < j_{l-1} \leq n} \omega_{j_1, j_2, \dots, j_{l-1}} dx_{j_1} \wedge dx_{j_2} \wedge \dots \wedge dx_{j_{l-1}}, \quad (2.36)$$

we can write  $u(x)$  as

$$u(x) = \sum_{1 \leq \alpha \leq n, \alpha \neq j_1, j_2, \dots, j_{l-1}} \sum_{1 \leq j_1 < j_2 < \dots < j_{l-1} \leq n} u_{\alpha, j_1, j_2, \dots, j_{l-1}} dx_\alpha \wedge dx_{j_1} \wedge \dots \wedge dx_{j_{l-1}}. \quad (2.37)$$

Hence,

$$\omega_{j_1, j_2, \dots, j_{l-1}}(x) = S_i u(x)(e_{j_1}, e_{j_2}, \dots, e_{j_{l-1}}). \quad (2.38)$$

Taking  $\xi = (e_{j_1}, e_{j_2}, \dots, e_{j_{l-1}})$  in (2.32), we obtain

$$\omega_{j_1, j_2, \dots, j_{l-1}}(x) = \int_{\Omega} \sum_{1 \leq \alpha \leq n, \alpha \neq j_1, \dots, j_{l-1}} K_{i\alpha}(z, x-z) u_{\alpha, j_1, j_2, \dots, j_{l-1}}(z) dz. \quad (2.39)$$



Now define  $u(x) = 0$  if  $x \in \mathbb{R}^n \setminus \Omega$ . Since  $K_{i\alpha}$  satisfies the conditions of Calderón-Zygmund kernel on  $\mathbb{R}^n \times \mathbb{R}^n$  for each  $\alpha$ , in view of Lemma 2.5,

$$\|\omega_{j_1, j_2, \dots, j_{l-1}}\|_{L^{p(x)}(\Omega)} \leq C(n, p) \sum_{1 \leq \alpha \leq n, \alpha \neq j_1, \dots, j_{l-1}} \|u_{\alpha, j_1, j_2, \dots, j_{l-1}}\|_{L^{p(x)}(\Omega)}. \quad (2.40)$$

So that

$$\|S_i u\|_{L^{p(x)}(\Omega, \Lambda^l)} \leq C(n, p) \|u\|_{L^{p(x)}(\Omega, \Lambda^l)}. \quad (2.41)$$

By (2.30), (2.35), and (2.41), we have

$$\|\nabla T u\|_{L^{p(x)}(\Omega)} \leq C(n, p, \Omega) \|u\|_{L^{p(x)}(\Omega, \Lambda^l)}. \quad (2.42)$$

□

Now define another norm

$$\|\omega\|_{\mathfrak{R}^{1,p(x)}(\Omega, \Lambda^l)} = \|\omega\|_{L^{p(x)}(\Omega, \Lambda^l)} + \|\nabla \omega\|_{L^{p(x)}(\Omega)}. \quad (2.43)$$

*Remark 2.10.* Replacing  $u$  with  $du$  in (2.34), we get by the definition of  $u_\Omega$

$$\begin{aligned} \|\nabla(u - u_\Omega)\|_{L^{p(x)}(\Omega)} &= \|\nabla T du\|_{L^{p(x)}(\Omega)} \\ &\leq C(n, p) \mu(\Omega) \|du\|_{L^{p(x)}(\Omega, \Lambda^l)} \\ &= C(n, p) \mu(\Omega) \|d(u - u_\Omega)\|_{L^{p(x)}(\Omega, \Lambda^l)}. \end{aligned} \quad (2.44)$$

Therefore  $\|\cdot\|_{\mathfrak{R}^{1,p(x)}(\Omega, \Lambda^l)}$  is equivalent to  $\|\cdot\|_{\mathfrak{R}^{1,p(x)}(\Omega, \Lambda^l)}$ .

In this paper we also need the following two lemmas.

**Lemma 2.11** (see[15]). *Let  $p(x)$  satisfies (1.3). Then the embedding  $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$  is compact.*

**Lemma 2.12** (see[15]). *Suppose that  $p \in L^\infty(\Omega)$ . Let  $\{u_k\}_{k=1}^\infty$  be bounded in  $L^{p(x)}(\Omega)$ . If  $u_k \rightharpoonup u$  a.e. on  $\Omega$ , then  $u_k \rightharpoonup u$  weakly in  $L^{p(x)}(\Omega)$ .*

*Remark 2.13.* Let  $\mathfrak{R}_0^{1,p(x)}(\Omega, \Lambda^l)$  be the completion of  $C_0^\infty(\Omega, \Lambda^l)$  in  $\mathfrak{R}^{1,p(x)}(\Omega, \Lambda^l)$ . Then from Remark 2.10 and Lemma 2.11, the embedding  $\mathfrak{R}_0^{1,p(x)}(\Omega, \Lambda^l) \hookrightarrow L^{p(x)}(\Omega, \Lambda^l)$  is compact.

*Remark 2.14.* Suppose  $p(x)$  satisfies (1.3), Lemma 2.12 also holds on space  $L^{p(x)}(\Omega, \Lambda^l)$ .

### 3. Weak Solutions of Dirichlet Problems for the A-Harmonic Equations with Variable Growth

**Theorem 3.1.** Under conditions (H1)–(H5), the Dirichlet problem (1.1) has at least one weak solution in  $\mathfrak{R}_0^{1,p(x)}(\Omega, \Lambda^l)$ , that is to say, there exists at least one  $u = \vartheta - \vartheta_\Omega \in \mathfrak{R}_0^{1,p(x)}(\Omega, \Lambda^l)$  satisfying

$$\int_{\Omega} \langle A(x, du(x)), d\varphi(x) \rangle + \langle B(x, u(x)), \varphi(x) \rangle dx = 0, \quad (3.1)$$

for all  $\varphi \in W_0^{1,p(x)}(\Omega, \Lambda^{l-1})$ . Here,  $\vartheta \in W^{1,p(x)}(\Omega, \Lambda^{l-1})$  and  $p(x)$  satisfies (1.3).

Let  $V = W_0^{1,p(x)}(\Omega, \Lambda^{l-1})$  and  $\mathfrak{R}_0 = \mathfrak{R}_0^{1,p(x)}(\Omega, \Lambda^l)$ . For  $u \in V$ , define  $\mathfrak{A} : V \rightarrow V^*$  in the following way: for each  $\varphi \in V$

$$(\mathfrak{A}u, \varphi) = \int_{\Omega} \langle A(x, du(x)), d\varphi(x) \rangle + \langle B(x, u(x)), \varphi(x) \rangle dx. \quad (3.2)$$

Now we need only to show that there exists  $u \in \mathfrak{R}_0$  such that  $(\mathfrak{A}u, \varphi) = 0$  for all  $\varphi = \sum \varphi_I(x) dx_I \in V$ .

