Expansion formulas in terms of integer-order derivatives for the Hadamard fractional integral and derivative*

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Abstract

We obtain series expansion formulas for the Hadamard fractional integral and fractional derivative of a smooth function. When considering finite sums only, an upper bound for the error is given. Numerical simulations show the efficiency of the approximation method.

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1 Introduction

In general terms, Fractional Calculus allows us to define integrals and derivatives of arbitrary real order, and can be seen as a generalization of ordinary calculus. A fractional derivative of order $\alpha > 0$, when α is integer, is in some sense a derivative of order $n \in \mathbb{N}$ of such function, while a fractional integral is a *n-folfed* integral. Although there exist in the literature a large number of definitions for fractional operators (integrals and derivatives), the Riemann-Liouville and Caputo are the most common for fractional derivatives, and for fractional integrals the usual one is the Riemann-Liouville definition. In our work we will consider the Hadamard fractional integral and fractional derivative. Although the definitions go back to the works of Hadamard in 1892 [5], this type of operators are not yet well studied and much exists to be done.

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One difficulty that arises when dealing with problems involving fractional operators, such as fractional differential equations, control problems, etc, is that solving them analytically has proven to be hard or even impossible. To overcome this, numerical methods are being employed. What is usually done is seing these operators as special types of integrals, and using partitions on the domains one writes them as finite sums and solve the problem in this way, with some error for the final result. Our approach is distinct from this, in the sense that we seek expansion formulas for the Hadamard fractional operators with integer-order derivatives, and in this way we can rewrite any initial problem that depends on fractional operators as a new one that involves integer derivatives only, and thus successfully apply the already known methods to obtain the desired solution. Also, in some cases, it can be easier to find the fractional derivative or integral using those expansions, instead applying the direct definitions. In contrast with [2], where an expansion for the Riemann-Liouville fractional derivative is given, our expansions of the Hadamard fractional derivative and integral do not omit the first derivative, allowing one to obtain considerable better accuracy in computation.

The paper is organized in the following way. In Section 2 we review some concepts of fractional calculus. Decomposition formulas for the left and right Hadamard fractional integrals are given in Section 3, together with approximation formulas and error estimations. Following the same approach, similar formulas are obtained for the left and right Hadamard fractional derivatives in Section 4. In Section 5 we test the efficiency of such approximations with some examples, comparing the analytical solution with the numerical approximation.

2 Preliminaries

In this section we review some necessary definitions for our present work, namely the Hadamard fractional integral and derivative. For more on fractional calculus, we refer the interested reader to [8, 10, 11, 13]. For related work on Hadamard fractional operators, see [3, 4, 6, 7, 9, 12].

Let a, b be two reals with $0 < a < b$ and $x : [a, b] \to \mathbb{R}$ a function. The left and right Hadamard fractional integrals of order $\alpha > 0$ are defined by

$$ {}_a\mathcal{I}_t^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} \frac{x(\tau)}{\tau} d\tau, \qquad t \in]a, b[, $$

and

$$ {}_t\mathcal{I}_b^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\ln \frac{\tau}{t}\right)^{\alpha-1} \frac{x(\tau)}{\tau} d\tau, \qquad t \in]a, b[, $$

respectively. These integrals were introduced by Hadamard in [5] in the special case $a = 0$.

When $\alpha = m$ is integer, these fractional integrals are *m-folfed* integrals:

$$ {}_a\mathcal{I}_t^m x(t) = \int_a^t \frac{d\tau_1}{\tau_1} \int_a^{\tau_1} \frac{d\tau_2}{\tau_2} \dots \int_a^{\tau_{m-1}} \frac{x(\tau_m)}{\tau_m} d\tau_m $$

and

$$ {}_t\mathcal{I}_b^m x(t) = \int_t^b \frac{d\tau_1}{\tau_1} \int_{\tau_1}^b \frac{d\tau_2}{\tau_2} \dots \int_{\tau_{m-1}}^b \frac{x(\tau_m)}{\tau_m} d\tau_m. $$

For fractional derivatives, we also consider the left and right derivatives. For $\alpha > 0$, the left and right Hadamard fractional derivatives of order α are defined by

$$ {}_a\mathcal{D}_t^\alpha x(t) = \left(t\frac{d}{dt}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\ln\frac{t}{\tau}\right)^{n-\alpha-1} \frac{x(\tau)}{\tau} d\tau, \qquad t \in]a,b[, $$

and

$$ {}_t\mathcal{D}_b^\alpha x(t) = \left(-t\frac{d}{dt}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_t^b \left(\ln\frac{\tau}{t}\right)^{n-\alpha-1} \frac{x(\tau)}{\tau} d\tau, \qquad t \in]a,b[, $$

respectively, with $n = [\alpha] + 1$. When $\alpha = m$ is integer, we have

$$ {}_a\mathcal{D}_t^m x(t) = \left(t\frac{d}{dt}\right)^m x(t) \text{ and } {}_t\mathcal{D}_b^m x(t) = \left(-t\frac{d}{dt}\right)^m x(t). $$

Hadamard's fractional integrals and derivatives can be seen as inverse operations of each other (see Property 2.28 and Theorem 2.3 of [8]).

When $\alpha \in (0,1)$ and $x \in AC[a,b]$, the Hadamard fractional derivatives may be expressed by formulas

$$ {}_a\mathcal{D}_t^\alpha x(t) = \frac{x(a)}{\Gamma(1-\alpha)} \left(\ln\frac{t}{a}\right)^{-\alpha} + \frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\ln\frac{t}{\tau}\right)^{-\alpha} \dot{x}(\tau) d\tau, \tag{1} $$

and

$$ {}_t\mathcal{D}_b^\alpha x(t) = \frac{x(b)}{\Gamma(1-\alpha)} \left(\ln\frac{b}{t}\right)^{-\alpha} - \frac{1}{\Gamma(1-\alpha)} \int_t^b \left(\ln\frac{\tau}{t}\right)^{-\alpha} \dot{x}(\tau) d\tau. \tag{2} $$

We refer to Theorem 3.2 in [7] with a formula for an arbitrary $\alpha > 0$. We also mention Theorem 17 in [4], where expansion formulas for the Hadamard fractional integrals and derivatives in terms of integer-order derivatives are proved, for functions that admit derivatives of any order:

$$ {}_0\mathcal{I}_t^\alpha x(t) = \sum_{k=0}^\infty S(-\alpha,k) t^k x^{(k)}(t) $$

and

$$ {}_0\mathcal{D}_t^\alpha x(t) = \sum_{k=0}^\infty S(\alpha,k) t^k x^{(k)}(t), $$

where $S(\alpha,k)$ is the Stirling function.

