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Research Article

Adaptive Stabilization of the Korteweg-de Vries-Burgers Equation with Unknown Dispersion

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This paper studies the adaptive control problem of the Korteweg-de Vries-Burgers equation. Using the Lyapunov function method, we prove that the closed-loop system including the parameter estimator as a dynamic component is globally L^2 stable. Furthermore, we show that the state of the system is regulated to zero by developing an alternative to Barbalat's lemma which cannot be used in the present situation. The closed-loop system is shown to be well posed.

1. Introduction

In this paper, we are concerned with the problem of boundary adaptive control of the KdVB equation:

$$\begin{gathered} u_t - \varepsilon u_{xx} + u u_x + \delta u_{xxx} = 0\ , \quad 0 < x < 1,\ t > 0, \\ u_x(0,t) = u_x(1,t) = 0, \quad t > 0, \\ u_{xx}(0,t) = \varphi_0, \quad t > 0, \\ u_{xx}(1,t) = \varphi_1, \quad t > 0, \\ u(x,0) = u^0(x), \quad 0 < x < 1, \end{gathered} \tag{1.1}$$

where the viscosity parameter $\varepsilon > 0$. The dispersion parameter $\delta > 0$ is unknown, φ_0 and φ_1 are control inputs, and $u^0(x)$ is an initial state in an appropriate function space. When $\varepsilon = 0$, the KdVB equation becomes the KdV equation; when $\delta = 0$, it becomes the Burgers equation.

The problem of control of the Burgers, KdV, and KdVB equations has received extensive attention for several decades [1–7]. In [2], Liu and Krstic obtained the adaptive control of the Burgers equation. Up to now, it seems not to have many discussions on the adaptive control of the KdV and KdVB equation. In this paper we establish a Barbalat-like lemma [8] and use the Lyapunov function method to prove that the system of the KdVB equation is globally L^2 stable under the boundary conditions. Using Banach fixed point theorem, we proved the well-posedness of the KdVB equation under the given boundary condition.

The rest of the paper is organized as follows. We present our main results in Section 2. In Section 3, we establish the alternative to Barbalat's lemma. In Section 4, we prove that the KdVB equation with the previous adaptive boundary feedbacks is globally L^2 stable. By the alternative to Barbalat's lemma, we show the regulation of the solution. In Section 5, we establish the global existence and uniqueness of the solution with help of the Banach fixed point theorem.

We now introduce some notations used throughout the paper. $H^s(0,1)$ denotes the usual Sobolev space [9] for any $s \in R$. For $s \geq 0$, $H_0^s(0,1)$ denotes the completion of $C_0^\infty(0,1)$ in $H^s(0,1)$, where $C_0^\infty(0,1)$ denotes the space of all infinitely differentiable functions on $(0,1)$ with compact support in $(0,1)$. The H^m norm is defined in the usual way, $m = 0,1,2,\ldots$. The norm on $L^2(0,1)$ is denoted by $\|\cdot\|$. It is easy to see that

$$\|u\|^2 \leq 2\|u\|_{H^2}^2. \tag{1.2}$$

Let X be the Banach space and $T > 0$. We denote by $C^n([0,T];X)$ the space of n times continuously differentiable functions defined on $[0,T]$ with values in X. We denote by $(\cdot,\cdot)$ the scalar product of $L^2(0,1)$.

2. Main Result

For notational convenience, in what follows, we denote

$$\widetilde{\eta}_0 = \eta_0 - \frac{1}{6\delta}, \qquad \widetilde{\eta}_1 = \eta_1 - \frac{1}{6\delta}, \tag{2.1}$$

where η_0 and η_1 will be used as estimates of $1/6\delta$.

Consider the system

$$\begin{gathered}
u_t - \varepsilon u_{xx} + u u_x + \delta u_{xxx} = 0, \\
u_x(0,t) = u_x(1,t) = 0, \\
u_{xx}(0,t) = -k\left[u(0,t) + u(0,t)^7\right] - \eta_0\left[u(0,t) + u(0,t)^3\right], \\
u_{xx}(1,t) = k\left[u(1,t) + u(1,t)^7\right] + \eta_1\left[u(1,t) + u(1,t)^3\right],
\end{gathered}$$

$$\dot{\eta}_0 = \gamma\left[u(0,t)^2 + u(0,t)^4\right],$$
$$\dot{\eta}_1 = \gamma\left[u(1,t)^2 + u(1,t)^4\right],$$
$$u(x,0) = u^0(x), \qquad \eta_0(0) = \eta_0^0, \qquad \eta_1(0) = \eta_1^0 \tag{2.2}$$

satisfies the following theorem.

