Bootstrap mechanisms and unitary symmetries

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BOOTSTRAP MECHANISMS

AND

UNITARY SYMMETRIES

by

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PREFACE

The work presented in this thesis was carried out in the Department of Mathematics, University of Durham, during the period October 1964 to December 1966, under the supervision of Professor E.J.Squires. The author expresses his thanks to Professor Squires for the continued help, guidance and the keen interest he showed in his work. The author is also grateful to him for his kindly reading through the manuscript. He also wishes to express his thanks to Dr. D.B.Fairlie for many useful discussions.

Except where mentioned in the text, the work described in this thesis is original and has not been submitted in this or any other university for any other degree. The thesis is mainly based on a paper by the author and two other works carried out in collaboration with Dr. M. Huq. The author is grateful to him for his permission to include the joint works in this thesis. Finally, the author wishes to express his acknowledgement to the Pakistan Atomic Energy Commission and the British Ministry for Overseas Development for a financial grant under the Colombo Plan.

ABSTRACT

We study the consequences of the applications of the 'Bootstrap' hypothesis to the Unitary Symmetries. The groups SU(3), SU(6), U(6,6) and their applications to the strong Interactions of the Hadrons are discussed in the first Chapter. In the second Chapter, we discuss some of the methods that have been used in the past in dynamical (bootstrap) calculations.

In the third Chapter, we consider the P-wave quark-pseudoscalar meson Octet scattering and investigate whether the existence of the three quarks, Q, which belong to the spinor representation of SU(3) and are supposed to have fractional charges, can be explained in a self-consistent scheme. The calculation shows that there exists a reciprocal bootstrap relationship between quarks, Q and some other particles, Q* which have the baryon number 1/3, spin 3/2 and belong to the 15-dimensional representation of SU(3). Using the determinantal method the self-consistent values we have obtained are: $M_Q \approx 2429$ Mev., $M_{Q*} \approx 5251$ Mev., $g_1^2 \approx 22$ and $g_2^2 \approx 32$, where M_Q, M_{Q*} and g_1^2, g_2^2 are respectively the masses and couplings of Q and Q*.

In the fourth Chapter, we consider the meson-baryon scattering in the context of U(6,6) symmetry and study the mass-splittings of the baryon Octet and Decouplet by N/D method. It is assumed that the SU(3) symmetry is approximately exact so that the masses of the baryon Octet and Decouplet obtained by using the U(6,6) vertices in the calculation should correspond roughly to their respective SU(3) degenerate masses. Although the results are very much cut off-dependent, the calculation shows that by varying the cut off, S_c and U(6,6) coupling, g^2 parameters it is possible to obtain the mass-splittings in the right direction. Considering the very much involved nature of the calculations, one may conclude that the results agree reasonably well with the known experimental facts.

INTRODUCTION

The S-matrix theory first proposed by Heisenberg[1] has played a very important role in explaining the dynamics of the strong interaction of hadrons. A considerable number of strongly interacting particles is now known. In the early attempts to deal with these particles theoretically, one usually followed the line of attack that proved so successful in quantum electrodynamics. In such a treatment, one usually chooses a simple Lagrangian with renormalised couplings and given masses and other physical observables are calculated by a perturbation expansion. This power series method has been successful in electrodynamics where the coupling is small but with the stronger interaction the theory runs into difficulties which arise due to the divergence of the power series expansion. In fact, when some of the particles are resonances or bound states by analogy with the nuclear physics, the perturbation expansion does not converge at all. In view of these difficulties that one encounters while dealing with strong interaction, the need for a modified approach was strongly felt. Such an approach which works even where the perturbation expansions fail, was provided by the S-matrix theory. The main reason for the success of the S-matrix approach lies in the fact that it provides a meeting ground between theory and experiment. As a matter of fact, all the experimental information is related to the scattering matrix, S.