3 An expansion formula for the Hadamard fractional integral

In this section we consider a class of differentiable functions up to order $n+1$, $x \in C^{n+1}[a,b]$, and deduce expansion formulas for the Hadamard fractional integrals in terms of $x^{(i)}(\cdot)$, for $i \in 0,\dots,n$. Before presenting the result in its full extension, we will briefly explain the techniques involved for the particular case $n = 2$. To that purpose, let $x \in C^3[a,b]$. Integrating by parts three times, we obtain

$$\begin{aligned}
{}_a\mathcal{I}_t^\alpha x(t) &= -\frac{1}{\Gamma(\alpha)}\int_a^t -\frac{1}{\tau}\left(\ln\frac{t}{\tau}\right)^{\alpha-1} x(\tau)d\tau \\
&= \frac{1}{\Gamma(\alpha+1)}\left(\ln\frac{t}{a}\right)^{\alpha} x(a) - \frac{1}{\Gamma(\alpha+1)}\int_a^t -\frac{1}{\tau}\left(\ln\frac{t}{\tau}\right)^{\alpha} \tau\dot{x}(\tau)d\tau \\
&= \frac{1}{\Gamma(\alpha+1)}\left(\ln\frac{t}{a}\right)^{\alpha} x(a) + \frac{1}{\Gamma(\alpha+2)}\left(\ln\frac{t}{a}\right)^{\alpha+1} a\dot{x}(a) \\
&\quad -\frac{1}{\Gamma(\alpha+2)}\int_a^t -\frac{1}{\tau}\left(\ln\frac{t}{\tau}\right)^{\alpha+1} (\tau\dot{x}(\tau)+\tau^2\ddot{x}(\tau))d\tau \\
&= \frac{1}{\Gamma(\alpha+1)}\left(\ln\frac{t}{a}\right)^{\alpha} x(a) + \frac{1}{\Gamma(\alpha+2)}\left(\ln\frac{t}{a}\right)^{\alpha+1} a\dot{x}(a) \\
&\quad +\frac{1}{\Gamma(\alpha+3)}\left(\ln\frac{t}{a}\right)^{\alpha+2} (a\dot{x}(a)+a^2\ddot{x}(a)) \\
&\quad +\frac{1}{\Gamma(\alpha+3)}\int_a^t \left(\ln\frac{t}{\tau}\right)^{\alpha+2} (\dot{x}(\tau)+3\tau\ddot{x}(\tau)+\tau^2\dddot{x}(\tau))d\tau
\end{aligned}.$$

On the other hand, using the binomial formula, we have

$$\begin{aligned}
\left(\ln\frac{t}{\tau}\right)^{\alpha+2} &= \left(\ln\frac{t}{a}\right)^{\alpha+2}\left(1-\frac{\ln\frac{\tau}{a}}{\ln\frac{t}{a}}\right)^{\alpha+2} \\
&= \left(\ln\frac{t}{a}\right)^{\alpha+2}\sum_{p=0}^{\infty}\frac{\Gamma(p-\alpha-2)}{\Gamma(-\alpha-2)p!}\cdot\frac{(\ln\frac{\tau}{a})^p}{(\ln\frac{t}{a})^p}.
\end{aligned}$$

This series converges since $\tau \in [a,t]$ and $\alpha + 2 > 0$. Combining these formulas, we get

$$\begin{aligned}
{}_a\mathcal{I}_t^\alpha x(t) = \frac{1}{\Gamma(\alpha+1)}\left(\ln\frac{t}{a}\right)^{\alpha} x(a) + \frac{1}{\Gamma(\alpha+2)}\left(\ln\frac{t}{a}\right)^{\alpha+1} a\dot{x}(a) + \frac{1}{\Gamma(\alpha+3)}\left(\ln\frac{t}{a}\right)^{\alpha+2} (a\dot{x}(a)+a^2\ddot{x}(a)) \\
+\frac{1}{\Gamma(\alpha+3)}\left(\ln\frac{t}{a}\right)^{\alpha+2}\sum_{p=0}^{\infty}\frac{\Gamma(p-\alpha-2)}{\Gamma(-\alpha-2)p!\left(\ln\frac{t}{a}\right)^p}\int_a^t\left(\ln\frac{\tau}{a}\right)^p(\dot{x}(\tau)+3\tau\ddot{x}(\tau)+\tau^2\dddot{x}(\tau))d\tau.
\end{aligned}$$

Now, split the series into the two cases $p = 0$ and $p = 1\dots\infty$, and integrate by parts the second one. We obtain

$$\begin{aligned}
{}_a\mathcal{I}_t^\alpha x(t) &= \frac{1}{\Gamma(\alpha+1)}\left(\ln\frac{t}{a}\right)^{\alpha} x(a) + \frac{1}{\Gamma(\alpha+2)}\left(\ln\frac{t}{a}\right)^{\alpha+1} a\dot{x}(a) \\
&\quad +\frac{1}{\Gamma(\alpha+3)}\left(\ln\frac{t}{a}\right)^{\alpha+2}(t\dot{x}(t)+t^2\ddot{x}(t))\left[1+\sum_{p=1}^{\infty}\frac{\Gamma(p-\alpha-2)}{\Gamma(-\alpha-2)p!}\right] \\
&\quad +\frac{1}{\Gamma(\alpha+2)}\left(\ln\frac{t}{a}\right)^{\alpha+2}\sum_{p=1}^{\infty}\frac{\Gamma(p-\alpha-2)}{\Gamma(-\alpha-1)(p-1)!\left(\ln\frac{t}{a}\right)^p}\int_a^t\left(\ln\frac{\tau}{a}\right)^{p-1}(\dot{x}(\tau)+\tau\ddot{x}(\tau))d\tau.
\end{aligned}$$