**Lemma 3.2.**  $\mathfrak{A}$  is strong-weakly continuous on  $V$ .

*Proof.* Let  $\{u_k : u_k(x) = \sum_I u_{kI}(x) dx_I\} \subset V$  be a sequence strongly convergent to an element  $u(x) = \sum u_I(x) dx_I \in V$  in  $V$ . Let  $du_k(x) = \sum_J \omega_{kJ}(x) dx_J$  and  $du(x) = \sum_J \omega_J(x) dx_J$ . Then

(h<sub>1</sub>)  $\|u_k\|_V \leq C$  for some constant  $C$ ,

(h<sub>2</sub>)  $\{\omega_{kJ}(x)\}$  is a sequence strongly convergent to  $\omega_J(x)$  in  $L^{p(x)}(\Omega)$  for each  $J$ .

In view of (H2) and (h<sub>1</sub>), we know that  $A(x, du_k) = \sum A_{kJ}(x) dx_J$  and  $B(x, u_k) = \sum B_{kI}(x) dx_I$  are uniformly bounded in  $L^{p'(x)}(\Omega, \Lambda^l)$  and  $L^{p'(x)}(\Omega, \Lambda^{l-1})$ , respectively. Hence,  $A_{kJ}(x)$  and  $B_{kI}(x)$  are uniformly bounded in  $L^{p'(x)}(\Omega)$ . On the other hand, by (h<sub>2</sub>), there exists a subsequence of  $\{\omega_{kJ}(x)\}$  (still denoted by  $\{\omega_{kJ}(x)\}$ ) such that

$$\lim_{k \rightarrow \infty} \omega_{kJ}(x) = \omega_J(x), \quad \text{a.e. } x \in \Omega, \text{ for each } J. \quad (3.3)$$

Then there exists a subsequence of  $\{u_k(x)\}$  (still denoted by  $\{u_k(x)\}$ ) such that

$$\lim_{k \rightarrow \infty} u_k(x) = u(x), \quad \lim_{k \rightarrow \infty} du_k(x) = du(x), \quad \text{a.e. } x \in \Omega. \quad (3.4)$$

In view of (H1), we obtain

$$\begin{aligned} \lim_{k \rightarrow \infty} A(x, du_k) &= A(x, du), \quad \text{a.e. } x \in \Omega, \\ \lim_{k \rightarrow \infty} B(x, u_k) &= B(x, u), \quad \text{a.e. } x \in \Omega. \end{aligned} \quad (3.5)$$

Let  $d\varphi(x) = \sum \varphi_J(x)dx_J$ ,  $A(x, du) = \sum A_J(x)dx_J$ , and  $B(x, u) = \sum B_I(x)dx_I$ , then  $\varphi_J(x) \in L^{p(x)}(\Omega)$ , in the meantime

$$\lim_{k \rightarrow \infty} A_{kJ}(x) = A_J(x), \quad \text{a.e. } x \in \Omega, \quad (3.6)$$

$$\lim_{k \rightarrow \infty} B_{kI}(x) = B_I(x), \quad \text{a.e. } x \in \Omega, \quad (3.7)$$

for each  $J$  and  $I$ .

Now by Lemma 2.12, we can show that  $\int_{\Omega} A_{kJ}(x)\varphi_J(x)dx \rightarrow \int_{\Omega} A_J(x)\varphi_J(x)dx$  and  $\int_{\Omega} B_{kI}(x)\varphi_I(x)dx \rightarrow \int_{\Omega} B_I(x)\varphi_I(x)dx$  as  $k \rightarrow \infty$ . Therefore,

$$\begin{aligned} (\mathfrak{A}u_k, \varphi) &= \int_{\Omega} \langle A(x, du_k), d\varphi \rangle + \langle B(x, u_k), \varphi \rangle dx \\ &\rightarrow \int_{\Omega} \langle A(x, du), d\varphi \rangle + \langle B(x, u), \varphi \rangle dx \\ &= (\mathfrak{A}u, \varphi), \end{aligned} \quad (3.8)$$

that is to say,  $\mathfrak{A}$  is strong-weakly continuous on  $V$ . □

**Lemma 3.3.**  $\mathfrak{A}$  is coercive on  $\mathfrak{K}_0$ , that is,

$$\lim_{\|u\|_{\mathfrak{R}} \rightarrow \infty} \frac{(\mathfrak{A}u, u)}{\|u\|_{\mathfrak{R}}} = +\infty, \quad \forall u \in \mathfrak{K}_0. \quad (3.9)$$

*Proof.* By (H3) and (H4),

$$\begin{aligned} (\mathfrak{A}u, u) &= \int_{\Omega} \langle A(x, du), du \rangle + \langle B(x, u), u \rangle dx \\ &\geq \int_{\Omega} \left( a|du|^{p(x)} - |h(x)| + \bar{a}|u|^{p(x)} - |\bar{h}(x)| \right) dx \\ &\geq \int_{\Omega} a|du|^{p(x)} dx - C(h, \bar{h}). \end{aligned} \quad (3.10)$$

By  $d\vartheta_{\Omega} = 0$  and Lemma 2.6, we have

$$\|u\|_{L^{p(x)}(\Omega, \Lambda^{l-1})} = \|Td\vartheta\|_{L^{p(x)}(\Omega, \Lambda^{l-1})} \leq 2^n C(n, p) \mu(\Omega) (\text{diam } \Omega) \|du\|_{L^{p(x)}(\Omega, \Lambda^l)}, \quad (3.11)$$

for all  $u = \vartheta - \vartheta_{\Omega} \in \mathfrak{K}_0$ . Then  $\|du\|_{L^{p(x)}(\Omega, \Lambda^l)} \rightarrow \infty$ , as  $\|u\|_{\mathfrak{R}} \rightarrow \infty$ . Taking

$$\delta = \frac{1}{2} \|du\|_{L^{p(x)}(\Omega, \Lambda^l)} > 1, \quad (3.12)$$

we have

$$\begin{aligned} \frac{\int_{\Omega} |du|^{p(x)} dx}{\|du\|_{L^{p(x)}(\Omega, \Lambda^l)}} &= \int_{\Omega} \left( \frac{|du|}{\|du\|_{L^{p(x)}(\Omega, \Lambda^l)} - \delta} \right)^{p(x)} \frac{(\|du\|_{L^{p(x)}(\Omega, \Lambda^l)} - \delta)^{p(x)}}{\|du\|_{L^{p(x)}(\Omega, \Lambda^l)}} dx \\ &\geq \frac{(\|du\|_{L^{p(x)}(\Omega, \Lambda^l)} - \delta)^{p_*}}{\|du\|_{L^{p(x)}(\Omega, \Lambda^l)}} = \left(\frac{1}{2}\right)^{p_*} \|du\|_{L^{p(x)}(\Omega, \Lambda^l)}^{p_*-1}. \end{aligned} \quad (3.13)$$

Therefore,

$$\frac{\int_{\Omega} |du|^{p(x)} dx}{\|du\|_{L^{p(x)}(\Omega, \Lambda^l)}} \longrightarrow \infty \quad \text{as } \|du\|_{L^{p(x)}(\Omega, \Lambda^l)} \longrightarrow \infty. \quad (3.14)$$

Then it is immediate to obtain that

$$\frac{(\mathfrak{A}u, u)}{\|u\|_{\mathfrak{R}}} \longrightarrow \infty \quad \text{as } \|u\|_{\mathfrak{R}} \longrightarrow \infty. \quad (3.15)$$