Theorem 2.1. *Suppose that $k > 0$, $\gamma > 0$, the initial condition $u^0 \in H^2(0,1)$, and $\eta_0^0 \geq 0$, $\eta_1^0 \geq 0$. If the problem* (2.2) *has a global solution (u, η_0, η_1), then one has the equilibrium $u(x) \equiv 0$, $\tilde{\eta}_0 = \tilde{\eta}_1 = 0$ is globally L^2-stable, that is:*

$$\|u(t)\|^2 + \frac{\delta}{\gamma}\tilde{\eta}_0(t)^2 + \frac{\delta}{\gamma}\tilde{\eta}_1(t)^2 \leq \left\|u^0\right\|^2 + \frac{\delta}{\gamma}\tilde{\eta}_0(0)^2 + \frac{\delta}{\gamma}\tilde{\eta}_1(0)^2, \quad \forall t \geq 0 \tag{2.3}$$

and u is regulated to zero in L^2 sense:

$$\lim_{t\to\infty}\|u(t)\| = 0. \tag{2.4}$$

3. The Alternative to Barbalat's Lemma

Recently, the Barbalat's lemma has more and more important applications in control theory, especially in the adaptive control theory. It is easy to connect with Lyapunov method to analyze the stability and convergence of the system. In this section, we establish the following alternative to Barbalat's lemma [8].

Lemma 3.1. *Suppose that the function $f(t)$ defined on $[0,\infty)$ satisfies the following conditions:*

(i) *$f(t) \geq 0$ for all $t \in [0,\infty)$,*

(ii) *$f(t)$ is differentiable on $[0,\infty)$ and there exists a constant M such that $f'(t) \leq M$, for all $t \geq 0$,*

(iii) *$\int_0^\infty f(t)dt < \infty$.*

Then one has

$$\lim_{t\to\infty} f(t) = 0. \tag{3.1}$$

Proof. Since $f'(t) \leq M$, for all $t \geq 0$, we have $f(t)$ is uniformly continuous, and

$$\left|\left|f_2(t)\right| - \left|f_1(t)\right|\right| \leq \left|f_2(t) - f_1(t)\right|, \tag{3.2}$$

therefore $|f(t)|$ is uniformly continuous.

Setting

$$F(t) = \int_0^t \left|f(\tau)\right| d\tau, \quad t \geq 0, \tag{3.3}$$

we have $\dot{F}(t) = |f(t)|$, such that $\dot{F}(t)$ is also uniformly continuous.

By the standard Barbalat's lemma, we have $\lim_{t\to\infty}\dot{F}(t) = 0$, then

$$\lim_{t\to\infty} f(t) = 0. \tag{3.4}$$

□

4. Proof of Stabilization

In this section, we prove our main result by the Lyapunov method. Now we present the proof of Theorem 2.1; first we prove the stability of the system, and then we prove the exisetence and uniqueness of the solution.

Step 1. Stability (2.3). We follow the Lyapunov approach: to this end, we introduce the energy function

$$E = \int_0^1 u^2 dx \tag{4.1}$$

and the Lyapunov function

$$V = E + \frac{\delta}{\gamma}\left(\tilde{\eta}_0^2 + \tilde{\eta}_1^2\right), \tag{4.2}$$

