

Robust MIMO Precoding for the Schatten Norm Based Channel Uncertainty Set

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Abstract—In this paper, we consider robust MIMO precoding designs against deterministic imperfect channel state information at the transmitter (CSIT). In contrast to the existing works based on one or two particular channel uncertainty models, we consider a general uncertainty set defined by a generic matrix norm, called the Schatten norm, which include most deterministic imperfect CSIT as special cases. Adopting the worst-case robustness, the robust MIMO precoding design is formulated as a maximin problem to maximize the worst-case received signal-to-noise ratio or minimize the worst-case error probability. We show that the robust precoder admits a channel-diagonalizing structure for the Schatten norm based uncertainty set, and then simplify the complex-matrix problem to a real-vector power allocation problem. We further show that the simplified power allocation problem can be analytically solved in a waterfilling manner, thus leading to a fully closed-form solution to the robust precoding design. Finally, we also investigate the robustness of beamforming and uniform-power transmission.

Index Terms—Imperfect CSIT, maximin, MIMO, minimax, Schatten norm, worst-case robustness.

I. INTRODUCTION

The full potential of multi-input multi-output (MIMO) communication systems relies on exploiting channel state information at the transmitter (CSIT) and adopting proper precoding techniques [1]. Given perfect CSIT, MIMO precoding designs have been well studied [2], [3]. However, in practice, CSIT is often subject to some uncertainty due to, e.g., inaccurate channel estimation, quantization of CSI, erroneous or outdated feedback, and time delays or frequency offsets between the reciprocal channels. Therefore, the imperfection of CSIT has to be considered, which calls for robust MIMO precoding designs to fully utilize CSIT and meanwhile combat against various channel uncertainty.

To characterize imperfect CSI, one common way regards that the actual channel lies in the neighborhood, often called the uncertainty set or region, of a nominal channel known by the transmitter [4]–[18]. The size of this set represents the amount of uncertainty on the channel, i.e., the larger the set is the more uncertainty there is. In this case, a precoding design is said to be robust if it can achieve the best performance in the worst channel within the uncertainty set, or equivalently can guarantee a performance level for any channel in the uncertainty set. Such robust precoding designs can be obtained

by optimizing the worst-case performance, thereby called the worst-case robustness [4].

Applying the worst-case robustness to MIMO precoding designs is an intensively scrutinized subject. Specifically, the worst-case robust minimum mean square error (MMSE) precoder was studied in [9] and later was generalized in [10] by incorporating transmit power constraints. In [11] and [12], the authors tried to maximize the worst-case received signal-to-noise ratio (SNR) but only focused on a simplified power allocation problem by fixing the transmit directions. Interestingly, it was found in [13] and [14] that the transmit directions imposed in [11], [12] are optimal in some situations, leading to fully analytical robust precoders along with some insights. The worst-case robust precoding was also studied for MIMO multiaccess channels [15], broadcasting channels [16], and cognitive radio systems [17], [18].

In these works, the channel uncertainty region is usually modelled as a sphere or ellipsoid set defined by a matrix norm, where the shape of the uncertainty set is determined by which norm is used. Two most common norms are the Frobenius norm [9]–[13], [15]–[17] and the spectral norm [8], [14], [19]. Other methods, e.g., the Kullback-Leibler divergence [7], can also be used to model an uncertainty set. Despite different types of imperfect CSIT that may be encountered in practice, most existing works only focused on one or two particular uncertainty sets, mainly based on the Frobenius and spectral norms due to their tractability.

In this paper we consider a worst-case robust MIMO precoding design, formulated as a maximin problem, to maximize the worst-case received SNR or to minimize the worst-case pairwise error probability (PEP) if a space-time block code (STBC) [20] is used. In contrast with the existing works [5]–[10], [12]–[17], [19] based on one or two particular imperfect CSI models, we try to take into account various channel uncertainty by considering a general uncertainty set defined by a generic matrix norm, termed the Schatten norm. Such a general uncertainty set contains many deterministic imperfect CSIT models as special cases. Therefore, the previous works, e.g., [11]–[14], are included as special cases in our framework.

We show that, for the Schatten norm based uncertainty set, the optimal robust precoder results in a favorable channel-diagonalizing structure, and thus simplifies the complex-matrix

problem to a power allocation problem without any loss of optimality. We further show that, solution of the simplified power problem can be analytically obtained via a convenient waterfilling fashion. Based on these results, we provide a fully closed-form solution to the robust precoder for a class of uncertainty sets. Finally, we also investigate the robustness of beamforming and equal power transmission.

