

OPTIMAL CONVERGENCE RATES AND ONE-TERM EDGEWORTH EXPANSIONS FOR MULTIDIMENSIONAL FUNCTIONALS OF GAUSSIAN FIELDS

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ABSTRACT. We develop techniques for determining the exact asymptotic speed of convergence in the multidimensional normal approximation of smooth functionals of Gaussian fields. As a by-product, our findings yield exact limits and often give rise to one-term generalized Edgeworth expansions increasing the speed of convergence. Our main mathematical tools are Malliavin calculus, Stein's method and the Fourth Moment Theorem. This work can be seen as an extension of the results of [NP09a] to the multi-dimensional case, with the notable difference that in our framework covariances are allowed to fluctuate. We apply our findings to exploding functionals of Brownian sheets, vectors of Toeplitz quadratic functionals and the Breuer-Major Theorem.

1. INTRODUCTION

Let X be an isonormal Gaussian process on some real, separable Hilbert space \mathcal{H} and (F_n) be a sequence of centered, real-valued functionals of X with converging covariances. Moreover, assume that $F_n \xrightarrow{\mathcal{L}} Z$, where Z is a centered Gaussian random variable and $\xrightarrow{\mathcal{L}}$ denotes convergence in law. In [NP09b], Nourdin and Peccati used a combination of Stein's method (see [NP12], [CGS11], [CS05], [Rei05], [Ste86], [Ste72]) and Malliavin calculus (see [NP12], [Nua06], [Jan97]) to derive the bound

$$(1.1) \quad |\mathbb{E}[g(F_n)] - \mathbb{E}[g(Z)]| \leq M(g) \varphi(F_n)$$

and used it to prove estimates for several probabilistic distances $d(F_n, Z)$ (among them the Fortet-Mourier, Kolmogorov and Wasserstein distances). The quantity $\varphi(F_n)$ in the bound (1.1) is defined by

$$\varphi(F_n) = \sqrt{\text{Var} \langle DF_n, -DL^{-1}F_n \rangle_{\mathfrak{H}} + |\mathbb{E}[F_n^2] - \mathbb{E}[Z^2]|},$$

where g is a sufficiently smooth function, $M(g)$ is a constant depending on g and the random variable $\langle DF_n, -DL^{-1}F_n \rangle_{\mathfrak{H}}$ involves the Malliavin derivative operator D and the pseudo-inverse L^{-1} of the Ornstein Uhlenbeck generator L (see for example [NP12] or [Nua06] for definitions).

Date: November 12, 2013.

2010 Mathematics Subject Classification. 60F05, 62E20, 60H07.

Key words and phrases. Malliavin calculus, Stein's method, Gaussian approximation, Edgeworth expansion, optimality, multiple integral, contraction.

Part of this research was supported by the Fonds National de la Recherche Luxembourg under the accompanying measure AM2b.

This approach was pushed further by the same two authors in [NP09a]. Disregarding technicalities, they showed that if

$$\left(F_n, \frac{\langle DF_n, -DL^{-1}F_n \rangle_{\mathfrak{H}} - \mathbb{E} \langle DF_n, -DL^{-1}F_n \rangle_{\mathfrak{H}}}{\sqrt{\text{Var} \langle DF_n, -DL^{-1}F_n \rangle_{\mathfrak{H}}}} \right)$$

jointly converges in law to a Gaussian random vector (Z_1, Z_2) , it holds that

$$(1.2) \quad \frac{P(F_n \leq z) - \Phi(z)}{\varphi(F_n)} = \frac{\mathbb{E} [1_{(-\infty, z]}(F_n)] - \Phi(z)}{\varphi(F_n)} \rightarrow \frac{\rho}{3} \Phi^{(3)}(z),$$

where $\rho = \mathbb{E} [Z_1 Z_2]$ and Φ is the cumulative distribution function of the Gaussian random variable Z and $\Phi^{(3)}$ denotes its third derivative, thus providing exact asymptotics for the difference $P(F_n \leq z) - \Phi(z)$.

Recently, Nourdin, Peccati and Réveillac showed in [NPR10b] that the bound (1.1) also has a multidimensional version. It can still be written as

$$(1.3) \quad |\mathbb{E} [g(F_n)] - \mathbb{E} [g(Z)]| \leq M(g) \varphi(F_n)$$

but now the functionals F_n and the normal Z are \mathbb{R}^d -valued and the function g has to be in $\mathcal{C}^2(\mathbb{R}^d)$ with bounded first and second derivatives. Moreover, the quantities $\varphi(F_n)$ are now given by $\varphi(F_n) = \Delta_\Gamma(F_n) + \Delta_C(F_n)$, where

$$\Delta_\Gamma(F_n) = \sqrt{\sum_{i,j=1}^d \text{Var} \Gamma_{ij}(F_n)},$$

$$\Delta_C(F_n) = \sqrt{\sum_{i,j=1}^d (\mathbb{E} [F_{i,n} F_{j,n}] - \mathbb{E} [Z_i Z_j])^2}$$

and $\Gamma_{ij}(F_n) = \langle DF_{i,n}, -DL^{-1}F_{j,n} \rangle_{\mathfrak{H}}$. As every Lipschitz function can be approximated by \mathcal{C}^2 functions with bounded derivatives up to order two, (1.3) yields an upper bound for the Wasserstein-distance (see [NPR10b]), which, to the knowledge of the author, is the strongest distance that has been achieved via an approach based on Stein's method (see the discussion before Theorem 4 in [CM08]). One should note that, using methods of Malliavin calculus, it is however possible to prove that, in several cases, the central limit theorems implied by the bound (1.3) take place in the total variation distance (see [NP13b, Theorem 5.2]). One should also note that another bound for the difference on the left hand side of (1.1) is given by the maximum of the third and fourth cumulants of F_n (see [BBNP12]) and that this bound is in fact optimal in total variation distance, if the sequence (F_n) lives in a fixed Wiener chaos (see [NP13a]).

The main result of this paper is Theorem 3.2, which provides exact asymptotics for the difference

$$(1.4) \quad \frac{\mathbb{E} [g(F_n)] - \mathbb{E} [g(Z_n)]}{\varphi(F_n)},$$

where (Z_n) is a sequence of Gaussian random vectors that has the same covariance structure as (F_n) . Analogously to the one-dimensional case, the random sequences $(F_n, \tilde{\Gamma}_{ij}(F_n))_{n \geq 1}$, where $\tilde{\Gamma}_{ij}(F_n)$ is a normalized version of $\Gamma_{ij}(F_n)$, will play a crucial role.

Assuming converging covariances, we are able to obtain an exact and explicit limit for the quantity (1.4), where the Gaussian sequence (Z_n) is replaced by a single Gaussian vector Z . This is Theorem 3.4, which can be seen as a multi-dimensional analogue to (1.2). As a by-product, we obtain the optimality of $\varphi(F_n)$ for the Wasserstein distance d_W , by which we mean the existence of positive constants c_1 and c_2 such that

$$c_1 \leq \frac{d_W(F_n, Z)}{\varphi(F_n)} \leq c_2$$

for $n \geq n_0$. Note that the mere existence of these constants is not hard to prove. Indeed, a suitable upper bound c_2 can always be obtained from (1.3) and by choosing g in (1.3) to depend only on one coordinate, the problem of finding lower bounds can essentially be reduced to the one-dimensional findings of [NP09a].

Taking these results a step further, we provide one-term generalized Edgeworth expansions that speed up the convergence of $(\mathbb{E}[g(F_n)] - \mathbb{E}[g(Z_n)])$ (or the respective sequence with Z_n replaced by Z in the converging variances case) towards zero.

As an important special case, we apply Theorems 3.2 and 3.4 and their implications to random sequences (F_n) , whose components are elements of some Wiener chaos (that can vary by component). In this case, the sufficient conditions for our results simplify substantially and can exclusively be expressed in terms of contractions of the respective kernels (or even cumulants in the case of the second chaos). In many cases, the only contractions one has to look at are those where all kernels are taken from the same component of F_n , in the spirit of part (B) of the Fourth Moment Theorem 2.8.

The remainder of the paper is organized as follows. In the preliminary Section 2, we introduce the necessary mathematical theory and gather some results from the existing literature. Our main results in a general framework are presented in Section 3. In the following Section 4, these results are then specialized to the case where all components of (F_n) are multiple integrals. We conclude by applying our methods to several examples, namely step functions, exploding integrals of Brownian sheets, continuous time Toeplitz quadratic functionals and the Breuer-Major Theorem.

2. PRELIMINARIES

2.1. Metrics for probability measures and asymptotic normality. We fix a positive integer d and denote by $\mathcal{P}(\mathbb{R}^d)$ the set of all probability measures on \mathbb{R}^d . If X is a \mathbb{R}^d -valued random vector, we denote its law by P_X . If $(P_n) \subset \mathcal{P}(\mathbb{R}^d)$ is a sequence of probability measures, weakly converging to some limit P , we can always find an almost surely converging sequence (X_n) of \mathbb{R}^d -valued random vectors, such that X_n has law P_n . This is the well-known Skorokhod representation theorem, which we will state here for convenience.

Theorem 2.1 (Skorokhod representation theorem, [Sko56]). *Let $(P_n)_{n \geq 0} \subset \mathcal{P}(\mathbb{R}^d)$ be a sequence of probability measures such that $P_n \xrightarrow{\mathcal{L}} P_0$. Then there exists a sequence $(X_n)_{n \geq 0}$ of \mathbb{R}^d -valued random vectors, defined on some common probability space $(\Omega^*, \mathcal{F}^*, P^*)$, such that $P_{X_n} = P_n$ and $X_n \rightarrow X$ P -almost surely.*

Given a metric γ on $\mathcal{P}(\mathbb{R}^d)$, we say that γ metrizes the weak convergence on $\mathcal{P}(\mathbb{R}^d)$, if for all $P \in \mathcal{P}(\mathbb{R}^d)$ and sequences $(P_n) \subseteq \mathcal{P}(\mathbb{R}^d)$ the following

equivalence holds:

$$\gamma(P_n, P) \rightarrow 0 \quad \Leftrightarrow \quad P_n \xrightarrow{\mathcal{L}} P.$$

Two prominent examples are the Prokhorov metric ρ and the Fortet-Mourier metric β , defined by

$$\rho(P, Q) = \inf \left\{ \varepsilon > 0: P(A) \leq Q(A^\varepsilon) + \varepsilon \quad \text{for every Borel set } A \subseteq \mathbb{R}^d \right\},$$

and

$$\beta(P, Q) = \sup \left\{ \left| \int f \, d(P - Q) \right| : \|f\|_\infty + \|f\|_L \leq 1 \right\}.$$

Here, $A^\varepsilon = \{x: \|x - y\| \leq \varepsilon \text{ for some } y \in A\}$, $\|\cdot\|$ is the ε -hull with respect to the Euclidean norm and $\|\cdot\|_L$ denotes the Lipschitz seminorm. For double sequences of probability measures whose elements are asymptotically close with respect to one of these two metrics, a result similar to the Skorokhod Representation Theorem 2.1 holds.

Theorem 2.2 ([Dud02], Th. 11.7.1). *Let $(P_n)_{n \geq 1}$, $(Q_n)_{n \geq 1} \subseteq \mathcal{P}(\mathbb{R}^d)$ be two sequences of probability measures. Then the following three conditions are equivalent.*

- a) $\beta(P_n, Q_n) \rightarrow 0$
- b) $\rho(P_n, Q_n) \rightarrow 0$
- c) *There exist two sequences (X_n) and (Y_n) of \mathbb{R}^d -valued random vectors, defined on some common probability space $(\Omega^*, \mathcal{F}^*, P^*)$, such that $P_{X_n} = P_n$ and $P_{Y_n} = Q_n$ for $n \geq 1$ and $X_n - Y_n \rightarrow 0$ P -almost surely.*

Note that the Skorokhod Representation Theorem 2.1 is not a simple corollary of Theorem 2.2. Also, the distances β and ρ can not easily be replaced by other metrics (see [Dud02, p.418] for details and counterexamples). Theorem 2.2 is the motivation for our following definition of asymptotically close normality.

Definition 2.3. Let (X_n) be a sequence of \mathbb{R}^d -valued random vectors with finite first and second moments. We say that (X_n) is *asymptotically close to normal* (in short: ACN), if

$$\beta(P_{X_n}, P_{Z_n}) \rightarrow 0,$$

where the probability measures P_{Z_n} are laws of d -dimensional Gaussian random variables Z_n , whose first and second moments coincide with those of X_n .

Note that we consider (almost surely) constant random vectors as being “degenerated” Gaussians. Thus, by the above definition, all sequences of random vectors whose second moments eventually vanish are ACN. By Theorem 2.2, we could of course replace the Fortet-Mourier metric β with the Prokhorov metric ρ . It is clear that if (X_n) is ACN, the same is true for any of its components $(X_{i,n})$. Furthermore, if all first and second moments of (X_n) converge (or, as a special case, are equal), being ACN is equivalent to converging in law to a Gaussian random variable Z (with the limiting moments as parameters). Indeed, the triangle inequality gives

$$\rho(P_{X_n}, P_Z) \leq \rho(P_{X_n}, P_{Z_n}) + \rho(P_{Z_n}, P_Z).$$

We will use the following asymptotic notation for two positive sequences (a_n) and (b_n) throughout the text. We write $(a_n) \preceq (b_n)$, if there exists a positive constant c such that $a_n \leq c b_n$ for $n \geq n_0$ and $(a_n) \asymp (b_n)$, if $(a_n) \preceq (b_n)$ and

$(b_n) \asymp (a_n)$ holds. For brevity, we often drop the braces and just write $a_n \asymp b_n$, $a_n \asymp b_n$, etc.

2.2. Hermite polynomials, integration by parts and the transformation $U_{g,C}$. Fix a positive integer d . Elements of the set \mathbb{N}_0^d , where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, will be called (d -dimensional) multi-indices. We define $|\alpha| = \sum_{i=1}^d \alpha_i$ and call this sum the order of α . For a d -dimensional vector $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we define $x^\alpha := \prod_{i=1}^d x_i^{\alpha_i}$. Multi-indices of order one will sometimes be denoted by e_i , where the index i marks the position of the non-zero entry. Thus, for example, $x^{e_i} = x_i$. It is clear that every multi-index α can be written as a sum of $|\alpha|$ multi-indices $l_1, \dots, l_{|\alpha|}$ of order one, and that this sum is unique up to the order of the summands. We will call the set $\{l_1, \dots, l_{|\alpha|}\}$ of these multi-indices the *elementary decomposition* of α . For example, the elementary decomposition for the multi-index $(2, 0, 1)$ is $\{(1, 0, 0), (1, 0, 0), (0, 0, 1)\}$.

For any multi-index α , the multidimensional *Hermite polynomials* $H_\alpha(x, \mu, C)$ are defined by

$$(2.1) \quad H_\alpha(x, \mu, C) = \frac{(-1)^{|\alpha|} \partial_\alpha \phi_d(x, \mu, C)}{\phi_d(x, \mu, C)},$$

where $\phi_d(x, \mu, C)$ denotes the density of a d -dimensional Gaussian random variable with mean vector μ and positive definite covariance matrix C (see for example [McC87, Section 5.4]). Note that in the case $\mu = 0$ and $d = C = 1$, this definition yields the well known one-dimensional Hermite polynomials. The first few multidimensional Hermite polynomials are given by $H_0(x, \mu, C) = 1$,

$$H_{e_i}(x, \mu, C) = \sum_{k=1}^d c_{ik} (x_k - \mu_k)$$

and

$$H_{e_i+e_j}(x, \mu, C) = H_{e_i}(x, \mu, C) H_{e_j}(x, \mu, C) - c_{ij},$$

where $1 \leq i, j \leq d$ and $C^{-1} = (c_{ij})_{1 \leq i, j \leq d}$ denotes the inverse of C .

The polynomial $H_\alpha(x, \mu, C)$ is of order $|\alpha|$ and one can show that for fixed μ and C , the family $\{H_\alpha(x, \mu, C) : \alpha \in \mathbb{N}_0^d\}$ is orthogonal in $L^2(\mathbb{R}^d, \phi_d(x, \mu, C))$. Furthermore, by integration by parts, we obtain the identity

$$(2.2) \quad \mathbb{E} [\partial_i f(Z) H_\alpha(Z, \mu, C)] = \mathbb{E} [f(Z) H_{\alpha+e_i}(Z, \mu, C)],$$

where $1 \leq i \leq d$, $\alpha \in \mathbb{N}_0^d$, $f \in \text{Lip}(\mathbb{R}^d)$ with at most polynomial growth and Z is a Gaussian random variable with mean μ and covariance matrix C . Note that the left hand side of (2.2) is well-defined by Rademacher's theorem, which guarantees the differentiability of the Lipschitz continuous function f almost everywhere. We will also make use of another integration by parts formula, which can be verified by direct calculation, namely

$$(2.3) \quad \mathbb{E} [f(Z) Z_i] = \sum_{j=1}^d \mathbb{E} [Z_i Z_j] \mathbb{E} [\partial_j f(Z)],$$

where $1 \leq i \leq d$, f as above and Z a d -dimensional Gaussian random variable (with possibly singular covariance matrix).