where γ is a positive constant. Using (1.1) and integrating by parts, we obtain

$$\begin{aligned}
\dot{V} &= 2\int_0^1 u(\varepsilon u_{xx} - uu_x - \delta u_{xxx})dx + \frac{2\delta}{\gamma}\left(\eta_0 - \frac{1}{6\delta}\right)\dot{\eta}_0 + \frac{2\delta}{\gamma}\left(\eta_1 - \frac{1}{6\delta}\right)\dot{\eta}_1 \\
&= 2\varepsilon uu_x|_0^1 - 2\varepsilon\int_0^1 u_x^2 dx - \frac{2}{3}u^3\Big|_0^1 - 2\delta uu_{xx}|_0^1 + \delta u_x^2\Big|_0^1 + \frac{2\delta}{\gamma}\left(\eta_0 - \frac{1}{6\delta}\right)\dot{\eta}_0 + \frac{2\delta}{\gamma}\left(\eta_1 - \frac{1}{6\delta}\right)\dot{\eta}_1 \\
&= -2\varepsilon\int_0^1 u_x^2 dx - \frac{2}{3}\left(u(1,t)^3 - u(0,t)^3\right) + 2\delta u(0,t)\varphi_0 - 2\delta u(1,t)\varphi_1 \\
&\quad + \frac{2\delta}{\gamma}\left(\eta_0 - \frac{1}{6\delta}\right)\dot{\eta}_0 + \frac{2\delta}{\gamma}\left(\eta_1 - \frac{1}{6\delta}\right)\dot{\eta}_1 \\
&\le 2\delta u(0,t)\varphi_0 - 2\delta u(1,t)\varphi_1 - 2\varepsilon\int_0^1 u_x^2 dx + \frac{1}{3}\left(u(0,t)^4 + u(0,t)^2 + u(1,t)^4 + u(1,t)^2\right) \\
&\quad + \frac{2\delta}{\gamma}\left(\eta_0 - \frac{1}{6\delta}\right)\dot{\eta}_0 + \frac{2\delta}{\gamma}\left(\eta_1 - \frac{1}{6\delta}\right)\dot{\eta}_1 \\
&= 2\delta u(0,t)\varphi_0 - 2\delta u(1,t)\varphi_1 - 2\varepsilon\int_0^1 u_x^2 dx + 2\delta\left(\frac{1}{6\delta} - \eta_0 + \eta_0\right)\left(u(0,t)^4 + u(0,t)^2\right) \\
&\quad + 2\delta\left(\frac{1}{6\delta} - \eta_1 + \eta_1\right)\left(u(1,t)^4 + u(1,t)^2\right) + \frac{2\delta}{\gamma}\left(\eta_0 - \frac{1}{6\delta}\right)\dot{\eta}_0 + \frac{2\delta}{\gamma}\left(\eta_1 - \frac{1}{6\delta}\right)\dot{\eta}_1
\end{aligned}$$

$$
\begin{aligned}
= & -2\varepsilon\int_0^1 u_x^2 dx + 2\delta u(0,t)\left[\varphi_0 + \eta_0\left(u(0,t)^3 + u(0,t)\right)\right] \\
& -2\delta u(1,t)\left[\varphi_1 - \eta_1\left(u(1,t)^3 + u(1,t)\right)\right] + 2\delta\left(\frac{1}{6\delta} - \eta_0\right)\left(u(0,t)^4 + u(0,t)^2 - \frac{\dot{\eta}_0}{\gamma}\right) \\
& + 2\delta\left(\frac{1}{6\delta} - \eta_1\right)\left(u(1,t)^4 + u(1,t)^2 - \frac{\dot{\eta}_1}{\gamma}\right).
\end{aligned}
\tag{4.3}
$$

This leads us to select the adaptive feedback control:

$$
\begin{aligned}
\dot{\eta}_0 &= \gamma\left[u(0,t)^2 + u(0,t)^4\right], \\
\dot{\eta}_1 &= \gamma\left[u(1,t)^2 + u(1,t)^4\right], \\
\varphi_0 &= -k\left[u(0,t) + u(0,t)^7\right] - \eta_0\left[u(0,t) + u(0,t)^3\right], \\
\varphi_1 &= k\left[u(1,t) + u(1,t)^7\right] + \eta_1\left[u(1,t) + u(1,t)^3\right],
\end{aligned}
\tag{4.4}
$$

where k is any positive constant. By this control, we obtain

$$
\dot{V} \le -2\varepsilon\int_0^1 u_x^2 dx - 2\delta k\left[u(0,t)^2 + u(0,t)^8 + u(1,t)^2 + u(1,t)^8\right]. \tag{4.5}
$$

Thus

$$
\|u(t)\|^2 + \frac{\delta}{\gamma}\tilde{\eta}_0(t)^2 + \frac{\delta}{\gamma}\tilde{\eta}_1(t)^2 \le \left\|u^0\right\|^2 + \frac{\delta}{\gamma}\tilde{\eta}_0(0)^2 + \frac{\delta}{\gamma}\tilde{\eta}_1(0)^2, \quad t \ge 0. \tag{4.6}
$$

This shows that (2.3) holds.