Ignoring some possible parameters which one can introduce into S-matrix but which cannot be determined by S-matrix theory, all strongly interacting particles are assumed to be dynamical composites of each other in S-matrix theory. One may, then, try to determine the properties of these particles by studying directly the properties of the S-matrix. It is assumed that the scattering amplitudes are analytic functions of the energy and the momentum transfers except for the singularities that are associated with the unitarity conditions in the three channels. (The other approach of considering the scattering matrix as analytic functions of the angular momentum has led to the proposition of the theory of Regge poles and from the view-point of this theory all strongly interacting particles are assumed to lie on Regge trajectories[2].) The singularities in the physical region of any of the three channels are connected with the unitarity of the S-matrix in the physical region of the channel concerned. The forces which are supposed to be responsible for causing the scattering arise due to the exchanges of particles in the two other crossed channels and consequently the singularities in the unphysical regions of a scattering channel are connected with the unitarity conditions in the physical regions of the relevant crossed channels. The 'nearby' singularities arise from the lighter systems that can be exchanged in the related crossed channels and the 'faraway' singularities correspond to the exchanges of the heavier or the multiparticle configurations. The range of the forces is roughly inversely proportional to the masses of the systems that are exchanged and the strength of the forces corresponds to the discontinuities of the amplitude across these branch cuts or to the residues of the amplitude evaluated at the positions of the poles if the singularities arise due to the occurrence of the poles. Thus, by studying the analytical properties of the scattering amplitude, one can determine, at least in principle, all the properties of the strong interaction and indeed of the strongly interacting particles.

Following the above procedure, one may try to determine the couplings and the masses of all the particles that may exist in nature but as the above programme is very complicated, one has to make some approximation in the calculations. These approximations usually consist in the considerations of the two-particle states in the unitarity relation and of the exchanges of single particle or the lightest possible systems in the crossed channels. Bootstrap mechanism is one of such approximate methods in which one imposes the self consistency requirement in order to evaluate the couplings and the masses of the particles. In other words, one assumes in such calculations that all the strongly interacting particles are dynamical composites of each other with the binding forces coming from the exchange in the relevant crossed channels of the particle themselves and in consequence one deals with a self-supporting mechanism which makes the calculation much simpler. This idea of bootstrap mechanism arose in the early work of Chew and Mandelstam[3] on $\pi\pi$-scattering. The most striking feature of the $\pi\pi$-scattering is the ρ-meson, the $I = 1$, $J = 1^-$ resonance at about 760 Mev with a width of about 110 Mev. If we consider $\pi\pi$-scattering, the lightest system that can be exchanged in the crossed channel is that of two pions. As the low energy $\pi\pi$-scattering is dominated by $I = 1$, $J = 1^-$ resonance, one may assume that the two-pion system in the crossed channel also prefers to be in the above resonant state. Forgetting that ρ-meson is unstable, one may consider the contribution of the ρ-meson exchange and thereby use the mass and coupling of this particle as the respective input values in the calculation. Assuming that there is no other particle in the state $I = 1$, $J = 1^-$, one further imposes the condition that the values of the input mass and coupling of the particle under consideration be equal to those of the output mass and the coupling. This is what is demanded by the self consistency requirement. Following the above procedure, the self-consistent values obtained by Zachariasen[4] for the mass and width of ρ-meson were roughly 350 Mev and 110 Mev respectively. The above calculation was carried out a bit further by Zachariasen and Zemach[5] who also considered the effects of the $\pi\omega$-channel and the results they obtained for the mass and the couplings of the ρ-meson were in reasonably good agreement with the experimental ones.

The bootstrap hypothesis was further extended to reciprocal bootstrap by Chew[6] in connection with the N-N* problem. In this hypothesis, the nucleon N is assumed to be a pion-nucleon composite with the dominant force coming from the N* exchange and vice versa. This problem has been investigated, in some detail, by Abers and Zemach[7] who have calculated the mass and coupling of N_{33} resonance. In spite of the limitation of the calculational method used, the results they have obtained are in reasonably good agreement with the experimental ones.