For a given Lipschitz function $g: \mathbb{R}^d \rightarrow \mathbb{R}$ and a positive semi-definite and symmetric matrix C of dimension $d \times d$, we define $U_{g,C}: \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$(2.4) \quad U_{g,C}(x) = \int_a^b \frac{v'(t)}{v(t)} \left(\mathbb{E}[g(N)] - \mathbb{E} \left[g \left(v(t)x + \sqrt{1-v^2(t)}N \right) \right] \right) dt,$$

where N is a d -dimensional centered Gaussian random variable with covariance C , $-\infty \leq a < b \leq \infty$ and $v: (a, b) \rightarrow (0, 1)$ is a diffeomorphism with $\lim_{t \rightarrow a^+} v(t) = 0$ (and therefore $\lim_{t \rightarrow b^-} v(t) = 1$). From the change of variables $v(t) = s$, we see that $U_{g,C}$ does not depend on the particular choice of v and by choosing $v(t) = e^{-t}$ on the interval $(0, \infty)$, we can write

$$U_{g,C}(x) = \int_0^\infty (P_t g(x) - P_\infty g(x)) dt,$$

where $P_t g(x) = \mathbb{E} \left[g \left(e^{-t}x + \sqrt{1-e^{-2t}}N \right) \right]$, $P_\infty g(x) := \lim_{t \rightarrow \infty} P_t g(x) = \mathbb{E}[g(N)]$ and N is defined as above. The operators P_t form the well-known Ornstein-Uhlenbeck semigroup on \mathbb{R}^d (see [NP12, Chapter 1] for details).

Before stating some properties of $U_{g,C}$, let us introduce some more notation. If $f \in C^k(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}_0^d$ is a multi-index with elementary decomposition $\{l_1, l_2, \dots, l_{|\alpha|}\}$, we write $\partial_\alpha f$ or $\partial_{l_1 l_2 \dots l_{|\alpha|}} f$ instead of the more cumbersome $\frac{\partial^{|\alpha|} f}{\partial x_{l_1} \partial x_{l_2} \dots \partial x_{l_{|\alpha|}}}$.

Lemma 2.4. *Let $g: \mathbb{R}^d \rightarrow \mathbb{R}$ be a Lipschitz-function with at most polynomial growth. Furthermore, let Z be a centered, d -dimensional Gaussian random variable with covariance matrix C and define $U_{g,C}$ via (2.4). Then the following is true.*

a) *The function $U_{g,C}$ satisfies the multidimensional Stein equation*

$$\langle C, \text{Hess } U_{g,C}(x) \rangle_{\text{H.S.}} - \langle x, \nabla U_{g,C}(x) \rangle_{\mathbb{R}^d} = g(x) - \mathbb{E}[g(Z)].$$

b) *If g is k -times differentiable with bounded derivatives up to order k , the same is true for $U_{g,C}$. In this case, for any $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$, the derivatives are given by*

$$(2.5) \quad \partial_\alpha U_{g,C}(x) = \int_a^b v'(t) v^{|\alpha|-1}(t) \mathbb{E} \left[\partial_\alpha g \left(v(t)x + \sqrt{1-v^2(t)}N \right) \right] dt$$

and it holds that

$$(2.6) \quad |\partial_\alpha U_{g,C}(x)| \leq \frac{\|\partial_\alpha g\|_\infty}{|\alpha|}$$

and

$$(2.7) \quad \mathbb{E}[\partial_\alpha U_{g,C}(Z)] = \frac{1}{|\alpha|} \mathbb{E}[\partial_\alpha g(Z)].$$

Proof. For a proof of part a) see [NPR10b]. Repeated differentiation under the integral sign (the first one being justified by the Lipschitz property of g) shows formula (2.5), of which the bound (2.6) is an immediate consequence. To show (2.7), we again use formula (2.5) and the fact that $v(t)Z + \sqrt{1-v^2(t)}N$ has the same

law as Z . This gives

$$\begin{aligned} \mathbb{E} [\partial_\alpha U_{g,C}(Z)] &= \int_a^b v'(t)v^{|\alpha|-1}(t) \mathbb{E} \left[\partial_\alpha g \left(v(t)Z + \sqrt{1-v^2(t)}N \right) \right] dt \\ &= \int_a^b v'(t)v^{|\alpha|-1}(t) dt \mathbb{E} [\partial_\alpha g(Z)] \\ &= \frac{1}{|\alpha|} \mathbb{E} [\partial_\alpha g(Z)]. \end{aligned}$$

□

2.3. Isonormal Gaussian processes and Wiener chaos. For a detailed discussion of the notions introduced in this section, we refer to [NP12] or [Nua06].

Fix a real separable Hilbert space \mathfrak{H} and a family $X = \{X(h) : h \in \mathfrak{H}\}$ of centered Gaussian random variables, defined on some probability space (Ω, \mathcal{F}, P) , such that the isometry property $\mathbb{E}[X(g)X(h)] = \langle g, h \rangle_{\mathfrak{H}}$ holds for $g, h \in \mathfrak{H}$. Such a family X is called an *isonormal Gaussian process* over \mathfrak{H} . Without loss of generality, we assume that the σ -field \mathcal{F} is generated by X . For $q \geq 1$, we denote the q th tensor product of \mathfrak{H} by $\mathfrak{H}^{\otimes q}$ and the q th symmetric tensor product of \mathfrak{H} by $\mathfrak{H}^{\odot q}$. Furthermore, we define \mathcal{H}_q , the *Wiener chaos of order q* (with respect to X), to be the closed linear subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the set $\{H_q(X(h)) : h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where $H_q(x) = H_q(x, 1)$ denotes the q th Hermite polynomial, defined by (2.1). The mapping $I_q(h^{\otimes q}) = q!H_q(X(h))$, where $\|h\|_{\mathfrak{H}} = 1$, can be extended to a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot q}$, equipped with the modified norm $\sqrt{q!} \|\cdot\|_{\mathfrak{H}^{\otimes q}}$, and the q th Wiener chaos \mathcal{H}_q . Wiener chaoses of different orders are orthogonal. More precisely, if $f_i \in \mathfrak{H}^{\odot q_i}$ and $f_j \in \mathfrak{H}^{\odot q_j}$ for $q_i, q_j \geq 1$ it holds that

$$(2.8) \quad \mathbb{E} [I_{q_i}(f_i) I_{q_j}(f_j)] = \begin{cases} q_i! \langle f_i, f_j \rangle_{\mathfrak{H}} & \text{if } q_i = q_j \\ 0 & \text{if } q_i \neq q_j. \end{cases}$$

Furthermore, the *Wiener chaos decomposition* tells us that the space $L^2(\Omega, \mathcal{F}, P)$ can be decomposed into the infinite orthogonal sum of the \mathcal{H}_q . As a consequence, any square-integrable random variable $F \in L^2(\Omega, \mathcal{F}, P)$ can be written as

$$(2.9) \quad F = \mathbb{E}[F] + \sum_{q=1}^{\infty} I_q(f_q).$$

where the kernels $f_q \in \mathfrak{H}^{\odot q}$ are uniquely defined. This identity is called the *chaos expansion* of F .

If $\{\psi_k : k \geq 1\}$ is a complete orthonormal system in \mathfrak{H} , $f_i \in \mathfrak{H}^{\odot q_i}$, $f_j \in \mathfrak{H}^{\odot q_j}$ and $r \in \{0, \dots, q_i \wedge q_j\}$, the *contraction* $f_i \otimes_r f_j$ of f_i and f_j of order r is the element of $\mathfrak{H}^{\odot (q_i + q_j - 2r)}$ defined by

$$(2.10) \quad f_i \otimes_r f_j = \sum_{l_1, \dots, l_r=1}^{\infty} \langle f_i, \psi_{l_1} \otimes \dots \otimes \psi_{l_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle f_j, \psi_{l_1} \otimes \dots \otimes \psi_{l_r} \rangle_{\mathfrak{H}^{\otimes r}}.$$

The contraction $f_i \otimes_r f_j$ is not necessarily symmetric. We denote its canonical symmetrization by $f_i \tilde{\otimes}_r f_j \in \mathfrak{H}^{\odot q_i + q_j - 2r}$. Note that $f_i \otimes_0 f_j$ is equal to the

usual tensor product $f_i \otimes f_j$ of f_i and f_j . Furthermore, if $q_i = q_j$, we have that $f_i \otimes_{q_i} f_j = \langle f_i, f_j \rangle_{\mathfrak{H}^{\otimes q_i}}$. The well known multiplication formula

$$(2.11) \quad I_{q_i}(f_i) I_{q_j}(f_j) = \sum_{r=0}^{q_i \wedge q_j} \beta_{q_i, q_j}(r) I_{q_i + q_j - 2r}(f_i \widetilde{\otimes}_r f_j),$$

where

$$(2.12) \quad \beta_{a,b}(r) = r! \binom{a}{r} \binom{b}{r},$$

gives us the chaos expansion of the product of two multiple integrals.

When $\mathfrak{H} = L^2(A, \mathcal{A}, \nu)$, where (A, \mathcal{A}) is a Polish space, \mathcal{A} is the associated Borel σ -field and the measure ν is positive, σ -finite and non-atomic, one can identify the symmetric tensor product $\mathfrak{H}^{\odot q}$ with the Hilbert space $L_s^2(A^q, \mathcal{A}^q, \nu^{\otimes q})$, which is defined as the collection of all $\nu^{\otimes q}$ -almost everywhere symmetric functions on A^q , that are square-integrable with respect to the product measure $\nu^{\otimes q}$. In this case, the random variable $I_q(h)$, $h \in \mathfrak{H}^{\odot q}$, coincides with the multiple Wiener-Itô integral of order q of h with respect to the Gaussian measure $B \mapsto X(1_B)$, where $B \in \mathcal{A}$ and $\nu(A) < \infty$. Furthermore, the contraction (2.10) can be written as

$$(2.13) \quad \begin{aligned} & (f_i \otimes_r f_j)(t_1, \dots, t_{q_i + q_j - 2r}) \\ &= \int_{A^r} f_i(t_1, \dots, t_{q_i - r}, s_1, \dots, s_r) f_j(t_{q_i - r + 1}, \dots, t_{q_i + q_j - 2r}, s_1, \dots, s_r) \\ & \quad d\nu(s_1) \dots d\nu(s_r). \end{aligned}$$

2.4. Operators from Malliavin calculus. In this section, we introduce the operators D , L and L^{-1} from Malliavin calculus, which will appear in the statements of our main results. This exposition is by no means complete, most notably, we do not introduce the divergence operator. Again, we refer to [NP12] or [Nua06] for a full discussion.

If \mathcal{S} is the set of all cylindrical random variables of the type

$$F = g(X(h_1), \dots, X(h_k)),$$

where $k \geq 1$, $h_i \in \mathfrak{H}$ for $1 \leq i \leq k$ and $g: \mathbb{R}^k \rightarrow \mathbb{R}$ is an infinitely differentiable function with compact support, the Malliavin derivative DF with respect to X is the element of $L^2(\Omega, \mathfrak{H})$ defined by

$$DF = \sum_{i=1}^k \partial_i(X(h_1), \dots, X(h_k)) h_i.$$

Iterating this procedure, we obtain higher derivatives $D^m F$ for any $m \geq 2$, which are elements of $L^2(\Omega, \mathfrak{H}^{\odot m})$. For $m, p \geq 1$, $\mathbb{D}^{m,p}$ denotes the closure of \mathcal{S} with respect to the norm $\|\cdot\|_{m,p}$, which is defined by

$$\|F\|_{m,p}^p = \mathbb{E} [|F|^p] + \sum_{i=1}^m \mathbb{E} [\|D^i F\|_{\mathfrak{H}^{\otimes i}}^p].$$

If $\mathfrak{H} = L^2(A, \mathcal{A}, \nu)$, with ν non-atomic, the Malliavin derivative of a random variable F having the chaos expansion (2.9) can be identified with the element of

$L^2(A \times \Omega)$ given by

$$(2.14) \quad D_t F = \sum_{q=1}^{\infty} q I_{q-1}(f_q(\cdot, t)), \quad t \in A.$$

The *Ornstein-Uhlenbeck generator* L is defined by $L = \sum_{q=0}^{\infty} -q J_q$. Here, J_q denotes the orthogonal projection onto the q th Wiener chaos. The domain of L is $\mathbb{D}^{2,2}$. Similarly, we define its pseudo-inverse L^{-1} by $L^{-1} = \sum_{q=1}^{\infty} -\frac{1}{q} J_q$. This pseudo-inverse is defined on $L^2(\Omega)$ and for any $F \in L^2(\Omega)$ it holds that $L^{-1}F$ lies in the domain of L . The name pseudo-inverse is justified by the relation

$$LL^{-1}F = F - \mathbb{E}[F],$$

valid for any $F \in L^2(\Omega)$.

2.5. Cumulants. Recall the multi-index notation introduced in the first paragraph of Section 2.2.

Let $F = (F_1, \dots, F_d)$ be a \mathbb{R}^d -valued random vector. For a multi-index α , we set $F^\alpha = \prod_{k=1}^d F_k^{\alpha_k}$ and, with slight abuse of notation, $|F|^\alpha = \prod_{k=1}^d |F_k|^{\alpha_k}$. The moments $\mu_\alpha(F)$ of F of order $|\alpha|$ are then defined by $\mu_\alpha(F) = \mathbb{E}[F^\alpha]$, provided that the expectation on the right hand side is finite. Analogously, one defines the absolute moments $\mu_\alpha(|F|)$. We denote by $\phi_F(t) = \mathbb{E}[\exp(i \langle t, F \rangle_{\mathbb{R}^d})]$ the characteristic function of F . If $\mu_\alpha(|F|) < \infty$, the *joint cumulant* $\kappa_\alpha(F)$ of order $|\alpha|$ of F is defined by

$$\kappa_\alpha(F) = (-i)^{|\alpha|} \partial_\alpha \log \phi_F(t)|_{t=0}.$$

Given all joint cumulants $\kappa_\alpha(F)$ up to some order exist, we can compute the moments up to the same order by Leonov and Shiryaev's formula (see [PT11, Proposition 3.2.1])

$$(2.15) \quad \mu_\alpha(F) = \sum_{\pi} \kappa_{b_1}(F) \kappa_{b_2}(F) \cdots \kappa_{b_m}(F),$$

where the sum is taken over all partitions $\pi = \{B_1, \dots, B_m\}$ of the elementary decomposition of α and the multi-indices b_k are defined by $b_k = \sum_{l_j \in B_k} l_j$. For example, if $1 \leq i, j, k \leq d$, we get $\mu_{e_i}(F) = \kappa_{e_i}(F)$, $\mu_{e_i+e_j}(F) = \kappa_{e_i+e_j}(F) + \kappa_{e_i}(F)\kappa_{e_j}(F)$ and

$$\begin{aligned} \mu_{e_i+e_j+e_k}(F) &= \kappa_{e_i+e_j+e_k}(F) \\ &+ \kappa_{e_i}(F)\kappa_{e_j+e_k}(F) + \kappa_{e_j}(F)\kappa_{e_i+e_k}(F) + \kappa_{e_k}(F)\kappa_{e_i+e_j}(F) \\ &+ \kappa_{e_i}(F)\kappa_{e_j}(F)\kappa_{e_k}(F) \end{aligned}$$

for the moments of order one, two and three, respectively. Note that if F is centered, all moments of order less than four coincide with the respective cumulants.

2.6. Generalized Edgeworth expansions. Let F_1 and F_2 be two \mathbb{R}^d -valued random vectors with finite absolute moments up to some order $m \in \mathbb{N}_0 \cup \{\infty\}$ and consider the problem of approximating F_1 in terms of F_2 . The classical Edgeworth expansion provides such an approximation in terms of formal "moments", which we will now describe.

For every multi-index α of order at most m , we define formal "cumulants" $\tilde{\kappa}_\alpha(F_1, F_2)$ by $\tilde{\kappa}_\alpha(F_1, F_2) = \kappa_\alpha(F_1) - \kappa_\alpha(F_2)$ and use Shiryaev's formula (2.15) to define corresponding formal "moments" $\tilde{\mu}_\alpha(F_1, F_2)$. Two things are important

to note at this point. In general, $\tilde{\mu}_\alpha(F_1, F_2) \neq \mu_\alpha(F_1) - \mu_\alpha(F_2)$ and the collection $\{\tilde{\kappa}_\alpha(F_1, F_2): |\alpha| \leq m\}$ can not be represented as cumulants associated with some random variable. If we now assume that F_1 and F_2 both have densities, say f_1 and f_2 , the classical Edgeworth expansion of order m for the density f_1 then reads

$$(2.16) \quad f_1(x) \sim f_2(x) + \sum_{1 \leq |\alpha| \leq m} \frac{(-1)^{|\alpha|}}{|\alpha|!} \tilde{\mu}_\alpha(F_1, F_2) \partial_\alpha f_2(x).$$

In the most prominent example where this is the case, F_1 is a normalized sum of iid random variables and F_2 is Gaussian. In this case, the Edgeworth expansion can be used to improve the speed of convergence in the classical central limit theorem. For details, we refer to [Hal92], [McC87, Chapter5] and [BRR86].