Repeating this procedure two more times, we obtain the following:

$$\begin{aligned}
{}_a\mathcal{I}_t^\alpha x(t) &= \frac{1}{\Gamma(\alpha+1)}\left(\ln\frac{t}{a}\right)^\alpha x(t)\left[1+\sum_{p=3}^\infty \frac{\Gamma(p-\alpha-2)}{\Gamma(-\alpha)(p-2)!}\right] \\
&+\frac{1}{\Gamma(\alpha+2)}\left(\ln\frac{t}{a}\right)^{\alpha+1} t\dot{x}(t)\left[1+\sum_{p=2}^\infty \frac{\Gamma(p-\alpha-2)}{\Gamma(-\alpha-1)(p-1)!}\right] \\
&+\frac{1}{\Gamma(\alpha+3)}\left(\ln\frac{t}{a}\right)^{\alpha+2} (t\dot{x}(t)+t^2\ddot{x}(t))\left[1+\sum_{p=1}^\infty \frac{\Gamma(p-\alpha-2)}{\Gamma(-\alpha-2)p!}\right] \\
&+\frac{1}{\Gamma(\alpha)}\left(\ln\frac{t}{a}\right)^{\alpha+2}\sum_{p=3}^\infty \frac{\Gamma(p-\alpha-2)}{\Gamma(-\alpha+1)(p-3)!\left(\ln\frac{t}{a}\right)^p}\int_a^t\left(\ln\frac{\tau}{a}\right)^{p-3}\frac{x(\tau)}{\tau}d\tau,
\end{aligned}$$

or, in a more concise way,

$$\begin{aligned}
{}_a\mathcal{I}_t^\alpha x(t) &= A_0(\alpha)\left(\ln\frac{t}{a}\right)^\alpha x(t)+A_1(\alpha)\left(\ln\frac{t}{a}\right)^{\alpha+1} t\dot{x}(t) \\
&+A_2(\alpha)\left(\ln\frac{t}{a}\right)^{\alpha+2}(t\dot{x}(t)+t^2\ddot{x}(t))+\sum_{p=3}^\infty B(\alpha,p)\left(\ln\frac{t}{a}\right)^{\alpha+2-p} V_p(t)
\end{aligned}$$

with

$$\begin{aligned}
A_0(\alpha) &= \frac{1}{\Gamma(\alpha+1)}\left[1+\sum_{p=3}^\infty \frac{\Gamma(p-\alpha-2)}{\Gamma(-\alpha)(p-2)!}\right], \\
A_1(\alpha) &= \frac{1}{\Gamma(\alpha+2)}\left[1+\sum_{p=2}^\infty \frac{\Gamma(p-\alpha-2)}{\Gamma(-\alpha-1)(p-1)!}\right], \\
A_2(\alpha) &= \frac{1}{\Gamma(\alpha+3)}\left[1+\sum_{p=1}^\infty \frac{\Gamma(p-\alpha-2)}{\Gamma(-\alpha-2)p!}\right],
\end{aligned}$$

$$B(\alpha,p)=\frac{\Gamma(p-\alpha-2)}{\Gamma(\alpha)\Gamma(1-\alpha)(p-2)!} \tag{3}$$

and

$$V_p(t)=\int_a^t (p-2)\left(\ln\frac{\tau}{a}\right)^{p-3}\frac{x(\tau)}{\tau}d\tau. \tag{4}$$

Remark 3.1. *When useful, namely on fractional differential equations problems, we can define V_p as in (4) by the the solution of the system*

$$\begin{cases} \dot{V}_p(t)=(p-2)\left(\ln\frac{t}{a}\right)^{p-3}\frac{x(t)}{t} \\ V_p(a)=0 \end{cases}$$

for all $p=3,4,\ldots$

We now discuss the convergence of the series involved in the definitions of $A_i(\alpha)$, for $i \in \{0,1,2\}$. Simply observe that

$$\sum_{p=3-i}^{\infty} \frac{\Gamma(p-\alpha-2)}{\Gamma(-\alpha-i)(p-2+i)!} = {}_1F_0(-\alpha-i,1)-1,$$

and ${}_1F_0(a,x)$ converges absolutely when $|x|=1$ if $a<0$ ([1, Theorem 2.1.2]).

For numerical purposes, only finite sums are considered, and thus the Hadamard left fractional integral is approximated by the decomposition

$$\begin{aligned}
{}_a\mathcal{I}_t^\alpha x(t) \quad &\approx A_0(\alpha,N)\left(\ln\frac{t}{a}\right)^{\alpha} x(t) + A_1(\alpha,N)\left(\ln\frac{t}{a}\right)^{\alpha+1} t\dot{x}(t)\\
&+A_2(\alpha,N)\left(\ln\frac{t}{a}\right)^{\alpha+2}(t\dot{x}(t)+t^2\ddot{x}(t)) + \sum_{p=3}^{N} B(\alpha,p)\left(\ln\frac{t}{a}\right)^{\alpha+2-p} V_p(t)
\end{aligned} \tag{5}$$

with

$$\begin{aligned}
A_0(\alpha,N) &= \frac{1}{\Gamma(\alpha+1)}\left[1+\sum_{p=3}^{N}\frac{\Gamma(p-\alpha-2)}{\Gamma(-\alpha)(p-2)!}\right],\\
A_1(\alpha,N) &= \frac{1}{\Gamma(\alpha+2)}\left[1+\sum_{p=2}^{N}\frac{\Gamma(p-\alpha-2)}{\Gamma(-\alpha-1)(p-1)!}\right],\\
A_2(\alpha,N) &= \frac{1}{\Gamma(\alpha+3)}\left[1+\sum_{p=1}^{N}\frac{\Gamma(p-\alpha-2)}{\Gamma(-\alpha-2)p!}\right],
\end{aligned}$$

$B(\alpha,p)$ and $V_p(t)$ as in (3)-(4), and $N \geq 3$. We proceed with an estimation for the error on such approximation. We have proven before that

$$\begin{aligned}
{}_a\mathcal{I}_t^\alpha x(t) = &\frac{1}{\Gamma(\alpha+1)}\left(\ln\frac{t}{a}\right)^{\alpha} x(a) + \frac{1}{\Gamma(\alpha+2)}\left(\ln\frac{t}{a}\right)^{\alpha+1} a\dot{x}(a) + \frac{1}{\Gamma(\alpha+3)}\left(\ln\frac{t}{a}\right)^{\alpha+2}(a\dot{x}(a)+a^2\ddot{x}(a))\\
&+\frac{1}{\Gamma(\alpha+3)}\left(\ln\frac{t}{a}\right)^{\alpha+2}\int_a^t \sum_{p=0}^{\infty}\frac{\Gamma(p-\alpha-2)}{\Gamma(-\alpha-2)p!}\frac{\left(\ln\frac{\tau}{a}\right)^p}{\left(\ln\frac{t}{a}\right)^p}(\dot{x}(\tau)+3\tau\ddot{x}(\tau)+\tau^2\dddot{x}(\tau))d\tau.
\end{aligned}$$