That is to say,  $\mathfrak{A}$  is coercive on  $\mathfrak{R}_0$ . □

**Lemma 3.4** (see [16]). *Suppose  $g = A(x)$  is a mapping from  $\mathbb{R}^m$  into itself such that*

$$\lim_{|x| \rightarrow \infty} \frac{A(x) \cdot x}{|x|} = \infty. \quad (3.16)$$

*Then the range of  $A$  is the whole of  $\mathbb{R}^m$ .*

**Lemma 3.5.** *There exists a sequence  $\{u_k\} \subset \mathfrak{R}_0$  and  $u_0 \in \mathfrak{R}_0$ , such that*

$$(\mathfrak{A}u_k, u_k - u_0) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \quad (3.17)$$

*Proof.* By Lemmas 2.7 and 2.8, we can choose a Schauder basis  $\{\omega_s\}$  of  $\mathfrak{R}_0$  such that the union of subspace finitely generated from  $\omega_s$  is dense in  $\mathfrak{R}_0$ . Let  $\mathfrak{R}_0^k$  be the subspace of  $\mathfrak{R}_0$  generated by  $\omega_1, \omega_2, \dots, \omega_k$ . Since  $\mathfrak{R}_0^k$  is topologically isomorphic to  $\mathbb{R}^k$ . By Lemmas 3.3, and 3.4, there exists  $u_k \in \mathfrak{R}_0^k$  such that

$$(\mathfrak{A}u_k, \omega) = 0 \quad \forall \omega \in \mathfrak{R}_0^k. \quad (3.18)$$

By Lemma 3.3 again, we know that  $\|u_k\|_{\mathfrak{R}} \leq C$ , where  $C$  is independent of  $k$ . Since  $\mathfrak{R}_0$  is reflexive, by Remark 2.14 and (H1), we can extract a subsequence of  $\{u_k\}$  (still denoted by  $\{u_k\}$ ) such that

$$u_k \rightharpoonup u_0 \quad \text{weakly in } \mathfrak{R}_0, \quad \mathfrak{A}u_k \rightharpoonup \xi \quad \text{weakly* in } \mathfrak{R}_0^*, \quad (\xi, \omega) = 0, \quad (3.19)$$

where  $\omega$  is in a dense subset of  $\mathfrak{K}_0$ . For fixed  $\xi$ , by the continuity of  $(\xi, \cdot)$ , we get  $(\xi, \omega) = 0$  for all  $\omega \in \mathfrak{K}_0$ . For  $(\mathfrak{A}u_k, u_k - u_0)$ , we have

$$(\mathfrak{A}u_k, u_k - u_0) = (\mathfrak{A}u_k, u_k) - (\mathfrak{A}u_k, u_0) = -(\mathfrak{A}u_k, u_0) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \quad (3.20)$$

This completes the proof of Lemma 3.5. □

Set  $v_k = u_k - u_0 = \sum v_{kI} dx_I$ . Then

$$v_k \rightharpoonup 0 \quad \text{weakly in } \mathfrak{K}_0 \quad \text{as } k \longrightarrow \infty. \quad (3.21)$$

Consider  $(\mathfrak{A}u_k, u_k - u_0)$  once more, then

$$(\mathfrak{A}u_k, u_k - u_0) = \int_{\Omega} \langle A(x, du_0 + dv_k), dv_k \rangle + \langle B(x, u_0 + v_k), v_k \rangle dx \longrightarrow 0, \quad (3.22)$$

as  $k \rightarrow \infty$ . By Remark 2.13, we get

$$v_k \longrightarrow 0 \quad \text{strongly in } L^{p(x)}(\Omega, \Lambda^{l-1}). \quad (3.23)$$

In view of (3.23) and (H2), it is immediate that

$$\int_{\Omega} \langle B(x, u_0 + v_k), v_k \rangle dx \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \quad (3.24)$$

that is to say,

$$\int_{\Omega} \langle A(x, du_0 + dv_k), dv_k \rangle dx \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \quad (3.25)$$

Now if we can prove that there exists a subsequence of  $\{v_k\}$  which is strongly convergent in  $\mathfrak{K}_0$ , then from the strong-weakly continuity of  $\mathfrak{A}$ , we get  $\mathfrak{A}u_k \rightharpoonup \mathfrak{A}u_0 = \xi$  weakly in  $\mathfrak{K}_0$  as  $k \rightarrow \infty$  and  $u_0$  will be a weak solution of (1.1). We need the following lemmas.

*Definition 3.6.* Let  $\Omega$  be an open subset of  $\mathbb{R}^n$  provided with the Lebesgue measure. The mapping  $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$  is said Carathéodory function if for almost all  $x \in \Omega$ ,  $f(x, \cdot)$  is continuous on  $\mathbb{R}^N$ , for all  $\xi \in \mathbb{R}^N$  is measurable on  $\Omega$ .

**Lemma 3.7** (see[17]). *A mapping  $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$  is a Carathéodory function if and only if for all compact sets  $K \subset \Omega$  and all  $\varepsilon > 0$ , there exists a compact subset  $K_\varepsilon \subset K$  such that  $\text{meas}(K - K_\varepsilon) < \varepsilon$  for with the restriction of  $f$  to  $K_\varepsilon \times \mathbb{R}^N$  is continuous.*

**Lemma 3.8** (see[15]). Let  $\{f_k\}$  be a sequence of bounded function in  $L^1(\mathbb{R}^n)$ . For each  $\varepsilon > 0$  there exists  $(A_\varepsilon, \delta, N)$  (where  $A_\varepsilon$  is measurable and  $\text{meas}(A_\varepsilon) < \varepsilon, \delta > 0, N$  is an infinite subset of natural numbers set  $\mathbb{N}$ ) such that for each  $k \in N$ ,

$$\int_B |f_k(x)| dx < \varepsilon, \quad (3.26)$$

where  $B$  and  $A_\varepsilon$  are disjoint and  $\text{meas}(B) < \delta$ .

*Definition 3.9.* For  $u \in C_0^1(\mathbb{R}^n)$ , define

$$(M^*u)(x) = (Mu)(x) + \sum_{\alpha=1}^n \left( M \frac{\partial u}{\partial x_\alpha} \right)(x). \quad (3.27)$$

**Lemma 3.10** (see[18]). If  $u \in C_0^\infty(\mathbb{R}^n)$ , then  $M^*u \in C^0(\mathbb{R}^n)$  and for all  $x \in \mathbb{R}^n$ ,

$$|u(x)| + \sum_{\alpha=1}^n \left| \frac{\partial u}{\partial x_\alpha}(x) \right| \leq (M^*u)(x). \quad (3.28)$$

Furthermore, if  $p > 1$ , then

$$\|M^*u\|_{L^p(\mathbb{R}^n)} \leq C(n, p) \|u\|_{W^{1,p}(\mathbb{R}^n)}, \quad (3.29)$$

and if  $p = 1$ , then

$$\text{meas}(\{x \in \mathbb{R}^n : (M^*u)(x) > \lambda\}) \leq \frac{C(n)}{\lambda} \|u\|_{W^{1,1}(\mathbb{R}^n)}, \quad (3.30)$$

for all  $\lambda > 0$ .

