Hindawi Publishing Corporation
International Journal of Differential Equations
Volume 2011, Article ID 346298, 13 pages
doi:10.1155/2011/346298

Research Article

On a Fractional Master Equation

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Received 9 February 2011; Accepted 25 August 2011

Academic Editor: Peiguang Wang

A fractional order time-independent form of the wave equation or diffusion equation in two dimensions is obtained from the standard time-independent form of the wave equation or diffusion equation in two-dimensions by replacing the integer order partial derivatives by fractional Riesz-Feller derivative and Caputo derivative of order $\alpha, \beta, 1 < \Re(\alpha) \le 2$ and $1 < \Re(\beta) \le 2$ respectively. In this paper, we derive an analytic solution for the fractional time-independent form of the wave equation or diffusion equation in two dimensions in terms of the Mittag-Leffler function. The solutions to the fractional Poisson and the Laplace equations of the same kind are obtained, again represented by means of the Mittag-Leffler function. In all three cases, the solutions are represented also in terms of Fox's H-function.

1. Introduction

The standard time-independent form of the wave equation or diffusion equation in two-dimensions

$$\nabla^2\Psi(x,y) + k^2\Psi(x,y) = 0, \tag{1.1}$$

where $k > 0$ is the wave number, is mathematically considered as the master equations to different classes of partial differential equations, namely, Poisson equation and Laplace equation. It represents the time-independent form of the wave equation or diffusion equation obtained while applying the technique of separation of variables to reduce the complexities of the solution procedure of the original equations. This equation appears in physical phenomena and engineering applications such as heat conduction, acoustic radiation, water wave propagation, and even in biology. For estimating the geodesic sea floor properties, the proper prediction of acoustic propagation in shallow water as well as at low frequencies is very essential. It also provides the solution to such problems, refer Liu et al. [1]. It also solves the problems in pattern formation in animal coating, see Murray and Myerscough [2].

In electromagnetics, the two-dimensional time-independent form of the wave equation or diffusion equation appears as the governing equation for waveguide problems. There is huge mathematical and engineering interest in electromagnetic wave scattering problems driven by many applications such as modeling radar, sonar, acoustic noise barriers, atmospheric particle scattering, and ultrasound since both the incident and scattered electric field satisfy the two-dimensional time-independent form of the wave equation or diffusion equation which is also known as the scalar Helmholtz equation (see Budiarto and Takada [3]). This paper introduces a new fractional-model time-independent form of the wave equation or diffusion equation in two-dimensions, in which both the space variables x and y are allowed to take fractional order changes. Such models are described in Mainardi et al. [4] and are defined as

$$ {}_xD_\theta^\alpha E(x,y) + {}_0D_y^\beta E(x,y) + k^2 E(x,y) = \Phi(x,y) \tag{1.2} $$

$k > 0$, $x \in \Re$, $y \in \Re^+$, $1 < \Re(\alpha) \le 2$, $1 < \Re(\beta) \le 2$, where k is the wave number given by $k = 2\pi/\lambda$ where λ is the wavelength, $E(x,y)$ is the field variable of interest, which could be acoustic pressure, wave elevation, or electromagnetic potential, among many other possibilities, and $\Phi(x,y)$ is a nonlinear function in the field. ${}_xD_\theta^\alpha$ is the Riesz-Feller space fractional derivative of order α and asymmetry parameter (skewness) θ, and ${}_yD_*^\beta$ is the Caputo fractional derivative of order β. These fractional derivatives are integrodifferential operators, and are defined in the section on mathematical preliminaries.

The Mittag-Leffler function is a special function having an essential role in the solutions of fractional order integral and differential equations. Recently, this function is frequently used in modeling phenomena of fractional order appearing in physics, biology, engineering and applied sciences. After being introduced and studied by Mittag-Leffler [5], Wiman [6] and Agarwal [7], the Mittag-Leffler function, in its two forms:

$$ E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}, \quad \alpha \in \mathbb{C}, \quad \Re(\alpha) > 0, \tag{1.3} $$

$$ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)}, \alpha,\beta \in \mathbb{C}, \quad \Re(\alpha) > 0, \quad \Re(\beta) > 0, \tag{1.4} $$

has been studied in details by Dzherbashyan [8]. Both functions (1.2)-(1.3) are entire functions of order $\rho = 1/\alpha$ and type $\sigma = 1$. In 1920, Hille and Tamarkin [9] have presented a solution of the Abel-Volterra type integral equation

$$ \phi(x) - \frac{\lambda}{\Gamma(\alpha)} \int_0^x \frac{\phi(t)}{(x-t)^{1-\alpha}} \mathrm{d}t = f(x), \quad 0 < x < 1, \tag{1.5} $$

in terms of Mittag-Leffler functions. Fox [10] used the H-function to give the most generalized symmetrical Fourier kernel. This function is defined in terms of Mellin-Barnes integrals and is a generalization of the Meijer G-function. Most of the special functions are available as the special cases of this function (Mathai et al. [11]).