II. SYSTEM MODEL

Consider a point-to-point MIMO communication system equipped with N transmit and M receive antennas. Mathematically, the system can be represented by a linear model

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} \quad (1)$$

where $\mathbf{x} \in \mathbb{C}^N$ and $\mathbf{y} \in \mathbb{C}^M$ are the transmitted and received signals, respectively, $\mathbf{H} \in \mathbb{C}^{M \times N}$ is the channel matrix, and $\mathbf{n} \in \mathbb{C}^M$ is a circularly symmetric complex Gaussian noise vector with zero mean and covariance matrix $\sigma_n^2 \mathbf{I}$, i.e., $\mathbf{n} \sim \mathcal{CN}(\mathbf{0}, \sigma_n^2 \mathbf{I})$. The transmit strategy or precoder is determined by the transmit covariance matrix $\mathbf{Q} = E\{\mathbf{x}\mathbf{x}^H\}$. Indeed, via decomposing $\mathbf{Q} = \mathbf{F}\mathbf{F}^H$, the transmitted symbol vector \mathbf{s} , with $E\{\mathbf{s}\mathbf{s}^H\} = \mathbf{I}$, can be linearly precoded by \mathbf{F} , resulting in $\mathbf{x} = \mathbf{F}\mathbf{s}$. In practice, the transmitter should satisfy the power constraint $\mathbf{Q} \in \mathcal{Q}$ where

$$\mathcal{Q} \triangleq \{\mathbf{Q} : \mathbf{Q} \succeq \mathbf{0}, \text{Tr}(\mathbf{Q}) \leq P\} \quad (2)$$

and $P \geq 0$ is the budget on the total transmit power.

Under the assumption of perfect CSIT, i.e., the channel \mathbf{H} is perfectly known at the transmitter, the optimal MIMO precoding has been well studied for various criteria [2], [3]. However, due to many practical issues, CSIT is seldom perfect, which thus calls for robust precoding designs that can utilize CSIT and at the same time combat against its imperfection. To model imperfect CSIT, we consider a compound channel model [21] assuming that \mathbf{H} belongs to a known set \mathcal{H} , often called an uncertainty set, of possible values but otherwise unknown. In the literature, this imperfect channel model has been widely used in robust designs, and the philosophy behind these robust designs is the so-called worst-case robustness [4], which is achieved by optimizing the system performance for the worst channel in \mathcal{H} [4]–[18].

Specifically, we denote the system performance measure by a utility or payoff function $\Psi(\mathbf{Q}, \mathbf{H})$. Then, the worst-case robust transmit strategy is given by the solution to the following maximin problem:

$$\max_{\mathbf{Q} \in \mathcal{Q}} \min_{\mathbf{H} \in \mathcal{H}} \Psi(\mathbf{Q}, \mathbf{H}) \quad (3)$$

which, namely, offers the best performance for the worst channel within \mathcal{H} . As a counterpart of the maximin problem, we also introduce the following minimax problem:

$$\min_{\mathbf{H} \in \mathcal{H}} \max_{\mathbf{Q} \in \mathcal{Q}} \Psi(\mathbf{Q}, \mathbf{H}) \quad (4)$$

which is, namely, to find the worst channel for the best one of all possible transmit strategies. We will show later that the

maximin problem (3) and the minimax problem (4) are closely related.

In this paper, we assume perfect CSI at the receiver (CSIR) and adopt the following payoff or utility function:

$$\Psi(\mathbf{Q}, \mathbf{H}) = \text{Tr}(\mathbf{H}\mathbf{Q}\mathbf{H}^H) \quad (5)$$

which is proportional to the received SNR. It can be verified (see [13, Sec. II]) that maximizing $\Psi(\mathbf{Q}, \mathbf{H})$ corresponds to: 1) maximizing the received SNR; 2) minimizing the pairwise error probability (PEP) if a space-time block code (STBC) [20] is used at the transmitter; 3) maximizing a low-SNR approximation of the mutual information; 4) minimizing a low-SNR approximation of the MSE if a linear MMSE equalizer is used at the receiver.