**Lemma 3.11** (see[19]). Let  $u \in C_0^\infty(\mathbb{R}^n)$  and  $\lambda > 0$ . Set

$$H^\lambda = \{x \in \mathbb{R}^n : (M^*u)(x) < \lambda\}. \quad (3.31)$$

Then for all  $x, y \in H^\lambda$ , we have

$$|u(y) - u(x)| \leq C(n)\lambda|y - x|. \quad (3.32)$$

**Lemma 3.12** (see[16]). Let  $X$  be a metric space,  $E$  be a subspace of  $X$ , and  $k$  be a positive number. Then any  $k$ -Lipchitz mapping from  $E$  into  $\mathbb{R}$  can be extended to a  $k$ -Lipchitz mapping from  $X$  into  $\mathbb{R}$ .

*Proof of Theorem 3.1.* We need only to show that there exists subsequence of  $\{v_k\}$  which is strongly convergent in  $\mathfrak{K}_0$ .

For each measurable set  $S \subset \Omega$ , define

$$F(v, S) = \int_S \langle A(x, du_0 + dv), dv \rangle dx, \tag{3.33}$$

where  $v \in \mathfrak{K}_0$ . Similar to the proof of Lemma 3.2,  $F(\cdot, S)$  is strongly continuous on  $\mathfrak{K}_0$ . Since  $C_0^\infty(\Omega, \Lambda^{l-1})$  is dense in  $\mathfrak{K}_0$ , there exists  $h_k \in C_0^\infty(\Omega, \Lambda^{l-1})$  such that

$$\|h_k - v_k\|_{\mathfrak{K}} < \frac{1}{k}, \quad |F(h_k, \Omega) - F(v_k, \Omega)| < \frac{1}{k}. \tag{3.34}$$

So we can suppose that  $\{v_k\} \subset C_0^\infty(\Omega, \Lambda^{l-1})$  is bounded in  $\mathfrak{K}_0$ .

Next define

$$v_k(x) = 0 \quad \text{when } x \in \mathbb{R}^n \setminus \Omega. \tag{3.35}$$

In this way, we extend the domain of  $v_k$  to  $\mathbb{R}^n$  and  $\text{supp } v_k \subset \Omega$ .

Let  $\beta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a continuous increasing function satisfying  $\beta(0) = 0$  and for each measurable set  $D \subset \Omega$ ,

$$\int_D \left( |G(x)|^{p'(x)} + |h(x)| + (C_1 + 1)|du_0|^{p(x)} \right) dx \leq \beta(\text{meas}(D)), \tag{3.36}$$

where  $C_1$  is the constant in (H2).

Let  $\{\varepsilon_j\}$  be a positive decreasing sequence with  $\varepsilon_j \rightarrow 0$  as  $j \rightarrow \infty$ . For  $\varepsilon_1$ , by Lemma 3.8, we get a subsequence  $\{k_1\}$  of  $\{k\}$ , a set  $A_{\varepsilon_1} \subset \Omega$  satisfying  $\text{meas}(A_{\varepsilon_1}) < \varepsilon_1$ , and a real number  $\delta_1 > 0$  such that

$$\int_B (M^*v_{k_1 I})^{p(x)} dx < \varepsilon_1, \tag{3.37}$$

for each  $k_1, I$  and  $B \subset \Omega \setminus A_{\varepsilon_1}$  satisfying  $\text{meas}(B) < \delta_1$ . By Lemma 3.10, we can choose  $\lambda > 1$  so large that for all  $I$  and  $k_1$ ,

$$\text{meas}(\{x \in \mathbb{R}^n : (M^*v_{k_1 I})(x) \geq \lambda\}) \leq \min\{\varepsilon_1, \delta_1\}. \tag{3.38}$$

For each  $I$  and  $k_1$ , define

$$H_{k_1 I}^\lambda = \{x \in \mathbb{R}^n : (M^*v_{k_1 I})(x) < \lambda\}, \quad H_{k_1}^\lambda = \bigcap_I H_{k_1 I}^\lambda. \tag{3.39}$$

In view of Lemma 3.11, we have

$$\frac{|v_{k_1 I}(y) - v_{k_1 I}(x)|}{|y - x|} \leq C(n)\lambda \quad \forall x, y \in H_{k_1}^\lambda \text{ and } I. \tag{3.40}$$

Form Lemma 3.12, there exists a Lipschitz function  $g_{k_1 I}$  which extends  $v_{k_1 I}$  outside  $H_{k_1}^\lambda$  and Lipschitz constant of  $g_{k_1 I}$  is no more than  $C(n)\lambda$ . As  $H_{k_1}^\lambda$  is an open set, we have  $g_{k_1 I} = v_{k_1 I}$  and  $\nabla g_{k_1 I}(x) = \nabla v_{k_1 I}(x)$  for all  $x \in H_{k_1}^\lambda$ , and  $\|\nabla g_{k_1 I}\|_{L^\infty(\mathbb{R}^n)} \leq C(n)\lambda$ . We can further suppose that

$$\|g_{k_1 I}\|_{L^\infty(\mathbb{R}^n)} \leq \|v_{k_1 I}\|_{L^\infty(H_{k_1}^\lambda)} \leq \lambda, \quad \|g_{k_1 I}\|_{W^{1,\infty}(\Omega)} \leq C(n)\lambda. \quad (3.41)$$

By the uniformly boundedness of  $\{\|g_{k_1 I}\|_{W^{1,\infty}(\Omega)}\}$ , there exists a subsequence of  $\{g_{k_1 I}\}$  (still denoted by  $\{g_{k_1 I}\}$ ) such that

$$g_{k_1 I} \rightharpoonup \omega_I \text{ weakly}^* \text{ in } W^{1,\infty}(\Omega) \text{ as } k_1 \rightarrow \infty \forall I. \quad (3.42)$$

Set  $\omega = \sum_I \omega_I dx_I$  and  $g_{k_1} = \sum_I g_{k_1 I} dx_I$ . We have

$$F(v_{k_1}, \Omega) = F(g_{k_1}, \Omega \setminus A_{\varepsilon_1}) - F(g_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^\lambda) + F(v_{k_1}, A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^\lambda)). \quad (3.43)$$

Next we estimate  $F(v_{k_1}, \Omega)$  in four steps.

(1) *The estimate of  $F(g_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^\lambda)$  and  $F(v_{k_1}, A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^\lambda))$ .* Since

$$\text{meas}\left((\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^\lambda\right) \leq \sum_I \text{meas}\left((\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1 I}^\lambda\right) \leq C_n^{l-1} \min\{\varepsilon_1, \delta_1\}, \quad (3.44)$$

where  $C_n^{l-1} = n(n-1)\cdots(n-l+2)/(l-1)(l-2)\cdots 1$ , from (H2), (H3), and the choose of  $A_{\varepsilon_1}$ , we have

$$\begin{aligned} & \left| F\left(g_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^\lambda\right) \right| \\ & \leq \int_{(\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^\lambda} \left( C_1 |du_0 + dg_{k_1}|^{p(x)-1} |dg_{k_1}| + |G(x)| |dg_{k_1}| \right) dx \\ & \leq \int_{(\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^\lambda} \left( C_1 2^{p^*-1} (|du_0|^{p(x)} + |dg_{k_1}|^{p(x)}) + C_1 |dg_{k_1}|^{p(x)} + |G(x)|^{p'(x)} + |dg_{k_1}|^{p(x)} \right) dx \\ & \leq 2^{p^*-1} \beta \left( \text{meas}\left((\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^\lambda\right) \right) + 2^{p^*} (C_1 + 1) \int_{(\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^\lambda} |dg_{k_1}|^{p(x)} dx \\ & \leq 2^{p^*-1} \beta \left( \text{meas}\left((\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^\lambda\right) \right) + 2^{p^*} (C_1 + 1) \int_{(\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^\lambda} \left( \sum_I |\nabla g_{k_1 I}| \right)^{p(x)} dx \end{aligned}$$