The objective of this paper is to develop a solution of the fractional time-independent form of the wave equation or diffusion equation (1.1) in terms of Mittag-Leffler function and then in Fox's H-function, using the Laplace and Fourier transforms and their inverse transforms. Mathematically, the Poisson and the Laplace equations are the two special cases of the two-dimensional time-independent form of the wave equation or diffusion equation. We apply this fact to fractional case also.

This paper is divided as follows. Section 2 is devoted to mathematical preliminaries used to solve the Cauchy problems. In Section 3, we derive the solution of the fractional Laplace equation in Mittag-Leffler function and then in Fox's H-function. Section 4 is devoted to the fractional Poisson equation. The solution of the fractional master equation is given in Section 5. Its proof and the convergence and the series representation of the H-function are given in the appendix. The Mellin-Barnes representation and the series representation of the special functions in the fundamental solutions are given in the appendix.

2. Mathematical Preliminaries

The space fractional Riesz-Feller derivative ${}_xD_\theta^\alpha$ of order α and skewness θ is defined as

$$\begin{aligned}{}_xD_\theta^\alpha f(x) = {}& \frac{\Gamma(1+\alpha)}{\pi}\left[\sin\left[(\alpha+\theta)\frac{\pi}{2}\right]\int_0^\infty \frac{f(x+\xi)-f(x)}{\xi^{1+\alpha}}\mathrm{d}\xi\right] \\ &+ \frac{\Gamma(1+\alpha)}{\pi}\left[\sin\left[(\alpha-\theta)\frac{\pi}{2}\right]\int_0^\infty \frac{f(x-\xi)-f(x)}{\xi^{1+\alpha}}\mathrm{d}\xi\right],\end{aligned} \tag{2.1}$$

where $0<\alpha\le 2$, $|\theta|\le\min(\alpha,2-\alpha)$ and the Caputo derivative of order α with respect to t, $t>0$,

$${}_tD_*^\alpha N(x,t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t \frac{N^{(m)}(x,u)}{(t-u)^{\alpha-m+1}}\mathrm{d}u, & m-1<\Re(\alpha)<m, \\ \dfrac{\mathrm{d}^m}{\mathrm{d}t^m}N(x,t), \quad \alpha=m, m\in N, & \end{cases} \tag{2.2}$$

where $[\cdots]$ is the integer part. The main results on Mittag-Leffler functions of (1.3), (1.4) are available in the handbook of Erdélyi et al. [12, 13], the monographs by Dzherbashyan [8, 14], some recent books by Mathai et al. [11], Kiryakova [15], Podlubny [16], Kilbas et al. [17], and Mainardi [18]. The H-function is defined by means of a Mellin-Barnes type integral in the following manner (Mathai and Haubold [19]):

$$H_{p,q}^{m,n}(z)\left[z\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = \frac{1}{2\pi i}\int_L \frac{\left\{\prod_{j=1}^m\Gamma(b_j+B_js)\right\}\left\{\prod_{j=1}^n\Gamma(1-b_j-B_js)\right\}}{\left\{\prod_{j=m+1}^q\Gamma(1-b_j-B_js)\right\}\left\{\prod_{j=n+1}^p\Gamma(a_j+A_js)\right\}}z^{-s}\mathrm{d}s, \tag{2.3}$$

and an empty product is always interpreted as unity; $m,n,p,q\in N_0$ with $0\le n\le p$, $1\le m\le q$, $A_j,B_j\in\Re_+$, $a_i,b_j\in\mathbb{C}$, $i=1,\ \ldots,\ p$; $j=1,\ \ldots,\ q$ such that $A_i(b_j+k)\ne B_j(a_i-l-1)$,

$k, l \in N_0$; $i = 1, \ldots, n$; $j = 1, \ldots, m$, where we employ the usual notations: $N_0 = (0, 1, \ldots,)$; $\Re = (-\infty, \infty)$, $\Re_+ = (0, \infty)$; $\mathbb{C}$ being the complex number field. For the details about the contour L and existence conditions see; Mathai et al. [11] and Kilbas and Saigo [20].