When we consider finite sums up to order N, the error is given by

$$|E_{tr}(t)| = \left|\frac{1}{\Gamma(\alpha+3)}\left(\ln\frac{t}{a}\right)^{\alpha+2}\int_a^t R_N(\tau)(\dot{x}(\tau)+3\tau\ddot{x}(\tau)+\tau^2\dddot{x}(\tau))d\tau\right|,$$

with

$$R_N(\tau) = \sum_{p=N+1}^{\infty}\frac{\Gamma(p-\alpha-2)}{\Gamma(-\alpha-2)p!}\frac{\left(\ln\frac{\tau}{a}\right)^p}{\left(\ln\frac{t}{a}\right)^p}.$$

Since $\tau \in [a,t]$, we have

$$\begin{aligned}
|R_N(\tau)| &\leq \sum_{p=N+1}^{\infty}\left|\binom{\alpha+2}{p}\right| \leq \sum_{p=N+1}^{\infty}\frac{e^{(\alpha+2)^2+\alpha+2}}{p^{\alpha+3}}\\
&\leq \int_N^\infty \frac{e^{(\alpha+2)^2+\alpha+2}}{p^{\alpha+3}}dp = \frac{e^{(\alpha+2)^2+\alpha+2}}{(\alpha+2)N^{\alpha+2}}.
\end{aligned}$$

Therefore,

$$|E_{tr}(t)| \leq \frac{1}{\Gamma(\alpha+3)} \left(\ln \frac{t}{a}\right)^{\alpha+2} \frac{e^{(\alpha+2)^2+\alpha+2}}{(\alpha+2)N^{\alpha+2}}[(t-a)L_1(t) + 3(t-a)^2 L_2(t) + (t-a)^3 L_3(t)],$$

where

$$L_i(t) = \max_{\tau\in[a,t]} |x^{(i)}(\tau)|, \quad i \in \{1,2,3\}.$$

Following similar arguments as done for $n = 2$, we can prove the general case with an expansion up to the derivative of order n. First, we introduce a notation. Given $k \in \mathbb{N} \cup \{0\}$, we define the sequences $x_{k,0}(t)$ and $x_{k,1}(t)$ recursively by the formulas

$$x_{0,0}(t) = x(t) \text{ and } x_{k+1,0}(t) = t\frac{d}{dt}x_{k,0}(t), \text{ for } k \in \mathbb{N} \cup \{0\},$$

and

$$x_{0,1}(t) = \dot{x}(t) \text{ and } x_{k+1,1}(t) = \frac{d}{dt}(t x_{k,1}(t)), \text{ for } k \in \mathbb{N} \cup \{0\}.$$

Theorem 3.2. *Let $n \in \mathbb{N}$, $0 < a < b$ and $x : [a,b] \to \mathbb{R}$ a function of class C^{n+1}. Then,*

$${}_a\mathcal{I}_t^\alpha x(t) = \sum_{i=0}^{n} A_i(\alpha) \left(\ln \frac{t}{a}\right)^{\alpha+i} x_{i,0}(t) + \sum_{p=n+1}^{\infty} B(\alpha,p) \left(\ln \frac{t}{a}\right)^{\alpha+n-p} V_p(t)$$

with

$$\begin{aligned} A_i(\alpha) &= \frac{1}{\Gamma(\alpha+i+1)} \left[1 + \sum_{p=n-i+1}^{\infty} \frac{\Gamma(p-\alpha-n)}{\Gamma(-\alpha-i)(p-n+i)!}\right], \\ B(\alpha,p) &= \frac{\Gamma(p-\alpha-n)}{\Gamma(\alpha)\Gamma(1-\alpha)(p-n)!}, \\ V_p(t) &= \int_a^t (p-n) \left(\ln \tfrac{\tau}{a}\right)^{p-n-1} \tfrac{x(\tau)}{\tau} d\tau. \end{aligned}$$

Moreover, if we consider the approximation

$${}_a\mathcal{I}_t^\alpha x(t) \approx \sum_{i=0}^{n} A_i(\alpha,N) \left(\ln \frac{t}{a}\right)^{\alpha+i} x_{i,0}(t) + \sum_{p=n+1}^{N} B(\alpha,p) \left(\ln \frac{t}{a}\right)^{\alpha+n-p} V_p(t)$$

with $N \geq n+1$ and

$$A_i(\alpha,N) = \frac{1}{\Gamma(\alpha+i+1)} \left[1 + \sum_{p=n-i+1}^{N} \frac{\Gamma(p-\alpha-n)}{\Gamma(-\alpha-i)(p-n+i)!}\right],$$

the error is bounded by the expression

$$|E_{tr}(t)| \leq L_n(t) \frac{e^{(\alpha+n)^2+\alpha+n}}{\Gamma(\alpha+n+1)(\alpha+n)N^{\alpha+n}} \left(\ln \frac{t}{a}\right)^{\alpha+n} (t-a),$$

where

$$L_n(t) = \max_{\tau\in[a,t]} |x_{n,1}(\tau)|.$$

Proof. Applying integration by parts repeatedly and the binomial formula, we arrive to

$$\begin{aligned} {}_a\mathcal{I}_t^\alpha x(t) &= \sum_{i=0}^{n} \frac{1}{\Gamma(\alpha+i+1)} \left(\ln\frac{t}{a}\right)^{\alpha+i} x_{i,0}(a) \\ &+ \frac{1}{\Gamma(\alpha+n+1)} \left(\ln\frac{t}{a}\right)^{\alpha+n} \sum_{p=0}^{\infty} \frac{\Gamma(p-\alpha-n)}{\Gamma(-\alpha-n)p!\left(\ln\frac{t}{a}\right)^p} \int_a^t \left(\ln\frac{\tau}{a}\right)^p x_{n,1}(\tau)d\tau. \end{aligned}$$