The reciprocal bootstrap hypothesis of Chew has also been applied to SU(3) symmetry[8]. In the SU(3) symmetry scheme, the eight spin ½ baryons and the ten spin 3/2 baryon resonances belong respectively to the eight and ten dimensional irreducible representations of SU(3). In the pseudoscalar octet and baryon octet scattering both the initial and the final states of both the direct (s) channel and the crossed (u) channel consist of the irreducible representations which are obtained from the reduction of the direct product of two octets and these are as follows: $\underline{8} \otimes \underline{8} = \underline{1} + \underline{8}s + \underline{8}A + \underline{10} + \underline{10}^* + \underline{27}$. It is, therefore, evident from the above reduction that a reciprocal bootstrap relationship between the baryon octet and the decouplet may exist. This aspect of SU(3) symmetry has been investigated by a number of authors[9] who have shown that such a reciprocal bootstrap relationship between the baryon octet and the decouplet does indeed exist.

Being encouraged by the success of the above and other[10] SU(3) bootstrap calculations, we have carried out an investigation in order to examine whether the existence of the quarks[11], which in the simplest scheme belong to the three dimensional representation of SU(3) and have fractional charges, can be explained in a self-consistent scheme. This problem has been discussed in chapter III. We consider the quark and the pseudo-scalar meson octet scattering and adopt the 'bootstrap' hypothesis in which all the strongly interacting particles are supposed to be composites of each other. In particular, we have used the analogy with the N-N* bootstrap of Chew and its SU(3) extension discussed above.

In chapter IV we consider the meson-baryon scattering in the context of U(6,6) symmetry[12] and investigate the mass-splitting between the baryon octet and the decouplet by using the well_known N/D method of Chew and Mandelstam[13]. It is assumed that the SU(3) symmetry is exact so that the mass of the baryon octet corresponds to the average mass of the eight spin $\frac{1}{2}^+$ baryons and that of the baryon decouplet to the average mass of the ten spin

$3/2^+$ baryon resonances. As these SU(3) multiplets belong to the same 364-dimensional irreducible representation of U(6,6) symmetry, they are supposed to have the same mass from the view-point of U(6,6) theory. Now, if the U(6,6) vertices are used in the calculations, then it is expected that by using the N/D method, as has been used by a number of authors[10,14] in order to obtain the mass-splittings of SU(3) multiplets, we shall, to a reasonable extent, get the SU(3) degenerate masses of the baryon octet and decouplet. In our calculations, we have followed the hypothesis of N-N* bootstrap of Chew and consequently considered that the forces responsible for the binding of the baryon octet and decouplet come predominantly from the exchanges in the crossed (u) channels of these SU(3) multiplets themselves. This problem, is in fact, a multi-channel one. The method of constructing the exchange Born terms and other complicated aspects of the problem have been discussed in the above mentioned chapter.

In Chapter I, we have discussed the SU(3), SU(6) and U(6,6) symmetry theories. Special emphasis has been given on the discussion of how the baryons and mesons are assigned to the irreducible representations of these groups. In particular, we have discussed in detail how the U(6,6) baryon-meson vertices, which we use in our calculation, are constructed. Chapter II has been devoted to the discussion of some of the methods that have been used in dynamical and, in particular, in the bootstrap calculations. We have discussed the advantages and disadvantages of the different methods and specially mentioned the details of the method we have used in our calculations.

In any calculation involving higher symmetry, particularly, the SU(3) symmetry, one needs to fix the phases of the eigenstates of the multiplets of the symmetry concerned. This is required for writing down the SU(3) vertices which can be used in calculating the SU(3) coupling coefficients. The Appendices A and B have been devoted to that end. In the Appendices C and D we have mentioned the details of the helicity formalisms and given some useful relations which facilitate the calculations of the exchange Born terms which have been used in the investigation of baryon octet and decouplet mass-splitting.