III. CHANNEL UNCERTAINTY MODEL

In the literature [4]–[18], the uncertainty set \mathcal{H} is often modeled as a neighborhood of a nominal channel $\hat{\mathbf{H}}$ known by the transmitter, where the nominal channel $\hat{\mathbf{H}}$ could be an estimate or feedback of the actual channel \mathbf{H} . By defining the channel error Δ as the difference between the nominal channel and the actual channel as $\Delta \triangleq \hat{\mathbf{H}} - \mathbf{H}$, the uncertainty $\mathbf{H} \in \mathcal{H}$ can be equally described by $\Delta \in \mathcal{E}$ for some set \mathcal{E} . Correspondingly, we can rewrite the utility function in (5) based on Δ as

$$\Psi(\mathbf{Q}, \Delta) \triangleq \text{Tr}\left((\hat{\mathbf{H}} - \Delta)\mathbf{Q}(\hat{\mathbf{H}} - \Delta)^H\right). \quad (6)$$

and thus the maximin and minimax problems (3) and (4) based on \mathcal{H} can be expressed as

$$\max_{\mathbf{Q} \in \mathcal{Q}} \min_{\Delta \in \mathcal{E}} \Psi(\mathbf{Q}, \Delta) \quad (7)$$

and

$$\min_{\Delta \in \mathcal{E}} \max_{\mathbf{Q} \in \mathcal{Q}} \Psi(\mathbf{Q}, \Delta) \quad (8)$$

based on \mathcal{E} , respectively.

The uncertainty set \mathcal{E} provides a convenient way to characterize different types of imperfect CSIT. However, most existing works on worst-case robust MIMO precoding designs, e.g., [5]–[9], [12]–[17], [19], considered only one or several particular choices of the uncertainty set \mathcal{E} . In this paper, we would like to consider a general uncertainty set based on the Schatten norm that includes many common uncertainty sets as special cases.

Definition 1. ([22]) Let $\mathbf{A} \in \mathbb{C}^{m \times n}$ with $r = \min\{m, n\}$ and $p \in [1, \infty]$. Then, the p -Schatten norm $\|\cdot\|_{\sigma p}$ is defined as

$$\|\mathbf{A}\|_{\sigma p} \triangleq \begin{cases} (\sum_{i=1}^r \sigma_i^p(\mathbf{A}))^{1/p}, & 1 \leq p < \infty \\ \sigma_{\max}(\mathbf{A}), & p = \infty. \end{cases} \quad (9)$$

Based on the p -Schatten norm $\|\cdot\|_{\sigma p}$, we define the uncertainty set

$$\mathcal{E}_{\sigma p} \triangleq \{\Delta : \|\Delta\|_{\sigma p} \leq \varepsilon\} \quad (10)$$

where, to avoid a trivial solution, we assume that $\varepsilon < \|\hat{\mathbf{H}}\|_{\sigma p}$. Different p -Schatten norms are related through

$$\|\mathbf{A}\|_{\sigma \infty} \leq \|\mathbf{A}\|_{\sigma q} \leq \|\mathbf{A}\|_{\sigma p} \leq \|\mathbf{A}\|_{\sigma 1} \quad (11)$$

where $p, q \in [1, \infty]$ and $p \leq q$, and therefore we have

$$\mathcal{E}_{\sigma_1} \subseteq \mathcal{E}_{\sigma_p} \subseteq \mathcal{E}_{\sigma_q} \subseteq \mathcal{E}_{\sigma_\infty}. \quad (12)$$

Some well-known examples of the Schatten norm include:

1) *The nuclear norm (also known as the trace norm)*

$$\|\mathbf{A}\|_* = \text{Tr} \left((\mathbf{A}^H \mathbf{A})^{1/2} \right) = \sigma_1(\mathbf{A}) + \dots + \sigma_r(\mathbf{A}) = \|\mathbf{A}\|_{\sigma_1}. \quad (13)$$

The nuclear norm is viewed as a convex approximation of the rank of a matrix, and widely used in rank minimization for sparse signal processing [23]. Hence, \mathcal{E}_{σ_1} approximately describes the uncertainty on the rank of the channel error matrix Δ . Note that \mathcal{E}_{σ_1} is the smallest one of all \mathcal{E}_{σ_p} .

2) *The Frobenius norm*

$$\begin{aligned} \|\mathbf{A}\|_F &= (\text{Tr}(\mathbf{A}^H \mathbf{A}))^{1/2} = (\sigma_1^2(\mathbf{A}) + \dots + \sigma_r^2(\mathbf{A}))^{1/2} \\ &= \|\mathbf{A}\|_{\sigma_2}. \end{aligned} \quad (14)$$

As the most frequently used model in the literature [9], [11]–[13], [15], [16], \mathcal{E}_{σ_2} represents the uncertainty on the total “power” of all elements of Δ . Meanwhile, from the probabilistic point of view, $\|\Delta\|_F^2 = \|\mathbf{H} - \hat{\mathbf{H}}\|_F^2$ is in fact a closed-form expression of the Kullback-Leibler divergence between the actual and nominal channel models with Gaussian noise [7].