The Wright's generalized hypergeometric function, Wright ([21, 22]); is defined by the series representation

$$ {}_p\psi_q(z) = {}_p\psi_q\left[z\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = \sum_{r=0}^{\infty} \frac{\left[\prod_{j=1}^{p} \Gamma(a_j + A_j s)\right] z^r}{\left[\prod_{j=1}^{q} \Gamma(b_j + B_j s)\right] r!}, \tag{2.4} $$

where $z \in \mathbb{C}$, $a_j, b_j \in \mathbb{C}$, $A_j, B_j \in \Re_+$; $i = 1, \ldots, p$; $j = 1, \ldots, q$; $\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} A_j > -1$; $\mathbb{C}$ is the complex number field. The Mittag-Leffler function is a special case of this function,

$$ E_{\alpha,\beta}(z) = {}_1\psi_1\left[z\Big|_{(\beta,\alpha)}^{(1,1)}\right] = H_{1,2}^{1,1}\left[-z\Big|_{(0,1)(1-\beta,\alpha)}^{(0,1)}\right]. \tag{2.5} $$

The Laplace transform of the function $N(x,t)$ with respect to t is

$$ \mathcal{L}[N(x,t)] = \int_0^{\infty} e^{-st} N(x,t) \mathrm{d}t = N^*(x,s), \quad x \in \Re,\ \Re(s) > 0, \tag{2.6} $$

and its inverse transform with respect to s is given by

$$ \mathcal{L}^{-1}[N^*(x,s)] = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} N^*(x,s) \mathrm{d}s = N(x,t), \tag{2.7} $$

γ being a fixed real number. The Fourier transform of a function $N(x,t)$ with respect to x is defined as

$$ \mathcal{F}[N(x,t)] = \int_{-\infty}^{\infty} e^{ipx} N(x,t) \mathrm{d}x = \widehat{N}(p,t), \quad x \in \Re, \tag{2.8} $$

the inverse Fourier transform with respect to p:

$$ \mathcal{F}^{-1}\left[\widehat{N}(p,t)\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ipx} \widehat{N}(p,t) \mathrm{d}p = N(x,t). \tag{2.9} $$

The space of functions for the above two transforms is $\mathcal{LF} = \mathcal{L}(\Re_+) \times \mathcal{F}(\Re)$, where $\mathcal{L}(\Re_+)$ is the space of summable functions on $\Re_+$ with norm $\|f\| = \int_0^{\infty} |f(t)| \mathrm{d}t < \infty$ and $\mathcal{F}(\Re)$ is the space of summable functions on $\Re$ with norm $\|f\| = \int_{-\infty}^{\infty} |f(t)|\mathrm{d}t < \infty$. From Mathai et al. [11], and Prudnikov et al. [23] the cosine transform of the H-function is defined as

$$\begin{aligned}
&\int_0^{\infty} t^{\rho-1}\cos(kt) H_{p,q}^{m,n}\left[at^{\mu}\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}t \\
&\quad = \frac{2^{\rho-1}\pi}{k^{\rho}} H_{p+2,q}^{m,n+1}\left[a\left(\frac{2}{k}\right)^{\mu}\Big|_{(b_q,B_q)}^{(2-\rho/2,\mu/2),(a_p,A_p),((1-\rho)/2,\mu/2)}\right],
\end{aligned} \tag{2.10}$$

where

$$\begin{gathered}
\Re(\rho) + \mu \min_{1\le j\le m} \Re\left[\frac{b_j}{B_j}\right] > 0, \qquad \Re(\rho) + \mu \max_{1\le j\le n}\left[\frac{(a_j-1)}{A_j}\right] < 1, \quad \left|\arg a\right| < \frac{1}{2}\pi\theta, \\
\theta = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j > 0.
\end{gathered} \tag{2.11}$$

The Laplace transform of the Caputo fractional derivative (see, e.g., Podlubny [16])

$$\mathcal{L}\left[{}_0D_t^{\alpha} N(x,t)\right] = s^{\alpha}N^*(x,s) - \sum_{r=1}^{n} s^{r-1}{}_0D_t^{\alpha-r}N(x,t)\big|_{t=0}, \quad n-1 < \Re(\alpha) \le n,\ n \in N. \tag{2.12}$$

From Mainardi et al. [4], the Fourier transform of the Riesz-Feller derivative is given by

$$\mathcal{F}\left\{ {}_xD_{\theta}^{\alpha} f(x); p\right\} = -\psi_{\alpha}^{\theta}\widehat{f}(p), \tag{2.13}$$

where $\psi_{\alpha}^{\theta}\widehat{f}(p) = |p|^{\alpha} e^{i(\operatorname{sign} p)\theta\pi/2}$, $p \in \Re$, and $\widehat{f} = \mathcal{F}[f(x);p] = \int_{-\infty}^{+\infty} e^{ipx} f(x)\mathrm{d}x$.