CHAPTER I

Unitary Symmetries

1. The Group SU(3) and its Applications to the Particle Physics.

a. The Spinor Representations and the Generators of SU(3):

The basic representation of the group SU(3) is formed by three quarks[11], q_1, q_2, q_3 whose wave function we denote by ψ_α, $\alpha = 1, 2, 3$. In analogy with Sakata Model[15], we take q_1 and q_2 as isospinor with zero strangeness and q_3 isosinglet with minus one strangeness. The assignment of minus one strangeness quantum number to the isosinglet necessarily follows from the fact that we are to construct the states of the strange particles from the fields of the basic ones. We then assume the invariance of the strong interaction under a transformation

$$\psi \rightarrow u\psi \tag{1.1}$$

where the transformation matrices u are taken to be unitary as well as unimodular. The set of such matrices u form a unitary unimodular group, denoted by SU(3) and, in particular, these matrices form the 3-dimensional representation of SU(3). The unitary and the unimodularity properties of these transformation matrices are respectively given by,

$$u^{+}u = uu^{+} = 1 \tag{1.2a}$$

$$\det u = 1 \tag{1.2b}$$

In SU(3) there is an inequivalent three dimentional representation, called the contragradient representation and is denoted by $\bar{3}$. The three antiquarks, $\bar{q}_1$, $\bar{q}_2$, $\bar{q}_3$, are taken to form the basis of this contragradient representation. Denoting the wave-function of the antiquarks by $\bar{\psi}^{\alpha}(\alpha = 1, 2, 3)$, the transformation properties of the antiquarks are,

$$\bar{\psi} \rightarrow \bar{\psi}u^{-1} \tag{1.3}$$

where u^{-1} is the inverse of the transformation matrix u. The relation (1.3) follows from the transformation properties of the contragradient vectors which we, here, associate with the anti-particles.

The group SU(3) is a Lie Group, and, as it is evident from (1.2a) and (1.2b), involve 8 real parameters which vary in a continuous fashion thus giving rise to the elements of the group which lie infinitesimally close to any given one. Moreover, the group SU(3) is also compact Lie Group. From the last property of SU(3) it follows that any finite dimensional representation of SU(3) is, by Weyl's[16)] famous theorem, equivalent to a unitary representation. Therefore, the transformation matrix u can be expressed in the form,

$$u = \exp(i\,\epsilon^{\mu} F_{\mu}) \tag{1.4}$$

where the summation over the repeated index is implied. The ϵ^{μ} are real and the F_{μ} are the generators of SU(3). In SU(3) there are 8 such hermitian and traceless operators, and in this particular representation these are, 3×3 hermitian traceless matrices, the traceless property being followed from the unimodularity condition of the transformation matrices u. Gell-Mann[8)] has given such a set of 8 hermitian traceless matrices which are the following;

$$F_1 = \frac{1}{2}\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad F_2 = \frac{1}{2}\begin{vmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad F_3 = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$F_4 = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} \quad F_5 = \frac{1}{2}\begin{vmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{vmatrix} \quad F_6 = \frac{1}{2}\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$F_7 = \frac{1}{2}\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{vmatrix} \quad F_8 = \frac{1}{2\sqrt{3}}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{vmatrix} \tag{1.5}$$

where we have chosen the normalisation such that $\mathrm{Tr}\,F_i^2 = 1/2$. The set of above matrices satisfy the commutation relation,

$$[F_i, F_j] = i f_{ijk} F_k \tag{1.6}$$

where f_{ijk} is completely antisymmetric in its indices and vanishes whenever two indices are equal. The values of the f_{ijk} have been given by Gell-Mann[8]. Corresponding to an arbitrary N-dimensional representation of SU(3), the above generators F_i, ($i = 1, ..., 8$) are $N \times N$ matrices and also satisfy the same commutation relation (1.6) with the same structure constants f_{ijk}. Sometimes it is convenient to work with a set of nine real hermitian and traceless operators which in SU(3) space has the following representations,

$$(A^{\mu}_{\nu})_{ij} = \delta_{\mu i}\, \delta_{\nu j} - \tfrac{1}{3}\, \delta_{\mu\nu}\, \delta_{ij} \qquad (1.7)$$