For our framework, however, the classical Edgeworth expansion is too rigid, as we cannot assume the existence of (smooth) densities. Therefore, instead of expanding the density f_1 in terms of f_2 and its derivatives, we pass to the distributional operators $g \mapsto \mathbb{E}[g(F_1)]$ and $g \mapsto \mathbb{E}[g(F_2)]$, defined on the space of infinitely differentiable functions with compact support. The expansion (2.16) becomes

$$(2.17) \quad \mathbb{E}[g(F_1)] \sim \mathbb{E}[g(F_2)] + \sum_{1 \leq |\alpha| \leq m} \frac{\tilde{\mu}_\alpha(F_1, F_2)}{|\alpha|!} \mathbb{E}[\partial_\alpha g(F_2)].$$

Note that in the case of existing smooth densities it holds that $\mathbb{E}[g(F_1)] = \int_{-\infty}^{\infty} g(x) f_1(x) dx$ and, by integration by parts,

$$\mathbb{E}[\partial_\alpha g(F_2)] = \int_{-\infty}^{\infty} \partial_\alpha g(x) f_2(x) dx = (-1)^{|\alpha|} \int_{-\infty}^{\infty} g(x) \partial_\alpha f_2(x) dx,$$

so that (2.17) is obtained in a natural way from (2.16), by multiplying with the test function g and integrating on both sides.

This leads us to the following definition of a generalized Edgeworth expansion.

Definition 2.5 (Generalized Edgeworth expansion). If g is m -times differentiable and has bounded derivatives up to order m , we define the *generalized m th order Edgeworth expansion* $\mathcal{E}_m(F_1, F_2, g)$ of $\mathbb{E}[g(F_1)]$ around $\mathbb{E}[g(F_2)]$ by

$$(2.18) \quad \mathcal{E}_m(F_1, F_2, g) = \mathbb{E}[g(F_2)] + \sum_{1 \leq |\alpha| \leq m} \frac{\tilde{\mu}_\alpha(F_1, F_2)}{|\alpha|!} \mathbb{E}[\partial_\alpha g(F_2)].$$

If Z is a d -dimensional centered normal with covariance matrix C (the case that we will exclusively consider in the sequel), formula (2.2) yields

$$\mathcal{E}_m(F_1, Z, g) = \mathbb{E}[g(Z)] + \sum_{1 \leq |\alpha| \leq m} \frac{\tilde{\mu}_\alpha(F_1, F_2)}{|\alpha|!} \mathbb{E}[g(Z) H_\alpha(Z, C)],$$

where the Hermite polynomials $H_\alpha(x, C)$ are defined by (2.1) (recall our convention that we drop the mean as an argument if it is zero).

2.7. Cumulant formulas for chaotic random vectors. When dealing with functionals of an isonormal Gaussian process, their cumulants can be generalized in terms of Malliavin operators. This (among other things) is the content of [NP10] and [NN11] (see also [NP12, Chapter8]), which we will summarize here.

Let $F = (F_1, \dots, F_d)$ be a \mathbb{R}^d -valued random vector whose components are functionals of some isonormal Gaussian process X and let l_1, l_2, \dots be a sequence of d -dimensional multi-indices of order one. If $F^{l_1} \in \mathbb{D}^{1,2}$, we set $\Gamma_{l_1}(F) = F_i$. Inductively, if $\Gamma_{l_1, l_2, \dots, l_k}(F)$ is a well-defined element of $L^2(\Omega)$ for some $k \geq 1$, we define

$$(2.19) \quad \Gamma_{l_1, \dots, l_{k+1}}(F) = \langle DF_{l_{k+1}}, -DL^{-1}\Gamma_{l_1, \dots, l_k}(F) \rangle_{\mathfrak{H}}.$$

The question of existence is answered by the following lemma.

Lemma 2.6 (Noredidine, Nourdin [NN11]). *With the notation as above, fix an integer $j \geq 1$ and assume that $F_i \in \mathbb{D}^{j, 2^j}$ for $1 \leq i \leq d$. Then it holds that $\Gamma_{l_1, \dots, l_k}(F)$ is a well-defined element of $\mathbb{D}^{j-k+1, 2^{j-k+1}}$ for all $k = 1, \dots, j$. In particular, $\Gamma_{l_1, \dots, l_j}(F) \in \mathbb{D}^{1,2} \subset L^2(\Omega)$ and the quantity $\mathbb{E}[\Gamma_{l_1, \dots, l_j}(F)]$ is well-defined and finite.*

Using these random elements, we can now state a formula for the cumulants of F .

Theorem 2.7 (Noredidine, Nourdin [NN11]). *Let α be a d -dimensional multi-index with elementary decomposition $\{l_1, \dots, l_{|\alpha|}\}$. If $F_i \in \mathbb{D}^{|\alpha|, 2^{|\alpha|}}$ for $1 \leq i \leq d$, then*

$$(2.20) \quad \kappa_\alpha(F) = \sum_{\sigma} \mathbb{E} \left[\Gamma_{l_1, l_{\sigma(2)}, l_{\sigma(3)}, \dots, l_{\sigma(|\alpha|)}}(F) \right],$$

where the sum is taken over all permutations σ of the set $\{2, 3, \dots, |\alpha|\}$.

We again stress that – as the labeling of the elementary decomposition is arbitrary – we can freely choose the fixed first element l_1 . For the case $d = 1$, this formula has been proven in [NP10].

To simplify notation, we will frequently write $\Gamma_{i_1 i_2 \dots i_k}(F)$ instead of the more cumbersome $\Gamma_{e_{i_1}, e_{i_2}, \dots, e_{i_k}}(F)$. For example, the random variable $\Gamma_{e_1, e_2}(F) = \langle DF_1, -DL^{-1}F_2 \rangle_{\mathfrak{H}}$ will also be denoted by $\Gamma_{12}(F)$.

If all components of F are elements of (possibly different) Wiener chaoses, formula (2.20) can be stated in terms of contractions. We state two special cases here and refer to Noredidine and Nourdin [NN11] for a general formula. As a first special case, assume that $F_i = I_{q_i}(f_i)$ where $q_i \geq 1$ and $f_i \in \mathfrak{H}^{\odot q_i}$ for $1 \leq i \leq d$. In this case, for $1 \leq i, j, k \leq d$, the third-order cumulants are given by

$$(2.21) \quad \kappa_{e_i + e_j + e_k} = \begin{cases} c \langle f_i \tilde{\otimes}_r f_j, f_k \rangle_{\mathfrak{H}^{\otimes q_k}} & \text{if } r := \frac{q_i + q_j - q_k}{2} \in \{1, 2, \dots, q_i \wedge q_j\}, \\ 0 & \text{otherwise,} \end{cases}$$

where c is some positive constant depending on the chaotic orders q_i, q_j and q_k . As a second special case, assume that the components F_i are all of the form $F_i = I_2(f_i)$, with $f_i \in \mathfrak{H}^{\odot 2}$ for $1 \leq i \leq d$. In this case, for any $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \geq 2$ it holds that

$$(2.22) \quad \kappa_\alpha(F) = 2^{|\alpha|-1} \sum_{\sigma} \left\langle (\dots (f_{i_1} \tilde{\otimes}_1 f_{i_{\sigma(1)}}) \tilde{\otimes}_1 f_{i_{\sigma(2)}}) \dots) \tilde{\otimes}_1 f_{i_{\sigma(|\alpha|-1)}}, f_{i_{\sigma(|\alpha|)}} \right\rangle_{\mathfrak{H}^{\otimes 2}},$$

where the sum is taken over all permutations σ of the set $\{2, 3, \dots, |\alpha|\}$ and the indices $i_1, \dots, i_{|\alpha|}$ are defined as follows: If $\{l_1, \dots, l_{|\alpha|}\}$ is the elementary

decomposition of α , we set $i_j = k$ if $l_j = e_k$, $j = 1, \dots, |\alpha|$. To illustrate this formula, we have for example

$$(2.23) \quad \kappa_{(2,1)}(F) = 8 \langle f_1 \tilde{\otimes}_1 f_1, f_2 \rangle_{\mathfrak{H}^{\otimes 2}},$$

or, with a different labelling of the elementary decomposition,

$$(2.24) \quad \kappa_{(2,1)}(F) = 8 \left(\langle f_1 \tilde{\otimes}_1 f_2, f_1 \rangle_{\mathfrak{H}^{\otimes 2}} + \langle f_2 \tilde{\otimes}_1 f_1, f_1 \rangle_{\mathfrak{H}^{\otimes 2}} \right).$$

One can verify by direct computations that the right hand sides of (2.23) and (2.24) are indeed equal.

2.8. Limit theorems for vectors of multiple integrals. In this section, we gather two results from the literature which we will use extensively in the sequel. The first is a version of the so called *Fourth Moment Theorem* (see [Pec07, Theorem 3]) for fluctuating covariances, that is based on the findings in [NP05], [NOL08] and [PT05] for the converging covariance case.

Theorem 2.8 (Fourth Moment Theorem, [Pec07]).

(A) Let $q \geq 1$ and $(F_n)_{n \geq 1} = (I_q(f_n))_{n \geq 1}$ be a sequence of multiple integrals and assume that there exists a constant M such that $\mathbb{E}[F_n^2] \leq M$ for $n \geq 1$. Then the following conditions are equivalent.

- (i) $(F_n)_{n \geq 1}$ is ACN
- (ii) $\mathbb{E}[F_n^4] - 3\mathbb{E}[F_n^2]^2 \rightarrow 0$
- (iii) For $1 \leq r \leq q-1$ it holds that $\|f_n \otimes_r f_n\|_{\mathfrak{H}^{\otimes 2(q-r)}} \rightarrow 0$
- (iv) $\text{Var}(\Gamma_{11}(F_n)) \rightarrow 0$

If the variance of F_n converges to some limit c , conditions (i)-(iv) are equivalent to

- (i') $F_n \xrightarrow{d} Z$, where Z is a centered normal with variance c .

(B) Let $(F_n)_{n \geq 1} = (F_{1,n}, \dots, F_{d,n})_{n \geq 1}$ be a random sequence such that $F_{i,n} = I_{q_i}(f_{i,n})$, $q_i \geq 1$, for $1 \leq i \leq d$ and the covariances of (F_n) are uniformly bounded. Then (F_n) is ACN if and only if $(F_{i,n})$ is ACN for $1 \leq i \leq d$.

Secondly, we will make use of the following central limit theorem for the case where one component of the random vectors F_n has a finite chaos expansion. As this result is an immediate consequence of the findings in [Pec07], we omit the proof.

Lemma 2.9. Let $(F_n)_{n \geq 1} = (F_{1,n}, \dots, F_{d,n})_{n \geq 1}$ be a sequence of random vectors such that $F_{i,n} = I_{q_i}(F_{i,n})$ for $n \geq 1$ and $1 \leq i \leq d$. Furthermore, let $G_n = \sum_{k=1}^M I_k(g_{k,n})$ for $n \geq 1$. If

$$(2.25) \quad \sum_{i=1}^d \sum_{r=1}^{q_i-1} \|f_{i,n} \otimes_r f_{i,n}\|_{\mathfrak{H}^{\otimes 2(q_i-r)}} \rightarrow 0$$

and

$$(2.26) \quad \sum_{k=2}^M \sum_{s=1}^{k-1} \|g_{k,n} \otimes_s g_{k,n}\|_{\mathfrak{H}^{\otimes 2(q_k-s)}} \rightarrow 0,$$

then $(F_n, G_n)_{n \geq 1}$ is ACN.

3. MAIN RESULTS

In this section, for some fixed positive integer d , we denote by $(F_n)_{n \geq 1} = (F_{1,n}, F_{2,n}, \dots, F_{d,n})_{n \geq 1}$ a sequence of centered, \mathbb{R}^d -valued random vectors such that $F_{i,n} \in \mathbb{D}^{1,4}$ for $1 \leq i \leq d$. We also introduce a normalized sequence $(\tilde{\Gamma}_{ij}(F_n))_{n \geq 1}$ for $1 \leq i, j \leq d$, which is defined by

$$\tilde{\Gamma}_{ij}(F_n) = \frac{\Gamma_{ij}(F_n) - \mathbb{E}[\Gamma_{ij}(F_n)]}{\sqrt{\text{Var} \Gamma_{ij}(F_n)}} = \frac{\Gamma_{ij}(F_n) - \mathbb{E}[F_{i,n}F_{j,n}]}{\sqrt{\text{Var} \Gamma_{ij}(F_n)}},$$

where $\Gamma_{ij}(F_n)$ is given by (2.19). Furthermore, for $1 \leq i, j \leq d$, let $(Z_n)_{n \geq 1} = (Z_{1,n}, \dots, Z_{d,n})_{n \geq 1}$ be a centered sequence of Gaussian random variables such that Z_n has the same covariance as F_n for $n \geq 1$. The following crucial identity is the starting point of our investigations.

Theorem 3.1 ([NPR10a]). *Let $g \in C^2(\mathbb{R}^d)$ and Z be a d -dimensional normal vector with covariance matrix C . Then, for every $n \geq 1$, it holds that*

$$(3.1) \quad \mathbb{E}[g(F_n)] - \mathbb{E}[g(Z)] = \sum_{i,j=1}^d \mathbb{E}[\partial_{ij} U_{g,C}(F_n) (\Gamma_{ij}(F_n) - C_{ij})],$$

where $U_{g,C}$ is defined by (2.4).

Identity (3.1) has been derived in [NPR10a] by the so called “smart path method” and Malliavin calculus. If the covariance matrix C is positive definite, one can give an alternative proof by using Stein’s method (see [NPR10b, proof of Theorem 3.5]).

A straightforward application of the Cauchy-Schwarz inequality to identity (3.1) yields the bound

$$(3.2) \quad |\mathbb{E}[g(F_n)] - \mathbb{E}[g(Z)]| \leq \frac{\sqrt{d}}{2} \left(\sup_{|\alpha|=2} \|\partial_{\alpha} g\|_{\infty} \right) \varphi_C(F_n),$$

where $\varphi_C(F_n) = \Delta_{\Gamma}(F_n) + \Delta_C(F_n)$ and the quantities $\Delta_{\Gamma}(F_n)$ and $\Delta_C(F_n)$, that already appeared in the Introduction, are defined by

$$(3.3) \quad \Delta_{\Gamma}(F_n) = \|\Gamma(F_n) - \text{Cov}(F_n)\|_{H.S.} = \sqrt{\sum_{i,j=1}^d \text{Var} \Gamma_{ij}(F_n)}$$

and

$$(3.4) \quad \Delta_C(F_n) = \|\text{Cov}(F_n) - \text{Cov}(Z)\|_{H.S.} = \sqrt{\sum_{i,j=1}^d (\mathbb{E}[F_i F_j] - C_{ij})^2}.$$

Here, $\|\cdot\|_{H.S.}$ denotes the Hilbert-Schmidt matrix norm. Note that $\Delta_C(F_n)$ is equal to zero if and only if F_n has covariance matrix C and that $\Delta_{\Gamma}(F_n)$ is equal to zero if F_n is Gaussian. The latter follows from the fact that $|\Gamma_{ij}(F_n)|$ is constant if $F_{i,n}$ and $F_{j,n}$ are Gaussian, which can be seen by applying the bound (3.2) to the vector $(F_{i,n}, F_{j,n})$ and a centered Gaussian vector (Z_1, Z_2) with the same covariance.

Assume now that $\varphi_C(F_n)$ converges to zero. For the one-dimensional case $d = 1$, an adaptation of the arguments in [NP09a] provides conditions under

which the ratio

$$\frac{\mathbb{E}[g(F_n)] - \mathbb{E}[g(Z)]}{\varphi_C(F_n)}$$

converges to some real number. If this number is non-zero, this implies in particular that the rate $\varphi_C(F_n)$ is optimal, in the sense that there exist positive constants c_1, c_2 and n_0 such that

$$c_1 \leq \frac{|\mathbb{E}[g(F_n)] - \mathbb{E}[g(Z)]|}{\varphi_C(F_n)} \leq c_2$$

for $n \geq n_0$. As already mentioned in the introduction, by approximating a Lipschitz function by functions with bounded derivatives, this implies that $\varphi_C(F_n)$ is optimal for the one-dimensional Wasserstein distance (see [NP09a] for details). By considering coordinate projections, optimality in multiple dimensions case can immediately be reduced to the one-dimensional case. However, obtaining exact asymptotics is a much more involved task, as the next two theorems show.

Theorem 3.2 (Exact asymptotics for the fluctuating variance case). *Assume that $\Delta_\Gamma(F_n) \rightarrow 0$ and let $g: \mathbb{R}^d \rightarrow \mathbb{R}$ be three times differentiable with bounded derivatives up to order three. If, for $1 \leq i, j \leq d$, the random sequences $(F_n, \tilde{\Gamma}_{ij}(F_n))_{n \geq 1}$ are ACN whenever $\sqrt{\text{Var } \Gamma_{ij}(F_n)} \asymp \Delta_\Gamma(F_n)$ it holds that*

$$(3.5) \quad \frac{1}{\Delta_\Gamma(F_n)} \left(\mathbb{E}[g(F_n)] - \mathbb{E}[g(Z_n)] - \frac{1}{3} \sum_{i,j,k=1}^d \sqrt{\text{Var } \Gamma_{ij}(F_n)} \rho_{ijk,n} \mathbb{E}[\partial_{ijk} g(Z_n)] \right) \rightarrow 0.$$

Here, the constants $\rho_{ijk,n}$ are defined by

$$\rho_{ijk,n} = \mathbb{E} \left[\tilde{Z}_{ij,n} Z_{k,n} \right]$$

whenever $\sqrt{\text{Var } \Gamma_{ij}(F_n)} \asymp \Delta_\Gamma(F_n)$ holds and $(F_n, \tilde{\Gamma}_{ij,n})_{n \geq 1}$ is ACN with corresponding Gaussian sequence $(Z_n, \tilde{Z}_{ij,n})$, and $\rho_{ijk,n} = 0$ otherwise.