3) *The spectral norm (also known as the 2-norm)*

$$\|\mathbf{A}\|_2 = \lambda_{\max}^{1/2}(\mathbf{A}^H \mathbf{A}) = \sigma_{\max}(\mathbf{A}) = \|\mathbf{A}\|_{\sigma_\infty}. \quad (15)$$

Intuitively, $\mathcal{E}_{\sigma_\infty}$ models the maximum uncertainty on each eigenmode of the channel [8], [14], [19]. Indeed, we know from (12) that, given the same error radius ε , $\mathcal{E}_{\sigma_p} \subseteq \mathcal{E}_{\sigma_\infty}$ for $p \in [1, \infty]$. Hence, $\mathcal{E}_{\sigma_\infty}$ is the most conservative one among all \mathcal{E}_{σ_p} , representing the largest channel error.

As a result of the generality of the Schatten norm, any result for \mathcal{E}_{σ_p} will be applicable to various matrix norm based uncertainty sets. In the next, we will analytically characterize the optimal robust precoder for \mathcal{E}_{σ_p} .

IV. ROBUST MIMO PRECODER

A. Eigenmode Transmission

In this subsection, we show that, for the uncertainty set \mathcal{E}_{σ_p} in (10), the optimal transmit directions, i.e., the eigenvectors of the optimal transmit covariance matrix, are just the right singular vectors of the nominal channel, meaning that eigenmode transmission is still optimal from the perspective of the worst-case robustness. Consequently, the complex-matrix robust precoding design can be simplified into a real-vector power allocation problem without loss of any optimality.

Before stating our result, we would like to introduce some notations and definitions. Denote the eigenvalue decomposition (EVD) of \mathbf{Q} by $\mathbf{Q} = \mathbf{U}_q \Lambda_q \mathbf{U}_q^H$ with eigenvalues $\mathbf{p} \triangleq \{p_i\}_{i=1}^N$. Denote the singular-value decomposition (SVD) of $\hat{\mathbf{H}}$ by $\hat{\mathbf{H}} = \mathbf{U}_h \Sigma_h \mathbf{V}_h^H$ with singular values $\{\gamma_i\}_{i=1}^N$, where $\gamma_i = 0$ for $i > \min\{M, N\}$. Denote the SVD of Δ by

$\Delta = \mathbf{U}_\delta \Sigma_\delta \mathbf{V}_\delta^H$ with singular values $\delta = \sigma(\Delta) \triangleq \{\delta_i\}_{i=1}^N$, where $\delta_i = 0$ for $i > \min\{M, N\}$. Define the set \mathcal{P} as

$$\mathcal{P} \triangleq \{\mathbf{p} : \mathbf{p} \geq \mathbf{0}, \mathbf{1}^T \mathbf{p} \leq P\}. \quad (16)$$

Define the function $f_{\sigma_p}(\delta)$ as

$$f_{\sigma_p}(\delta) \triangleq \begin{cases} \left(\sum_{i=1}^N \delta_i^p \right)^{1/p}, & 1 \leq p < \infty \\ \max_i \{\delta_i\}, & p = \infty \end{cases} \quad (17)$$

and the set \mathcal{D}_{σ_p} as

$$\mathcal{D}_{\sigma_p} \triangleq \{\delta : f_{\sigma_p}(\delta) \leq \varepsilon\}. \quad (18)$$

Then, we have the following result, whose proof is omitted due to the space limitation (the interested reader is referred to [24] for the detailed proof).

Theorem 1. *Let $\mathcal{E} = \mathcal{E}_{\sigma_p}$. Then, the following statements hold.*

- 1) *There exists a solution \mathbf{Q}^* to the maximin problem (7) such that $\mathbf{U}_q^* = \mathbf{V}_h$ and \mathbf{p}^* is the solution to the following maximin problem:*

$$\max_{\mathbf{p} \in \mathcal{P}} \min_{\delta \in \mathcal{D}_{\sigma_p}} \sum_{i=1}^N (\gamma_i - \delta_i)^2 p_i. \quad (19)$$

- 2) *There exists a solution Δ^* to the minimax problem (8) such that $\mathbf{U}_\delta^* = \mathbf{U}_h$, $\mathbf{V}_\delta^* = \mathbf{V}_h$, and δ^* is the solution to the following minimax problem:*

$$\min_{\delta \in \mathcal{D}_{\sigma_p}} \max_{\mathbf{p} \in \mathcal{P}} \sum_{i=1}^N (\gamma_i - \delta_i)^2 p_i. \quad (20)$$

Theorem 1 reveals that, for $\mathcal{E} = \mathcal{E}_{\sigma_p}$, both the robust transmit covariance matrix and the worst channel error align with the nominal channel, resulting in a fully channel-diagonalizing structure. In this case, the complex-matrix maximin and minimax problems (7) and (8) can be simplified respectively into the real-vector maximin and minimax problems (19) and (20) without loss of any optimality. Consequently, searching the complex-matrix robust precoder (or worst channel error) reduces to searching its eigenvalues (or singular values), which significantly decreases the computational complexity.