where $(A^i{}_k)^+ = A^k_i$ and $A^1_1 + A^2_2 + A^3_3 = 0$ so that only 8 of them are independent. The commutation relations which the above operators (1.7) satisfy are very simple and given by,

$$[A^{\mu}_{\nu}, A^{\lambda}_{\rho}] = \delta^{\lambda}_{\nu}\, A^{\mu}_{\rho} - \delta^{\mu}_{\rho}\, A^{\lambda}_{\nu} \qquad (1.8)$$

where, $\mu, \nu, \lambda, \rho = 1, 2, 3$.

We now use the familiar technique of considering the infinitesimal generators of SU(3) to be operators which have physical significance. Firstly, we note that the group SU(3) is of rank 2. Thus, we can construct only <u>two independent and mutually commuting operators</u> from the infinitesimal generators of SU(3). These operators can, therefore, be simultaneously diagonalised in any representation. Again, we know two operators, namely the hypercharge and a component of the isotopic spin operator which commute with each other. Thus, we can choose two of the diagonal elements to be the hypercharge Y and the third component of the isotopic spin I_3. The choice of I_3 is purely conventional and just fixes the direction of quantisation of the isotopic spin. The other components of the isotopic spin do, in fact, commute with Y but not with I_3. Thus, they cannot be simultaneously diagonalised with I_3.

Let us now consider the eigenvalues of I_3 and Y in the triplet representation. We have already seen (1.5) that there are two operators F_3 and F_8 which are diagonal and commute with each other. F_3 is already in the form which account for the isotopic spin content of the triplet. For the hypercharge Y we redefine F_8 in analogy with the Octet model[8] as $Y = \frac{2}{\sqrt{3}} F_8$. Following these prescriptions we write down the eigenvalues of the basis vectors of the triplet representation in table 1.1 in which we also mention the baryon number and charges which we calculate by using the Gell-Mann Nishijima formula[17]. Eigenvalues of these operators corresponding to the contragradient representation are also mentioned in the same Table 1.1.

Table 1.1

Particles	Basic Vectors	I_3	S	N	Y	Q
q_1	ψ_1	1/2	0	1/3	1/3	2/3
q_2	ψ_2	-1/2	0	1/3	1/3	-1/3
q_3	ψ_3	0	-1	1/3	-2/3	-1/3
$\bar{q}_1$	ψ^1	-1/2	0	-1/3	-1/3	-2/3
$\bar{q}_2$	ψ^2	1/2	0	-1/3	-1/3	1/3
$\bar{q}_3$	ψ^3	0	1	-1/3	2/3	1/3

From the six non-diagonal operators in (1.5) we can obtain a set of six isotopic spin and hypercharge changing operators. As a result, we can establish the relations between the operators in (1.7) with those in (1.5) as follows:

$$I_+ = A^1_2 = F_1 + iF_2 \; ; \quad I_- = A^2_1 = F_1 - iF_2 \; ;$$

$$I_3 = \tfrac{1}{2}(A^1_1 - A^2_2) \; ; \quad U_+ = A^2_3 = F_6 + iF_7 \; ; \quad U_- = F_6 - iF_7 \; ;$$

$$U_3 = \tfrac{1}{2}(-I_3 + \tfrac{3}{2}Y); \quad V_+ = A^1_3 = F_4 + iF_5; \quad V_- = F_4 - iF_5 \; ;$$

$$V_3 = \tfrac{1}{2}(I_3 + \tfrac{3}{2}Y); \quad Y = -A^3_3 \qquad (1.9)$$

From (1.8) and (1.9) we obtain the following commutation relations:

$$[I_3, I_\pm] = \pm I_\pm \;; \quad [I_3, U_\pm] = \mp \tfrac{1}{2} U_\pm \;; \quad [I_3, V_\pm] = \pm \tfrac{1}{2} V_\pm$$

$$[Y\, I_\pm] = 0; \quad [Y\, U_\pm] = \pm U_\pm; \quad [Y, V_\pm] = \pm V_\pm \qquad (1.10a)$$