To achieve the expansion formula, we repeat the same procedure as for the case $n=2$: we split the sum into two parts (the first term plus the remainings) and integrate by parts the second one. The convergence of the series $A_i(\alpha)$ is ensured by the relation

$$\sum_{p=n-i+1}^{\infty} \frac{\Gamma(p-\alpha-n)}{\Gamma(-\alpha-i)(p-n+i)!} = {}_1F_0(-\alpha-i,1)-1.$$

The error on the approximation is given by

$$|E_{tr}(t)| = \left| \frac{1}{\Gamma(\alpha+n+1)} \left(\ln\frac{t}{a}\right)^{\alpha+n} \int_a^t R_N(\tau) x_{n,1}(\tau)d\tau \right|$$

with

$$R_N(\tau) = \sum_{p=N+1}^{\infty} \frac{\Gamma(p-\alpha-n)}{\Gamma(-\alpha-n)p!} \frac{\left(\ln\frac{\tau}{a}\right)^p}{\left(\ln\frac{t}{a}\right)^p}.$$

Also, for $\tau \in [a,t]$,

$$|R_N(\tau)| \leq \sum_{p=N+1}^{\infty} \left| \binom{\alpha+n}{p} \right| \leq \frac{e^{(\alpha+n)^2+\alpha+n}}{(\alpha+n)N^{\alpha+n}}.$$

□

We remark that the error formula tends to zero as N increases. Similarly to what was done with the left fractional integral, we can also expand the right Hadamard fractional integral.

Theorem 3.3. *Let $n \in \mathbb{N}$, $0<a<b$ and $x:[a,b]\to\mathbb{R}$ a function of class C^{n+1}. Then,*

$${}_t\mathcal{I}_b^\alpha x(t) = \sum_{i=0}^{n} A_i(\alpha) \left(\ln\frac{b}{t}\right)^{\alpha+i} x_{i,0}(t) + \sum_{p=n+1}^{\infty} B(\alpha,p) \left(\ln\frac{b}{t}\right)^{\alpha+n-p} W_p(t)$$

with

$$\begin{aligned} A_i(\alpha) &= \frac{(-1)^i}{\Gamma(\alpha+i+1)} \left[1 + \sum_{p=n-i+1}^{\infty} \frac{\Gamma(p-\alpha-n)}{\Gamma(-\alpha-i)(p-n+i)!}\right], \\ B(\alpha,p) &= \frac{\Gamma(p-\alpha-n)}{\Gamma(\alpha)\Gamma(1-\alpha)(p-n)!}, \\ W_p(t) &= \int_t^b (p-n) \left(\ln\frac{b}{\tau}\right)^{p-n-1} \frac{x(\tau)}{\tau} d\tau. \end{aligned}$$

Remark 3.4. *Analogous to what was done for the left fractional integral, one can consider an approximation for the right fractional integral by considering finite sums in the expansion obtained in Theorem 3.3. In this case, the error is bounded by*

$$|E_{tr}(t)| \leq L_n(t) \frac{e^{(\alpha+n)^2+\alpha+n}}{\Gamma(\alpha+n+1)(\alpha+n)N^{\alpha+n}} \left(\ln \frac{b}{t}\right)^{\alpha+n} (b-t),$$

where

$$L_n(t) = \max_{\tau \in [t,b]} |x_{n,1}(\tau)|.$$

4 An expansion formula for the Hadamard fractional derivative

Starting with formulas (1) and (2), and applying similar techniques as presented in Section 3, we are able to present expansion formulas, and respectively approximation formulas with an error estimation, for the left and right Hadamard fractional derivatives. Due the restrictions on the number of pages, we will omit the details here and just exhibit the results.

Given $n \in \mathbb{N}$ and $x \in C^{n+1}[a,b]$, we have

$$\begin{aligned} {}_a\mathcal{D}_t^\alpha x(t) &= \frac{1}{\Gamma(1-\alpha)} \left(\ln \frac{t}{a}\right)^{-\alpha} x(t) + \sum_{i=1}^{n} A_i(\alpha) \left(\ln \frac{t}{a}\right)^{i-\alpha} x_{i,0}(t) \\ &+ \sum_{p=n+1}^{\infty} \left[B(\alpha,p) \left(\ln \frac{t}{a}\right)^{n-\alpha-p} V_p(t) + \frac{\Gamma(p+\alpha-n)}{\Gamma(1-\alpha)\Gamma(\alpha)(p-n)!} \left(\ln \frac{t}{a}\right)^{-\alpha} x(t) \right] \end{aligned}$$

with

$$\begin{aligned} A_i(\alpha) &= \frac{1}{\Gamma(i+1-\alpha)} \left[1 + \sum_{p=n-i+1}^{\infty} \frac{\Gamma(p+\alpha-n)}{\Gamma(\alpha-i)(p-n+i)!} \right], \quad i \in \{1,\ldots,n\}, \\ B(\alpha,p) &= \frac{\Gamma(p+\alpha-n)}{\Gamma(-\alpha)\Gamma(1+\alpha)(p-n)!}, \quad p \in \{n+1,\ldots\}, \\ V_p(t) &= \int_a^t (p-n) \left(\ln \frac{\tau}{a}\right)^{p-n-1} \frac{x(\tau)}{\tau} d\tau, \quad p \in \{n+1,\ldots\}. \end{aligned}$$

When we consider finite sums,

$${}_a\mathcal{D}_t^\alpha x(t) \approx \sum_{i=0}^{n} A_i(\alpha,N) \left(\ln \frac{t}{a}\right)^{i-\alpha} x_{i,0}(t) + \sum_{p=n+1}^{N} B(\alpha,p) \left(\ln \frac{t}{a}\right)^{n-\alpha-p} V_p(t)$$

with

$$A_i(\alpha,N) = \frac{1}{\Gamma(i+1-\alpha)} \left[1 + \sum_{p=n-i+1}^{N} \frac{\Gamma(p+\alpha-n)}{\Gamma(\alpha-i)(p-n+i)!} \right], \quad i \in \{0,\ldots,n\},$$

and the error is bounded by

$$|E_{tr}(t)| \leq L_n(t) \frac{e^{(n-\alpha)^2+n-\alpha}}{\Gamma(n+1-\alpha)(n-\alpha)N^{n-\alpha}} \left(\ln \frac{t}{a}\right)^{n-\alpha} (t-a),$$

where

$$L_n(t) = \max_{\tau \in [a,t]} |x_{n,1}(\tau)|.$$

Remark 4.1. *The series involved in the definition of A_i are convergent, for all $i \in \{1, \ldots, n\}$. This is due to the fact that*

$$\sum_{p=n-i+1}^{\infty} \frac{\Gamma(p+\alpha-n)}{\Gamma(\alpha-i)(p-n+i)!} = {}_1F_0(\alpha-i,1) - 1$$

and ${}_1F_0(\alpha-i,1)$ converges ([1, Theorem 2.1.1]), since $\alpha - i < 0$.