$$\begin{aligned}
 &\leq 2^{p^*-1}\beta\left(C_n^{l-1}\varepsilon_1\right) + 2^{p^*}C(C_1, n, l) \int_{(\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1}^l} \lambda^{p(x)} dx \\
 &\leq 2^{p^*-1}\beta\left(C_n^{l-1}\varepsilon_1\right) + 2^{p^*}C(C_1, n, l) \sum_I \int_{(\Omega \setminus A_{\varepsilon_1}) \setminus H_{k_1 I}^l} (M^*v_{k_1 I})^{p(x)} dx \\
 &\leq 2^{p^*-1}\beta\left(C_n^{l-1}\varepsilon_1\right) + 2^{p^*}C(C_1, n, l)\varepsilon_1 \leq O(\varepsilon_1),
 \end{aligned} \tag{3.45}$$

$$\begin{aligned}
 &F\left(v_{k_1}, A_{\varepsilon_1} \cup \left(\Omega \setminus H_{k_1}^l\right)\right) \\
 &= \int_{A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^l)} \langle A(x, du_0 + dv_{k_1}), du_0 + dv_{k_1} \rangle - \langle A(x, du_0 + dv_{k_1}), du_0 \rangle dx \\
 &\geq \int_{A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^l)} \left( a|du_0 + dv_{k_1}|^{p(x)} - h(x) \right) - \left( C_1|du_0 + dv_{k_1}|^{p(x)-1}|du_0| + |G(x)||du_0| \right) dx \\
 &\geq \int_{A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^l)} \left( \left( a2^{-(p^*-1)} - C_1\mu 2^{p^*-1} \right) |dv_{k_1}|^{p(x)} - |h(x)| - |G(x)|^{p'(x)} \right) \\
 &\quad - \left( -a2^{-(p^*-1)} + C_1\mu 2^{p^*-1} + C_1C(\mu) + 1 \right) |du_0|^{p(x)} dx \\
 &\geq \left( a2^{-(p^*-1)} - C_1\mu 2^{p^*-1} \right) \int_{A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^l)} |dv_{k_1}|^{p(x)} dx - C(a, p, C_1, \mu)\beta\left(\text{meas}\left(A_{\varepsilon_1} \cup \left(\Omega \setminus H_{k_1}^l\right)\right)\right) \\
 &\geq a2^{-p^*} \int_{A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^l)} |dv_{k_1}|^{p(x)} dx - O(\varepsilon_1),
 \end{aligned} \tag{3.46}$$

where  $\mu > 0$  is small enough.

From (3.43)–(3.46), we get

$$F(v_{k_1}, \Omega) \geq F(g_{k_1}, \Omega \setminus A_{\varepsilon_1}) + a2^{-p^*} \int_{A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^l)} |dv_{k_1}|^{p(x)} dx - O(\varepsilon_1). \tag{3.47}$$

(2) *The estimate of  $F(g_{k_1}, \Omega \setminus A_{\varepsilon_1})$ .* Set  $f_{k_1 I} = g_{k_1 I} - \omega_I$ , where  $\omega_I$  is defined by (3.42). Then

$$\begin{aligned}
 &f_{k_1 I} \rightharpoonup 0 \text{ weakly}^* \text{ in } W^{1,\infty}(\Omega) \text{ as } k_1 \rightarrow \infty \quad \forall I, \\
 &\|f_{k_1 I}\|_{L^\infty(\Omega)} \leq 2\lambda, \quad \|df_{k_1 I}\|_{L^\infty(\Omega, \Lambda^l)} \leq 2C(n)\lambda.
 \end{aligned} \tag{3.48}$$

Let  $G = \bigcup_I G_I$  with  $G_I = \{x \in \Omega : \omega_I(x) \neq 0\}$ . According to Acerbi and Fusco [19], we have  $\text{meas}(G) \leq (C_n^{l-1} + 1)\varepsilon_1$  where  $C_n^{l-1} = n(n-1) \cdots (n-l+2)/(l-1)(l-2) \cdots 1$ , and set  $f_{k_1} = \sum_I f_{k_1 I} dx_I$ , then

$$\begin{aligned} F(g_{k_1}, \Omega \setminus A_{\varepsilon_1}) &= F(f_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \setminus G) \\ &\quad + F(v_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \cap H_{k_1}^\lambda \cap G) \\ &\quad + F(g_{k_1}, (\Omega \setminus A_{\varepsilon_1}) \cap (G \setminus H_{k_1}^\lambda)). \end{aligned} \quad (3.49)$$

Define

$$\begin{aligned} \Omega_1^{\varepsilon_1, k_1} &= A_{\varepsilon_1} \cup (\Omega \setminus H_{k_1}^\lambda), & \Omega_2^{\varepsilon_1} &= (\Omega \setminus A_{\varepsilon_1}) \setminus G, \\ \Omega_3^{\varepsilon_1, k_1} &= (\Omega \setminus A_{\varepsilon_1}) \cap H_{k_1}^\lambda \cap G, & \Omega_4^{\varepsilon_1, k_1} &= (\Omega \setminus A_{\varepsilon_1}) \cap (G \setminus H_{k_1}^\lambda). \end{aligned} \quad (3.50)$$

Similar to the proof of (3.46), we get

$$F(v_{k_1}, \Omega_3^{\varepsilon_1, k_1}) \geq a2^{-p^*} \int_{\Omega_3^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx - O(\varepsilon_1). \quad (3.51)$$

Since on  $\Omega_4^{\varepsilon_1, k_1}$  we have

$$\int_{\Omega_4^{\varepsilon_1, k_1}} |dg_{k_1}|^{p(x)} dx \leq C(n, p)(C_n^{l-1} + 1)\varepsilon_1, \quad (3.52)$$

then similar to the proof of (3.45), we get

$$\left| F(g_{k_1}, \Omega_4^{\varepsilon_1, k_1}) \right| \leq O(\varepsilon_1). \quad (3.53)$$

By (3.49)–(3.53), we have

$$F(g_{k_1}, \Omega \setminus A_{\varepsilon_1}) \geq F(f_{k_1}, \Omega_2^{\varepsilon_1}) + a2^{-p^*} \int_{\Omega_3^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx - O(\varepsilon_1). \quad (3.54)$$

Thus, we have

$$F(v_{k_1}, \Omega) \geq F(f_{k_1}, \Omega_2^{\varepsilon_1}) + a2^{-p^*} \int_{\Omega_3^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx - O(\varepsilon_1), \quad (3.55)$$

where  $\Omega_5^{\varepsilon_1, k_1} = \Omega_1^{\varepsilon_1, k_1} \cup \Omega_3^{\varepsilon_1, k_1}$ .