We also need the property of H-function (Mathai et al. [11])

$$\begin{aligned}
&H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\ldots(b_{q-1},B_{q-1}),(a_1,A_1)}^{(a_1,A_1),\ldots(a_p,A_p)}\right] \\
&\quad = H_{p-1,q-1}^{m,n-1}\left[z\Big|_{(b_1,B_1)\ldots(b_{q-1},B_{q-1})}^{(a_2,A_2),\ldots(a_p,A_p)}\right], \quad m \ge 1,\ q > m,
\end{aligned} \tag{2.14}$$

and the following result (Podlubny, [16]):

$$\mathcal{L}^{-1}\left(\frac{s^{\beta-1}}{s^{\alpha}-a}; t\right) = t^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(at^{\alpha}), \quad \Re(s) > 0, \quad \Re(\alpha-\beta) > -1, \quad \left|\frac{a}{s^{\alpha}}\right| < 1. \tag{2.15}$$

3. The Exact Solution of the Fractional Laplace Equation

The exact solution of the fractional Laplace equation (1.2) in terms of three special functions namely, Mittag-Leffler function, Fox's H-function, and Mainardi function, respectively, are derived in the following subsections.

3.1. The Exact Solution in Terms of Mittag-Leffler Function

Theorem 3.1. *Consider the fractional Laplace equation*

$$ {}_xD_\theta^\alpha E(x,y) + {}_yD_*^\beta E(x,y) = 0, \quad x \in \Re, \ y \geqslant 0, \tag{3.1}$$

$1 < \Re(\alpha) \le 2$, $|\theta| \le \min(\alpha, 2-\alpha)$, $1 < \Re(\beta) \le 2$ *with initial conditions* $E(x,0) = f(x)$, $\partial/\partial y E(x,y)|_{y=0} = g(x)$, $x \in \Re$, $\lim_{x\to\pm\infty} E(x,y) = 0$, *where* ${}_xD_\theta^\alpha$ *is the Riesz-Feller fractional derivative of order* α *and symmetry* θ *and* ${}_yD_*^\beta$ *is the Caputo fractional derivative of order* β*. Then the solution of* (3.1) *subject to the initial conditions is*

$$ E(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(p) E_{\beta,1}\left[\left(\psi_\alpha^\theta(p)\right)y^\beta\right] e^{-ipx} dp + \frac{y}{2\pi}\int_{-\infty}^{\infty} \hat{g}(p) E_{\beta,2}\left[\left(\psi_\alpha^\theta(p)\right)y^\beta\right] e^{-ipx} dp, \tag{3.2}$$

where $\hat{}$ *indicates the Fourier transform with respect to the space variable* x*.*

Proof. If we apply Laplace transform with respect to the space variable y and use (2.12), (3.1) becomes

$$ {}_xD_\theta^\alpha E^*(x,s) + s^\beta E^*(x,s) - s^{\beta-1} f(x) - s^{\beta-2} g(x) = 0. \tag{3.3}$$

Now applying the Fourier transform with respect to the space variable x to obtain

$$\begin{gathered} \left[s^\beta - \psi_\alpha^\theta(p)\right]\hat{E}^*(p,s) = s^{\beta-1}\hat{f}(p) + s^{\beta-2}\hat{g}(p), \\ \hat{E}^*(p,s) = \frac{s^{\beta-1}\hat{f}(p)}{s^\beta - \psi_\alpha^\theta(p)} + \frac{s^{\beta-2}\hat{g}(p)}{s^\beta - \psi_\alpha^\theta(p)}. \end{gathered} \tag{3.4}$$

Using the result (2.15), we get

$$ \hat{E}(p,y) = \hat{f}(p) E_{\beta,1}\left[\psi_\alpha^\theta(p) y^\beta\right] + \hat{g}(p) y E_{\beta,2}\left[\psi_\alpha^\theta(p) y^\beta\right]. \tag{3.5}$$

Taking the inverse Fourier transform on both sides, we obtain the solution

$$ E(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(p) E_{\beta,1}\left[\left(\psi_\alpha^\theta(p)\right)y^\beta\right] e^{-ipx} dp + \frac{y}{2\pi}\int_{-\infty}^{\infty} \hat{g}(p) E_{\beta,2}\left[\left(\psi_\alpha^\theta(p)\right)y^\beta\right] e^{-ipx} dp. \tag{3.6}$$

Hence Theorem 3.1 follows.