Remark 3.3. Clearly, the condition $\sqrt{\text{Var } \Gamma_{ij}(F_n)} \asymp \Delta_\Gamma(F_n)$ in the above Theorem expresses the fact that we can neglect those summands of $\Delta_\Gamma(F_n)$ that vanish “too fast” and therefore do not contribute to the overall speed of convergence of $\Delta_\Gamma(F_n)$.

Proof of Theorem 3.2. By applying Theorem 3.1, we get

$$(3.6) \quad \frac{\mathbb{E}[g(F_n)] - \mathbb{E}[g(Z_n)]}{\Delta_\Gamma(F_n)} = \sum_{i,j=1}^d \frac{\sqrt{\text{Var } \Gamma_{ij}(F_n)}}{\Delta_\Gamma(F_n)} \mathbb{E} \left[\partial_{ij} U_{g,C_n}(F_n) \tilde{\Gamma}_{ij}(F_n) \right].$$

The bound (2.6) for the derivatives of U_{g,C_n} and the fact that $\tilde{\Gamma}_{ij}(F_n)$ has unit variance immediately implies that the expectations occurring in the sum on the right hand side of (3.6) are bounded. Therefore, we only have to examine those summands in the same sum, for which (??) is true (as all others vanish in the limit). Now choose $1 \leq i, j \leq d$ such that (??) holds. Due to our assumption,

Theorem 2.2 implies the existence of random vectors $(F_n^*, \tilde{\Gamma}_{ij}(F_n)^*)$ and Gaussian random variables $(Z_n^*, \tilde{Z}_{ij,n}^*)$, defined on some common probability space, such that $(F_n^*, \tilde{\Gamma}_{ij}(F_n)^*)$ has the same law as $(F_n, \tilde{\Gamma}_{ij}(F_n))$, $(Z_n^*, \tilde{Z}_{ij,n}^*)$ has the same law as $(Z, \tilde{Z}_{ij,n})$ and $(F_n^* - Z_n^*, \tilde{\Gamma}_{ij}(F_n)^* - \tilde{Z}_{ij,n}^*) \rightarrow 0$ almost surely. Thus we can write

$$(3.7) \quad \mathbb{E} \left[\partial_{ij} U_{g,C_n}(F_n) \tilde{\Gamma}_{ij}(F_n) \right] = \eta_{ij,n}^1 + \eta_{ij,n}^2 + \eta_{ij,n}^3,$$

where

$$\begin{aligned} \eta_{ij,n}^1 &= \mathbb{E} \left[(\partial_{ij} U_{g,C_n}(F_n^*) - \partial_{ij} U_{g,C_n}(Z_n^*)) \tilde{\Gamma}_{ij}(F_n)^* \right], \\ \eta_{ij,n}^2 &= \mathbb{E} \left[\partial_{ij} U_{g,C_n}(Z_n^*) (\tilde{\Gamma}_{ij}(F_n)^* - \tilde{Z}_{ij,n}^*) \right] \end{aligned}$$

and

$$\eta_{ij,n}^3 = \mathbb{E} \left[\partial_{ij} U_{g,C_n}(Z_n^*) \tilde{Z}_{ij,n}^* \right].$$

The integration by parts formula (2.3) and Lemma 2.4b) yield

$$\eta_{ij,n}^3 = \frac{1}{3} \sum_{k=1}^d \mathbb{E} \left[Z_{k,n}^* \tilde{Z}_{ij,n}^* \right] \mathbb{E} [\partial_{ijk} U_{g,C_n}(Z_n^*)],$$

so that the proof is finished as soon as we have established that

$$\eta_{ij,n}^1 \rightarrow 0 \quad \text{and} \quad \eta_{ij,n}^2 \rightarrow 0.$$

But this is an immediate consequence of the Lipschitz continuity of $\partial_{ij} U_{g,C_n}$, the fact that $\tilde{\Gamma}_{ij}(F_n)^*$ has unit variance (implying uniform integrability of the sequences $(\tilde{\Gamma}_{ij}(F_n)^*)_{n \geq 0}$ and $(\tilde{\Gamma}_{ij}(F_n)^* - \tilde{Z}_{ij,n}^*)_{n \geq 0}$) and the bound (2.6). \square

Theorem 3.4 (Exact asymptotics for the converging variance case). *Assume that $\Delta_\Gamma(F_n) \rightarrow 0$ and let $g: \mathbb{R}^d \rightarrow \mathbb{R}$ be three times differentiable with bounded derivatives up to order three. If there exists a covariance matrix C such that $\Delta_C(F_n) \rightarrow 0$ and, for $1 \leq i, j \leq d$, the random sequences $(F_n, \tilde{\Gamma}_{ij}(F_n))_{n \geq 1}$ converge in law to a centered Gaussian random vector (Z, \tilde{Z}_{ij}) whenever*

$$(3.8) \quad \sqrt{\text{Var } \Gamma_{ij}(F_n)} + |\mathbb{E}[F_{i,n} F_{j,n}] - C_{ij}| \asymp \varphi_C(F_n)$$

it holds that

$$(3.9) \quad \frac{1}{\varphi_C(F_n)} \left(\mathbb{E}[g(F_n)] - \mathbb{E}[g(Z)] - \frac{1}{2} \sum_{i,j=1}^d (\mathbb{E}[F_{i,n} F_{j,n}] - C_{ij}) \mathbb{E}[\partial_{ij} g(Z)] - \frac{1}{3} \sum_{i,j,k=1}^d \sqrt{\text{Var } \Gamma_{ij}(F_n)} \rho_{ijk} \mathbb{E}[\partial_{ijk} g(Z)] \right) \rightarrow 0.$$

Here, the constants ρ_{ijk} are defined by $\rho_{ijk} = \mathbb{E}[\tilde{Z}_{ij} Z_k]$ whenever (3.8) is true and $\rho_{ijk} = 0$ otherwise.

Proof. Theorem 3.1 implies

$$(3.10) \quad \frac{\mathbb{E}[g(F_n)] - \mathbb{E}[g(Z)]}{\varphi_C(F_n)} = \sum_{i,j=1}^d \left(\frac{\mu_{ij}(F_n) - C_{ij}}{\varphi_C(F_n)} \mathbb{E}[\partial_{ij}U_{g,C}(F_n)] + \frac{\sqrt{\text{Var} \Gamma_{ij}(F_n)}}{\varphi_C(F_n)} \mathbb{E}[\partial_{ij}U_{g,C}(F_n) \tilde{\Gamma}_{ij}(F_n)] \right).$$

Arguing as in the proof of Theorem 3.2, we see that all expectations occurring in the sum on the right hand side of (3.10) are bounded. Therefore, we can choose $1 \leq i, j \leq d$ and assume that (3.8) is true (as otherwise the corresponding summand would vanish in the limit). By the boundedness of the second derivatives of $U_{g,C}$ (see (2.6)) and our assumption of convergence in law, we get

$$\mathbb{E}[\partial_{ij}U_{g,C}(F_n)] \rightarrow \mathbb{E}[\partial_{ij}U_{g,C}(Z)]$$

and

$$\mathbb{E}[\partial_{ij}U_{g,C}(F_n) \tilde{\Gamma}_{ij}(F_n)] \rightarrow \mathbb{E}[\partial_{ij}U_{g,C}(Z) \tilde{Z}_{ij}].$$

The integration by parts formula (2.3) and Lemma 2.4b) now yield

$$\mathbb{E}[\partial_{ij}U_{g,C}(Z)] = \frac{1}{2} \mathbb{E}[\partial_{ij}g(Z)]$$

and

$$\mathbb{E}[\partial_{ij}U_{g,C}(Z) \tilde{Z}_{ij}] = \frac{1}{3} \sum_{k=1}^d \mathbb{E}[\tilde{Z}_{ij} Z_k] \mathbb{E}[\partial_{ijk}Z],$$

finishing the proof. \square

Remark 3.5. If the covariance C of the Gaussian random variable Z is positive definite, the Hermite polynomials $H_\alpha(x, C)$ form an orthonormal basis for the space $L^2(\mathbb{R}^d, \gamma_C)$, where γ_C is the density of Z , so that an expansion of the form $g(x) = \sum_\alpha H_\alpha(x, C)$ exists for all $x \in \mathbb{R}^d$. Thus, the integration by parts formula

$$\mathbb{E}[\partial_\alpha g(Z)] = \mathbb{E}[g(Z) H_\alpha(Z, C)],$$

valid for any multi-index α up to order three, yields a necessary condition for the limit to be non-zero: g must not be orthogonal (in $L^2(\mathbb{R}^d, \gamma_C)$) to all second- and third-order Hermite polynomials.

An immediate consequence of the Theorems 3.2 and 3.4 is the following corollary.

Corollary 3.6 (Sharp bounds and exact limits).

a) *In the setting of Theorem 3.2, the lim inf and lim sup of the sequence*

$$\left(\frac{|\mathbb{E}[g(F_n)] - \mathbb{E}[g(Z_n)]|}{\Delta_\Gamma(F_n)} \right)_{n \geq 1}$$

coincide with those of the sequence

$$\left(\frac{1}{3\Delta_\Gamma(F_n)} \sum_{i,j,k=1}^d \sqrt{\text{Var} \Gamma_{ij}(F_n)} \rho_{ik,n} \mathbb{E}[\partial_{ijk}g(Z_n)] \right)_{n \geq 1}.$$

b) In the setting of Theorem 3.4, the \liminf and \limsup of the sequence

$$(3.11) \quad \left(\frac{\mathbb{E}[g(F_n)] - \mathbb{E}[g(Z)]}{\varphi_C(F_n)} \right)_{n \geq 1}$$

coincide with those of

$$(3.12) \quad \left(\frac{1}{2} \sum_{i,j=1}^d \frac{\mathbb{E}[F_{i,n}F_{j,n}] - C_{ij}}{\varphi_C(F_n)} \mathbb{E}[\partial_{ij}g(Z)] \right. \\ \left. + \frac{1}{3} \sum_{i,j,k=1}^d \frac{\sqrt{\text{Var } \Gamma_{ij}(F_n)}}{\varphi_C(F_n)} \rho_{ijk} \mathbb{E}[\partial_{ijk}g(Z)] \right)_{n \geq 1}.$$

In particular, if the sequence (3.12) converges, it provides the exact limit of the sequence (3.11).

If $d = 1$ and the F_n all have identical variances, the assumptions of Corollary 3.6b) are always satisfied and one obtains an analogue of Theorem 3.1 in [NP09a].

If the third-order moments $\mu_\alpha(F_n)$ of the random vectors F_n exist, the third order Edgeworth expansion $\mathcal{E}_3(F_n, Z, g)$, introduced in section 2.5, is well-defined for any Gaussian random vector Z and any three-times differentiable function g with bounded derivatives up to order three. The next theorem shows how these expansions can be used to increase the speed of convergence.

Theorem 3.7 (One-term Edgeworth expansions). *Let $g: \mathbb{R}^d \rightarrow \mathbb{R}$ be three times differentiable with bounded derivatives up to order three and assume that F_n has finite moments up to order three for $n \geq 1$, and moreover*

$$(3.13) \quad \left(\frac{\sum_{k=1}^d \mu_{ijk}(F_n)}{\sqrt{\text{Var } \Gamma_{ij}(F_n)}} \right)_{n \geq 1}$$

is bounded whenever $\sqrt{\text{Var } \Gamma_{ij}(F_n)} / \Delta_\Gamma(F_n) \rightarrow 0$.

a) If all assumptions of Theorem 3.2a) are satisfied, it holds that

$$(3.14) \quad \frac{\mathbb{E}[g(F_n)] - \mathcal{E}_3(F_n, Z_n, g)}{\Delta_\Gamma(F_n)} \rightarrow 0.$$

b) If all assumptions of Theorem 3.2b) are satisfied, it holds that

$$(3.15) \quad \frac{\mathbb{E}[g(F_n)] - \mathcal{E}_3(F_n, Z, g)}{\varphi_C(F_n)} \rightarrow 0.$$

Remark 3.8. The third order Edgeworth expansion $\mathcal{E}_3(F_n, Z_n, g)$ in (3.14) takes the explicit form

$$(3.16) \quad \mathcal{E}_3(F_n, Z_n, g) = \mathbb{E}[g(Z_n)] + \sum_{i,j,k=1}^d \frac{\mu_{ijk}(F_n)}{3!} \mathbb{E}[\partial_{ijk}g(Z_n)]$$

whereas $\mathcal{E}_3(F_n, Z, g)$ in (3.15) is given by

$$(3.17) \quad \mathbb{E}[g(Z)] + \sum_{i,j=1}^d \frac{C_{ij} - \mu_{ij}(F_n)}{2} \mathbb{E}[\partial_{ij}g(Z)] + \sum_{i,j,k=1}^d \frac{\mu_{ijk}(F_n)}{3!} \mathbb{E}[\partial_{ijk}g(Z)]$$

Proof of Theorem 3.7. By Theorem 3.2a), it is sufficient to show that

$$(3.18) \quad \frac{1}{3} \sum_{i,j,k=1}^d \frac{\sqrt{\text{Var } \Gamma_{ij}(F_n)}}{\Delta_\Gamma(F_n)} \left(\rho_{ijk,n} - \frac{\mu_{ijk}(F_n)}{2\sqrt{\text{Var } \Gamma_{ij}(F_n)}} \right) \mathbb{E} [\partial_{ijk} g(Z_n)] = 0,$$

Fix $1 \leq i, j \leq d$. If

$$\frac{\sqrt{\text{Var } \Gamma_{ij}(F_n)}}{\Delta_\Gamma(F_n)} \rightarrow 0,$$

the quantity

$$\frac{\mu_{ijk}(F_n)}{2\sqrt{\text{Var } \Gamma_{ij}(F_n)}}$$

is bounded by assumption, so that the corresponding summand in the sum (3.18) vanishes in the limit. If $\limsup \sqrt{\text{Var } \Gamma_{ij}(F_n)}/\Delta_\Gamma(F_n)$ is positive, the sequence $(F_n, \tilde{\Gamma}_{ij,n})_{n \geq 1}$ is ACN. Thus, for $n \geq 1$, there exists Gaussian random variables $(Z_n, \tilde{Z}_{ij,n})$ with the same covariance as $(F_n, \tilde{\Gamma}_{ij,n})$. By definition, we get

$$\rho_{ijk,n} = \mathbb{E} [Z_{k,n} \tilde{Z}_{ij,n}] = \frac{\mathbb{E} [F_{k,n} \Gamma_{ij}(F_n)]}{\sqrt{\text{Var } \Gamma_{ij}(F_n)}} = \frac{\mathbb{E} [\Gamma_{ijk}(F_n)]}{\sqrt{\text{Var } \Gamma_{ij}(F_n)}}.$$

The cumulant formula (2.20) and the fact that $\text{Var } \Gamma_{ij}(F_n) = \text{Var } \Gamma_{ji}(F_n)$ now yields

$$\rho_{ijk,n} + \rho_{jik,n} = \frac{\kappa_{ijk}(F_n)}{\sqrt{\text{Var } \Gamma_{ij}(F_n)}} = \frac{\mu_{ijk}(F_n)}{\sqrt{\text{Var } \Gamma_{ij}(F_n)}}$$

so that (3.18) follows.

Likewise, by Theorem 3.4, it is sufficient for the proof of assertion b) to show that

$$(3.19) \quad \frac{1}{3} \sum_{i,j,k=1}^d \frac{\sqrt{\text{Var } \Gamma_{ij}(F_n)}}{\varphi_C(F_n)} \left(\rho_{ijk} - \frac{\mu_{ijk}(F_n)}{2\sqrt{\text{Var } \Gamma_{ij}(F_n)}} \right) \mathbb{E} [\partial_{ijk} G(Z)] \rightarrow 0,$$

Again, by assumption, if $1 \leq i, j \leq d$ is such that $\sqrt{\text{Var } \Gamma_{ij}(F_n)}/\varphi_C(F_n) \rightarrow 0$, the corresponding summand in (3.19) vanishes in the limit. If, on the other hand, $\sqrt{\text{Var } \Gamma_{ij}(F_n)} \asymp \varphi_C(F_n)$, the sequence $(F_n, \tilde{\Gamma}_{ij}(F_n))_{n \geq 1}$ converges in law to (Z, \tilde{Z}_{ij}) . Therefore,

$$(3.20) \quad \mathbb{E} [\tilde{\Gamma}_{ij,n} F_{k,n}] \rightarrow \mathbb{E} [\tilde{Z}_{ij} Z_k] = \rho_{ijk}$$

for $1 \leq k \leq d$. The cumulant formula (2.20) gives

$$\frac{\mu_{ijk}(F_n)}{\sqrt{\text{Var } \Gamma_{ij}(F_n)}} = \frac{\kappa_{ijk}(F_n)}{\sqrt{\text{Var } \Gamma_{ij}(F_n)}} = \mathbb{E} [\tilde{\Gamma}_{ij,n} F_{k,n}] + \mathbb{E} [\tilde{\Gamma}_{ji,n} F_{k,n}],$$

which together with (3.20) implies that

$$\frac{\mu_{ijk}(F_n)}{\sqrt{\text{Var } \Gamma_{ij}(F_n)}} \rightarrow \rho_{ijk} + \rho_{jik}.$$

This immediatiely yields (3.19), finishing the proof. \square

4. THE CASE OF MULTIPLE INTEGRALS

In this section, we specialize our results to the case where the components of the sequence $(F_n)_{n \geq 1}$ are vectors of multiple integrals. As in the previous section, we fix an integer $d \geq 1$ and study a sequence $(F_n)_{n \geq 1} = (F_{1,n}, \dots, F_{d,n})_{n \geq 1}$ of \mathbb{R}^d -valued random vectors, but now each component $F_{i,n}$ is a multiple integral of the form $F_{i,n} = I_{q_i}(f_{i,n})$ where $q_i \geq 1$ and $f_{i,n} \in \mathfrak{H}^{\odot q_i}$. Recall the definitions for the random variables $\Gamma_{ij}(F_n)$, $\tilde{\Gamma}_{ij}(F_n)$ and Z_n , which were given in the first paragraph of the previous section.

