Isoparametric Boundary Elements

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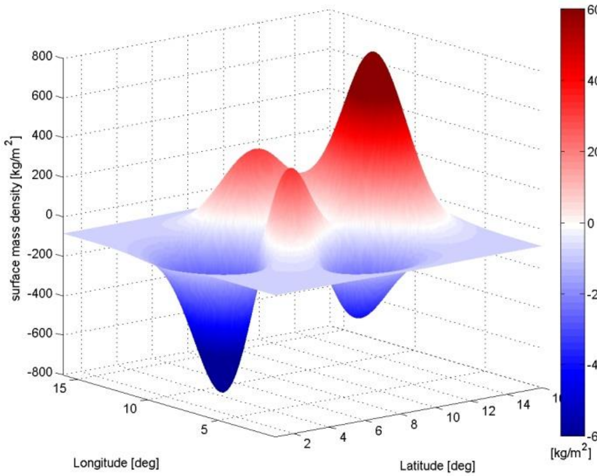
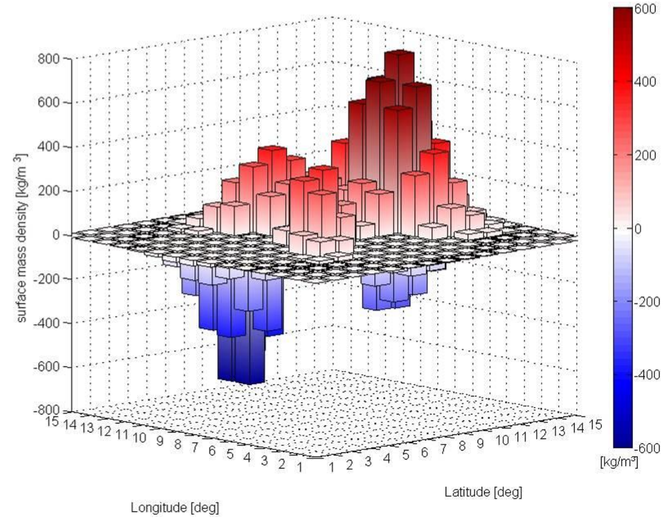
Application

Primary objective:	*local gravity field determination from CHAMP, GRACE (and GOCE)*
Interest:	*improved static and time-variable gravity field* *(make use of residual signal and data distribution)*
Input data:	*potential values along the orbit* *gravity gradient along the orbit* *relative gradient along the orbit*
Data type:	*currently only single data type*
However:	*no restrictions on the combination of different data types (combination on normal equation level)*

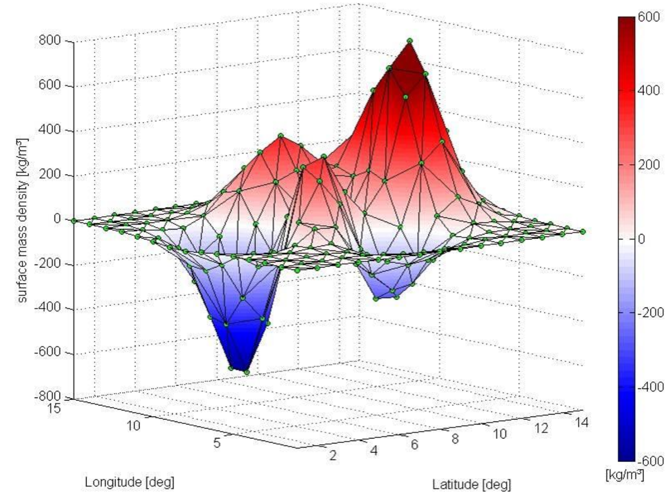
Basic principle

Decomposition of the surface into elements with finite extend (boundary elements)





e.g. blocks



e.g. triangles

Basic principle

Example: Consider the single layer potential

$$V = \int_{\Omega} \frac{\sigma\left(\vec{X}_Q\right)}{\|\vec{X} - \vec{X}_Q\|} d\vec{X}_Q$$

Separation of the surface into elements: $\Omega = \sum_{i=1}^{N} \Omega_i$

Assumption:

$$\vec{X} = \sum_{k=1}^{K} \Phi_{i,k}^{\vec{X}} \vec{X}_{i,k}$$

$$\sigma_i\left(\vec{X}_Q\right) = \sum_{k=1}^{K} \Phi_{i,k}^{\sigma} \sigma_{i,k}\left(\vec{X}_{i,k}\right)$$

The boundary elements are called ***isoparametric*** *if* $\Phi_{i,k}^{\vec{X}} = \Phi_{i,k}^{\sigma}$