From the above Cauchy problem, by introducing the initial conditions $f(x) = \delta(x)$ where $\delta(x)$ is the Dirac delta function and $g(x) = 0$, the exact solution or the Green function in Mittag-Leffler function is derived with the help of the Fourier convolution:

$$E(x,y) = \int_{-\infty}^{\infty} G_{\alpha,\beta}^{\theta}(x-\tau,y) f(\tau)\mathrm{d}\tau, \tag{3.7}$$

where the exact solution in terms of the Mittag-leffler function $G(x,y)$ is

$$G_{\alpha,\beta}^{\theta}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{\beta,1}\left[\left(\psi_{\alpha}^{\theta}(p)\right)y^{\beta}\right]e^{-ipx}\mathrm{d}p. \tag{3.8}$$

□

3.2. Exact Solution in Terms of Fox's H-Function

From the above theorem, exact solution in Fox's H-function for the fractional Laplace equation

$${}_{x}D_{\theta}^{\alpha}E(x,y) + {}_{0}D_{y}^{\beta}E(x,y) = 0, \quad x \in \Re, \quad y \geqslant 0, \tag{3.9}$$

$1 < \Re(\alpha) \leq 2$, $|\theta| \leq \min(\alpha, 2-\alpha)$, $1 < \Re(\beta) \leq 2$ with initial conditions $E(x,0) = f(x) = \delta(x)$, $\partial/\partial y E(x,y)|_{y=0} = 0$, $x \in \Re$, $\lim_{x\to\pm\infty} E(x,y) = 0$, is given by

$$E(x,y) = \int_{-\infty}^{\infty} G_{\alpha,\beta}^{\theta}(x-\tau,y) f(\tau)d\tau, \tag{3.10}$$

where

$$\begin{aligned}
G_{\alpha,\beta}^{\theta}(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{\beta,1}\left[\left(\psi_{\alpha}^{\theta}(p)\right)y^{\beta}\right]e^{-ipx}\mathrm{d}p, \\
&= \frac{1}{\pi}\int_{0}^{\infty} \cos(px) H_{1,2}^{1,1}\left[\left(-\psi_{\alpha}^{\theta}(p)\right)y^{\beta}\Big|_{(0,1),(0,\beta)}^{(0,1)}\right]\mathrm{d}p, \\
&= \frac{1}{\pi}\int_{0}^{\infty} \cos(px) H_{1,2}^{1,1}\left[-\left(|p|^{\alpha}y^{\beta}e^{i\theta(\pi/2)}\right)\Big|_{(0,1),(0,\beta)}^{(0,1)}\right] \mathrm{d}p, \\
&= \frac{1}{\sqrt{\pi}|x|} H_{3,2}^{1,2}\left[-2^{\alpha}e^{i\theta(\pi/2)}|x|^{-\alpha}y^{\beta}\Big|_{(0,1),(0,\beta)}^{(1/2,\alpha/2),(0,1),(0,\alpha/2)}\right], \qquad \Re(\beta) < 2, \\
&= \frac{1}{\sqrt{\pi}|x|} H_{2,1}^{1,1}\left[-2^{\alpha}e^{i\theta(\pi/2)}|x|^{-\alpha}y^{\beta}\Big|_{(0,\beta)}^{(1/2,\alpha/2),(0,\alpha/2)}\right], \qquad \text{using (2.14).}
\end{aligned} \tag{3.11}$$

4. Fractional Poisson Equation

As a generalization of the Cauchy problem 3.1, the Mittag-Leffler solution of the fractional Poisson equation

$$ {}_xD_\theta^\alpha E(x,y) + {}_0D_y^\beta E(x,y) = \Phi(x,y) \tag{4.1} $$

is given as.

4.1. Cauchy Problem

The solution to the fractional Poisson equation

$$ {}_xD_\theta^\alpha E(x,y) + {}_0D_y^\beta E(x,y) = \Phi(x,y), \quad y \geqslant 0, \quad x \in \Re, \quad 1 < \Re(\alpha) \leq 2, \tag{4.2} $$

where $\Phi(x,y)$ is a nonlinear function, $1 < \Re(\alpha) \leq 2$, $|\theta| \leq \min(\alpha, 2-\alpha)$, $1 < \Re(\beta) \leq 2$ with initial conditions $E(x,0) = f(x)$, $\partial/\partial y E(x,y)|_{y=0} = g(x)$, $x \in \Re$, $\lim_{x\to\pm\infty} E(x,y) = 0$ where ${}_xD_\theta^\alpha$ is the Riesz-Feller fractional derivative of order α and symmetry θ and ${}_yD_*^\beta$ is the Caputo fractional derivative of order β. Then, the solution of (4.2) subject to the initial conditions is

$$ \begin{aligned} E(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(p) E_{\beta,1}\left[\left(\psi_\alpha^\theta(p)\right) y^\beta\right] e^{-ipx} \mathrm{d}p \\ &+ \frac{y}{2\pi}\int_{-\infty}^{\infty} \hat{g}(p) E_{\beta,2}\left[\left(\psi_\alpha^\theta(p)\right) y^\beta\right] e^{-ipx} \mathrm{d}p \\ &+ \frac{1}{2\pi}\int_0^y \xi^{\beta-1}\int_{-\infty}^{\infty} \hat{\Phi}(p, y-\xi) E_{\beta,\beta}\left[\left(\psi_\alpha^\theta(p)\right) \xi^\beta\right] e^{-ipx} \mathrm{d}p\mathrm{d}\xi, \end{aligned} \tag{4.3} $$

where $\hat{}$ indicates the Fourier transform with respect to the space variable x.