$$[I_+\, I_-] = 2I_3 \;; \quad [U_+\, U_-] = -I_3 + \tfrac{3}{2} Y; \quad [V_+, V_-] = I_3 + \tfrac{3}{2} Y$$

$$[I_-, V_+] = U_+ \;; \quad [V_-\, I_+] = U_- \;; \quad [U_-\, I_-] = V_- \;;$$

$$[I_+, U_+] = V_+ \;; \quad [V_+, U_-] = I_+ \;; \quad [U_+, V_-] = I_- \qquad (1.10b)$$

The relations (1.9) and (1.10) are extremely useful in determining the relative phases (which we discuss in the Appendix A) among the eigenstates as well as in calculating the eigenvalues for the Casimir operators of SU(3) corresponding to an irreducible representation of SU(3).

We also note that there is a difficulty in the classification of the eigenstates in a representation with only Y and I_3 as diagonal, the reason being that in some representations more than one state may have the same Y and I_3. This incomplete classification is the result of the fact that the group SU(3) is of rank 2 and has 8 generators. Therefore, we need $\frac{1}{2}(8 - 3\times2)$

i.e. one more operator which commutes with Y and I_s and does not commute with all other generators of the group. But as the group SU(3) is of rank 2 we cannot have any other linear operator commuting with Y and I_s. Therefore, we have to consider the non-linear operator $|I|^2$ which commutes with Y and I_s and <u>not</u> with the other generators of SU(3). Thus, a complete identification of the eigenvectors in an irreducible representation of SU(3) can be obtained by specifying for each eigenvector the corresponding I, I_s, Y eigenvalues.

b. The Irreducible Representations for Mesons and Baryons:

In the last section we have discussed the two inequivalent spinor representations of SU(3). The bases of those two representations are respectively the three quarks and their anti-particles. The philosophy now is to construct all the states from the fields of the three quarks and their antiparticles. The states so constructed from the triplets, in general, form the bases of some irreducible representations of SU(3). The job is then to identify the states of a particular irreducible representation with a set of physical states, i.e. particles, resonances, etc. For that matter, we have to look for the sub-quantum numbers that a particular representation contains and find a set of known particles which possess those quantum numbers. As the transformations of SU(3) commute with the space-time transformation, the particles which we assign to a particular representation must have the same spatial properties, i.e. spin, parity, baryon number etc. In particular, the particles belonging to a SU_s supermultiplet must have the same mass. In other words, the particles forming a supermultiplet are indistinguishable from the point of view of exact symmetry. However, the group has a set of operators which allow us to allocate different symmetry quantum numbers to the various members of a supermultiplet. As we have discussed before, these quantum numbers are I, I_s, Y which are sufficient for the complete identification of the various states in an irreducible representation of SU(3).

Let us now go back to the quark-model and construct the various states which we shall identify with the set of known particles. First, we construct the states for the pseudoscalar and vector mesons. As these particles have zero baryon number their corresponding states can be constructed from the quarks and antiquarks combination. Thus, we expect a representation for either of the mesons in the direct product $\bar{3} \times 3$ representation. The technique of reducing this direct product representation is well known[18] so that we have,

$$\bar{\underset{\sim}{3}} \otimes \underset{\sim}{3} = \underline{1} \oplus \underset{\sim}{8} \qquad (1.11a)$$

In terms of the wave functions of the quarks and antiquarks we have,

$$\bar{\psi}^{\alpha}\psi_{\beta} = \tfrac{1}{3}\delta^{\alpha}_{\beta}\bar{\psi}^{\lambda}\psi_{\lambda} + \left\{\bar{\psi}^{\alpha}\psi_{\beta} - \tfrac{1}{3}\delta^{\alpha}_{\beta}\bar{\psi}^{\lambda}\psi_{\lambda}\right\}$$

$$= \tfrac{1}{3}\delta^{\alpha}_{\beta}\bar{\psi}^{\lambda}\psi_{\lambda} + \phi^{\alpha}_{\beta} \qquad (1.11b)$$