Among existing works, [11] and [12] imposed the same transmit directions but without knowing whether or when they were optimal, whereas [13] and [14] proved the optimality of the similar channel-diagonalizing structure but only for the uncertainty sets defined by the Frobenius and spectral norms, i.e., \mathcal{E}_{σ_2} and $\mathcal{E}_{\sigma_\infty}$, as the special cases of this work. We show that the eigenmode transmission is actually optimal in terms of worst-case robustness for a general class of uncertainty sets.

The goal of this paper is to find the robust precoder by solving the maximin problem (7) or (19). One may wonder what is the merit of the minimax problem (8) or (20). The following result answers this question by showing that the maximin and minimax problems are closely linked.

Theorem 2. Let δ^* be the optimal solution to the following convex problem:

$$\begin{aligned} & \underset{\delta \in \mathcal{D}_{\sigma_p, t}}{\text{minimize}} && Pt \\ & \text{subject to} && (\gamma_i - \delta_i)^2 \leq t, \quad i = 1, \dots, N \end{aligned} \quad (21)$$

and $\eta^* \triangleq \{\eta_i^*\}_{i=1}^N$ be the optimal Lagrange multipliers associated with the constraints $(\gamma_i - \delta_i)^2 \leq t$, $i = 1, \dots, N$. Then, δ^* is the optimal solution to the minimax problem (20) and η^* is the optimal solution to the maximin problem (19).

Theorem 2 implies that the robust precoder and the worst channel error can be simultaneously obtained by solving (21). Since \mathcal{D}_{σ_p} is a convex set, (21) is indeed a convex problem, meaning that it can be efficiently solved numerically. Nevertheless, in the next, we will show that the solution to (21) can be analytically obtained via a waterfilling manner.

B. Waterfilling Solution

From the definition of \mathcal{D}_{σ_p} , it is not difficult to see that a solution to the following problem is also a solution to (21):

$$\begin{aligned} & \underset{\delta}{\text{minimize}} && \max_{i=1, \dots, N} (\gamma_i - \delta_i) \\ & \text{subject to} && \delta \in \mathcal{D}, \quad 0 \leq \delta_i \leq \gamma_i, \quad \forall i \end{aligned} \quad (22)$$

where we assume without loss of generality (w.l.o.g.) that $\gamma_1 \geq \dots \geq \gamma_N$. We are particularly interested in characterizing the solution of (22) or equivalently (21) in the following two situations: 1) the coupled uncertainty set \mathcal{D}_{σ_p} for $1 \leq p < \infty$; 2) the decoupled uncertainty set $\mathcal{D}_{\sigma_\infty}$ for $p = \infty$.

Let us first consider the coupled case. Intuitively, to minimize $\max_i (\gamma_i - \delta_i)$, δ_1 should first compensate the difference $\gamma_1 - \gamma_2$, then δ_1 and δ_2 together compensate the difference $\gamma_2 - \gamma_3$ and so on. As shown in Fig. 1, the whole process is like pouring the bottle of water \mathcal{D}_{σ_p} into the container $\gamma \triangleq \{\gamma_i\}_{i=1}^N$, where the water level μ is given by $\mu = \gamma_1 - \delta_1$. Based on this waterfilling procedure, we can provide a closed-form solution to (21). For this purpose, we define $\gamma_{N+1} = 0$ and define for $k = 1, \dots, N + 1$

$$\begin{aligned} \mathbb{R}^N \ni \theta_k & \triangleq \underbrace{[\gamma_1 - \gamma_k, \gamma_2 - \gamma_k, \dots, \gamma_{k-1} - \gamma_k, 0, \dots, 0]^T}_{k-1} \\ & = \max\{\gamma - \gamma_k \mathbf{1}, \mathbf{0}\}. \end{aligned} \quad (23)$$

From this definition, one can see that $\theta_1 = \mathbf{0}$, $\theta_{N+1} = \gamma$, and $\theta_k \leq \theta_{k+1}$ so that $f_{\sigma_p}(\theta_k) \leq f_{\sigma_p}(\theta_{k+1})$. Then, we have the following result (the interested reader is referred to [24] for the detailed proof).