Solution. Applying the Laplace transform with respect to the space variable y and Fourier transform with respect to the space variable x and using initial conditions, we have

$$ \begin{aligned} \left[s^\beta - \psi_\alpha^\theta(p)\right]\hat{E}^*(p,s) &= s^{\beta-1}\hat{f}(p) + s^{\beta-2}\hat{g}(p) + \hat{\Phi}^*(p,s), \\ \hat{E}^*(p,s) &= \frac{s^{\beta-1}\hat{f}(p)}{s^\beta - \psi_\alpha^\theta(p)} + \frac{s^{\beta-2}\hat{g}(p)}{s^\beta - \psi_\alpha^\theta(p)} + \frac{\hat{\Phi}^*(p,s)}{s^\beta - \psi_\alpha^\theta(p)}. \end{aligned} \tag{4.4} $$

Taking the inverse Laplace transform and using Laplace convolution,

$$ \begin{aligned} \hat{E}(p,y) &= \hat{f}(p) E_{\beta,1}\left[\left(\psi_\alpha^\theta(p)\right) y^\beta\right] + \hat{g}(p) y E_{\beta,2}\left[\psi_\alpha^\theta(p)\xi^\beta\right] \\ &+ \int_0^\infty \hat{\Phi}(p, y-\xi) E_{\beta,\beta}\left[\left(\psi_\alpha^\theta(p)\right) y^\beta\right] \mathrm{d}\xi. \end{aligned} \tag{4.5} $$

Taking the inverse Fourier transform on both sides and using Fourier convolution, we obtain the solution

$$
\begin{aligned}
E(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(p)E_{\beta,1}\left[\left(\psi_{\alpha}^{\theta}(p)\right)y^{\beta}\right]e^{-ipx}\mathrm{d}p \\
&+ \frac{y}{2\pi}\int_{-\infty}^{\infty}\hat{g}(p)E_{\beta,2}\left[\left(\psi_{\alpha}^{\theta}(p)\right)y^{\beta}\right]e^{-ipx}\mathrm{d}p \\
&+ \frac{1}{2\pi}\int_{0}^{y}\xi^{\beta-1}\int_{-\infty}^{\infty}\hat{\Phi}(p,y-\xi)E_{\beta,\beta}\left[\left(\psi_{\alpha}^{\theta}(p)\right)\xi^{\beta}\right]e^{-ipx}\mathrm{d}p\mathrm{d}\xi.
\end{aligned}
\tag{4.6}
$$

5. The Fractional Master Equation

In this section, we solve the fractional master equation, namely, the fractional time-independent form of the wave or diffusion equation (fractional Helmholtz equation) (1.2) to generate a solution in terms of the Mittag-Leffler function, using the Laplace and Fourier transforms and their inverses. The wave number k is selected such a way that $k^2 > \psi_{\alpha}^{\theta}(p) = |p|^{\alpha}e^{i(\mathrm{sign}\ p)\theta\pi/2}$, $p \in \Re$. Thus, the model is made suitable to handle the scattering problems of electromagnetic waves of large wave number or short wavelength.

Theorem 5.1. *The analytical solution of the fractional time-independent form of the wave equation or diffusion equation*

$$
{}_{x}D_{\theta}^{\alpha}E(x,y) + {}_{y}D_{*}^{\beta}E(x,y) + k^{2}E(x,y) = \Phi(x,y),
\tag{5.1}
$$

$k > 0$, $x \in \Re$, $y \in \Re^{+}$, with the initial conditions $E(x,0) = f(x)$, $\partial/\partial y E(x,y)|_{y=0} = g(x)$, $\lim_{x\to\pm\infty} E(x,y) = 0$, is

$$
\begin{aligned}
E(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(p)E_{\beta,1}\left[-\left(k^{2}-\psi_{\alpha}^{\theta}(p)\right)y^{\beta}\right]\exp(-ipx)\mathrm{d}p \\
&+ \frac{y}{2\pi}\int_{-\infty}^{\infty}\hat{g}(p)E_{\beta,2}\left[-\left(k^{2}-\psi_{\alpha}^{\theta}(p)\right)y^{\beta}\right]\exp(-ipx)\mathrm{d}p \\
&+ \frac{1}{2\pi}\int_{0}^{\infty}\xi^{\beta-1}\int_{-\infty}^{\infty}\hat{\Phi}(p,y-\xi)E_{\beta,\beta}\left[-\left(k^{2}-\psi_{\alpha}^{\theta}(p)\right)\xi^{\beta}\right]\exp(-ipx)\mathrm{d}p\mathrm{d}\xi,
\end{aligned}
\tag{5.2}
$$

Where $\hat{}$ is the Fourier transform with respect to x.