And we have:

$$
\begin{aligned}
2\varepsilon\int_0^\infty\int_0^1 u_x^2 dx\, dt + 2\delta k\int_0^\infty\left[u(0,t)^2 + u(0,t)^8 + u(1,t)^2 + u(1,t)^8\right]dt \\
\le \left\|u^0\right\|^2 + \frac{\delta}{\gamma}\tilde{\eta}_0(0)^2 + \frac{\delta}{\gamma}\tilde{\eta}_1(0)^2.
\end{aligned}
\tag{4.7}
$$

Step 2. Regulation (2.4). To prove (2.4), it suffices to verify conditions (ii) and (iii) of Lemma 3.1.

By (1.2) and (4.7), we obtain

$$
\int_0^\infty \left\|u(t)^2\right\| dt \le C(\delta,\gamma,k)\left[\left\|u^0\right\|^2 + \tilde{\eta}_0(0)^2 + \tilde{\eta}_1(0)^2\right]. \tag{4.8}
$$

Here and in the sequel, $C = C(\delta, \gamma, k)$ denotes a generic positive constant depending on ε, γ, k, which may vary from line to line. Thus condition (iii) of Lemma 3.1 is fulfilled. On the other hand, using Young inequality and noting that $a^4 \le a^8 + a^2$, $a^2 \le a^6 + 1$, we have

$$\begin{aligned}
\frac{d}{dt}\left(\|u(t)\|^2\right) &= 2\int_0^1 u(\varepsilon u_{xx} - uu_x - \delta u_{xxx})dx \\
&= 2\varepsilon\, uu_x|_0^1 - 2\varepsilon \int_0^1 u_x^2 dx - \frac{2}{3}u^3\Big|_0^1 - 2\delta\, uu_{xx}|_0^1 + \delta\, u_x^2\Big|_0^1 \\
&= -2\delta u(0,t)\left\{k\left[u(0,t) + u(0,t)^7\right] + \eta_0\left[u(0,t) + u(0,t)^3\right]\right\} \\
&\quad - 2\delta u(1,t)\left\{k\left[u(1,t) + u(1,t)^7\right] + \eta_1\left[u(1,t) + u(1,t)^3\right]\right\} \\
&\quad - \frac{2}{3}\left[u(1,t)^3 - u(0,t)^3\right] - 2\varepsilon\int_0^1 u_x^2 dx \\
&\le -2\delta k\left[u(0,t)^2 + u(0,t)^8 + u(1,t)^2 + u(1,t)^8\right] \\
&\quad - 2\delta\left\{\eta_0\left[u(0,t)^2 + u(0,t)^4\right] + \eta_1\left[u(1,t)^2 + u(1,t)^4\right]\right\} \\
&\quad + \frac{1}{3}\left[u(0,t)^2 + u(0,t)^4 + u(1,t)^2 + u(1,t)^4\right] \\
&\le -2\delta k\left[u(0,t)^2 + u(0,t)^8 + u(1,t)^2 + u(1,t)^8\right] \\
&\quad + \delta k\left[u(0,t)^2 + u(0,t)^8 + u(1,t)^2 + u(1,t)^8\right] + C(\delta,k)\left(\eta_0^2 + \eta_1^2\right) \\
&\quad + \delta k\left[u(0,t)^2 + u(0,t)^8 + u(1,t)^2 + u(1,t)^8\right] + C(\delta,k) \\
&\le C(\delta,k)\left(1 + \eta_0^2 + \eta_1^2\right)
\end{aligned} \tag{4.9}$$

which, combining with (4.6), implies condition (ii) of Lemma 3.1.