Choose an open set  $\Omega' \subset \Omega$  which contains  $\Omega_2^{\varepsilon_1}$  such that

$$|F(f_{k_1}, \Omega') - F(f_{k_1}, \Omega_2^{\varepsilon_1})| < \varepsilon_1. \tag{3.56}$$

From (3.55), we get

$$F(v_{k_1}, \Omega) \geq F(f_{k_1}, \Omega') + a2^{-p^*} \int_{\Omega_5^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx - O(\varepsilon_1). \tag{3.57}$$

Approximate  $\Omega'$  by hypercubes with edges parallel to coordinate axes, that is, construct

$$\begin{aligned} H_j &\subset \Omega', \\ \text{meas}(\Omega' \setminus H_j) &\longrightarrow 0 \text{ as } j \longrightarrow \infty, \\ H_j &= \bigcup_{s=1}^{h_j} D_{j,s}, \\ \text{meas}(D_{j,s}) &= 1/2^{nj}, \quad 1 \leq s \leq h_j. \end{aligned} \tag{3.58}$$

Let  $j > 0$  be large enough such that for all  $k_1 > 0$ , we have

$$\begin{aligned} |F(f_{k_1}, \Omega') - F(f_{k_1}, H_j)| &< \varepsilon_1, \quad \int_{\Omega' \setminus H_j} |df_{k_1}|^{p(x)} dx < \varepsilon_1, \\ \text{meas}(\Omega' \setminus H_j) &< \min\{\varepsilon_1, \delta_1\}. \end{aligned} \tag{3.59}$$

Thus,

$$F(v_{k_1}, \Omega) \geq F(f_{k_1}, H_j) + a2^{-p^*} \int_{\Omega_5^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx - O(\varepsilon_1) - 2\varepsilon_1. \tag{3.60}$$

(3) *The estimate of  $F(f_{k_1}, H_j)$ .* Let  $\alpha > 0$  be large enough such that for  $E = \{x \in \Omega' : \eta(x) \leq \alpha\}$ . Then

$$\text{meas}(\Omega' \setminus E) < \frac{\varepsilon_1}{N}, \quad \int_{\Omega' \setminus E} \eta(x) dx < \varepsilon_1, \tag{3.61}$$

where  $\|df_{k_1}\|_{L^\infty(\Omega, \Lambda')} \leq 2C_n^{l-1} C(n)\lambda = N$  and  $\eta(x) = |G(x)|^{p'(x)} + 2^{p^*-1}(C_1 + 1)|du_0|^{p(x)}$ .

For  $x \in \Omega, \xi \in \Lambda^l(\Omega)$ , define

$$\psi(x, \xi) = \langle A(x, du_0(x) + \xi), \xi \rangle. \tag{3.62}$$

By Lemma 3.7 and (H1), there exists a compact subset  $K \subset H_j$  such that  $\psi(x, \xi)$  is continuous on  $K \times \Lambda^l(\Omega)$  and  $\text{meas}(H_j \setminus K) < \varepsilon_1 / (\alpha + N)$ . Hence,  $\psi(x, \xi)$  is uniformly continuous on bounded subsets of  $K \times \Lambda^l(\Omega)$ .

Divide each  $D_{j,s}$  into  $2^{nm}$  hypercubes  $Q_{t,j,s}^m$  with edge length  $2^{-jm}$ ,  $1 \leq t \leq 2^{nm}$ . For all  $j, s, m, t$ , take  $x_{t,j,s}^m \in Q_{t,j,s}^m \cap K \cap E$  (if this set is empty, take  $x_{t,j,s}^m \in Q_{t,j,s}^m$ ) such that

$$\eta(x_{t,j,s}^m) \text{meas}(Q_{t,j,s}^m) \leq \int_{Q_{t,j,s}^m} \eta(x) dx. \quad (3.63)$$

Then

$$\begin{aligned} & F(f_{k_1}, H_j) \\ &= F(f_{k_1}, H_j \cap K \cap E) + F(f_{k_1}, H_j \setminus E) + F(f_{k_1}, (H_j \cap E) \setminus K) \\ &\geq F(f_{k_1}, H_j \cap K \cap E) - \int_{H_j \setminus E} \eta(x) dx - \int_{(H_j \cap E) \setminus K} \eta(x) dx \\ &\quad - 2^{p^*} (C_1 + 1) \left( \int_{H_j \setminus E} |df_{k_1}|^{p(x)} dx + \int_{(H_j \cap E) \setminus K} |df_{k_1}|^{p(x)} dx \right) \\ &= F(f_{k_1}, H_j \cap K \cap E) - O(\varepsilon_1) \\ &= b_{k_1}^{m,j} + c_{k_1}^{m,j} + d_{k_1}^{m,j} - O(\varepsilon_1), \end{aligned} \quad (3.64)$$

where

$$\begin{aligned} b_{k_1}^{m,j} &= \sum_{t,s} \int_{Q_{t,j,s}^m \cap K \cap E} \left( \psi(x, df_{k_1}(x)) - \psi(x_{t,j,s}^m, df_{k_1}(x)) \right) dx, \\ c_{k_1}^{m,j} &= \sum_{t,s} \int_{Q_{t,j,s}^m} \psi(x_{t,j,s}^m, df_{k_1}(x)) dx, \\ d_{k_1}^{m,j} &= - \sum_{t,s} \int_{Q_{t,j,s}^m \setminus (K \cap E)} \psi(x_{t,j,s}^m, df_{k_1}(x)) dx. \end{aligned} \quad (3.65)$$

By (3.25), we have

$$\lim_{k_1 \rightarrow \infty} F(v_{k_1}, \Omega) = 0. \quad (3.66)$$

Note that if  $Q_{t,j,s}^m \cap K \cap E$  is an empty set, then

$$\int_{Q_{t,j,s}^m \cap K \cap E} \left[ \psi(x, df_{k_1}(x)) - \psi(x_{t,j,s}^m, df_{k_1}(x)) \right] dx = 0. \quad (3.67)$$

Now we only consider  $Q_{t,j,s}^m$  which satisfies  $Q_{t,j,s}^m \cap K \cap E \neq \emptyset$ . Since  $du_0(x)$  is uniformly continuous on  $H_j$ , then by the uniform continuity of  $\psi$  on bounded subsets of  $K \times \Lambda^l(\Omega)$ , we obtain that for  $x \in Q_{t,j,s}^m$ , there exists a constant  $L > 0$  such that

$$\begin{aligned} & \left| \psi(x, df_{k_1}(x)) - \psi(x_{t,j,s}^m, df_{k_1}(x)) \right| \\ &= \left| \left\langle A(x, du_0(x) + df_{k_1}(x)) - A(x_{t,j,s}^m, du_0(x_{t,j,s}^m) + df_{k_1}(x)), df_{k_1}(x) \right\rangle \right| \quad (3.68) \\ &< \frac{1}{\text{meas}(H_j)} \varepsilon_1 \end{aligned}$$

holds for all  $m > L$  and each  $k_1$ . Therefore,  $|b_{k_1}^{m,j}| < \varepsilon_1$  for all  $k_1$ .