Basic principle

Example: Consider the single layer potential

Including transformation to the normal triangle and spherical integration

$$V = \frac{GR^2}{4\pi} \sum_{i=1}^{N} \sum_{k=1}^{K} \sigma_{i,k} \int_{-1}^{1} \int_{-1}^{-\xi} \frac{J_i(\xi,\eta) \cdot \Phi_{i,k}(\xi,\eta) \cdot \cos\phi(\xi,\eta)}{\|\vec{X}(r,\phi,\lambda) - \vec{X}(R,\phi(\xi,\eta),\lambda(\xi,\eta))\|} d\eta d\xi$$

Integration by Gaussian quadrature

$$\int_{-1}^{1} \int_{-1}^{-\xi} \ldots d\eta d\xi \Rightarrow \sum_{l=1}^{L} \sum_{m=1}^{L} w_l w_m \ldots \qquad \text{with } w_l w_m = 0 \text{ for } m > l$$

Integration is exact for a polynomial of order 2L.

Mathematical properties

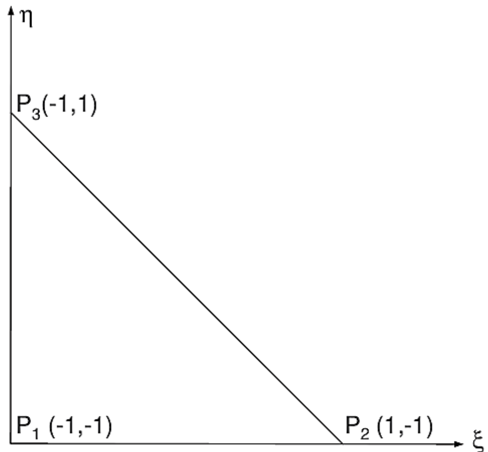
Base functions are

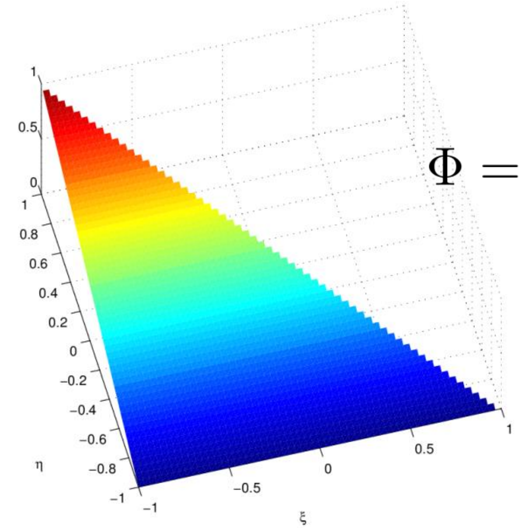
- *strictly space-limited (i.e. band-unlimited)*
- *compact*
- *continuous but not differentiable at the edges*
- *singular if the point of interest lies inside or on the corner of the element*

  - *Weak singularity for potential* $\left(\frac{1}{r}\right)$
  - *Strong singularity for gradient* $\left(\frac{1}{r^3}\right)$