Hence

$$\lim_{t\to\infty}\|u(t)\| = 0. \tag{4.10}$$

5. Well-Posedness

In this section, we use the Banach fixed point theorem to prove that the problem (2.2) is well posed. To this end, for any constant $T > 0$, we first consider the following linear boundary value problem for any fixed $\lambda \in C([0,1] \times [0,T])$:

$$
\begin{gathered}
u_t - \varepsilon u_{xx} + \delta u_{xxx} + \lambda u_x = 0, \\
u_x(0,t) = u_x(1,t) = 0, \\
u_{xx}(0,t) = -k\left[\lambda(0,t) + \lambda(0,t)^7\right] - \eta_0\left[\lambda(0,t) + \lambda(0,t)^3\right], \\
u_{xx}(1,t) = k\left[\lambda(1,t) + \lambda(1,t)^7\right] + \eta_1\left[\lambda(1,t) + \lambda(1,t)^3\right], \\
\dot{\eta}_0 = \gamma\left[\lambda(0,t)^2 + \lambda(0,t)^4\right], \\
\dot{\eta}_1 = \gamma\left[\lambda(1,t)^2 + \lambda(1,t)^4\right], \\
u(x,0) = u^0(x), \qquad \eta_0(0) = \eta_0^0, \qquad \eta_1(0) = \eta_1^0.
\end{gathered}
\tag{5.1}
$$

We introduce some notation as follows.

If u_1 and u_2 are two solutions of the problem (5.1) corresponding to λ_1 and λ_2 respectively, we set

$$
z = u_1 - u_2, \quad \omega = \lambda_1 - \lambda_2. \tag{5.2}
$$

For a general function $\phi = \phi(x,t)$, we set

$$
\begin{gathered}
\|\phi\|_\infty = \max_{\substack{0\le x\le 1\\ 0\le t\le T}}|\phi(x,t)| \\
H_0^2(0,1) = \left\{u \in H^2(0,1) : u_x(0,t) = u_x(1,t) = 0\right\}.
\end{gathered}
\tag{5.3}
$$

Lemma 5.1. *If $\lambda \in C([0,T];C[0,1])$ and the initial data $u^0(x) \in H_0^2(0,1)$, then the problem* (5.1) *has a unique weak solution u satisfying $u \in C([0,T];H_0^2(0,1))$. Moreover, one has*

$$
\|z_x(t)\|^2 \le T\|\omega\|_\infty^2 F\left(\|\lambda_1\|_\infty, \|\lambda_2\|_\infty, \left\|z_x^0\right\|\right), \tag{5.4}
$$

where $F(x_1,x_2,x_3)$ is a positive continuous function.

Proof. We use the standard Galerkin method. This method relies on a number of prior estimates which can be usually obtained by using Gronwall's inequality.

Step 1. Transformation to a Homogeneous Problem. In order to use the Galerkin meth-od, we transform the problem (5.1) into a homogenous boundary value problem. We first assume that λ is infinitely differentiable with respect to both x and t. Set

$$\begin{gathered}\psi = \frac{1}{6}(1-x)^3\left\{-k\left[\lambda(0,t)+\lambda(0,t)^7\right]-\eta_0\left[\lambda(0,t)+\lambda(0,t)^3\right]\right\} \\ +\frac{1}{6}x^3\left\{k\left[\lambda(1,t)+\lambda(1,t)^7\right]+\eta_1\left[\lambda(1,t)+\lambda(1,t)^3\right]\right\}, \\ v = u-\psi.\end{gathered} \tag{5.5}$$

Then it is clear that v satisfies the following equation:

$$\begin{gathered}v_t-\varepsilon v_{xx}+\delta v_{xxx}+\lambda v_x = f, \\ v_x(0,t)=v_x(1,t)=v_{xx}(0,t)=v_{xx}(1,t)=0, \\ v(x,0)=v^0(x),\end{gathered} \tag{5.6}$$

where

$$\begin{gathered}f(x,t) = -\psi_t+\varepsilon\psi_{xx}-\delta\psi_{xxx}-\lambda\psi_x, \\ v^0(x)=u^0(x)-\psi(0)\in H_0^2(0,1).\end{gathered} \tag{5.7}$$

Step 2. Approximate Problem. Set

$$H_0^3(0,1) = \left\{\varphi\in H^3(0,1):\varphi_x(0)=\varphi_x(1)=\varphi_{xx}(0)=\varphi_{xx}(1)=0\right\}. \tag{5.8}$$