Remark 4.2. *For the right Hadamard fractional derivative, the expansion reads as*

$$\begin{aligned} {}_t\mathcal{D}_b^{\alpha} x(t) &= \frac{1}{\Gamma(1-\alpha)} \left(\ln \frac{b}{t}\right)^{-\alpha} x(t) + \sum_{i=1}^{n} A_i(\alpha) \left(\ln \frac{b}{t}\right)^{i-\alpha} x_{i,0}(t) \\ &\quad + \sum_{p=n+1}^{\infty} \left[B(\alpha,p) \left(\ln \frac{b}{t}\right)^{n-\alpha-p} W_p(t) + \frac{\Gamma(p+\alpha-n)}{\Gamma(1-\alpha)\Gamma(\alpha)(p-n)!} \left(\ln \frac{b}{t}\right)^{-\alpha} x(t) \right] \end{aligned}$$

with

$$\begin{aligned} A_i(\alpha) &= \frac{(-1)^i}{\Gamma(i+1-\alpha)} \left[1 + \sum_{p=n-i+1}^{\infty} \frac{\Gamma(p+\alpha-n)}{\Gamma(\alpha-i)(p-n+i)!} \right], \quad i \in \{1, \ldots, n\}, \\ B(\alpha,p) &= \frac{\Gamma(p+\alpha-n)}{\Gamma(-\alpha)\Gamma(1+\alpha)(p-n)!}, \quad p \in \{n+1, \ldots\}, \\ W_p(t) &= \int_t^b (p-n) \left(\ln \frac{b}{\tau}\right)^{p-n-1} \frac{x(\tau)}{\tau} d\tau. \end{aligned}$$

5 Examples

We obtained approximation formulas for the Hadamard fractional integrals and derivatives, and an upper bound for the error on such decompositions. In this section we will study several cases, comparing the solution with the approximations. To gather more information on the accuracy, we will evaluate the error using the distance

$$\text{dist} = \sqrt{\int_a^b (X(t) - \tilde{X}(t))^2 dt},$$

where $X(t)$ is the exact formula and $\tilde{X}(t)$ the approximation. To begin with, we consider $\alpha = 0.5$, and functions $x_1(t) = \ln t$ and $x_2(t) = 1$, with $t \in [1, 10]$. Then

$${}_1\mathcal{I}_t^{0.5} x_1(t) = \frac{\sqrt{\ln^3 t}}{\Gamma(2.5)} \text{ and } {}_1\mathcal{I}_t^{0.5} x_2(t) = \frac{\sqrt{\ln t}}{\Gamma(1.5)}$$

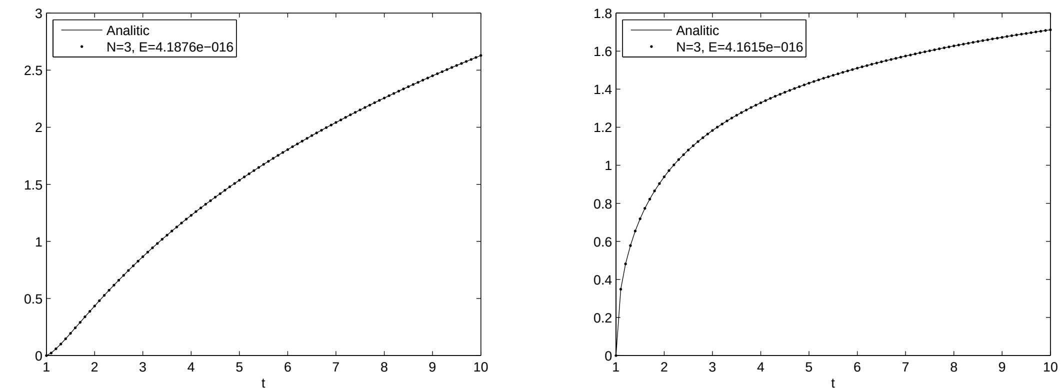
(cf. Property 2.24 [8]). We will consider the expansion formula for $n = 2$ as in (5) for both cases. We obtain then the approximations

$$ {}_1\mathcal{I}_t^{0.5} x_1(t) \approx \left[A_0(0.5,N) + A_1(0.5,N) + \sum_{p=3}^{N} B(0.5,p) \frac{p-2}{p-1} \right] \sqrt{\ln^3 t} $$

and

$$ {}_1\mathcal{I}_t^{0.5} x_2(t) \approx \left[A_0(0.5,N) + \sum_{p=3}^{N} B(0.5,p) \right] \sqrt{\ln t}. $$

The results are exemplified in Figures 1(a) and 1(b). As can be seen, the value $N = 3$ is enough in order to obtain a good accuracy.



(a) ${}_1\mathcal{I}_t^{0.5}(\ln t)$ (b) ${}_1\mathcal{I}_t^{0.5}(1)$

Figure 1: Analytic vs. numerical approximation for $n = 2$.