The proof of this theorem and the H-function solution of the fractional time-independent form of the wave equation or diffusion equation are discussed in the appendix.

6. Conclusion

The closed form solutions in terms of the Mittag-Leffler function and Fox's H-function are obtained for the fractional Laplace and the fractional Poisson equations. It is seen that the solutions in terms of the Mittag-Leffler function as well as in the H-function to the fractional time-independent form of the wave equation or diffusion equation or the fractional Helmholtz equation are the master solutions to the solutions of the fractional Laplace equation and the fractional Poisson equation.

Appendices

A. The Mittag-Leffler Solution of the Fractional Time-Independent form of the Wave Equation or Diffusion Equation

If we apply the Laplace transform with respect to the space variable y and use (2.12), (5.1) becomes

$$ {}_xD_\theta^\alpha E^*(x,s) + s^\alpha E^*(x,s) - f(x) - sg(x) + k^2 E^*(x,s) = \Phi^*(x,s). \tag{A.1}$$

Applying Fourier transform with respect to x,

$$ -\psi_\alpha^\theta(p)\widehat{E}^*(p,s) + s^\beta \widehat{E}^*(p,s) - s^{\beta-1}\widehat{f}(p) - s^{\beta-2}\widehat{g}(p) + k^2\widehat{E}^*(p,s) = \widehat{\Phi^*}(p,s). \tag{A.2}$$

Thus,

$$ \widehat{E}^*(p,s) = \frac{s^{\beta-1}}{s^\beta + \left[k^2 - \psi_\alpha^\theta(p)\right]}\widehat{f}(p) + \frac{s^{\beta-2}}{s^\beta + \left[k^2 - \psi_\alpha^\theta(p)\right]}\widehat{g}(p) + \frac{\widehat{\Phi^*}(p,s)}{s^\beta + \left[k^2 - \psi_\alpha^\theta(p)\right]}. \tag{A.3}$$

Taking Laplace inverse transform with respect to s,

$$\begin{aligned} \widehat{E}(p,y) = {} & \widehat{f}(p)E_{\beta,1}\left[-\left(k^2 - \psi_\alpha^\theta(p)\right)y^\beta\right] \\ & + \widehat{g}(p)yE_{\beta,2}\left[-\left(k^2 - \psi_\alpha^\theta(p)\right)y^\beta\right] \\ & + \widehat{\Phi}(p,y)y^{\beta-1}E_{\beta,\beta}\left[-\left(k^2 - \psi_\alpha^\theta(p)\right)y^\beta\right]. \end{aligned} \tag{A.4}$$

Using inverse Fourier transform with respect to p, the solution is

$$\begin{aligned} E(x,y) = {} & \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(p)E_{\beta,1}\left[-\left(k^2 - \psi_\alpha^\theta(p)\right)y^\beta\right]\exp(-ipx)\mathrm{d}p \\ & + \frac{y}{2\pi}\int_{-\infty}^{\infty} \widehat{g}(p)E_{\beta,2}\left[-\left(k^2 - \psi_\alpha^\theta(p)\right)y^\beta\right]\exp(-ipx)\mathrm{d}p \\ & + \frac{1}{2\pi}\int_0^y \xi^{\beta-1}\int_{-\infty}^{\infty} \widehat{\Phi}(p,y-\xi)E_{\beta,\beta}\left[-\left(k^2 - \psi_\alpha^\theta(p)\right)\xi^\beta\right]\exp(-ipx)\mathrm{d}p\mathrm{d}\xi. \end{aligned} \tag{A.5}$$

It is seen that the above solution is a master solution of the solutions to the fractional Laplace equation (3.6) and the fractional Poisson equation (4.6).

B. The Fox's H-Function Solution of the Fractional Time-Independent form of the Wave Equation or Diffusion Equation.

The Fox's H-function solution of the fractional time-independent form of the wave equation or diffusion equation is

$$ {}_xD_\theta^\alpha E(x,y) + {}_yD_*^\beta E(x,y) + k^2 E(x,y) = \Phi(x,y), \tag{B.1}$$