Let $\{\varphi_i\}_{i=1}^{\infty}$ be an orthonormal basis in $H_0^2(0,1)$ such that each function φ_i is in $H_0^3(0,1)$. Since $v^0(x)\in H_0^2(0,1)$, $v^0(x)$ can be expanded as

$$v^0(x)=\sum_{j=1}^{\infty}b_j\varphi_j(x). \tag{5.9}$$

Set

$$v_n(x,t)=\sum_{j=1}^{n}a_j(t)\varphi_j(x), \qquad v_n^0(x)=\sum_{j=1}^{n}b_j\varphi_j(x). \tag{5.10}$$

Then we have

$$\left\|v^0-v_n^0\right\|_{H^2}^2=\sum_{j=n+1}^{\infty}b_j^2\longrightarrow 0, \quad \text{as } n\longrightarrow\infty. \tag{5.11}$$

Consider the following approximate problem:

$$v_{nt} - \varepsilon v_{nxx} + \delta v_{nxxx} + \lambda v_{nx} = f, \tag{5.12}$$

$$v(x,0) = v^0(x). \tag{5.13}$$

Multiplying (5.12) by φ_i and integrating from 0 to 1, we obtain

$$\begin{gathered}\sum_{j=1}^{n}\left[\alpha_{ij}\dot{a}_j(t) + \beta_{ij}(t)a_j(t)\right] = \gamma_i(t), \\ a_i(0) = b_i\ , \quad i = 1,\dots,n,\end{gathered} \tag{5.14}$$

where

$$\begin{gathered}\alpha_{ij} = \left(\varphi_j, \varphi_i\right), \\ \beta_{ij} = -\varepsilon\left(\varphi_{jxx}, \varphi_i\right) + \delta\left(\varphi_{jxxx}, \varphi_i\right) + \left(\lambda\varphi_{jx}, \varphi_i\right), \\ \gamma_i(t) = \left(f, \varphi_i\right).\end{gathered} \tag{5.15}$$

Since the matrix α_{ij} is nonsingular, by the classical theory of ordinary differential equations, the linear problem (5.14) has a unique continuously differential solution on $[0,T]$. Therefore, the approximate problems (5.12) and (5.13) have a unique solution v_n with $v_n \in C^1$ $([0,T], H_0^3)$.

Step 3. A Priori Estimate on v_n. In what follows, we denote by $c = c(\varepsilon, \delta, T, \lambda, v^0)$ a positive generic constant, independent of n, which may vary from line to line. Multiplying (5.12) by v_n and integrating from 0 to 1, we have

$$\begin{aligned}\frac{d}{dt}\left(\|v_n(t)\|^2\right) &= 2\int_0^1 v_n\left(\varepsilon v_{nxx} - \delta v_{nxxx} - \lambda v_{nx} + f\right)dx \\ &= 2\varepsilon\, v_n v_{nx}|_0^1 - 2\varepsilon\int_0^1 v_{nx}^2 dx - 2\delta\, v_n v_{nxx}|_0^1 + \ 2\delta v_{nx}^2\Big|_0^1 - 2\lambda\int_0^1 v_n v_{nx} dx + 2\int_0^1 f v_n dx \\ &= -2\varepsilon\|v_{nx}(t)\|^2 - 2(\lambda v_{nx}, v_n) + 2\left(f, v_n\right) \\ &\le -2\varepsilon\|v_{nx}(t)\|^2 + \varepsilon\|v_{nx}(t)\|^2 + \frac{\|\lambda v_n(t)\|^2}{\varepsilon} + \left\|f(t)\right\|^2 + \|v_n(t)\|^2 \\ &\le \left\|f(t)\right\|^2 + \left(1 + \frac{\|\lambda\|_\infty^2}{\varepsilon}\right)\|v_n(t)\|^2.\end{aligned} \tag{5.16}$$

By Gronwall-Bellman's inequality and using inequality $\|v_n^0 - v^0\| \le \|v_n^0 - v^0\|_{H^2}$,

$$\begin{aligned}
\|v_n(t)\|^2 &\le \left[\left\|v_n^0\right\|^2 + \int_0^T \|f(t)\|^2 dt\right] \exp\left[\left(1 + \frac{\|\lambda\|_\infty^2}{\varepsilon}\right)t\right] \\
&\le \left[2\left\|v_n^0\right\|^2 + 2\left\|v_n^0 - v^0\right\|_{H^2}^2 + \int_0^T \|f(t)\|^2 dt\right] \exp\left[\left(1 + \frac{\|\lambda\|_\infty^2}{\varepsilon}\right)t\right] \\
&\le c.
\end{aligned} \tag{5.17}$$