In (1.11b) the first term is the trace of the mixed tensor of rank 2 and has only one component which transforms like a scalar under SU(3). The second term is a traceless mixed tensor and has only eight independent components. The additive quantum numbers I_3 and Y of the tensor ϕ^{α}_{β} can be calculated from the corresponding quantum numbers of the components of the constituent quarks. We then find that the eight dimensional irreducible representation formed by the traceless mixed tensor ϕ^{α}_{β} consists of two isotopic doublets (Y = 1, -1), one isotopic triplet (Y = 0) and one isotopic singlet (Y = 0) and all these states have zero baryon number. We can, therefore, associate them with the pseudoscalar as well as the vector mesons. For the pseudoscalar mesons we make the following identifications:

$$
\begin{aligned}
P^{2}_{1} &= \phi^{2}_{1} : \pi^{+} \\
P^{1}_{2} &= \phi^{1}_{2} : \pi^{-} \\
P^{3}_{1} &= \phi^{3}_{1} : K^{+} \\
P^{1}_{3} &= \phi^{1}_{3} : K^{0} \\
P^{2}_{3} &= \phi^{2}_{3} : \bar{K}^{0} \\
P^{3}_{2} &= \phi^{3}_{2} : K^{-} \\
\frac{1}{\sqrt{2}} (P^{1}_{1} - P^{2}_{2}) &= \frac{1}{\sqrt{2}} (\phi^{1}_{1} - \phi^{2}_{2}) = \pi^{0} \\
\frac{\sqrt{6}}{2} P^{3}_{3} &= \frac{\sqrt{6}}{2} \phi^{3}_{3} = -\eta
\end{aligned}
\tag{1.12}
$$

where the identifications of the last two in (1.12) can be obtained by using the traceless property of the mixed tenser and orthogonality condition of the states π^0 and η. The pseudoscalar meson octet can then be written in the following form:

$$P_\alpha^\beta = \begin{vmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{vmatrix} \qquad (1.12')$$

Similar associations can be made with the eight vector mesons and written in the same form as (1.12'). Thus, writing the mixed tensor for the eight vector mesons as V_α^β, we have

$$V_\alpha^\beta = \begin{vmatrix} \frac{\rho^0}{\sqrt{2}} + \frac{\omega_0}{\sqrt{6}} & \rho^+ & K^{*+} \\ \rho^- & -\frac{1}{\sqrt{2}}\rho_0 + \frac{1}{\sqrt{6}}\omega_0 & K^{*0} \\ K^{*-} & \bar{K}^{*0} & -\frac{2}{\sqrt{6}}\omega_0 \end{vmatrix} \qquad (1.13)$$

where ω_0 is a mixed state of the two physical isosinglets ϕ and ω.

This admixture of the two states appears when the pure SU(3) symmetry is broken[8]. We might as well speculate that the isosinglet member of the pseudoscalar octet could be an admixture of the isosinglet $\eta(550)$ and the recently discovered isosinglet $X^0(960)$. However, from the large difference in X^0, η masses, the amount of the mixing is assumed to be negligible.

We now construct the states for the baryon and baryon resonances. As the quarks have the baryon number 1/3, we need three or the multiple of three quarks to form the baryon states. Considering the product of three quarks' wave functions we have,

$$\psi_\alpha \phi_\beta \chi_\lambda = V_{[\alpha\beta\lambda]} + \psi_{[\alpha\beta]\lambda} + \psi_{[\beta\lambda]\alpha} + D_{\{\alpha\beta\lambda\}}$$

where $V_{[\alpha\beta\lambda]}$ is completely antisymmetric in its indices, $\psi_{[\alpha\beta]}$ are antisymmetric between the interchange of the indices $\alpha\ \beta$ and $D_{\{\alpha\beta\lambda\}}$ is completely symmetric. The tensor $\psi_{[\alpha\beta]\lambda}$ has eight independent components, and $D_{\{\alpha\beta\lambda\}}$ has 10 independent components. Now, using the Levi-Civita tensor $\epsilon^{\alpha\beta\delta}$ we can write,