$$\begin{aligned} |d_{k_1}^{m,j}| &\leq \sum_{t,s} \int_{Q_{t,j,s}^m \setminus (K \cap E)} \left| \psi(x_{t,j,s}^m, df_{k_1}(x)) \right| dx \\ &= \sum_{t,s} \int_{Q_{t,j,s}^m \setminus (K \cap E)} \left\langle A(x_{t,j,s}^m, du_0(x_{t,j,s}^m) + df_{k_1}(x)), df_{k_1}(x) \right\rangle dx \\ &\leq \sum_{t,s} \int_{Q_{t,j,s}^m \setminus (K \cap E)} C_1 \left| du_0(x_{t,j,s}^m) + df_{k_1}(x) \right|^{p(x)-1} |df_{k_1}(x)| + \left| G(x_{t,j,s}^m) \right| |df_{k_1}(x)| dx \\ &\leq \sum_{t,s} \int_{Q_{t,j,s}^m \setminus (K \cap E)} \left( \eta(x_{t,j,s}^m) + 2^{p^*} (C_1 + 1) N \right) dx \\ &\leq \int_{(H_j \cap E) \setminus K} \left( \eta(x_{t,j,s}^m) + 2^{p^*} (C_1 + 1) N \right) dx + C(C_1, p) \sum_{t,s} \int_{Q_{t,j,s}^m \setminus E} \left( \eta(x_{t,j,s}^m) + N \right) dx \\ &\leq C(\alpha, N, C_1, p) \text{meas}((H_j \cap E) \setminus K) + C(C_1, p) \int_{H_j \setminus E} [\eta(x) + N] dx \\ &\leq C(\alpha, N, C_1, p) \varepsilon_1 \leq O(\varepsilon_1). \end{aligned} \quad (3.69)$$

Now we suppose that  $m$  is large enough that  $|b_{k_1}^{m,j}| < \varepsilon_1$  for each  $k_1 > 0$  and there exists  $\bar{k}_1 > 0$  such that  $F(v_{k_1}, \Omega) < \varepsilon_1$  for  $k_1 > \bar{k}_1$ . Therefore, from (3.25), (3.60), and (3.64), we have

$$\begin{aligned} \varepsilon_1 &\geq F(v_{k_1}, \Omega) \\ &\geq c_{k_1}^{m,j} + a2^{-p^*} \int_{\Omega_5^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx - O(\varepsilon_1) - 3\varepsilon_1 - C(C_1, p) \varepsilon_1 \quad (3.70) \\ &= c_{k_1}^{m,j} + a2^{-p^*} \int_{\Omega_5^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx - O(\varepsilon_1). \end{aligned}$$

(4) The estimate of  $c_{k_1}^{m,j}$ . By  $f_{k_1 I} \rightharpoonup 0$  weakly\* in  $W^{1,\infty}(\Omega)$  as  $k_1 \rightarrow \infty$ , we obtain  $\|f_{k_1 I}\|_{L^\infty(\Omega)} \rightarrow 0$  as  $k_1 \rightarrow \infty$  for each  $I$ . Then

$$R_{t,s,j}^{k_1,m} = \| \|f_{k_1}\| \|_{L^\infty(Q_{t,s,j}^m)} \rightarrow 0 \quad \text{as } k_1 \rightarrow \infty \text{ for fixed } m. \quad (3.71)$$

Define a hypercube  $E_{t,s,j}^{k_1,m}$  contained in  $Q_{t,s,j}^m$  with edge length  $1/2^{jm} - 2R_{t,s,j}^{k_1,m}$  such that  $\text{dist}(\partial Q_{t,s,j}^m, E_{t,s,j}^{k_1,m}) = R_{t,s,j}^{k_1,m}$ .  
Next define

$$\begin{aligned} \varphi_{k_1}(x) &= 0, \quad x \in \partial Q_{t,s,j}^m \\ \varphi_{k_1}(x) &= f_{k_1}(x), \quad x \in E_{t,s,j}^{k_1,m}. \end{aligned} \quad (3.72)$$

Since  $\varphi_{k_1 I}$  is a Lipschitz mapping on set  $E_{t,s,j}^m \cup \partial Q_{t,s,j}^m$  and its Lipschitz constant is no more than  $2C(n)\lambda$ , by Lemma 3.12,  $\varphi_{k_1 I}$  can be extended to the whole  $Q_{t,s,j}^m$ , where it is also a Lipschitz mapping with the same Lipschitz constant. We still denote the extension by  $\varphi_{k_1 I}$  and suppose that it is defined on the whole  $H_j$ . Then by [20]

$$\nabla \varphi_{k_1 I} - \nabla f_{k_1 I} \rightarrow 0 \quad \text{a.e. on } H_j. \quad (3.73)$$

Thus, there exists a  $\bar{k}_1 > \bar{k}_1$  such that for all  $k_1 > \bar{k}_1$ , we have

$$\begin{aligned} \int_{H_j} |d\varphi_{k_1} - df_{k_1}|^{p(x)} dx &\leq \frac{\varepsilon_1}{2}, \\ \sum_{t,s} \left| \int_{Q_{t,j,s}^m} \psi(x_{t,j,s}^m, df_{k_1}(x)) - \psi(x_{t,j,s}^m, d\varphi_{k_1}(x)) dx \right| &\leq \frac{\varepsilon_1}{2}. \end{aligned} \quad (3.74)$$

In view of (H5), we obtain that

$$\begin{aligned} c_{k_1}^{m,j} &= \sum_{t,s} \int_{Q_{t,j,s}^m} \psi(x_{t,j,s}^m, df_{k_1}(x)) dx \\ &\geq \sum_{t,s} \int_{Q_{t,j,s}^m} \psi(x_{t,j,s}^m, d\varphi_{k_1}(x)) dx - \frac{\varepsilon_1}{2} \\ &= \sum_{t,s} \int_{Q_{t,j,s}^m} \langle A(x_{t,j,s}^m, du_0(x_{t,j,s}^m) + d\varphi_{k_1}(x)), d\varphi_{k_1}(x) \rangle dx - \frac{\varepsilon_1}{2} \\ &\geq \gamma \sum_{t,s} \int_{Q_{t,j,s}^m} |d\varphi_{k_1}|^{p(x)} dx - \frac{\varepsilon_1}{2} \\ &\geq \frac{\gamma}{2^{p^*-1}} \int_{H_j} |df_{k_1}|^{p(x)} dx - \frac{(\gamma+1)\varepsilon_1}{2}. \end{aligned} \quad (3.75)$$

Thus in (3.70) for  $k_1 > \overline{\overline{k_1}}$ , we obtain the estimate of  $F(v_{k_1}, \Omega)$  from the four steps above

$$\begin{aligned} \varepsilon_1 &\geq F(v_{k_1}, \Omega) \\ &\geq a2^{-p^*} \int_{\Omega_5^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx + \frac{\gamma}{2^{p^*-1}} \int_{H_j} |df_{k_1}|^{p(x)} dx - \frac{(\gamma + 1)\varepsilon_1}{2} - O(\varepsilon_1). \end{aligned} \tag{3.76}$$

Let  $K(\varepsilon_1) = (\gamma + 1)\varepsilon_1 / (2 + o(\varepsilon_1)) / \min\{a2^{-p^*}, \gamma/2^{p^*-1}\}$ . Then

$$\int_{\Omega_5^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx + \int_{H_j} |df_{k_1}|^{p(x)} dx \leq K(\varepsilon_1), \quad \text{for } k_1 > \overline{\overline{k_1}}. \tag{3.77}$$