We now test the approximation on the power functions $x_3(t) = t^4$ and $x_4(t) = t^9$, with $t \in [1, 2]$. Observe first that

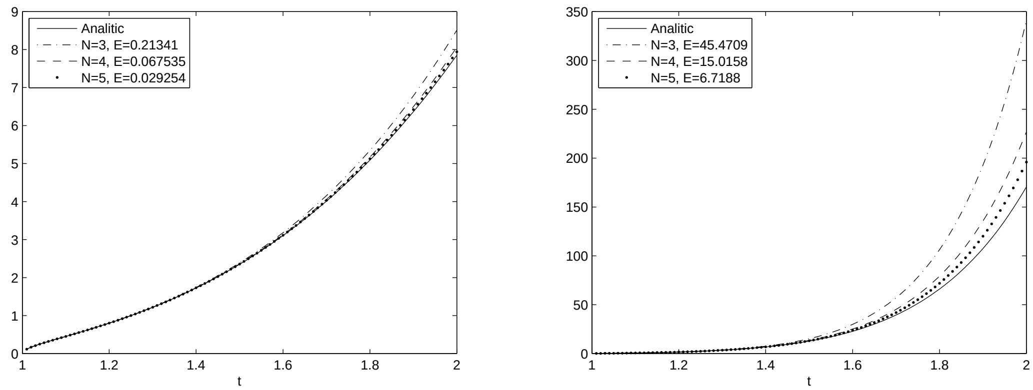
$$ {}_1\mathcal{I}_t^{0.5}(t^k) = \frac{1}{\Gamma(0.5)} \int_1^t \left(\ln \frac{t}{\tau} \right)^{-0.5} \tau^{k-1} d\tau = \frac{t^k}{\Gamma(0.5)} \int_0^{\ln t} \xi^{-0.5} e^{-\xi k} d\xi $$

by the change of variables $\xi = \ln \frac{t}{\tau}$. In our cases,

$$ {}_1\mathcal{I}_t^{0.5}(t^4) \approx \frac{0.8862269255}{\Gamma(0.5)} t^4 \text{erf}(2\sqrt{\ln t}) \text{ and } {}_1\mathcal{I}_t^{0.5}(t^9) \approx \frac{0.5908179503}{\Gamma(0.5)} t^9 \text{erf}(3\sqrt{\ln t}), $$

where $\text{erf}(\cdot)$ is the error function. In Figures 2(a) and 2(b) we show approximations for several values of N. We mention that, as N increases, the error decreases and thus we obtain a better approximation.

Another way to obtain different expansion formulas is to vary n. To exemplify, we will choose the previous test functions x_i, for $i = 1, 2, 3, 4$, and consider the cases $n = 2, 3, 4$ with



(a) ${}_1\mathcal{I}_t^{0.5}(t^4)$

(b) ${}_1\mathcal{I}_t^{0.5}(t^9)$

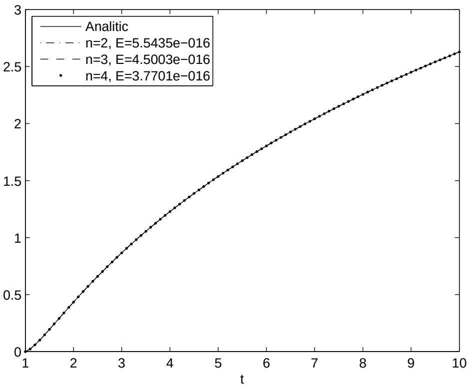
Figure 2: Analytic vs. numerical approximation for $n = 2$.

$N = 5$ fixed. The results are shown in Figures 3(a), 3(b), 3(c) and 3(d). Observe that as n increases, the error may increase. This can be easily explained by analysis of the error formula, and the values of the sequence $x_{(k,0)}$ involved. For example, for x_4 we have $x_{(k,0)}(t) = 9^k t^9$, for $k = 0 \ldots, n$. This suggests that, when we increase the value of n and the function grows fast, in order to obtain a better accuracy on the method, the value of N should also increase.

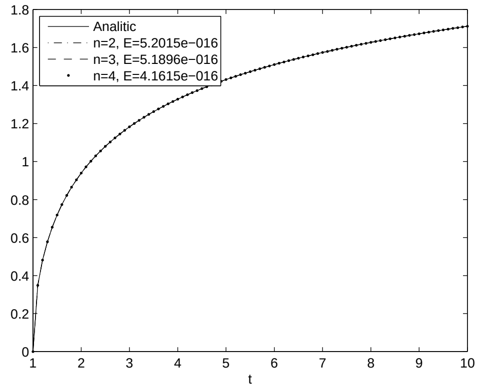
We now proceed with some examples for the Hadamard fractional derivatives. The test functions will be the same as before, and in Figures 4(a), 4(b), 4(c) and 4(d) we exemplify the results. In this case,

$$\begin{aligned}
{}_1\mathcal{D}_t^{0.5} x_1(t) &= \frac{\sqrt{\ln t}}{\Gamma(1.5)}, \\
{}_1\mathcal{D}_t^{0.5} x_2(t) &= \frac{1}{\Gamma(0.5)\sqrt{\ln t}}, \\
{}_1\mathcal{D}_t^{0.5} x_3(t) &\approx \frac{1}{\Gamma(0.5)\sqrt{\ln t}} + \frac{0.8862269255}{\Gamma(0.5)} 4t^4 \mathrm{erf}(2\sqrt{\ln t}), \\
{}_1\mathcal{D}_t^{0.5} x_4(t) &\approx \frac{1}{\Gamma(0.5)\sqrt{\ln t}} + \frac{0.5908179503}{\Gamma(0.5)} 9t^9 \mathrm{erf}(3\sqrt{\ln t}).
\end{aligned}$$

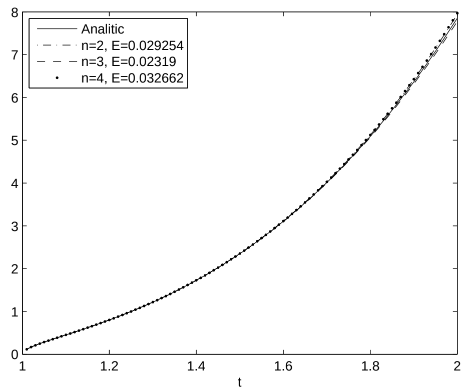
One main advantage of this method is that we can replace fractional integrals and fractional derivatives as a sum of integer derivatives, and by doing this we are rewriting the initial problem, that falls in the theory of fractional calculus, into a new one where we can apply the already known techniques (analytical or numerical) and thus solving it. For example, when in presence of a fractional integral or a fractional derivative, with a number of initial conditions, replace the fractional operator by the appropriate approximation, with the value of n given by the number of initial conditions.