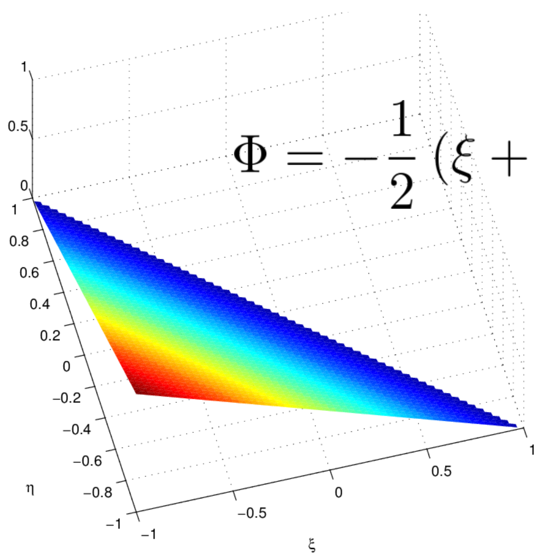
Base function

Example: linear triangle

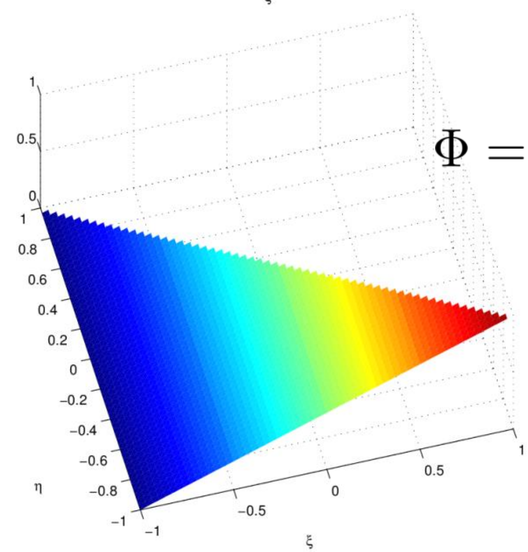




$$\Phi = \frac{1}{2}(\eta + 1)$$



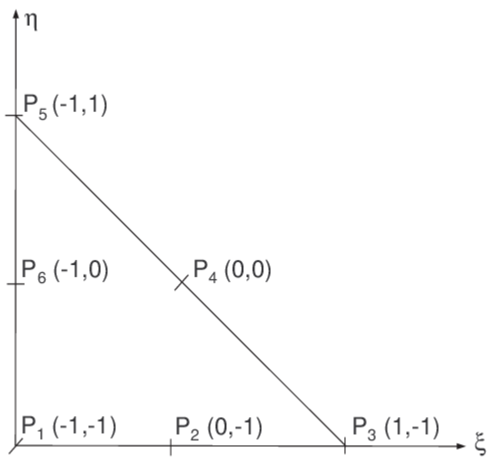
$$\Phi = -\frac{1}{2}(\xi + \eta)$$

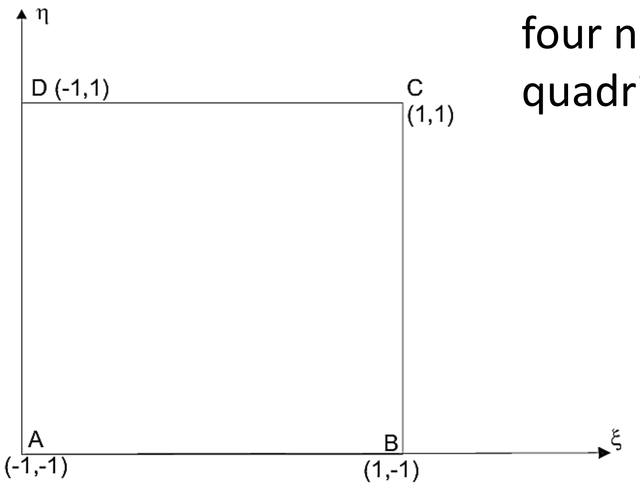


$$\Phi = \frac{1}{2}(\xi + 1)$$

Other base functions

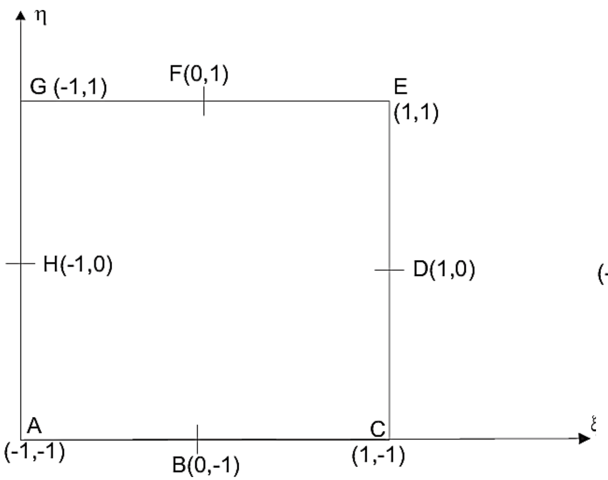
six node triangle



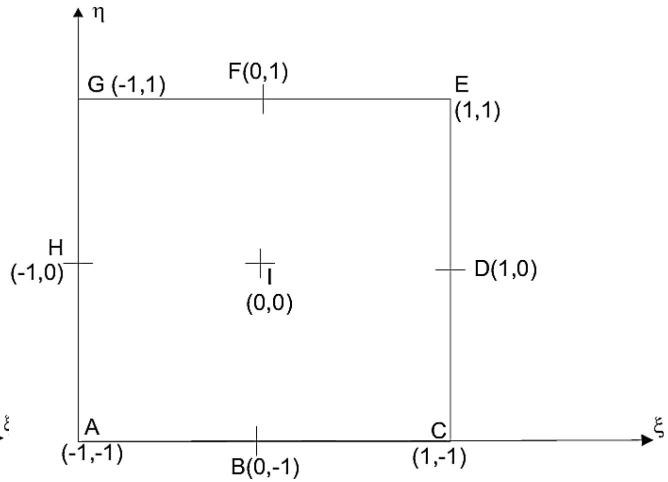


four node quadrilateral

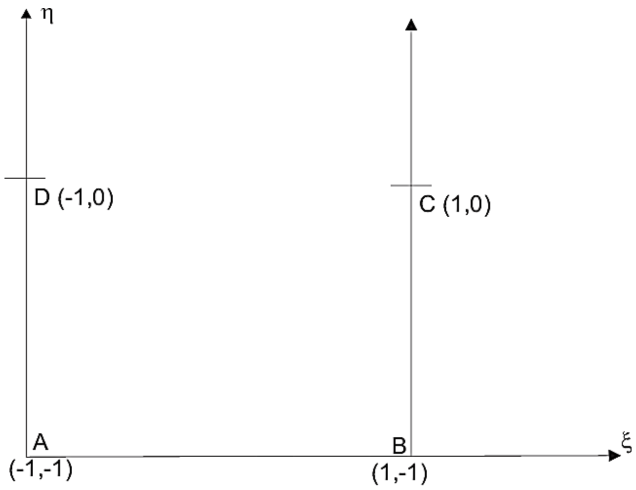
eight node quadrilateral:



nine node quadrilateral:



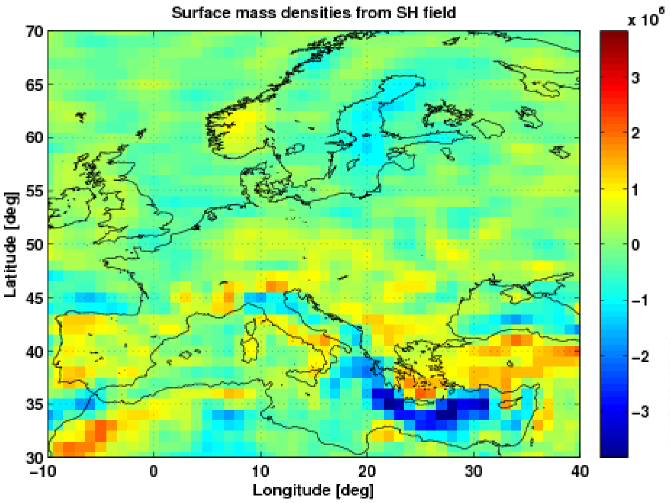
infinite quadrilateral:

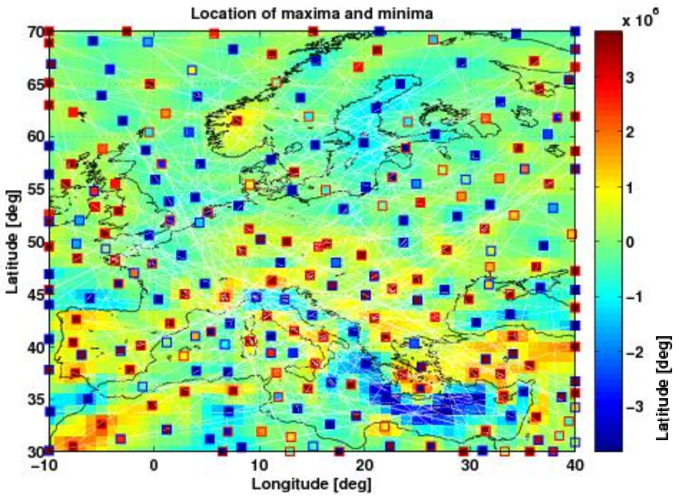


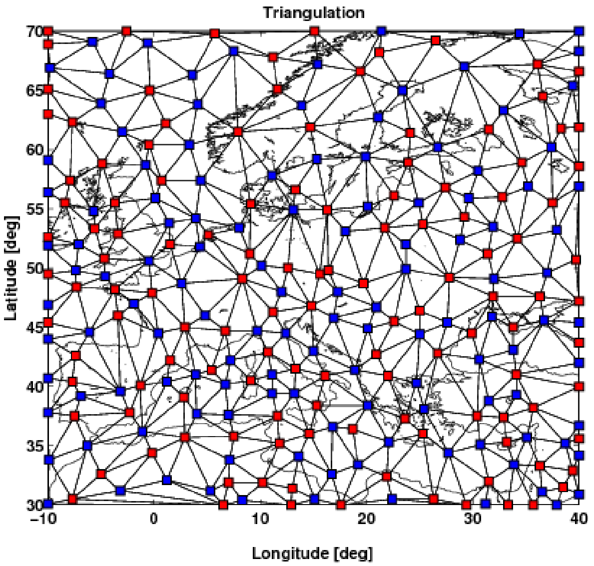
Point Grids

Point grid: *<u>any</u>* *(as long as a proper tessellation is possible)*

Currently based on the maxima and minima of an a priori field





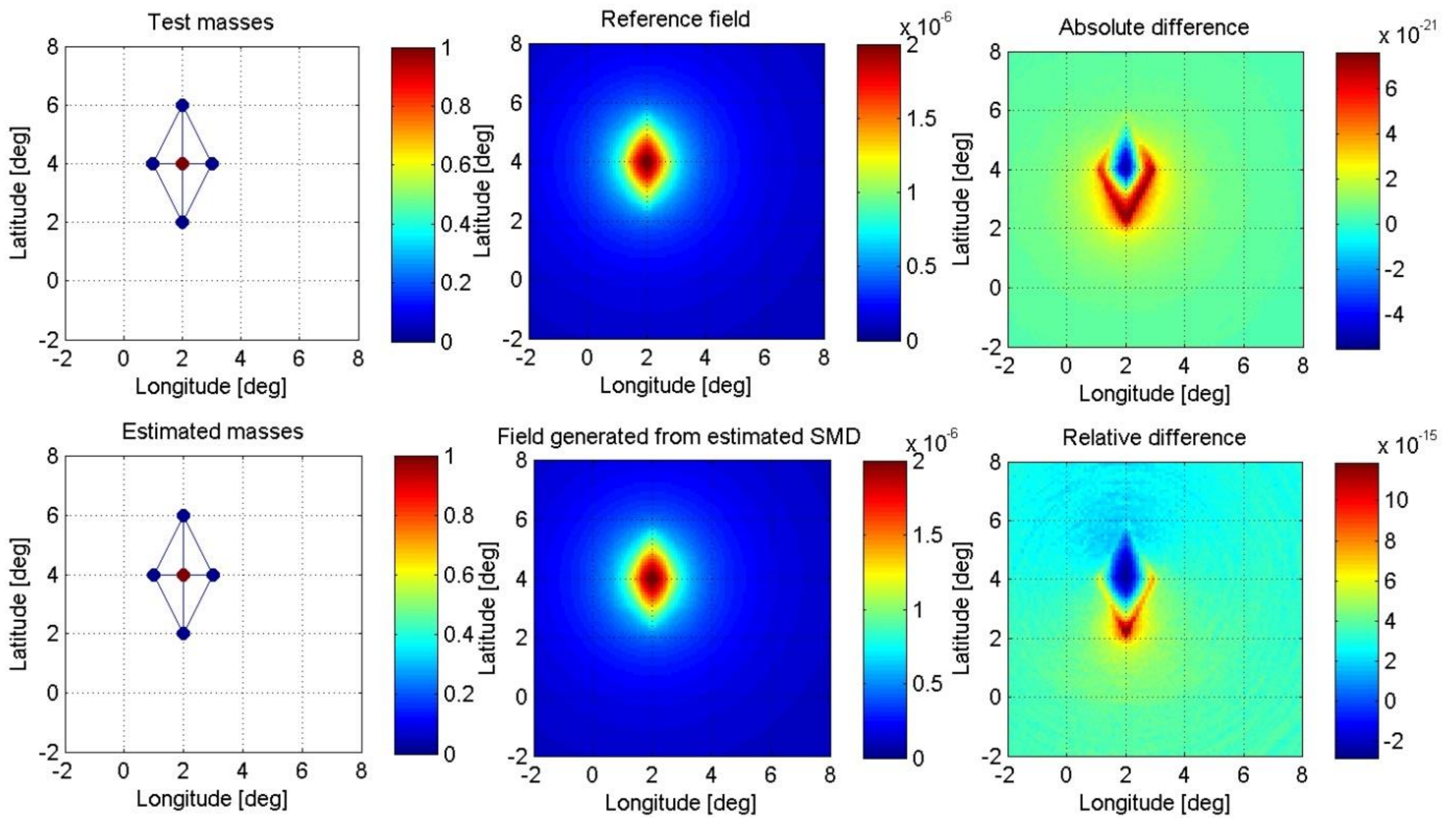


Solution approach

- *Search for maxima and minima of an a priori field*
- *Triangulation by Delaunay tessellation*
- *Least-squares adjustment*
  - *brute-force*
  - *assembly of the normal matrix (singularity!)*
  - *full consideration of the stochastic information*
- *No regularization*
  - *objective: avoid regularization by proper grid*
  - *iterative search for vertices*