Theorem 3. Let $\mathcal{E} = \mathcal{E}_{\sigma_p}$ and $\varepsilon < f_{\sigma_p}(\gamma)$ for $p \in [1, \infty)$. Then, the optimal solution δ^* to (21) is given by

$$\delta_i^* = \begin{cases} \gamma_i - \mu^*, & i \leq k \\ 0, & i > k \end{cases} \quad (24)$$

where k is an integer such that

$$f_{\sigma_p}(\theta_k) < \varepsilon \leq f_{\sigma_p}(\theta_{k+1}) \quad (25)$$

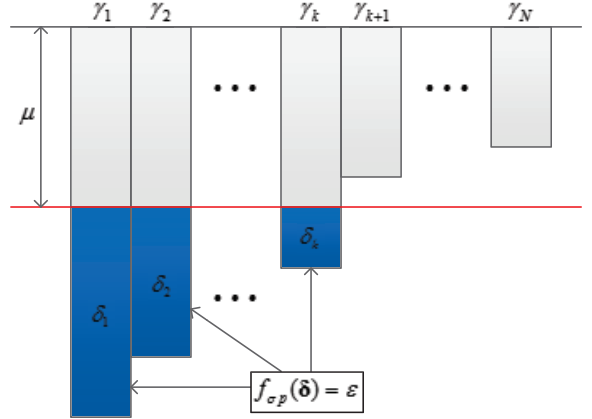


Figure 1. Waterfilling procedure for the coupled uncertainty set \mathcal{D}_{σ_p} with $p \in [1, \infty)$.

and $\mu^* \in [\gamma_{k+1}, \gamma_k)$ is the root of the equation $f_{\sigma_p}(\delta^*(\mu^*)) = \varepsilon$. The optimal power allocation \mathbf{p}^* is given by

$$p_i^* = \begin{cases} \frac{P \delta_i^{*p-1}}{\sum_{j \leq k} \delta_j^{*p-1}}, & i \leq k \\ 0, & i > k. \end{cases} \quad (26)$$

The integer k is the number of active eigenmodes and can be easily determined from (25). Since $f_{\sigma_p}(\delta)$ is an increasing function, $f_{\sigma_p}(\delta^*(\mu^*))$ is monotonically increasing in μ^* , meaning that the optimal water level μ^* can be efficiently found via the bisection method over $[\gamma_{k+1}, \gamma_k)$. In some situations μ^* may be obtained in a closed form (e.g., [13]). The assumption $\varepsilon < f_{\sigma_p}(\gamma)$ is to avoid a trivial solution, since if the uncertainty is too large, i.e., $\varepsilon \geq f_{\sigma_p}(\gamma)$, the best worst-case performance is zero.

Now we consider the decoupled case. It is easy to rewrite the constraint in $\mathcal{D}_{\sigma_\infty}$ as $\delta_i \leq \varepsilon$ for $\forall i$. The process of solving (22) is like pouring the water ε in each bottle i into each container γ_i independently. Therefore, we obtain the following result.

Theorem 4. Let $\mathcal{E} = \mathcal{E}_{\sigma_\infty}$ and $\varepsilon < f_{\sigma_\infty}(\gamma)$. The optimal solution δ^* to (21) is given by $\delta_i^* = \gamma_i$ for $i = 1, \dots, N$. The optimal power allocation \mathbf{p}^* is given by $p_1^* = P$ and $p_i^* = 0$ for $i > 1$.

From Theorems 3 and 4, we can obtain some interesting insights on the robust precoder. The following two corollaries concern the optimality of beamforming that transmits data only over one eigenmode, and the optimality of the equal power allocation.

Corollary 1. Let $\mathcal{E} = \mathcal{E}_{\sigma_p}$. The robust maximin MIMO precoding is beamforming over the largest eigenmode if either: 1) $p = \infty$; or 2) $p \in [1, \infty)$ and $\varepsilon \leq \gamma_1 - \gamma_2$.

Corollary 2. Let $\mathcal{E} = \mathcal{E}_{\sigma_p}$. The robust maximin MIMO precoding allocates power equally on the active eigenmodes if either: 1) $p = 1$; or 2) $p \in (1, \infty]$ and $\gamma_1 = \gamma_2 = \dots = \gamma_k$.