Form (3.59) and (3.77), we deduce that

$$\int_{\Omega_5^{\varepsilon_1, k_1}} |dv_{k_1}|^{p(x)} dx \leq K(\varepsilon_1), \quad \int_{\Omega'} |df_{k_1}|^{p(x)} dx \leq K(\varepsilon_1) + \varepsilon_1. \tag{3.78}$$

According to the definition of  $\Omega_2^{\varepsilon_1}$ , we have

$$\int_{\Omega_2^{\varepsilon_1}} |dg_{k_1}|^{p(x)} dx \leq K(\varepsilon_1) + \varepsilon_1. \tag{3.79}$$

Since  $dg_{k_1}(x) = dv_{k_1}(x)$  for each  $x \in H_{k_1}^\lambda$ , we get

$$\int_{\Omega_2^{\varepsilon_1} \cap H_{k_1}^\lambda} |dv_{k_1}|^{p(x)} dx \leq K(\varepsilon_1) + \varepsilon_1. \tag{3.80}$$

By the definitions of  $\Omega_2^{\varepsilon_1}$  and  $\Omega_5^{\varepsilon_1, k_1}$ , it is immediate that

$$\left(\Omega_2^{\varepsilon_1} \cap H_{k_1}^\lambda\right) \cup \Omega_5^{\varepsilon_1, k_1} = \Omega, \tag{3.81}$$

which implies that

$$\int_{\Omega} |dv_{k_1}|^{p(x)} dx \leq 2K(\varepsilon_1) + \varepsilon_1 \leq O(\varepsilon_1). \tag{3.82}$$

For  $\varepsilon_2 > 0$  and the sequence  $\{v_{k_1}\}$ , repeating the above arguments we can extract a subsequence  $\{v_{k_2}\}$  of  $\{v_{k_1}\}$  such that

$$\int_{\Omega} |dv_{k_2}|^{p(x)} dx \leq O(\varepsilon_2), \tag{3.83}$$

whenever  $k_2 > \overline{\overline{k_2}}$  for some  $\overline{\overline{k_2}}$ . If  $\{v_{k_n}\}$  has been obtained, repeating the above process, we can extract a subsequence  $\{k_{n+1}\}$  of  $\{k_n\}$  such that

$$\int_{\Omega} |dv_{k_{n+1}}|^{p(x)} dx \leq O(\varepsilon_{n+1}), \tag{3.84}$$

whenever  $k_{n+1} > \overline{\overline{k_{n+1}}}$  for some  $\overline{\overline{k_{n+1}}}$ . Finally, by a diagonal argument we get a subsequence  $\{v_{k_i}\}_{i=1}^{\infty}$  which satisfies

$$\int_{\Omega} |dv_{k_i}|^{p(x)} dx \rightarrow 0 \quad \text{as } i \rightarrow \infty. \tag{3.85}$$

Therefore,

$$\|dv_{k_i}\|_{L^{p(x)}(\Omega, \Lambda^l)} \rightarrow 0 \quad \text{as } i \rightarrow \infty, \tag{3.86}$$

and by (3.23),  $\{v_{k_i}\}_{i=1}^{\infty}$  strongly converges to zero in  $\mathfrak{K}_0$  as  $i \rightarrow \infty$ . This completes the proof of Theorem 3.1.  $\square$

### 4. Applications

In this section, we explore applications of our results developed in this paper.

Let  $\Omega \subset \mathbb{R}^n$  be a bounded and convex Lipschitz domain. Suppose that maps  $A : \Omega \times \Lambda^l(\Omega) \rightarrow \Lambda^l(\Omega)$  and  $B : \Omega \times \Lambda^{l-1}(\Omega) \rightarrow \Lambda^{l-1}(\Omega)$ , where  $l = 1, 2, \dots, n$ .

*Example 4.1.* If  $p(x)$  satisfies (1.3), let  $l = 1$ ,  $A(x, \xi) = \xi|\xi|^{p(x)-2}$  and  $B(x, \varsigma) = \varsigma|\varsigma|^{p(x)-2} - f(x)$ , where  $f(x) \in L^{p'(x)}(\Omega)$ . Then  $A, B$  satisfy the required conditions, and (1.1) reduce to the following  $p(x)$ -Laplacian equations:

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + |u|^{p(x)-2}u &= f(x), \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \partial\Omega. \end{aligned} \tag{4.1}$$

Now by Theorem 3.1, we deduce that the  $p(x)$ -Laplacian equations (4.1) have at least one weak solution in  $\mathfrak{K}^{1,p(x)}(\Omega)$  with  $u = 0$  on  $\partial\Omega$ .

*Example 4.2.* If  $l = 1$ ,  $A(x, \xi) = \sum_{i,j} A_{ij}(x)\xi_j dx_i$ ,  $B(x, \varsigma) = B(x)\varsigma - f(x)$ , where  $f(x) \in L^2(\Omega)$ , and  $A_{ij}(x), B(x)$  satisfy the following conditions:

$$A_{ij}(x) = A_{ji}(x), \quad \wedge |\xi|^2 \geq \sum_{i,j=1}^n A_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \lambda \leq B(x) \leq \Lambda, \tag{4.2}$$



for some constants  $\lambda, \Lambda > 0$ . Then  $A, B$  satisfy the required conditions, and (1.1) reduce to the following Divergence form equations:

$$\sum_{i,j=1}^n \nabla_j (A_{ij}(x) \nabla_i u(x)) + B(x)u(x) = f(x), \quad x \in \Omega, \quad (4.3)$$

$$u(x) = 0, \quad x \in \partial\Omega, \quad (4.4)$$

where  $\nabla_i = (\partial/\partial x_i)$ . Now by Theorem 3.1, we deduce that the divergence form (4.3) have at least one weak solution  $u(x)$  in  $\mathfrak{R}^{1,2}(\Omega)$  with  $u = 0$  on  $\partial\Omega$ . The comparison principles, the maximum principles, and the existence of weak solutions for divergence form equation (4.3) can be found in [21].

*Example 4.3.* If  $p(x)$  satisfies (1.3), let  $A(x, \xi) = \xi|\xi|^{p(x)-2}$  and  $B(x, \zeta) = \zeta|\zeta|^{p(x)-2} - f(x)$ , where  $f(x) \in L^{p'(x)}(\Omega, \Lambda^{-1})$ . Then  $A, B$  satisfy the required conditions, and (1.1) reduce to the following  $p(x)$ -harmonic equations for differential forms:

$$d^* (du|du|^{p(x)-2}) + u|u|^{p(x)-2} = f(x), \quad x \in \Omega, \quad (4.5)$$

$$u(x) = 0, \quad x \in \partial\Omega. \quad (4.6)$$

Now by Theorem 3.1, we deduce that (4.5) have at least one weak solution  $u(x)$  in  $\mathfrak{R}_0^{1,p(x)}(\Omega, \Lambda^{-1})$ . If  $p(x)$  is a constant  $q$  and  $1 < q < \infty$ , the equation (4.5) is called nonhomogeneous  $q$ -harmonic equation. In [2], Iwaniec and Lutoborski studied the  $L^q$  theory of weak solution for homogeneous  $q$ -harmonic equations.

## References

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