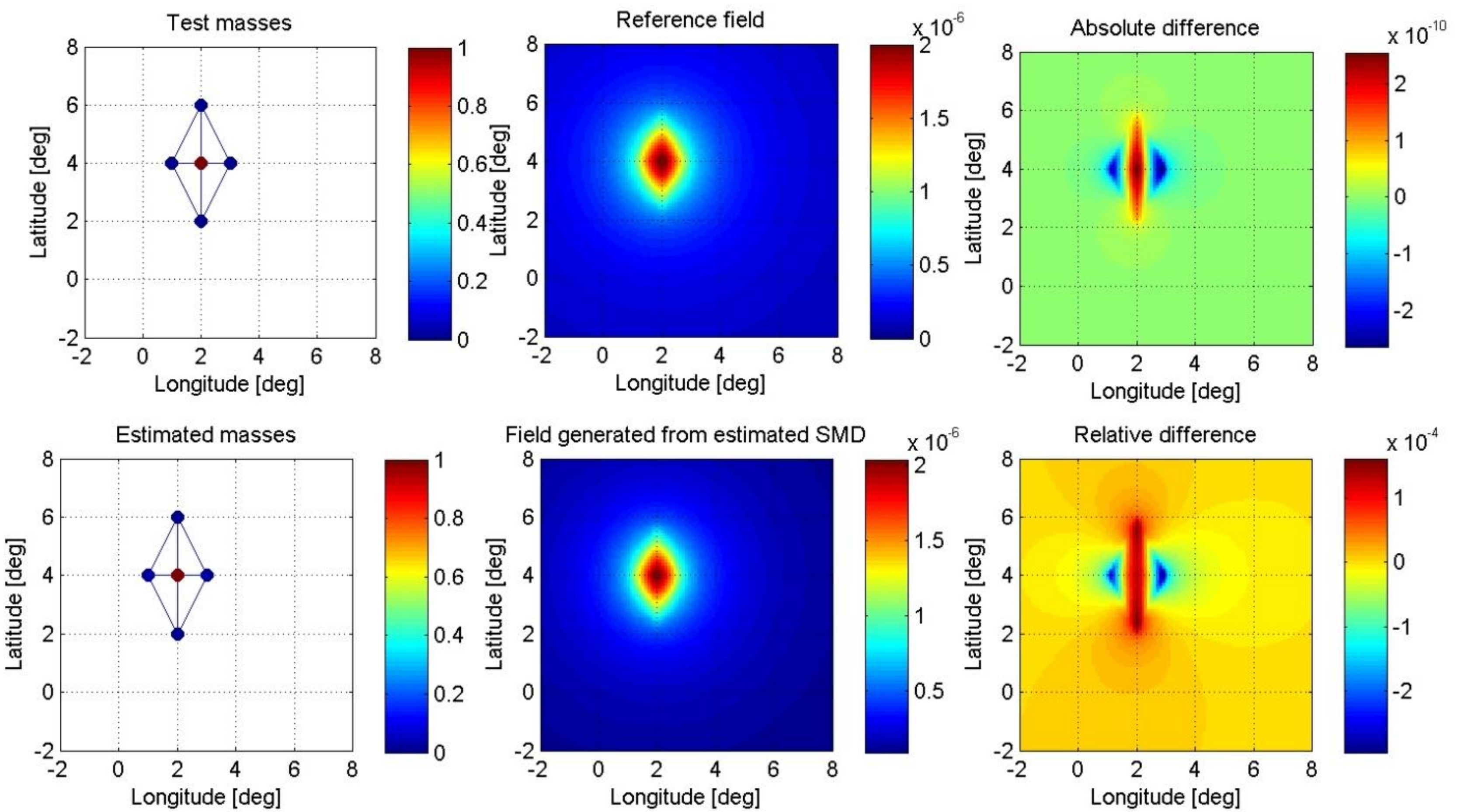
Simulation study

Closed loop simulation: noisefree and h=0km

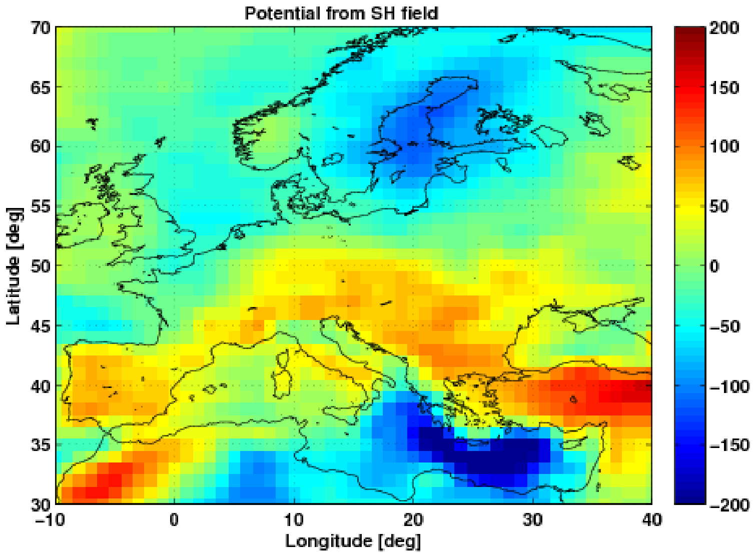


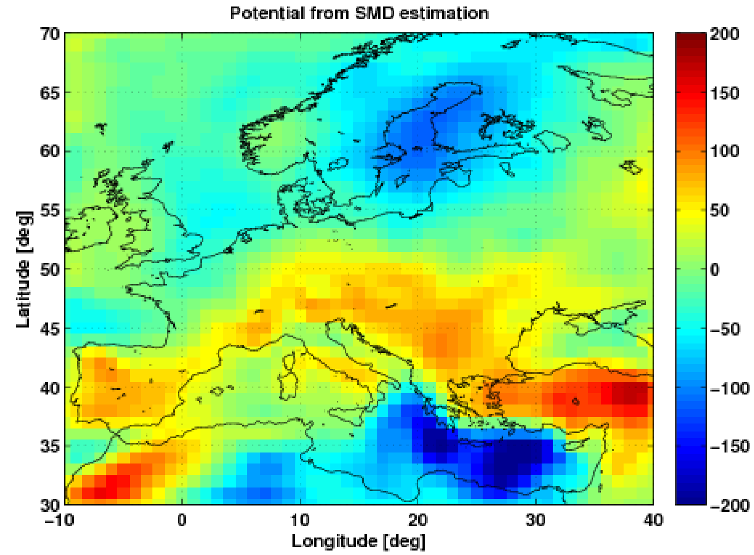
Simulation study

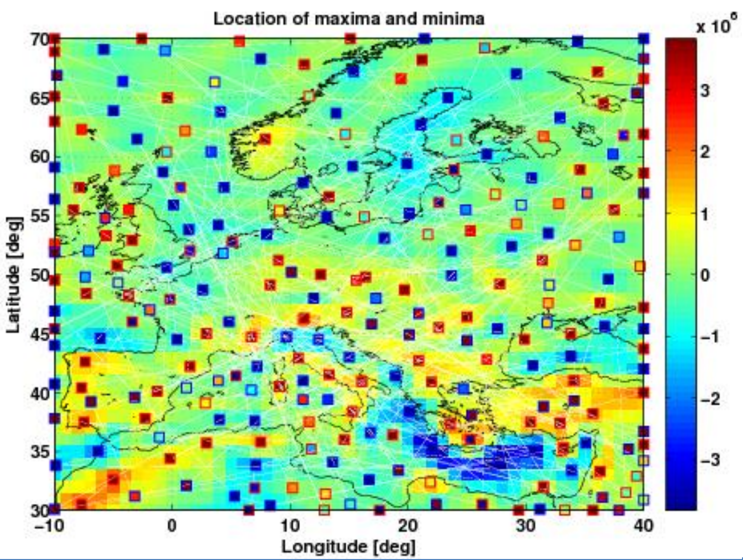
Closed loop simulation: noise=1% and h=400km

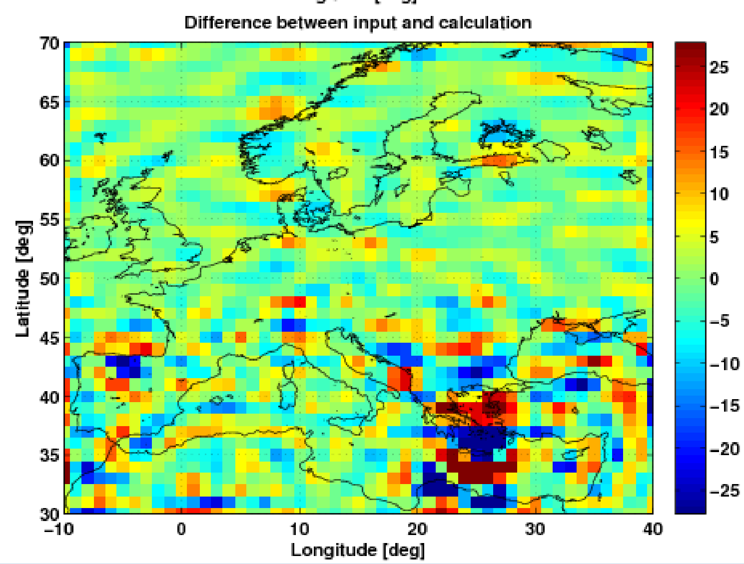


Simulation study









Open questions

- *Optimal placement of the surface masses*
  - *considering other quantities, e.g. curvature*
  - *considering quadratic base elements*
- *Optimal tessellation using also combinations of elements*
- *Edge effect*
  - *RCR technique*
  - *Infinite elements ?*
- *Regularization*
  - *avoidable by choosing the proper grid (?)*

ICCT study group JSG 0.6

Applicability of current GRACE solution strategies to the next generation of inter-satellite range observations