As one of the simplest transmit strategies, beamforming is often regarded to be sensitive to imperfect CSIT [2], [3]

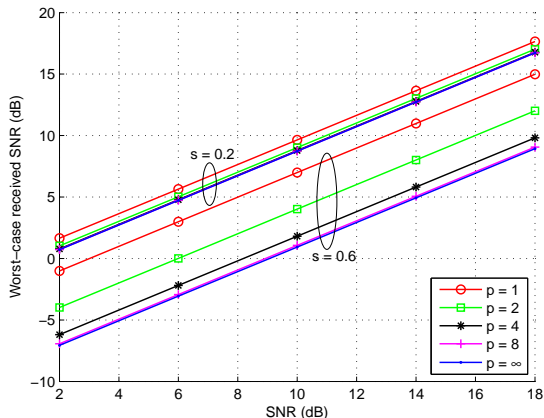


Figure 2. Average worst-case received SNRs of robust precoding strategies versus SNR for different \mathcal{E}_{σ_p} with $s = 0.2$ and 0.6 and $M = N = 4$.

because of its simplicity. However, our results reveal that beamforming is actually a robust solution if either the uncertainty set is $\mathcal{E}_{\sigma_\infty}$, or $\varepsilon \leq \gamma_1 - \gamma_2$, i.e., the uncertainty is small or the channel is nearly rank-one. As the most conservative one of \mathcal{E}_{σ_p} , $\mathcal{E}_{\sigma_\infty}$ defines the maximum uncertainty on each eigenmode independently, so the robust transmit strategy shall, intuitively, put all power on the strongest eigenmode. Furthermore, one can imagine that, when the channel uncertainty or the size of the channel matrix becomes smaller, the gap between $\mathcal{E}_{\sigma_\infty}$ and other uncertainty sets shall become smaller as well. Therefore, we can reasonably infer that beamforming, although might not be optimally robust, is a nearly robust transmit strategy when the channel uncertainty or the channel dimension is small.

The uncertainty set \mathcal{E}_{σ_1} represents another extreme case of \mathcal{E}_{σ_p} , as it is the smallest one and thus the least conservative one of \mathcal{E}_{σ_p} . Since \mathcal{E}_{σ_1} approximately models the uncertainty on the rank of the channel error, the robust transmit strategy may not distinguish the uncertainty on each eigenmode but treats all active eigenmodes equally, thus leading to an equal power allocation over the active eigenmodes. The number of active eigenmodes k , however, is determined by the total uncertainty, and especially $k \rightarrow \text{rank}(\hat{\mathbf{H}})$ as $\varepsilon \rightarrow f_{\sigma_p}(\gamma)$ for $p \in [1, \infty)$. For \mathcal{E}_{σ_p} with $p \in (1, \infty]$, the equal power allocation is generally not robust unless the channel gains of the active eigenmodes are all equal.

V. NUMERICAL RESULTS

In this section, we demonstrate the effect of the robust MIMO precoding through several numerical examples. According to the philosophy of worst-case robustness, different precoding strategies are compared via their worst-case performance, where the worst channel error for any given (either non-robust or robust) precoder can be obtained by solving the inner minimization of (7) for a fixed \mathbf{Q} (note that the robust strategy and its worst channel error can be simultaneously obtained according to this work). Moreover, to take into account different channels, the worst-case performance is averaged over the nominal channel $\hat{\mathbf{H}}$, whose elements

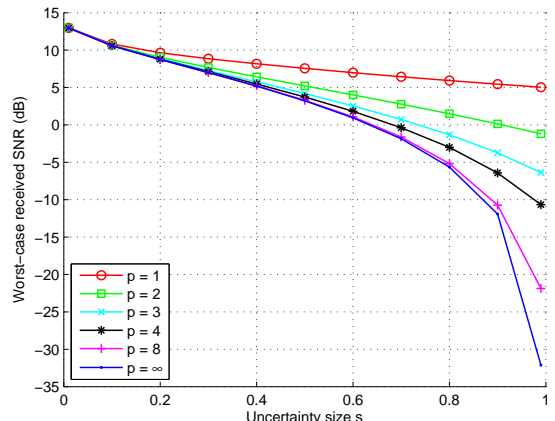


Figure 3. Average worst-case received SNRs of robust precoding strategies versus uncertainty set size s for different \mathcal{E}_{σ_p} with SNR = 10dB and $M = N = 4$.

are randomly generated according to zero-mean, unit-variance, i.i.d. Gaussian distributions.