Let us begin by deducing explicit representations of some of the crucial quantities of the last section. Using the product formula (2.11) and the orthogonality property (2.8) of multiple integrals, we see that

$$(4.1) \quad \Gamma_{ij}(F_n) = \frac{1}{q_j} \langle DF_{i,n}, DF_{j,n} \rangle_{\mathfrak{H}} \\ = \sum_{r=1}^{q_i \wedge q_j} q_i \beta_{q_i-1, q_j-1}(r-1) I_{q_i+q_j-2r}(f_{i,n} \tilde{\otimes}_r f_{j,n})$$

and

$$(4.2) \quad \text{Var } \Gamma_{ij}(F_n) \\ = \sum_{r=1}^{q_i \wedge q_j - \delta_{q_i q_j}} (q_i + q_j - 2r)! q_i^2 \beta_{q_i-1, q_j-1}^2(r-1) \|f_{i,n} \tilde{\otimes}_r f_{j,n}\|_{\mathfrak{H}^{\otimes(q_i+q_j-2r)}}^2,$$

where the positive constants $\beta_{a,b}(r)$ are defined by (2.12).

As all constants in the sum on the right hand side of (4.2) are positive, this implies that

$$(4.3) \quad \text{Var } \Gamma_{ij}(F_n) \asymp \sum_{r=1}^{q_i \wedge q_j - \delta_{q_i q_j}} \|f_{i,n} \tilde{\otimes}_r f_{j,n}\|_{\mathfrak{H}^{\otimes(q_i+q_j-2r)}}^2$$

and therefore

$$(4.4) \quad \Delta_{\Gamma}(F_n) \asymp \left(\sum_{i,j=1}^d \sum_{r=1}^{q_i \wedge q_j - \delta_{q_i q_j}} \|f_{i,n} \tilde{\otimes}_r f_{j,n}\|_{\mathfrak{H}^{\otimes(q_i+q_j-2r)}}^2 \right)^{1/2}.$$

If, for some integers i, j, k with $1 \leq i, j, k \leq d$, it holds that $r := \frac{q_i + q_j - q_k}{2} \in \{1, 2, \dots, q_i \wedge q_j\}$, formula (2.21) and the Cauchy-Schwarz inequality yield

$$\mu_{ijk}(F_n) = \kappa_{e_i + e_j + e_k} F_n \leq c \|f_{i,n} \tilde{\otimes}_r f_{j,n}\|_{\mathfrak{H}^{\otimes(q_i+q_j-2r)}} \|f_k\|_{\mathfrak{H}^{\otimes q_k}}.$$

Combining this with (4.3), we see that the quantity (3.13) from Theorem 3.7 is bounded and therefore one-term Edgeworth expansions are always possible whenever the corresponding ACN-conditions from Theorems 3.2 and 3.4 are verified. Thus we have proven the following proposition.

Proposition 4.1 (Exact asymptotics and Edgeworth expansions for multiple integrals). *In the above framework, let $g: \mathbb{R}^d \rightarrow \mathbb{R}$ be three times differentiable with bounded derivatives up to order three.*

a) If, for $1 \leq i, j \leq d$, the random sequence $(F_n, \tilde{\Gamma}_{ij}(F_n))_{n \geq 1}$ is ACN whenever

$$\limsup_n \sqrt{\text{Var } \Gamma_{ij}(F_n) / \Delta_\Gamma(F_n)} > 0,$$

it holds that

$$\frac{\mathbb{E}[g(F_n)] - \mathcal{E}_3(F_n, Z_n, g)}{\Delta_\Gamma(F_n)} \rightarrow 0.$$

b) If there exists a covariance matrix C such that $\Delta_C(F_n) \rightarrow 0$ and, for $1 \leq i, j \leq d$, the random sequence $(F_n, \tilde{\Gamma}_{ij}(F_n))_{n \geq 1}$ is ACN whenever

$$\limsup_n \sqrt{\text{Var } \Gamma_{ij}(F_n) + |\mathbb{E}[F_{i,n} F_{j,n}] - C_{ij}| / \varphi_C(F_n)} > 0,$$

it holds that

$$\frac{\mathbb{E}[g(F_n)] - \mathcal{E}_3(F_n, Z, g)}{\varphi_C(F_n)} \rightarrow 0.$$

Sufficient conditions for the sequences $(F_n, \tilde{\Gamma}_{ij}(F_n))_{n \geq 1}$ to be ACN are given by the following proposition.

Proposition 4.2. *The sequence $(F_n, \tilde{\Gamma}_{ij}(F_n))_{n \geq 1}$ is ACN if*

$$(4.5) \quad \sum_{i=1}^d \sum_{r=1}^{q_i-1} \|f_{i,n} \otimes_r f_{i,n}\|_{\mathfrak{H}^{\otimes 2(q_i-r)}} \rightarrow 0$$

and

$$(4.6) \quad \frac{\sum_{r=1}^{q_i \wedge q_j - \delta_{q_i, q_j}} \sum_{s=1}^{q_i + q_j - 2r - 1} \|(f_{i,n} \tilde{\otimes}_r f_{j,n}) \otimes_s (f_{i,n} \tilde{\otimes}_r f_{j,n})\|_{\mathfrak{H}^{\otimes 2(q_i + q_j - 2r - 1 - s)}}}{\sum_{r=1}^{q_i \wedge q_j - \delta_{q_i, q_j}} \|f_{i,n} \tilde{\otimes}_r f_{j,n}\|_{\mathfrak{H}^{\otimes (q_i + q_j - 2r)}}^2} \rightarrow 0.$$

Proof. This is a direct consequence of (4.1), (4.2) and Lemma 2.9. \square

If we assume that $F_i = I_2(f_i)$ for $f_i \in \mathfrak{H}^{\otimes 2}$, $1 \leq i \leq d$, we can state conditions for the ACN property of $(F_n, \tilde{\Gamma}_{ij}(F_n))_{n \geq 1}$ which only involve cumulants. This is due to the well known formula (see [FT87])

$$(4.7) \quad \begin{aligned} \kappa_{k \times e_i}(F) &= 2^{k-1} (k-1)! \text{Tr} \left(H_{f_i}^k \right) \\ &= 2^{k-1} (k-1)! \left\langle f_i \otimes_1^{(k-1)} f_i, f_i \right\rangle_{\mathfrak{H}} = 2^{k-1} (k-1)! \sum_{n=1}^{\infty} \lambda_{f_i, n}^k, \end{aligned}$$

where $H_f: \mathfrak{H} \rightarrow \mathfrak{H}$ is the Hilbert-Schmidt operator defined by $H_f(g) = f \otimes_1 g$ and $\{\lambda_{f, n}: n \geq 1\}$ are its eigenvalues.

In particular, we have

$$(4.8) \quad \begin{aligned} \kappa_{4e_i}(F) &= 2^3 3! \left\langle f_i \otimes_1^{(3)} f_i, f_i \right\rangle_{\mathfrak{H}} \\ &= 2^3 3! \left\langle f_i \otimes_1 f_i, f_i \otimes_1 f_i \right\rangle_{\mathfrak{H}} = 2^3 3! \|f_i \otimes_1 f_i\|_{\mathfrak{H}}^2 \end{aligned}$$

and

$$(4.9) \quad \kappa_{8e_i}(F) = 2^7 7! \left\langle f_i \otimes_1^{(7)} f_i, f_i \right\rangle_{\mathfrak{H}} = 2^7 7! \|(f_i \otimes_1 f_i) \otimes_1 (f_i \otimes_1 f_i)\|_{\mathfrak{H}}^2.$$

Using Propositions 4.1, 4.2 and the Cauchy-Schwarz inequality, we can now deduce the following Proposition. We will however omit these elementary calculations, as the Proposition will also follow as a special case from Theorem 4.6 of the forthcoming section.

Proposition 4.3. *Let $(F_n) = (F_{1,n}, \dots, F_{d,n})$ be a sequence of random vectors whose components are elements of the second chaos, $g: \mathbb{R}^d \rightarrow \mathbb{R}$ be three times differentiable with bounded derivatives up to order three and assume that*

$$(4.10) \quad \sum_{i=1}^d \kappa_{4e_i}(F_n) \rightarrow 0.$$

a) *If, for $1 \leq i \leq d$, it holds that $\kappa_{4e_i}(F_n) \asymp \sum_{i=1}^d \kappa_{4e_i}(F_n)$ implies*

$$(4.11) \quad \frac{\kappa_{8e_i}(F_n)}{\kappa_{4e_i}(F_n)^2} \rightarrow 0,$$

then

$$\frac{\mathbb{E}[g(F_n)] - \mathcal{E}_3(F_n, Z_n, g)}{\Delta_\Gamma(F_n)} \rightarrow 0.$$

b) *If there exists a covariance matrix C such that $\Delta_C(F_n) \rightarrow 0$ and, for $1 \leq i \leq d$, the convergence (4.11) is implied by*

$$\begin{aligned} & \kappa_{4e_i}(F_n) + |\kappa_{e_i+e_j}(F_n) - C_{ij}| \\ & \asymp \sum_{i=1}^d \kappa_{4e_i}(F_n) + \sqrt{\sum_{i,j=1}^d (\kappa_{e_i+e_j}(F_n) - C_{ij})^2}, \end{aligned}$$

then

$$\frac{\mathbb{E}[g(F_n)] - \mathcal{E}_3(F_n, Z, g)}{\varphi_C(F_n)} \rightarrow 0.$$

Note that in the case $d = 1$, part b) becomes a (weaker) version of Proposition 3.8 from [NPR10b]. Thus, in the second chaos, the joint speed of convergence can be completely characterized by the coordinate sequences.

4.1. Majorizing integrals and the role of mixed contractions. We now turn to the question whether the mixed contractions (i.e. those for which $i \neq j$) in the numerator and denominator of condition (4.6) are necessary to ensure that $(F_n, \tilde{\Gamma}_{ij}(F_n))_{n \geq 1}$ is ACN. It will turn out that in some cases, most notably the one where all kernels are non-negative, we can replace condition (4.6) by a similar fraction containing only non-mixed contractions. In these cases, the ‘‘interplay’’ of the different kernels thus has no influence on the speed of convergence.

To be able to develop our theory, we assume that $\mathfrak{H} = L^2(A, \mathcal{A}, \nu)$, where (A, \mathcal{A}) is a Polish space, \mathcal{A} is the associated Borel σ -field and the measure ν is positive, σ -finite and non-atomic. This can be done without loss of generality (see [NP05, section 2.2]). The components of the random vectors F_n under examination are still multiple integrals.

If $f_i \in \mathfrak{H}^{\odot q_i}$ and $f_j \in \mathfrak{H}^{\odot q_j}$ are two symmetric kernels and $1 \leq r \leq q_i \wedge q_j$, then according to formula (2.13) we can write

$$(4.12) \quad \|f_i \tilde{\otimes}_r f_j\|_{\mathfrak{H}^{\otimes q_i+q_j-2r}}^2 = \sum_{u=0}^{(q_i \wedge q_j) - r} c_u G_r(f_i, f_j, u),$$

where the c_u are some positive universal constants not depending on f_i and f_j and each $G_r(f_i, f_j, u)$ is an integral of the form

$$(4.13) \quad \int_{A^u} \int_{A^u} \int_{A^r} \int_{A^r} \int_{A^n} \int_{A^m} f_i(v, x, y) f_i(v, \tilde{x}, \tilde{y}) f_j(\tilde{x}, y, w) f_j(x, \tilde{y}, w) \\ d\mu^{\otimes m}(v) d\mu^{\otimes n}(w) d\mu^{\otimes r}(x) d\mu^{\otimes r}(\tilde{x}) d\mu^{\otimes u}(y) d\mu^{\otimes u}(\tilde{y}),$$

where $m = q_i - r - u$ and $n = q_j - r - u$. We can visualize each of these integrals by an integer weighted, undirected graph

$$(4.14) \quad \begin{array}{ccc} f_i & \xrightarrow{q_i - r - u} & f_i \\ \left| \begin{array}{c} r \\ \\ \\ r \end{array} \right. & \begin{array}{c} \diagdown \\ u \\ \diagup \\ u \end{array} & \left. \begin{array}{c} r \\ \\ \\ r \end{array} \right| \\ f_j & \xrightarrow{q_j - r - u} & f_j \end{array},$$

by identifying each kernel occuring in the integral with a vertex and drawing an edge with weight l between two functions, if l variables of these two functions coincide. For example, the edge with label $q_i - r - u$ in the above graph corresponds to the variable v in the integral (4.13). Due to the symmetry of the kernels involved, we can freely translate back and forth between the explicit notation (4.13) and the visual notation (4.14) without losing any information. To avoid cumbersome treatment of degenerate cases, we adopt the convention that edges with weight zero are non-existent.

Analogously, we can write

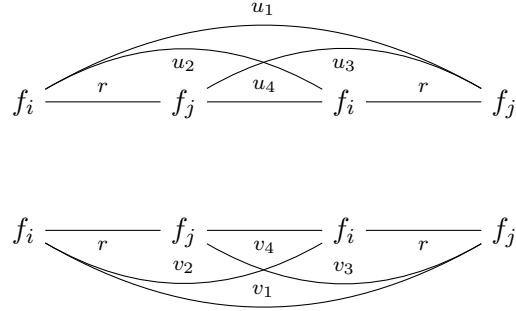
$$(4.15) \quad \|(f_i \tilde{\otimes}_r f_j) \otimes_s (f_i \tilde{\otimes}_r f_j)\|_{\mathfrak{S}^{\otimes 2(q_i + q_j - 2r - s)}}^2 = \sum_{\lambda \in \Lambda} c_\lambda G_{r,s}(f_i, f_j, \lambda),$$

where Λ is some finite index set, the c_λ are positive constants and the $G_{r,s}(f_i, f_j, \lambda)$ are integrals of the form

$$\int f_i(\cdot) f_i(\cdot) f_i(\cdot) f_i(\cdot) f_j(\cdot) f_j(\cdot) f_j(\cdot) f_j(\cdot)$$

involving four copies of the kernels f_i and f_j , respectively, which are obtained by first choosing $2(q_i + q_j)$ pairs of variables, then identifying variables that have been paired and finally integrating with respect to the $2(q_i + q_j)$ resulting variables. The only constraint one has to obey is that two variables that stem from the same kernel must not be paired. One could write this with a lot more rigour (using, for example, diagrams and partitions, see [PT11], or a visual method similar to ours, see [Mar08]) but for our purposes it is enough to know that each of this integrals can be visualized as a graph with eight vertices (four of them

labeled with f_i and f_j , respectively) that contains

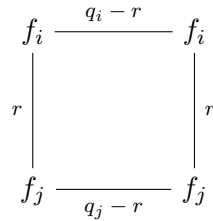


as a subgraph, where $\sum_i u_i = \sum_i v_i = s$ and some of the u_i and v_i can be zero (recall our convention that an edge with weight zero is non-existent). Note that in the original graph there are always edges (to be precise, exactly $2(q_i + q_j - r) - s$ of them) connecting the “upper” and “lower” groups of four edges.

If we are given an integral G occurring in the representations (4.12) or (4.15), we can arbitrarily divide the four or eight kernels appearing in the integrands into two sets A and B and use the Cauchy-Schwarz inequality to obtain a bound of the type $|G| \leq G_1^{1/2} G_2^{1/2}$, where the integrals G_1 and G_2 only involve kernels in the set A and B , respectively. Of course, G_1 and G_2 can also be visualized by graphs. In fact, these two graphs can be obtained without analytical detour by the following, purely visual “cut-mirror-merge”-operation on the graph of G :

- 1) Divide the vertices of G into two groups A and B .
- 2) Erase all edges that connect vertices of different groups, thus obtaining two subgraphs. We refer to a vertex adjacent to an edge that has been erased as a *bordering vertex*.
- 3) For each of these two subgraphs, take a copy of this subgraph and connect each bordering vertex of the subgraph with the corresponding vertex in the copy by an edge. The weight of this edge is equal to the sum of the weights of all erased edges that were adjacent to this vertex and have been erased in step two.