$k > 0$, $x \in \Re$, $y \in \Re^+$, with the initial conditions $E(x,0) = f(x) = \delta(x)$, $\partial/\partial y E(x,y)|_{y=0} = 0$, $\lim_{x\to\pm\infty} E(x,y) = 0$ is

$$E(x,y) = \int_{-\infty}^{\infty} G_{\alpha,\beta}^\theta(x-\tau,y) f(\tau) d\tau + \int_0^y (y-\xi)^{\alpha-1} \int_0^x g_{\alpha,\beta}^\theta(x-\tau, y-\xi)\Phi(\tau,\xi) \mathrm{d}\tau \mathrm{d}\xi, \tag{B.2}$$

where the exact solution in terms of the H-function $G_{\alpha,\beta}^\theta(x,y)$ is

$$G_{\alpha,\beta}^\theta(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{1,2}^{1,1}\left[\left(k^2 - \psi_\alpha^\theta(p)\right) y^\beta \Big|_{(0,1),(0,\beta)}^{(0,1)}\right] e^{-ipx} \mathrm{d}p, \tag{B.3}$$

$$g_{\alpha,\beta}^\theta(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{1,2}^{1,1}\left[\left(k^2 - \psi_\alpha^\theta(p)\right) y^\beta \Big|_{(0,1),(1-\beta,\beta)}^{(0,1)}\right] e^{-ipx} \mathrm{d}p. \tag{B.4}$$

C. Mellin-Barnes Representation and Region of Convergence of the Mittag-Leffler $E_{\beta,1}[(\psi_\alpha^\theta(p))y^\beta]$

For $0 < \Re(\beta) \le 2$, $E_{\beta,1}[(\psi_\alpha^\theta(p))y^\beta$ is represented by the Mellin-Barnes integral as

$$\begin{aligned} E_{\beta,1}\left[\left(\psi_\alpha^\theta(p)\right)y^\beta\right. &= \frac{1}{2\pi i}\int_L \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-\beta s)}\left(y^\beta \psi_\alpha^\theta(p)\right)^{-s} \mathrm{d}s, \\ &= H_{1,2}^{1,1}\left[y^\beta \psi_\alpha^\theta(p)\Big|_{(0,1),(0,\beta)}^{(0,1)}\right], \end{aligned} \tag{C.1}$$

where $|\arg(y^\beta \psi_\alpha^\theta(p))| < \pi$; the contour of integration is at $c - i\infty$ and ending at $c + i\infty$, $0 < c < 1$, separates all poles at $s = -k$, $k = 0, 1, \ldots$ to the left and all the poles at $s = n + 1$, $n = 0, 1, \ldots$ to the right. From Mathai et al. [11], the integral converges for all $z \neq 0$, where $z = y^\beta \psi_\alpha^\theta(p)$, $p \in \Re$.

D. Series Representation of $H_{2,1}^{1,1}[-z|_{(0,\beta)}^{(1/2,\,\alpha/2),(0,\,\alpha/2)}]$

By Mathai et al. [11], the series expansion of the function $H_{2,1}^{1,1}[-z|_{(0,\beta)}^{(1/2,\,\alpha/2),(0,\,\alpha/2)}]$ where $z = -2^{\alpha}e^{i\theta(\pi/2)}|x|^{-\alpha}y^{\beta}$ is given as follows. We have

$$H_{2,1}^{1,1}\left[-z|_{(0,\beta)}^{(1/2,\,\alpha/2),(0,\,\alpha/2)}\right] = \frac{1}{2\pi i}\int_L \frac{\Gamma(\beta s)\Gamma((1/2)-(\alpha/2)s)}{\Gamma(\alpha/2s)}[-z]^{-s}\mathrm{d}s, \tag{D.1}$$

where $z = 2^{\alpha}e^{i\theta(\pi/2)}|x|^{-\alpha}y^{\beta}$. Let us assume that the poles of the integrand are simple. The region of convergence is $L = L_{\infty}$ is a loop beginning and ending at ∞ and encircling all the poles of $\Gamma((1/2)-(\alpha/2)s)$ once in the negative direction but none of the poles of $\Gamma(\beta s)$. By calculating the residues at the poles of $\Gamma((1/2)-(\alpha/2)s)$ where the poles are given by $(1/2)-(\alpha/2)s = -\nu,\ \nu = 0,1,2,\ldots$, we will get the series representation as

$$H_{2,1}^{1,1}\left[-z|_{(0,\beta)}^{(1/2,\,\alpha/2),(0,\,\alpha/2)}\right] = \frac{2}{\alpha}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}\Gamma\left((\beta/\alpha)[2\nu+1]\right)}{\nu!\Gamma(\nu+1/2)}(-z)^{-(2\nu+1)/\alpha}. \tag{D.2}$$

The H-function exists for all $z,\ z\neq 0$ where $z = -2^{\alpha}e^{i\theta(\pi/2)}|x|^{-\alpha}y^{\beta}$.

Acknowledgments

The author acknowledges gratefully the encouragements given by Professor A. M. Mathai, Department of Mathematics and Statistics, McGill University, Montreal, Canada, and Dr. M. S. Samuel, Director, Department of Computer Science, MACFAST, Thiruvalla, Kerala, India, in this research work. The author would like to thank the Centre for Mathematical Sciences, Pala Campus, for providing facilities for carrying out this work.

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