Step 4. A Priori Estimate on v_{nx}. Multiplying (5.12) by v_{nxx} and integrating from 0 to 1, we have

$$\begin{aligned}
\frac{d}{dt}\left(\|v_{nx}(t)\|^2\right) &= 2\int_0^1 v_{nxx}(-\varepsilon v_{nxx} + \delta v_{nxxx} + \lambda v_{nx} - f)dx \\
&\le -2\varepsilon\|v_{nxx}(t)\|^2 + \varepsilon\|v_{nxx}(t)\|^2 + \frac{\|\lambda v_{nx}(t)\|^2}{\varepsilon} + \frac{\|f(t)\|^2}{\varepsilon} + \varepsilon\|v_{nxx}(t)\|^2 \\
&\le \frac{\|f(t)\|^2}{\varepsilon} + \frac{\|\lambda\|_\infty^2 \|v_{nx}(t)\|^2}{\varepsilon},
\end{aligned} \tag{5.18}$$

which implies

$$\begin{aligned}
\|v_{nx}(t)\|^2 &\le \left[2\left\|v_{nx}^0\right\|^2 + \frac{1}{\varepsilon}\int_0^T \|f(t)\|^2 dt\right] \exp\left[\frac{\|\lambda\|_\infty^2}{\varepsilon}\right] \\
&\le c.
\end{aligned} \tag{5.19}$$

Step 5. Existence and Uniqueness. By (5.17) and (5.19), we deduce that v_n and v_{nx} are bounded in $L^\infty([0,T], L^2(0,1))$. Consequently, there exists a subsequence of $\{v_n\}$ denoted by $\{v_{n_k}\}$ such that v_{n_k} converges to a function v in the weak-star topology of $L^\infty([0,T], L^2(0,1))$.

It is easy to see that v is the weak solution of the problem (5.6) satisfying

$$v \in C\left([0,T]; H_0^2(0,1)\right). \tag{5.20}$$

Therefore, for any differentiable function λ, the problem (5.1) has a unique weak solution:

$$u = v + \psi \in C\left([0,T]; H_0^2(0,1)\right). \tag{5.21}$$

The estimate (5.4) can be proved in the same way as in the proof of (5.19). Finally, the continuous differentiability assumption on λ can be relaxed by using (5.4). This completes the proof. □

Theorem 5.2. *For the initial data $u^0(x) \in H_0^2(0,1)$, the problem* (2.2) *has a unique solution u satisfying $u \in C([0,\infty]; H_0^2(0,1))$.*

Proof. Let $T > 0$ be any constant and $\lambda \in C([0,1] \times [0,T])$. By Lemma 5.1, the problem (5.1) has a unique solution u with $u \in C([0,1] \times [0,T])$. Hence we define the nonlinear mapping A by

$$A\lambda = u. \tag{5.22}$$

Set $R = 2\|u_x^0\|$. By (5.4), we deduce that if T is small enough, then A maps $B(0,R)$ into $B(0,R)$ and A is a contractive mapping, where

$$B(0,R) = \{\lambda \in C([0,1] \times [0,T]) : \|\lambda\|_\infty \leq R\}. \tag{5.23}$$

Therefore, by the Banach fixed point theorem, A has a unique fixed point u^*. So the problem (2.2) has a unique solution u^* for T small enough. Since u^* is also the solution of the linear problem (5.1), by Lemma 5.1, we deduce that $u^* \in C([0,T]; H_0^2(0,1))$. By Theorem 2.1, the solution u^* can be continued to the whole real line, that is, $u^* \in C([0,\infty]; H_0^2(0,1))$. This completes the proof. □

6. Conclusion

We have shown the adaptive boundary stabilization of the KdV-Burgers' equation by nonlinear boundary control. It seems yet not to be discussed. As for the adaptive case, we establish an extension to Barbalat's lemma to show the L^2 regulation of the solution of the KdV-Burgers' equation. Also, we prove that the solution of the KdV-Burgers equation is well posed. However, we want to get the H^1 stability which implies boundedness of the solution. We will solve this problem in the future.

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