For example, starting from the integral given by the graph (4.14), if we choose two identical sets consisting of one f_i and one f_j , respectively, the resulting graphs for G_1 and G_2 are identical as well and given by



Translated back into the language of integrals, this is just the well-known fact that

$$\|f_i \tilde{\otimes}_r f_j\|_{\mathcal{S}^{\otimes(q_i+q_j-2r)}} \leq \|f_i \otimes_r f_j\|_{\mathcal{S}^{\otimes(q_i+q_j-2r)}},$$

which can of course be proven much more concisely by a direct calculation. However, the advantage of working with graphs reveals itself when dealing with the

integrals on the right hand side of (4.15). We will see this when proving the forthcoming Majorizing Lemma, that plays a key role in this section.

If $f_i = f_j$, some integrals appearing in the sum on the right hand side of (4.12) are of special interest, as they dominate all others (in a sense that will be made clear in the sequel). We title them *majorizing integrals*. They are defined as follows.

Definition 4.4 (Majorizing integrals). For $f \in \mathfrak{H}^{\odot q}$, $1 \leq r \leq q-1$ and $0 \leq m \leq q-r$, the *majorizing integrals* $M_r(f, m)$ are given by the graph

$$(4.16) \quad \begin{array}{c} \begin{array}{cccc} f & & f & & f & & f \\ \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \\ f & & f & & f & & f \\ \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \\ f & & f & & f & & f \end{array} \\ \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \end{array}$$

The diagram shows a graph with 8 vertices arranged in a 2x4 grid. The top row vertices are labeled f and the bottom row vertices are labeled f . There are four vertical edges between corresponding top and bottom vertices, each labeled $q-r-m$. There are four horizontal edges between adjacent vertices in both the top and bottom rows, each labeled r . There are two curved edges between the first and second vertices in the top row, each labeled m . Similarly, there are two curved edges between the first and second vertices in the bottom row, each labeled m . The entire graph is enclosed in large square brackets on the left, with the label (4.16) to its left.

Observe that the integrals $M_r(f, m)$ are non-negative and appear in the expansion of the type (4.15) for the norm $\|(f \tilde{\otimes}_r f) \otimes_s (f \tilde{\otimes}_r f)\|_{\mathfrak{H}^{\otimes 4(q-r)-2s}}$. Also, by grouping the inner four vertices in the graph (4.16) and applying “cut-mirror-merge”, we see that

$$M_r(f, m) \leq \|f \otimes_r f\|_{\mathfrak{H}^{\otimes 2(q-r)}}^4,$$

with equality if $m \in \{0, q-r\}$.

Lemma 4.5 (Majorizing Lemma). Let q_i and q_j be two positive integers and $f_i \in \mathfrak{H}^{\odot q_i}$, $f_j \in \mathfrak{H}^{\odot q_j}$ be two symmetric kernels. For given integers r and s with $1 \leq r \leq q_i \wedge q_j - \delta_{q_i, q_j}$ and $1 \leq s \leq q_i + q_j - 2r - 1$, let $G_{r,s}(f_i, f_j, \lambda)$ be one of the summands in the representation (4.15). Then, for $1 \leq k \leq 4$, there exists integers $n_k \in \{0, 1, \dots, q_i - r\}$ and $m_k \in \{0, 1, \dots, q_j - r\}$ with $m_k + n_k = s$, such that

$$(4.17) \quad G_{r,s}(f_i, f_j, \lambda)^8 \leq \prod_{k=1}^4 (M_r(f_i, m_k) M_r(f_j, n_k)).$$

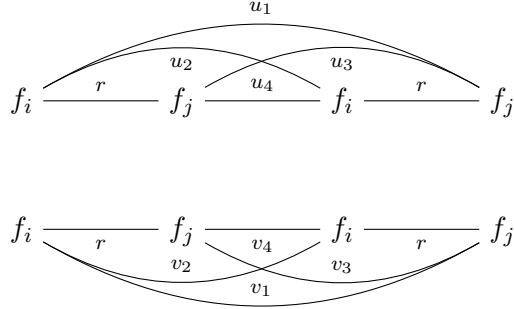
Proof. We will iteratively apply Cauchy-Schwarz (using the visual method developed above) to obtain the chain

$$(4.18) \quad G_{r,s}(f_i, f_j, \lambda)^8 \leq (G'_1 G''_1)^4 \leq (G'_{2,1} G'_{2,2} G''_{2,1} G''_{2,2})^2 \leq \prod_{k=1}^4 (M_r(f_i, m_k) M_r(f_j, n_k)),$$

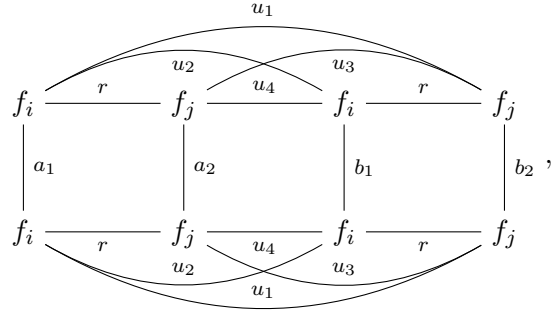
where all primed and double-primed quantities are integrals which will be described by their corresponding graphs.

As already mentioned, a graph associated with an integral $G_{r,s}(f_i, f_j, \lambda)$ in the sum (4.15) has eight vertices (four of them labeled with f_i , the other four with f_j)

and contains

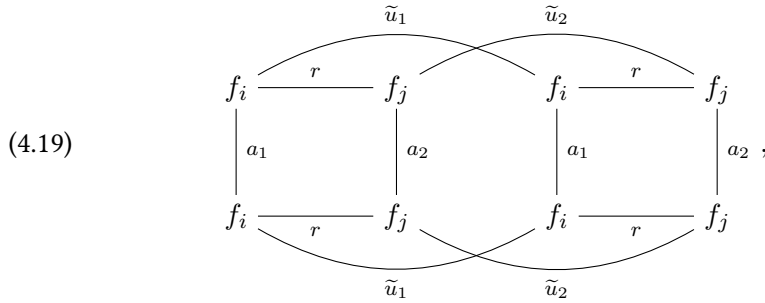


as a subgraph, where $\sum_i u_i = \sum_i v_i = s$ and some of the u_i and v_i can be zero (recall our convention that an edge with weight zero is non-existent). We now apply the Cauchy-Schwarz inequality for the first time, grouping the four vertices connected by the u_i - and v_i -edges. The resulting bounding integrals G'_1 and G''_1 are given by the graph



where $a_1 + a_2 + b_1 + b_2 = 2(q_i + q_j - r) - s > 0$ and the same graph with u_i replaced by v_i . We now continue to apply Cauchy-Schwarz to G'_1 . The exact same operations then have to be performed with G''_1 to obtain the final result.

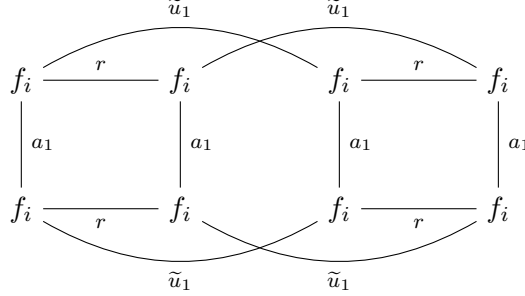
In the graph of G'_1 , we group the four vertices connected by a_i - and b_i -edges respectively and then apply Cauchy-Schwarz. This yields bounding integrals $G'_{2,1}$, given by a “cube” of the form



where $0 \leq a_1 + a_2 \leq 2(q_i + q_j - r) - s$, $0 \leq \tilde{u}_1, \tilde{u}_2 \leq s$ and $\tilde{u}_1 + \tilde{u}_2 = s$, and $G'_{2,2}$, given by the same graph with the a_i replaced by b_i .

From the graphs for $G'_{2,1}$ and $G'_{2,2}$, by grouping the four f_i - and f_j -vertices together and then applying Cauchy-Schwarz another time, we now obtain graphs that represent majorizing integrals. For example, starting from the graph (4.19)

for $G'_{2,1}$, we obtain



with $0 \leq \tilde{u}_1 \leq s$ and $a_1 = q_i - r - \tilde{u}_1$ and the same graph with f_i , \tilde{u}_1 and a_1 replaced by f_j , \tilde{u}_2 and a_2 , respectively.

Starting from G''_1 , the integrals $G''_{2,1}$ and $G''_{2,1}$ as well as the corresponding majorizing integrals are obtained analogously. Finally, a careful inspection of the single steps indicated above yields that the majorizing integrals obtained after the final application of the Cauchy-Schwarz inequality are indeed of the form stated in (4.17). \square

We are now ready to prove the main theorem of this section.

Theorem 4.6. *Let $g: \mathbb{R}^d \rightarrow \mathbb{R}$ be three times differentiable with bounded derivatives up to order three and assume that the following conditions are true.*

(i)

$$\sum_{i=1}^d \sum_{r=1}^{q_i-1} \|f_{i,n} \otimes_r f_{i,n}\|_{\mathfrak{H}^{\otimes 2(q_i-r)}} \rightarrow 0.$$

(ii) *For those $i, j \in \{1, 2, \dots, d\}$ for which $\sqrt{\text{Var } \Gamma_{ij}(F_n)} \asymp \Delta_\Gamma(F_n)$ it holds that*

$$(4.20) \quad \sum_{r=1}^{q_i-1} \|f_{i,n} \tilde{\otimes}_r f_{i,n}\|_{\mathfrak{H}^{\otimes 2(q_i-r)}} \asymp \sum_{r=1}^{q_i-1} \|f_{i,n} \otimes_r f_{i,n}\|_{\mathfrak{H}^{\otimes 2(q_i-r)}}.$$

and

$$(4.21) \quad \frac{\sum_{r=1}^{q_i-1} \sum_{s=1}^{q_i-r-1} M_r(f_{i,n}, s)}{\sum_{r=1}^{q_i-1} \|f_{i,n} \otimes_r f_{i,n}\|_{\mathfrak{H}^{\otimes 2(q_i-r)}}^4} \rightarrow 0.$$

Then it holds that

$$(4.22) \quad \frac{\mathbb{E}[g(F_n)] - \mathcal{E}_3(F_n, Z_n, g)}{\Delta_\Gamma(F_n)} \rightarrow 0.$$

If, in addition, there exists a covariance matrix C such that $\Delta_C(F_n) \asymp \Delta_\Gamma(F_n)$, then

$$(4.23) \quad \frac{\mathbb{E}[g(F_n)] - \mathcal{E}_3(F_n, Z, g)}{\varphi_C(F_n)} \rightarrow 0.$$

Proof. Let $i, j \in \{1, 2, \dots, d\}$ such that $\sqrt{\text{Var } \Gamma_{ij}(F_n)} \asymp \Delta_\Gamma(F_n)$ and assume, without loss of generality, that $q_i \leq q_j$.

We will show that

$$(4.24) \quad \text{Var } \Gamma_{ij}(F_n) \asymp \sqrt{\text{Var } \Gamma_{ii}(F_n) \text{Var } \Gamma_{jj}(F_n)}$$

and

$$(4.25) \quad \sum_{r=1}^{q_i \wedge q_j - \delta_{q_i q_j}} \sum_{s=1}^{q_i + q_j - 2r - 1} \|(f_{i,n} \tilde{\otimes}_r f_{j,n}) \otimes_s (f_{i,n} \tilde{\otimes}_r f_{j,n})\|_{\mathfrak{H}^{\otimes 2(q_i + q_j - 2r - s)}}^2 \preccurlyeq \sqrt{A_{1,n}} + \sqrt{A_{2,n}} + \sqrt{A_{3,n}},$$

where

$$A_{1,n} = \left(\sum_{r=1}^{q_i - 1} \sum_{s=1}^{2(q_i - r) - 1} M_r(f_{i,n}, s) \right) \left(\sum_{r=1}^{q_j - 1} \sum_{s=1}^{2(q_j - r) - 1} M_r(f_{j,n}, s) \right)$$

$$A_{2,n} = \left(\sum_{r=1}^{q_i - 1} \|f_{i,n} \otimes_r f_{i,n}\|_{\mathfrak{H}^{\otimes 2(q_i - r)}}^4 \right) \left(\sum_{r=1}^{q_j - 1} \sum_{s=1}^{2(q_j - r) - 1} M_r(f_{j,n}, s) \right)$$

and

$$A_{3,n} = \left(\sum_{r=1}^{q_j - 1} \|f_{j,n} \otimes_r f_{j,n}\|_{\mathfrak{H}^{\otimes 2(q_j - r)}}^4 \right) \left(\sum_{r=1}^{q_i - 1} \sum_{s=1}^{2(q_i - r) - 1} M_r(f_{i,n}, s) \right)$$

As a consequence, we get

$$\frac{\sum_{r=1}^{q_i \wedge q_j - \delta_{q_i q_j}} \sum_{s=1}^{q_i + q_j - 2r - 1} \|(f_{i,n} \tilde{\otimes}_r f_{j,n}) \otimes_s (f_{i,n} \tilde{\otimes}_r f_{j,n})\|_{\mathfrak{H}^{\otimes 2(q_i + q_j - 2r - s)}}^2}{\sum_{r=1}^{q_i \wedge q_j - \delta_{q_i q_j}} \|f_{i,n} \tilde{\otimes}_r f_{j,n}\|_{\mathfrak{H}^{\otimes (q_i + q_j - 2r)}}^2} \preccurlyeq \left(\frac{\sqrt{A_{1,n}} + \sqrt{A_{2,n}} + \sqrt{A_{3,n}}}{\sqrt{\text{Var } \Gamma_{ii}(F_n) \text{Var } \Gamma_{jj}(F_n)}} \right)^{1/2} \rightarrow 0,$$

where the convergence to zero is implied by assumptions (4.21) and (4.20). In view of Proposition 4.2, this proves (4.22). By the same argument we obtain (4.23), as $\Delta_C(F_n) \preccurlyeq \Delta_\Gamma(F_n)$ implies that

$$\sqrt{\text{Var } \Gamma_{ij}(F_n)} + |\mathbb{E}[F_{i,n} F_{j,n}] - C_{ij}| \asymp \varphi_C(F_n)$$

is equivalent to $\sqrt{\text{Var } \Gamma_{ij}(F_n)} \asymp \Delta_\Gamma(F_n)$.

To prove (4.24), note that the Cauchy-Schwarz inequality implies for $1 \leq r < q_i$ that

$$\begin{aligned} \|f_{i,n} \tilde{\otimes}_r f_{j,n}\|_{\mathfrak{H}^{\otimes (q_i + q_j - 2r)}}^2 &\leq \|f_{i,n} \otimes_r f_{j,n}\|_{\mathfrak{H}^{\otimes (q_i + q_j - 2r)}}^2 \\ &\leq \|f_{i,n} \otimes_r f_{i,n}\|_{\mathfrak{H}^{\otimes 2(q_i - r)}} \|f_{j,n} \otimes_r f_{j,n}\|_{\mathfrak{H}^{\otimes 2(q_j - r)}}. \end{aligned}$$

Therefore,

$$\begin{aligned}
& \sum_{r=1}^{q_i \wedge q_j - 1} \|f_{i,n} \tilde{\otimes}_r f_{j,n}\|_{\mathfrak{H}^{\otimes(q_i+q_j-2r)}}^2 \\
& \leq \sum_{r=1}^{q_i \wedge q_j - 1} \|f_{i,n} \otimes_r f_{i,n}\|_{\mathfrak{H}^{\otimes 2(q_i-r)}} \|f_{j,n} \otimes_r f_{j,n}\|_{\mathfrak{H}^{\otimes 2(q_j-r)}} \\
& \leq \left(\sum_{r=1}^{q_i-1} \|f_{i,n} \otimes_r f_{i,n}\|_{\mathfrak{H}^{\otimes 2(q_i-r)}} \right) \left(\sum_{r=1}^{q_j-1} \|f_{j,n} \otimes_r f_{j,n}\|_{\mathfrak{H}^{\otimes 2(q_j-r)}} \right)
\end{aligned}$$

Together with assumption (4.20), this gives

$$\begin{aligned}
\text{Var}_{ij}(F_n) &= \sum_{r=1}^{q_i \wedge q_j - \delta_{q_i q_j}} \|f_{i,n} \tilde{\otimes}_r f_{j,n}\|_{\mathfrak{H}^{\otimes(q_i+q_j-2r)}}^2 \\
&\asymp (1 - \delta_{q_i q_j}) \|f_{i,n} \tilde{\otimes}_{q_i} f_{j,n}\|_{\mathfrak{H}^{\otimes(q_j-q_i)}}^2 + \sqrt{\text{Var}_{ii}(F_n) \text{Var}_{jj}(F_n)} \\
&\asymp (1 - \delta_{q_i q_j}) \|f_{i,n} \tilde{\otimes}_{q_i} f_{j,n}\|_{\mathfrak{H}^{\otimes(q_j-q_i)}}^2 + \text{Var}_{ii}(F_n) + \text{Var}_{jj}(F_n) \\
&\asymp \Delta_\Gamma(F_n)^2,
\end{aligned}$$

where we set $\mathfrak{H}^{\otimes 0} = \mathbb{R}$. As $\sqrt{\text{Var}_{ij}(F_n)} \asymp \Delta_\Gamma(F_n)$, this shows

$$\text{Var}_{ij}(F_n) \asymp (1 - \delta_{q_i q_j}) \|f_{i,n} \tilde{\otimes}_{q_i} f_{j,n}\|_{\mathfrak{H}^{\otimes(q_j-q_i)}}^2 + \sqrt{\text{Var}_{ii}(F_n) \text{Var}_{jj}(F_n)}$$

and therefore (4.24).