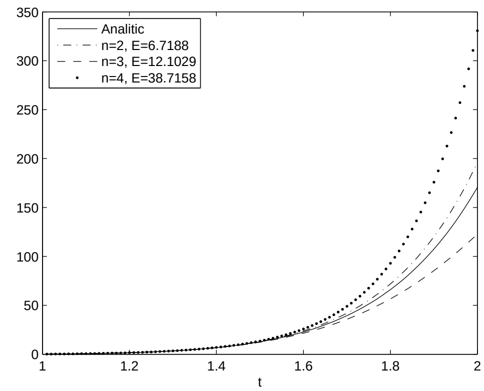
(a) ${}_1\mathcal{I}_t^{0.5}(\ln t)$



(b) ${}_1\mathcal{I}_t^{0.5}(1)$



(c) ${}_1\mathcal{I}_t^{0.5}(t^4)$



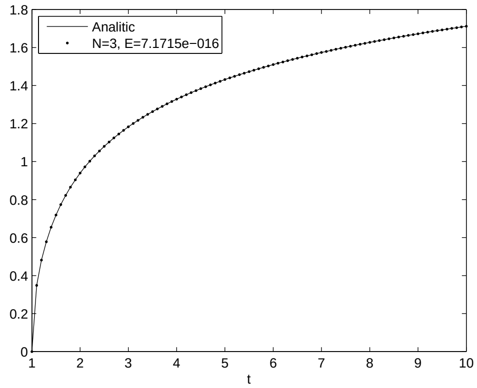
(d) ${}_1\mathcal{I}_t^{0.5}(t^9)$

Figure 3: Analytic vs. numerical approximation for $n = 2, 3, 4$ and $N = 5$.

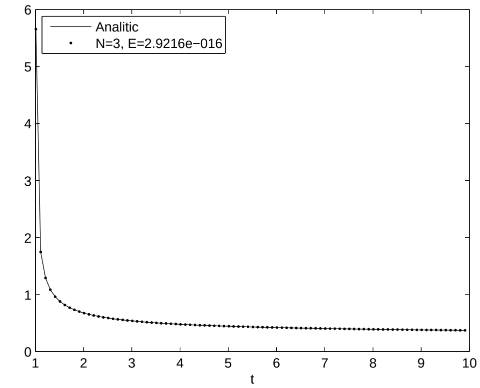
For example, consider the problem

$$\begin{cases} {}_1\mathcal{D}_t^{0.5} x(t) + x(t) = \dfrac{\sqrt{x(t)}}{\Gamma(1.5)} + \ln t \\ x(1) = 0. \end{cases} \tag{6}$$

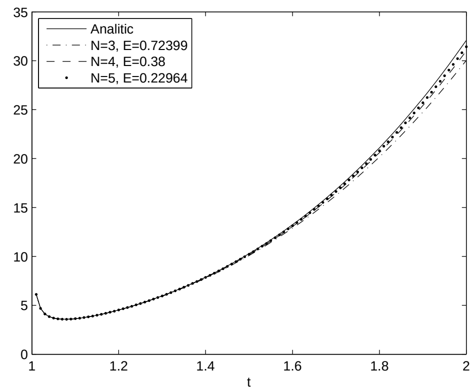
Obviously, $x(t) = \ln t$ is a solution for (6). Since we have only one initial condition, we replace



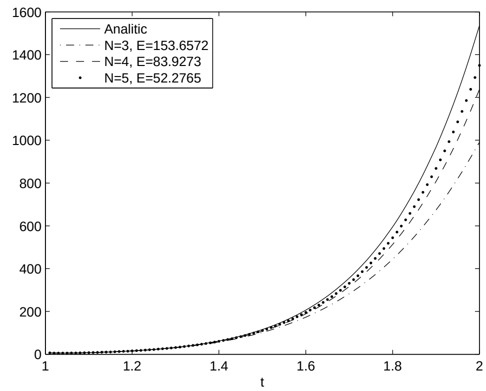
(a) ${}_1\mathcal{D}_t^{0.5}(\ln t)$



(b) ${}_1\mathcal{D}_t^{0.5}(1)$



(c) ${}_1\mathcal{D}_t^{0.5}(t^4)$



(d) ${}_1\mathcal{D}_t^{0.5}(t^9)$

Figure 4: Analytic vs. numerical approximation for $n = 2$.

the operator ${}_1\mathcal{D}_t^{0.5}(\cdot)$ by the expansion with $n = 1$ and thus obtaining

$$\begin{cases} [1 + A_0(0.5, N)(\ln t)^{-0.5}]x(t) + A_1(0.5, N)(\ln t)^{0.5} t\dot{x}(t) + \displaystyle\sum_{p=2}^{N} B(0.5, p)(\ln t)^{0.5-p} V_p(t) \\ = \dfrac{\sqrt{x(t)}}{\Gamma(1.5)} + \ln t, \\ \dot{V}_p(t) = (p-1)(\ln t)^{p-2}\dfrac{x(t)}{t}, \quad p = 2, 3, \ldots, N, \\ x(1) = 0, \\ V_p(1) = 0, \quad p = 2, 3, \ldots, N. \end{cases}$$

Below, on Figure 5, we compare the analytical solution of this FDE with the numerical result for $N = 2$.

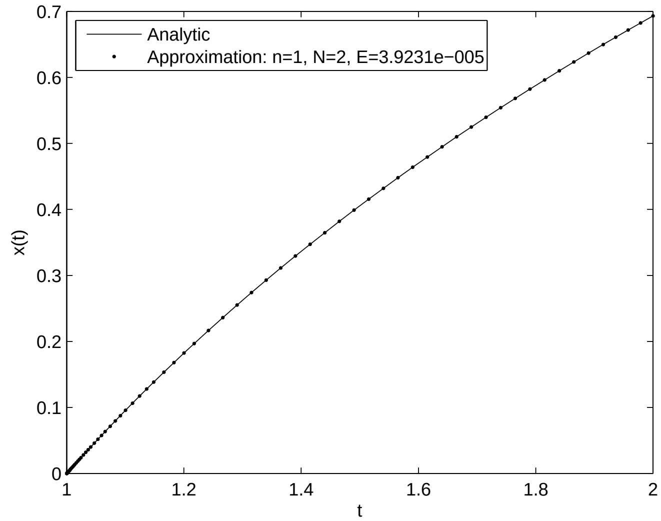


Figure 5: Analytic vs. numerical approximation for FDE with one initial condition.

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