Consider the channel uncertainty set \mathcal{E}_{σ_p} defined in (10) based on the Schatten norm, where $p \in [1, \infty]$ determines the shape of \mathcal{E}_{σ_p} , while the size of the uncertainty set is given by the error radius ε . As $\|\cdot\|_{\sigma_\infty}$ (or $\|\cdot\|_2$) is the smallest one among all p -Schatten norms, we set a common error radius for all \mathcal{E}_{σ_p} such that $\varepsilon^2 = s \|\hat{\mathbf{H}}\|_2^2$ with $s \in (0, 1)$, so they can be reasonably compared. As shown in Section III, given the same error radius, the larger the parameter p is, the bigger and thus the more conservative the uncertainty set \mathcal{E}_{σ_p} is.

Fig. 2 shows the average worst-case received SNRs, achieved by the robust precoding strategies for different \mathcal{E}_{σ_p} , versus SNR, while Fig. 3 displays the relation between the average worst-case received SNR and the uncertainty set size s . In Figs. 4 and 5, we plot the average worst-case symbol error rates (SERs), achieved by the robust precoding strategies for different \mathcal{E}_{σ_p} , versus SNR and the uncertainty set size s , respectively, where we have used an 1/2-rate complex OSTBC [20] and an ML decoder at the receiver.

From these figures, one can observe tradeoff between the conservativeness of the uncertainty model and the system performance, i.e., the more conservative the uncertainty model is, the lower the performance is. Among all \mathcal{E}_{σ_p} with $p \in [1, \infty]$, $\mathcal{E}_{\sigma_\infty}$ is the most conservative set, thus resulting in the lowest worst-case received SNR, whereas \mathcal{E}_{σ_1} is the least conservative one, thus leading to the highest performance. In practice, the choice of an uncertainty set depends on the prediction of channel errors—large errors correspond to more conservative uncertainty sets, while small errors correspond to less conservative sets.

VI. CONCLUSION

We have considered a robust MIMO precoding design, which was formulated as a maximin problem, to maximize the worst-case received SNR or minimize the worst-case PEP for STBCs with imperfect CSIT. To take into account various kinds of channel uncertainty, we have considered a

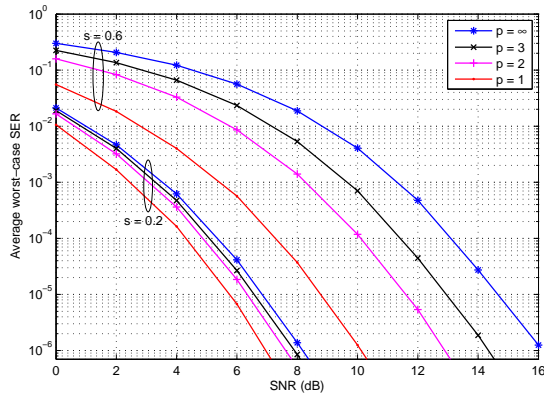


Figure 4. Average worst-case SERs of robust precoding strategies versus SNR for different $\mathcal{E}_{\sigma p}$ with QPSK modulation, 1/2-rate OSTBC, $s = 0.2$ and 0.6 , and $M = N = 4$.

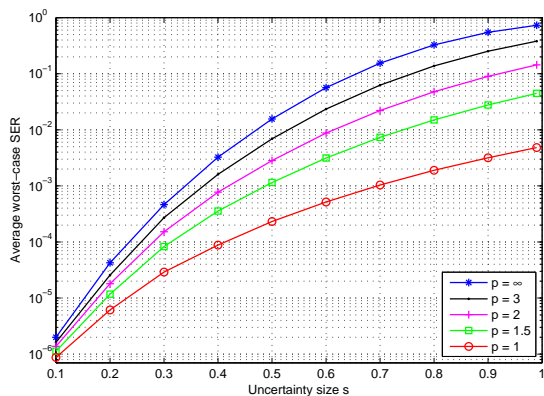


Figure 5. Average worst-case SERs of robust precoding strategies versus uncertainty set size s for different $\mathcal{E}_{\sigma p}$ with QPSK modulation, 1/2-rate OSTBC, SNR = 6dB, and $M = N = 4$.

class of uncertainty sets defined by the Schatten norm, which cover most commonly used uncertainty models as special cases. We have characterized the structure of the optimal robust precoder and simplified the matrix problem to a power allocation problem. We then related the maximin and minimax problems, and showed that the robust precoder and the worst channel error can be simultaneously obtained by solving a convex problem. Furthermore, we have provided a closed-form solution to the simplified power allocation problem through a waterfilling fashion, and also investigated the robustness of some common transmit strategies such as beamforming and equal power transmission.

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