Now let $1 \leq r \leq q_i \wedge q_j - \delta_{q_i q_j}$, $1 \leq s \leq q_i + q_j - 2r - 1$ and $G_{r,s}(f_{i,n}, f_{j,n}, \lambda)$ be an integral from the right hand side of representation (4.15). By the Majorizing Lemma 4.5, we can find integers $n_k \in \{0, 1, \dots, q_i - r\}$ and $m_k \in \{0, 1, \dots, q_j - r\}$ such that $m_k + n_k = s$ for $1 \leq k \leq 4$ and

$$G_{r,s}(f_{i,n}, f_{j,n}, \lambda) \leq \prod_{k=1}^4 (M_r(f_{i,n}, m_k) M_r(f_{j,n}, n_k))^{1/8}.$$

As clearly

$$M_r(f_{i,n}, m_k) M_r(f_{j,n}, n_k) \leq A_{1,n} + A_{2,n} + A_{3,n},$$

this gives

$$G_{r,s}(f_{i,n}, f_{j,n}, \lambda) \leq \prod_{k=1}^4 (A_{1,n} + A_{2,n} + A_{3,n})^{1/8} \leq \sqrt{A_{1,n}} + \sqrt{A_{2,n}} + \sqrt{A_{3,n}}.$$

The representation (4.15) thus yields

$$\|(f_{i,n} \tilde{\otimes}_r f_{j,n}) \otimes_s (f_{i,n} \tilde{\otimes}_r f_{j,n})\|_{\mathfrak{H}^{\otimes 2(q_i+q_j-2r-s)}}^2 \asymp \sqrt{A_{1,n}} + \sqrt{A_{2,n}} + \sqrt{A_{3,n}},$$

wich immediately implies (4.25). \square

Remark 4.7. a) Note that (4.20) is satisfied, if the kernels $f_{i,n}$ are either all non-negative or all non-positive for $n \geq n_0$.

b) In the case where $q_i = 2$ for $1 \leq i \leq d$, we obtain Proposition 4.3 as a special case of Theorem 4.6. Indeed, (4.20) is trivially satisfied, and, by (4.8) and (4.9), conditions (i) and (ii) are equivalent to (4.10) and (4.11), respectively.

5. EXAMPLES

In this section, we provide several examples that illustrate our techniques.

5.1. Step functions and matrix representations. We start with a counterexample, that in a way shows that the kernels involved can not be too “simple” in order for our techniques to work. Let $\mathfrak{H} = L^2([0, 1], \mu)$, where μ is the Lebesgue measure and partition $[0, 1]$ into N equidistant intervals $\alpha_1, \alpha_2, \dots, \alpha_N$ where $\alpha_k = [\frac{k-1}{N}, \frac{k}{N})$ for $k = 1, \dots, N$. Using this partition, we endow $[0, 1]^2$ with a grid and define a symmetric kernel $f \in \mathfrak{H}^{\otimes 2}$ that is constant on each sector by

$$(5.1) \quad f(x, y) = \sum_{i,j=1}^N a_{ij} 1_{\alpha_i}(x) 1_{\alpha_j}(y),$$

where the a_{ij} are real constants and $a_{ij} = a_{ji}$. Of course, f is uniquely determined by the symmetric matrix $A = (a_{ij})_{1 \leq i, j \leq N}$. If g is another kernel of the type (5.1), given by a matrix $B = (b_{ij})_{1 \leq i, j \leq N}$, we have

$$\begin{aligned} (f \otimes_1 g)(x, y) &= \int_0^1 \left(\sum_{i,j=1}^N a_{ij} 1_{\alpha_i}(t) 1_{\alpha_j}(x) \right) \left(\sum_{k,l=1}^N b_{kl} 1_{\alpha_k}(t) 1_{\alpha_l}(y) \right) d\mu(t) \\ &= \sum_{i,j,l=1}^N a_{ij} b_{jl} \mu(\alpha_j) 1_{\alpha_i}(x) 1_{\alpha_l}(y) \\ &= \frac{1}{N} \sum_{i,j,l=1}^N a_{ij} b_{jl} 1_{\alpha_i}(x) 1_{\alpha_l}(y) \end{aligned}$$

and

$$(f \tilde{\otimes}_1 g)(x, y) = \frac{1}{2N} \sum_{i,j,l=1}^N (a_{ij} b_{jl} + a_{lj} b_{ji}) 1_{\alpha_i}(x) 1_{\alpha_l}(y).$$

Therefore, $f \otimes_1 g$ can be identified with the matrix $C = \frac{1}{N} AB$ and $f \tilde{\otimes}_1 g$ by $\frac{1}{2}(C + C^T)$, where C^T denotes the transpose of C . Analogously, one can show that

$$\langle f, g \rangle_{\mathfrak{H}^{\otimes 2}} = \frac{1}{N^2} \langle A, B \rangle_{H.S.} = \frac{\text{tr}(AB^T)}{N^2}.$$

By formula (4.7), it is now easy to see that for any $m \geq 2$ we get

$$(5.2) \quad \kappa_m(I_2(f)) = 2^{m-1}(m-1)! \frac{\text{tr} A^m}{N^m}$$

and therefore, by the Cauchy-Schwarz inequality,

$$(5.3) \quad \frac{3!^2}{2 \times 7!} \frac{\kappa_8(I_2(f))}{\kappa_4(I_2(f))^2} = \frac{\text{tr}(A^8)}{\text{tr}(A^4)^2} = \left(1 + \frac{\sum_{1 \leq k, l \leq N, k \neq l} \lambda_k^4 \lambda_l^4}{\sum_{1 \leq k \leq N} \lambda_k^8} \right)^{-1} \geq \frac{1}{2},$$

where $(\lambda_k)_{1 \leq k \leq N}$ denotes the eigenvalue sequence of A . Now fix $d \geq 1$ and choose a sequence (N_n) of positive integers greater than two. For $n \geq 1$, define random vectors $F_n = (I_2(f_{1,n}), \dots, I_2(f_{d,n}))$, where the kernels $f_{i,n}$ are given by symmetric $(N_n \times N_n)$ -matrices $A_{i,n}$, $1 \leq i \leq d$, such that

$$\frac{\text{tr} A_{i,n}^4}{N_n^4} \rightarrow 0 \quad (n \rightarrow \infty)$$

for $1 \leq i \leq d$. Then, by the Fourth Moment Theorem 2.8, F_n converges in distribution to a d -dimensional, centered Gaussian random vector. However, by (5.3), condition (4.11) is never satisfied and thus Proposition 4.6 fails to provide optimal rates of convergence.

Remark 5.1. The following explicit example illustrates how symmetrization can drastically increase the speed of convergence (see [NR11], Remark 3.2(3) for another example): If $N = 2$ and the kernels f and g are represented by the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

respectively, it holds that $AB + BA = 0$ and $\text{tr}(ABBA) = 2$. Therefore, $\|f \otimes_1 g\|_{\mathfrak{H}^{\otimes 2}}^2 = 0$ while $\|f \otimes_1 g\|_{\mathfrak{H}^{\otimes 2}}^2 = \frac{1}{8}$.

5.2. Exploding integrals of Brownian sheets. Let $W = \{W(t_1, \dots, t_l) : 0 \leq t_1, \dots, t_l \leq 1\}$ be a standard Brownian sheet on $[0, 1]^l$, i.e. a centered Gaussian process such that

$$\mathbb{E}[W(s_1, \dots, s_l)W(t_1, \dots, t_l)] = \prod_{i=1}^l (s_i \wedge t_i)$$

for all $(s_1, \dots, s_l), (t_1, \dots, t_l) \in [0, 1]^l$. We can identify the Gaussian space generated by W with an isonormal process X on $\mathfrak{H} = L^2([0, 1]^l)$ via

$$W(t_1, \dots, t_l) \sim X \left(\prod_{i=1}^l 1_{[0, t_i]} \right).$$

For positive ε , we now define

$$(5.4) \quad F_\varepsilon = \int_{[0, 1]^l} \frac{W(t_1, \dots, t_l)^2}{(t_1 t_2 \cdots t_l)^{2-\varepsilon}} d(t_1, \dots, t_l).$$

An application of Jeulin's lemma (see [Jeu80, Lemma 1, p. 44]) shows that F_ε "explodes" in the limit, i.e. that P -almost surely

$$F_\varepsilon \rightarrow \infty \quad (\varepsilon \rightarrow 0).$$

However, for the normalized sequence \tilde{F}_ε , defined by

$$(5.5) \quad \tilde{F}_\varepsilon = \frac{F_\varepsilon - \mathbb{E}[F_\varepsilon]}{\sqrt{\mathbb{E}[F_\varepsilon^2]}},$$

the central limit theorem

$$(5.6) \quad \tilde{F}_\varepsilon \xrightarrow{\mathcal{L}} Z \sim \mathcal{N}(0, 1) \quad (\varepsilon \rightarrow 0).$$

holds. This is a consequence of the Fourth Moment Theorem 2.8 and the forthcoming formula (5.12) that provides asymptotics for the cumulants of \tilde{F}_ε . For slightly different exploding functionals of the above type, an analogous central limit theorem was established in [PY04] for the case $l = 1$, [DPY06] for the case $l = 2$ and [NP05] for the case $l > 2$. Exact asymptotics in the Kolmogorov distance were provided in [NP09a]. Here, we are interested in vectors of such functionals.

Routine calculations show that

$$\tilde{F}_\varepsilon = \frac{F_\varepsilon - \mu_\varepsilon}{\sigma_\varepsilon} = \frac{1}{\sigma_\varepsilon} I_2(f_\varepsilon),$$

where

$$(5.7) \quad f_\varepsilon(x_1, \dots, x_l, y_1, \dots, y_l) = \prod_{k=1}^l \int_0^1 \frac{1_{[0, t_k]}(x_k) 1_{[0, t_k]}(y_k)}{t_k^{2-\varepsilon}} dt_k,$$

$$\mu_\varepsilon = \mathbb{E} \left[\tilde{F}_\varepsilon \right], \text{ and } \sigma_\varepsilon^2 = \mathbb{E} \left[\tilde{F}_\varepsilon^2 \right].$$

From (5.7) we conclude that if $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k$ are k positive numbers, then

$$(5.8) \quad \langle (\cdots ((f_{\varepsilon_1} \otimes_1 f_{\varepsilon_2}) \otimes_1 f_{\varepsilon_3}) \otimes_1 \cdots) \otimes_1 f_{\varepsilon_{k-1}}, f_{\varepsilon_k} \rangle_{\mathfrak{H}} = C(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)^l,$$

where

$$(5.9) \quad C(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k) = \int_{[0,1]^k} \frac{(s_1 \wedge s_2)(s_2 \wedge s_3) \cdots (s_{k-1} \wedge s_k)(s_k \wedge s_1)}{s_1^{2-\varepsilon_1} s_2^{2-\varepsilon_2} \cdots s_k^{2-\varepsilon_k}} d(s_1, \dots, s_k).$$

For convenience, we will write $C_k(\varepsilon) = C(\varepsilon, \dots, \varepsilon)$, if all k arguments are equal. With this notation, we have $\mu_\varepsilon = C(\varepsilon)^l$ and $\sigma_\varepsilon^2 = C_2(\varepsilon)^l$. By partitioning the k -dimensional unit interval into simplexes, the integral (5.9) can be computed explicitly. These calculations yield that

$$(5.10) \quad C(\varepsilon_1, \dots, \varepsilon_k) = \tilde{c}(\varepsilon_1, \dots, \varepsilon_k) \times \frac{k!}{\varepsilon_1 + \dots + \varepsilon_k}.$$

Here, $\tilde{c}(\varepsilon_1, \dots, \varepsilon_k)$ is the canonical symmetrization of

$$c(\varepsilon_1, \dots, \varepsilon_k) = ((1 + \varepsilon_1)(1 + \varepsilon_1 + \varepsilon_2) \cdots (1 + \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{k-1}))^{-1}.$$

Observe that $0 < \tilde{c}(\varepsilon_1, \dots, \varepsilon_k) < 1$ and $\tilde{c}(\varepsilon_1, \dots, \varepsilon_k) \rightarrow 1$ if $\varepsilon_1, \dots, \varepsilon_k \rightarrow 0$.

Now fix $d \geq 1$ and, for $\varepsilon_1, \dots, \varepsilon_d > 0$, define $\tilde{F}_{(\varepsilon_1, \dots, \varepsilon_d)} = (\tilde{F}_{\varepsilon_1}, \dots, \tilde{F}_{\varepsilon_d})$. As the inner product on the left hand side of (5.8) does not depend on the order in which the k kernels $f_{\varepsilon_1}, f_{\varepsilon_2}, \dots, f_{\varepsilon_k}$ are contracted, the cumulant formula (2.22) yields that

$$(5.11) \quad \kappa_\alpha(\tilde{F}_{(\varepsilon_1, \dots, \varepsilon_d)}) = 2^{|\alpha|-1} (|\alpha| - 1)! C(\varepsilon_{l_1}, \varepsilon_{l_2}, \dots, \varepsilon_{l_{|\alpha|}})^l \prod_{k=1}^{|\alpha|} C_2(\varepsilon_{l_k})^{-l/2},$$

where $\alpha \in \mathbb{N}_0^d$ with elementary decomposition $\{l_1, \dots, l_{|\alpha|}\}$. Identity (5.10) shows that

$$(5.12) \quad \kappa_\alpha(\tilde{F}_n)^{-l} \asymp \frac{(\varepsilon_{l_1} \cdots \varepsilon_{l_{|\alpha|}})^{1/2}}{\varepsilon_{l_1} + \dots + \varepsilon_{l_{|\alpha|}}}.$$

This allows us to apply Proposition 4.6 to the vector $\tilde{F}_{(\varepsilon_1, \dots, \varepsilon_d)}$. We obtain the following explicit result.

Proposition 5.2. *For $\varepsilon_1, \dots, \varepsilon_d > 0$, let $\tilde{F}_{\varepsilon_1, \dots, \varepsilon_d} = (\tilde{F}_{\varepsilon_1}, \dots, \tilde{F}_{\varepsilon_d})$ be defined by (5.5) and let $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a three times differentiable function with bounded derivatives up to order three. Then it holds that*

$$(5.13) \quad \frac{\mathbb{E} \left[g(\tilde{F}_{\varepsilon_1, \dots, \varepsilon_d}) \right] - \mathcal{E}_3(\tilde{F}_{\varepsilon_1, \dots, \varepsilon_d}, Z_{\varepsilon_1, \dots, \varepsilon_d}, g)}{\sqrt{\sum_{i=1}^d \varepsilon_i^l}} \rightarrow 0, \quad (\varepsilon_1, \dots, \varepsilon_d \rightarrow 0),$$

where $Z_{\varepsilon_1, \dots, \varepsilon_d}$ is a centered Gaussian random variable with the same covariance matrix as $\tilde{F}_{(\varepsilon_1, \dots, \varepsilon_d)}$. If, in addition, for $1 \leq i, j \leq d$ it holds that

$$(5.14) \quad \frac{2}{\sqrt{\frac{\varepsilon_i}{\varepsilon_j}} + \sqrt{\frac{\varepsilon_j}{\varepsilon_i}}} \rightarrow C_{ij}^{1/l}, \quad (\varepsilon_1, \dots, \varepsilon_d \rightarrow 0),$$

and

$$(5.15) \quad \frac{\sum_{i,j=1}^d \left(\left(\sqrt{\frac{\varepsilon_i}{\varepsilon_j}} + \sqrt{\frac{\varepsilon_j}{\varepsilon_i}} \right)^{-l} - \frac{C_{ij}}{2^l} \right)^2}{\sum_{i=1}^d \varepsilon_i^l} \rightarrow 0, \quad (\varepsilon_1, \dots, \varepsilon_d \rightarrow 0),$$

where $C_{ij} \geq 0$, then

$$\frac{\mathbb{E} \left[g(\tilde{F}_{(\varepsilon_1, \dots, \varepsilon_d)}) \right] - \mathcal{E}_3(\tilde{F}_{(\varepsilon_1, \dots, \varepsilon_d)}, Z, g)}{\sqrt{\sum_{i=1}^d \varepsilon_i^l}} \rightarrow 0 \quad (\varepsilon_1, \dots, \varepsilon_d \rightarrow 0)$$

and

$$\begin{aligned} & \lim_{\varepsilon_1, \dots, \varepsilon_d \rightarrow 0} \frac{\mathbb{E} \left[g(\tilde{F}_{(\varepsilon_1, \dots, \varepsilon_d)}) \right] - \mathbb{E} [g(Z)]}{\varphi_C(\tilde{F}_{(\varepsilon_1, \dots, \varepsilon_d)})} \\ &= 4\sqrt{6} \lim_{\varepsilon_1, \dots, \varepsilon_d \rightarrow 0} \frac{\sum_{i,j,k=1}^d \left(\sqrt{\frac{\varepsilon_i}{\varepsilon_j \varepsilon_k}} + \sqrt{\frac{\varepsilon_j}{\varepsilon_i \varepsilon_k}} + \sqrt{\frac{\varepsilon_k}{\varepsilon_i \varepsilon_j}} \right)^{-l} \mathbb{E} [\partial_{ijk} g(Z)]}{\sqrt{\sum_{i,j=1}^d \left(\frac{1}{\varepsilon_i} + \frac{1}{\varepsilon_j} \right)^{-l}}}. \end{aligned}$$

Here, $\varphi_C(\tilde{F}_{(\varepsilon_1, \dots, \varepsilon_d)})^2 \asymp \sum_{i=1}^d \varepsilon_i^l$ and Z is a d -dimensional, centered Gaussian random variable with covariance $C = (C_{ij})_{i,j=1}^d$.