Chairs: Matthias Weigelt, Co-Chair Adrian Jäggi

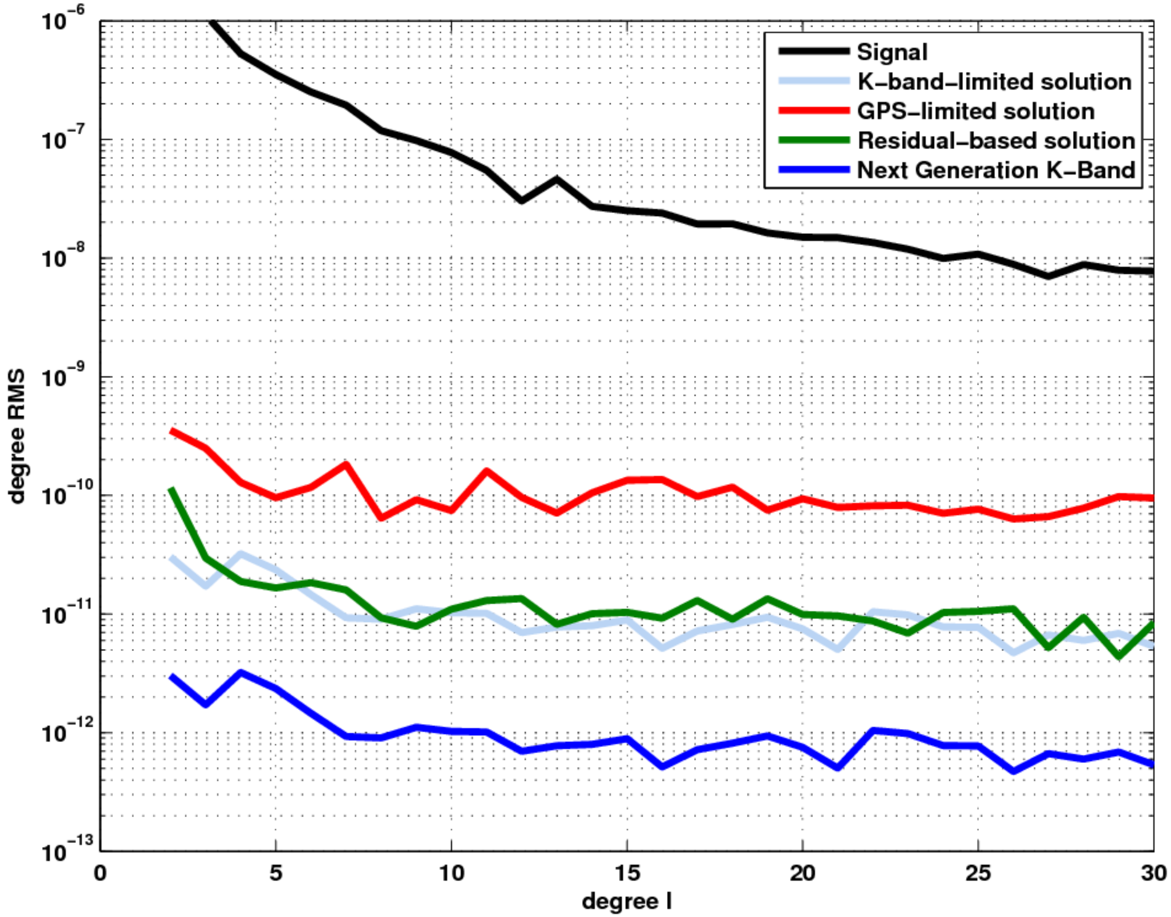
In collaboration with JSG 0.3

Comparison of methodologies for regional gravity field modelling

Chairs: M. Schmidt, Co-Chair: Ch. Gerlach

Background

$$\ddot{\rho} - \frac{1}{\rho}\left(\vec{\dot{X}}_{AB} \cdot \vec{\dot{X}}_{AB} - \dot{\rho}^2\right) = \nabla V_{AB} \cdot \vec{e}_{AB}$$



Approaches

Variational equations

$\rho, \dot{\rho}$ *Classical* (Reigber 1989, Tapley 2004)

$\rho, \dot{\rho}, \Delta\rho$ *Celestial mechanics approach* (Beutler et al. 2010, Jäggi 2007)

$\rho, \dot{\rho}$ Short-arc method (Mayer-Gürr 2006)

...

In-situ observations

Energy Integral (Han 2003, Ramillien et al. 2010) $\dot{\rho}$

Differential gravimetry (Liu 2010) $\ddot{\rho}$

LoS Gradiometry (Keller and Sharifi 2005) $\frac{\ddot{\rho}}{\rho}$

...

Objectives

The objectives of the study group are therefore to:

- *investigate each solution strategy, identify approximations and linearizations and test them for their permissibility to the next generation of inter-satellite range observations,*
- *identify limitations or the necessity for additional and/or more accurate measurements,*
- *quantify the sensitivity to error sources, e.g. in tidal or non-gravitational force modeling,*
- *investigate the interaction with global and local modeling,*
- *extend the applicability to planetary satellite mission, e.g. GRAIL*
- *establish a platform for the discussion and in-depth understanding of each approach and provide documentation.*