Note that by (5.11) and (5.10), the Edgeworth expansions $\mathcal{E}_3(\tilde{F}_{\varepsilon_1, \dots, \varepsilon_d}, Z_{\varepsilon_1, \dots, \varepsilon_d}, g)$ and $\mathcal{E}_3(\tilde{F}_{\varepsilon_1, \dots, \varepsilon_d}, Z, g)$ can be calculated explicitly.

To illustrate this, we choose a positive sequence $(a_n)_{n \geq 1}$ converging to zero and two positive numbers ξ and ζ . Then it holds that $\tilde{F}_{(\xi \cdot a_n, \zeta \cdot a_n)}$ has covariance

$$C = \begin{pmatrix} 1 & \rho^l \\ \rho^l & 1 \end{pmatrix}$$

for all $n \geq 1$, where

$$\rho = \frac{2}{\sqrt{\frac{\xi}{\zeta}} + \sqrt{\frac{\zeta}{\xi}}}.$$

Thus, the conditions (5.14) and (5.15) are trivially satisfied and all conclusions of Proposition 5.2 hold.

5.3. Continuous time Toeplitz quadratic functionals. Let $(X_t)_{t \geq 0}$ be a centered, real valued Gaussian process with a covariance function r of the form $r(t) = \mathbb{E} [X_u X_{u+t}] = \hat{f}(t)$, where $f: \mathbb{R} \rightarrow \mathbb{R}$ is an integrable, even function, customarily called the spectral density of the process (X_t) and \hat{f} denotes its Fourier

transform $t \mapsto \int_{-\infty}^{\infty} e^{ixt} f(x) dx$. If $h: \mathbb{R} \rightarrow \mathbb{R}$ is another integrable even function with Fourier transform \widehat{h} and $T > 0$, we define the Toeplitz functional $Q_{h,T}$ associated with h and T by

$$Q_{h,T} := \int_{[0,T]^2} \widehat{h}(t-s) X_t X_s d(s,t).$$

and denote a normalized version by

$$\widetilde{Q}_T = \frac{Q_{h,T} - \mathbb{E}[Q_{h,T}]}{\sqrt{T}}.$$

In the following, we want to apply our results to sequences of random vectors whose components are (normalized) Toeplitz functionals, analogous to the treatment in [NP09a] for the one-dimensional case.

For $T > 0$ and $\psi \in L^1(\mathbb{R})$, the *truncated Toeplitz operator* $B_T(\psi)$, defined on $L^2(\mathbb{R})$, is given by

$$B_T(\psi)(u)(t) = \int_0^T u(x) \widehat{\psi}(t-x) dx.$$

As usual, if $\psi_1, \dots, \psi_m \in L^1(\mathbb{R})$ and $j \geq 1$, we write

$$B_T(\psi_m) B_T(\psi_{m-1}) \cdots B_T(\psi_1) = B_T(\psi_m) \circ B_T(\psi_{m-1}) \circ \cdots \circ B_T(\psi_1)$$

and

$$(B_T(\psi_2) B_T(\psi_1))^j = \underbrace{B_T(\psi_2) B_T(\psi_1) \cdots B_T(\psi_2) B_T(\psi_1)}_{j \text{ times}}$$

for its operator products and powers, respectively. Explicitly, the above operator product takes the form

$$\begin{aligned} & (B_T(\psi_m) B_T(\psi_{m-1}) \cdots B_T(\psi_1)(u))(t) \\ &= \int_0^T \cdots \int_0^T \int_0^T u(x_1) \widehat{\psi}_1(x_2-x_1) \widehat{\psi}_2(x_3-x_2) \cdots \widehat{\psi}_m(t-x_m) dx_1 dx_2 \cdots dx_m. \end{aligned}$$

To adapt the setting to our framework, we introduce the Hilbert space of complex-valued, square integrable and even functions $h: \mathbb{R} \rightarrow \mathbb{C}$, equipped with the inner product $\langle h_1, h_2 \rangle_{\mathfrak{H}} = \int_{-\infty}^{\infty} h_1(x) \overline{h_2(x)} f(x) dx$. The process $(X_t)_{t \geq 0}$ can then be identified with an isonormal Gaussian process on the real subspace \mathfrak{H} generated by the family $\{x \mapsto e^{ixt}: t > 0\}$. This allows us to represent the normalized Toeplitz functional $\widetilde{Q}_{h,T}$ as a multiple integral of second order with kernel $\varphi_{h,T}$, which is given by

$$(5.16) \quad \varphi_{h,T}(x,y) = \int_{[0,T]^2} \widehat{h}(t-s) e^{i(sx+ty)} ds dt.$$

We now choose even functions $h_1, \dots, h_d \in L^1(\mathbb{R})$ be even functions and define a random vector $F_T = (F_{1,T}, \dots, F_{d,T})$ by setting $F_{i,T} = \widetilde{Q}_{h_i,T}$ for $1 \leq i \leq d$ and $T > 0$. The following Theorem collects some results from the literature.

Theorem 5.3. *Let $\alpha \in \mathbb{N}_0^d$ be a multi-index with $|\alpha| \geq 2$ and elementary decomposition $\{l_1, \dots, l_{|\alpha|}\}$. Then the following is true.*

a) The cumulant $\kappa_\alpha(F_T)$ is given by

$$(5.17) \quad \kappa_\alpha(F_T) = T^{-|\alpha|/2} 2^{|\alpha|-1} (|\alpha| - 1)! \operatorname{Tr} \left[B_T(f)^{|\alpha|} \prod_{i=1}^{|\alpha|} B_T(h_{l_i}) \right].$$

b) If $f \in L^1(\mathbb{R}) \cap L^{q_0}(\mathbb{R})$ and $h_i \in L^1(\mathbb{R}) \cap L^{q_i}(\mathbb{R})$ such that $1/q_0 + 1/q_i \leq 1/|\alpha|$ for $1 \leq i \leq d$, then

$$(5.18) \quad \lim_{T \rightarrow \infty} T^{|\alpha|/2-1} \kappa_\alpha(F_T) = 2^{|\alpha|-1} (|\alpha| - 1)! \int_{-\infty}^{\infty} f^{|\alpha|}(x) \prod_{i=1}^{|\alpha|} h_{l_i}(x) dx.$$

c) If $f \in L^1(\mathbb{R}) \cap L^{q_0}(\mathbb{R})$ and $h_i \in L^1(\mathbb{R}) \cap L^{q_i}(\mathbb{R})$ such that $1/q_0 + 1/q_i \leq 1/2$ for $1 \leq i \leq d$, then

$$(5.19) \quad F_T \xrightarrow{d} \mathcal{N}(0, C), \quad (T \rightarrow \infty),$$

where the covariance matrix $C = (C_{ij})_{1 \leq i, j \leq d}$ is given by

$$C_{ij} = 2 \int_0^\infty f^2(x) h_i(x) h_j(x) dx.$$

Proof. Part a) follows from a straightforward adaptation of the arguments in [GS84, Chapter 11] to multiple dimensions. Part b) follows from a) and [Gin94, Theorem 1a)]. Finally, part c) is a consequence of part b) and the Fourth Moment Theorem 2.8. For the one-dimensional case ($d = 1$), part c) was first proven in [Gin94]. Weaker conditions for the convergence (5.19) to take place can be found in [GS07]. \square

We are now able to prove the following Edgeworth expansion.

Proposition 5.4. *In the above framework, assume that $f \in L^1(\mathbb{R}) \cap L^{q_0}(\mathbb{R})$, $h_i \in L^1(\mathbb{R}) \cap L^{q_i}(\mathbb{R})$ such that $1/q_0 + 1/q_i \leq 1/8$ for $1 \leq i \leq d$ and let $g: \mathbb{R}^d \rightarrow \mathbb{R}$ be three times differentiable with bounded derivatives up to order three. Then it holds that*

$$(5.20) \quad \frac{\mathbb{E}[g(F_T)] - \mathcal{E}_3(F_T, Z, g)}{\varphi_C(F_T)} \rightarrow 0 \quad (T \rightarrow \infty),$$

where Z is a d -dimensional Gaussian random variable with covariance matrix $C = (C_{ij})_{1 \leq i, j \leq d}$ given by $C_{ij} = 2 \int_0^\infty f^2(x) h_i(x) h_j(x) dx$.

Proof. By Theorem 5.3b), we immediately verify the conditions of Proposition 4.6b) and obtain the result. \square

5.4. Edgeworth expansions for the Breuer-Major Theorem. Define $B = \{B_x: x \geq 0\}$ to be a fractional Brownian motion with Hurst index $H \in (0, \frac{1}{2})$, i.e. a centered Gaussian process with covariance

$$\mathbb{E}[B_x B_y] = \frac{1}{2} \left(x^{2H} + y^{2H} - |x - y|^{2H} \right), \quad x, y \geq 0.$$

For fixed $H \in (0, \frac{1}{2})$, the Gaussian space generated by B can be identified with an isonormal Gaussian process $X = \{X(h): h \in \mathfrak{H}\}$, where the real and separable Hilbert space \mathfrak{H} is the closure of the set of all \mathbb{R} -valued step functions on \mathbb{R}_+ with respect to the inner product

$$\langle 1_{[0,x]}, 1_{[0,y]} \rangle_{\mathfrak{H}} := \mathbb{E}[B_x B_y].$$

In particular, we have $B_x = X(1_{[0,x]})$ for $x > 0$. For more details on fractional Brownian motion see for example [NP12], [Nua06] or [Nou12]. We denote the covariance function of the stationary increment process $(B_{x+1} - B_x)_{x \geq 0}$ by

$$\rho(t) := \mathbb{E}[B_{x+1-x}(B_{x+1+t} - B_{x+t})] = \frac{1}{2} \left(|t+1|^{2H} + |t-1|^{2H} - 2|t|^{2H} \right).$$

Now choose d integers $q_i \geq 2$ and, for $T > 0$, define the centered random vectors $F_T = (F_{1,T}, \dots, F_{d,T})$ by

$$(5.21) \quad F_{i,T} = \frac{1}{\sqrt{T}} \int_0^T H_{q_i}(B_{u+1} - B_u) \, du.$$

We immediately see that the covariance matrix $C_T = (C_{ij,T})_{1 \leq i,j \leq d}$ of F_T is given by

$$C_{ij,T} = \mathbb{E}[F_{i,T} F_{j,T}] = \delta_{q_i q_j} \frac{q_i!}{T} \int_{[0,T]^2} \rho^{q_i}(u-v) \, d(u,v)$$

and converges to $C = (C_{ij})_{1 \leq i,j \leq d}$ for $T \rightarrow \infty$, where

$$C_{ij} = \delta_{q_i q_j} q_i! \int_{-\infty}^{\infty} \rho^{q_i}(x) \, dx.$$

It is well known (see for example [BM83] or [GS85]) that for each component $F_{i,T}$ the central limit theorem

$$F_{i,T} \xrightarrow{\mathcal{L}} \mathcal{N}(0, C_{ii}), \quad (T \rightarrow \infty)$$

holds and the Fourth Moment Theorem 2.8A therefore implies the joint convergence of F_T towards a centered d -dimensional Gaussian random vector Z with covariance C . By applying our methods, we are able to derive a fluctuating and non-fluctuating Edgeworth expansion for F_T , which in many cases yield exact asymptotics.

Theorem 5.5. *If, in the above setting, g is three times differentiable with bounded derivatives up to order three it holds that*

$$(5.22) \quad \frac{\mathbb{E}[g(F_T)] - \mathcal{E}_3(F_T, Z, g)}{1/\sqrt{T}} \rightarrow 0, \quad (T \rightarrow \infty).$$

Remark 5.6. For a non-trivial application of Theorem 5.5 at least one of the integers q_i should be even. Indeed, otherwise the Edgeworth expansion $\mathcal{E}_3(F_T, Z_T, g)$ (or $\mathcal{E}_3(F_T, Z, g)$, respectively) would merely reduce to the expectation $\mathbb{E}[g(Z_T)]$ (or $\mathbb{E}[g(Z)]$).

Proof of Theorem 5.5. We want to apply Theorem 4.6. Observe that by the well-known relation $H_{q_i}(B_{u+1} - B_u) = I_q(1_{[u,u+1]}^{\otimes q_i})$ we can represent each component $F_{i,T}$ by a multiple integral $I_{q_i}(f_{i,T})$, where the kernels $f_{i,T}$ are given by

$$f_{i,T} = \frac{1}{\sqrt{T}} \int_0^T 1_{[u,u+1]}^{\otimes q_i} \, du.$$

Straightforward calculations now yield

$$\begin{aligned}
& \|f_{i,T} \otimes_r f_{i,T}\|_{\mathfrak{S}^{\otimes 2(q_i-r)}}^2 \\
&= \frac{1}{T^2} \int_{[0,T]^4} \rho(v_1 - u_1)^r \rho(v_2 - u_2)^r \\
&\quad \rho(v_2 - v_1)^{q_i-r} \rho(u_2 - u_1)^{q_i-r} \mathrm{d}(u_1, u_2, v_1, v_2) \\
&\asymp \frac{1}{T} \int_{[-T,T]^3} \rho(x_1)^r \rho(x_2)^r \rho(x_3)^{q_i-r} \rho(x_1 - x_2 + x_3)^{q_i-r} \mathrm{d}(x_1, x_2, x_3),
\end{aligned}$$

where the integral in the last line, which has been obtained by a change of variables, converges for $T \rightarrow \infty$ (due to the restriction $H \in (0, 1/2)$). As symmetrizing only changes the exponents of the factors of the integrand, we see that

$$(5.23) \quad \|f_{i,T} \otimes_r f_{i,T}\|_{\mathfrak{S}^{\otimes 2(q_i-r)}} \asymp \|f_{i,T} \widetilde{\otimes}_r f_{i,T}\|_{\mathfrak{S}^{\otimes 2(q_i-r)}} \asymp \frac{1}{\sqrt{T}}.$$

Moreover, for $1 \leq r \leq q_i - 1$ and $1 \leq m \leq q_i - r - 1$, we see that the majorizing integrals $M_r(f_{i,T}, s)$ are given by

$$\begin{aligned}
M_r(f_{i,T}, s) &= \frac{1}{T^4} \int_{[0,T]^8} (\rho(v_1 - u_1)\rho(v_2 - u_2)\rho(v_3 - u_3)\rho(v_4 - u_4))^r \\
&\quad (\rho(u_2 - u_1)\rho(v_2 - v_1)\rho(u_4 - u_3)\rho(v_4 - v_3))^s \\
&\quad (\rho(u_3 - u_1)\rho(v_3 - v_1)\rho(u_4 - u_2)\rho(v_4 - v_2))^{q_i-r-s} \\
&\quad \mathrm{d}(u_1, \dots, u_4, v_1, \dots, v_4) \\
&\asymp \frac{1}{T^3} \int_{[-T,T]^7} (\rho(x_1)\rho(x_2)\rho(x_3)\rho(x_4))^r \\
&\quad (\rho(x_5)\rho(x_2 + x_5 - x_1)\rho(x_6)\rho(x_4 + x_6 - x_3))^s \\
&\quad (\rho(x_7)\rho(x_3 + x_7 - x_1)\rho(x_6 + x_7 - x_5) \\
&\quad \quad \rho(x_4 + x_6 + x_7 - x_5 - x_2))^{q_i-r-s} \\
&\quad \mathrm{d}(x_1, \dots, x_7).
\end{aligned}$$

By the same argument as before, the integral converges for $T \rightarrow \infty$ and we get

$$(5.24) \quad M_r(f_{i,T}, s) \asymp \frac{1}{T^3}.$$

Together with the asymptotic relation (5.23), this shows that conditions (i) and (ii) of Theorem 4.6 are satisfied. It remains to show that $\Delta_C(F_T) \asymp \Delta_\Gamma(F_T)$, so that (5.22) follows. by Theorem 4.6. A linear change of variables allows us to write

$$C_{ij,T} - C_{ij} = \delta_{q_i q_j} \frac{q_i!}{T} \int_0^T \int_{(-\infty, -v) \cup (T-v, \infty)} \rho^{q_i}(w) \mathrm{d}w \mathrm{d}v.$$

Using the well known asymptotic relation $\rho(t) \asymp t^{2(H-1)}$, we get

$$C_{ij,T} - C_{ij} \asymp \delta_{q_i q_j} T^{2(H-1)q_i+1}$$

and thus

$$\Delta_C(F_T) \asymp T^{1+2(H-1)q_{\min}}.$$

As, by (5.23), $\Delta_\Gamma(F_T) \asymp 1/\sqrt{T}$ and $1 + 2(H-1)q_{\min} < -1/2$, the proof is finished. \square

ACKNOWLEDGEMENTS

The author heartily thanks Giovanni Peccati for many fruitful discussions and remarks, Denis Serre for providing the elegant example in Remark 5.1 and an anonymous referee for the other elegant example hinted at in the same remark and for pointing out several mistakes in a previous version.

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