$$B_\lambda^\delta = \epsilon^{\alpha\beta\delta}\, \psi_{[\alpha\beta]\lambda}$$

where it can be shown that B_λ^δ is traceless. We can associate the 8 baryons namely, N, Λ, Σ, Ξ with the various components of the traceless tensor B_α^β. Consequently, the baryon octet can also be written in the matrix form,

$$B^{\nu}_{\mu} = \begin{vmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda^0 \end{vmatrix} \qquad (1.14)$$

The completely symmetric tensor $D_{\alpha\beta\gamma}$ can be expressed in terms of the wave-functions of the constituent quarks as follows:

$$D_{\alpha\beta\lambda} = \frac{1}{\sqrt{6}}\Big[\psi_\alpha \phi_\beta \chi_\lambda + \psi_\beta \phi_\lambda \chi_\alpha + \psi_\lambda \phi_\alpha \chi_\beta + \psi_\lambda \phi_\beta \chi_\alpha + \psi_\beta \phi_\alpha \chi_\lambda + \psi_\alpha \phi_\lambda \chi_\beta\Big] \qquad (1.15)$$

From (1.15) the normalisation constants for each of the ten independent states can be calculated. Thus, we obtain the following identifications:

$$N^{*++} = D_{111} \; ; \quad N^{*+} = \sqrt{3}\, D_{112} \; ; \quad N^{*0} = \sqrt{3}\, D_{221} \; ;$$

$$N^{*-} = D_{222} \; ; \quad Y^{*+} = \sqrt{3}\, D_{113} \; ; \quad Y^{*0} = \sqrt{6}\, D_{123} \; ;$$

$$Y^{*-} = \sqrt{3}\, D_{223} ; \quad \Xi^{*0} = \sqrt{3}\, D_{331} ; \quad \Xi^{*-} = \sqrt{3}\, D_{332} ;$$

$$\Omega^{-} = D_{333} . \qquad (1.16)$$

c. The Baryon-meson Yukawa type strong Interaction Lagrangians and F/D Ratio :

One of the consequences of the assignments of various particles to different SU(3) multiplets is that we can couple them together and thereby establish some relationships between their coupling constants. In order to see how these relationships are obtained we consider the strong interaction between the baryon octet B_{μ}^{ν} and the pseudoscalar octet P_{μ}^{ν} and write down the SU(3) invariant Yukawa type strong interaction Lagrangian between them. The vertex we are going to consider is of the form $\bar{B}BP$, where $\bar{B}$ is to be obtained from (1.14) by taking transpose of the matrix B_{μ}^{ν} with bar; this is necessary for the conservation of the electric change. Further, the interaction Lagrangian we construct has to be invariant under Lorentz transformation as well. However, we assume that these factors are always taken care of and confine our attention to only symmetry dependent part of the interaction.

Before we proceed to write down the baryon-meson vertex let us consider the general Yukawa type coupling of the form $M_1\ M_2\ M_3$ where these three M's belong possibly to different representations of SU(3). The method of constructing such an invariant Lagrangian is well_known in relation to the isotopic spin symmetry where we consider the Yukawa type coupling of the form $\bar{N}N\pi$, where N is the nucleon doublet and π the meson triplet. As the pion is an isovector in the isotopic spin space, we must construct a vector from $\bar{N}$ and N. Such a vector is of the form $\bar{N}\underset{\sim}{\tau}N$ where $\underset{\sim}{\tau}$ is the familiar Pauli matrices in the isotopic spin space. Then the SU(2) invariant interaction will be of the form,

$$i\,\bar{N}\,\tau\,N\,.\,\pi \tag{1.17}$$

where i has been added to make the interaction hermitian. Here, we note that there is only one method of constructing a scalar in the same way as there is only one way of constructing a vector from two spinors. The same procedure can be applied to the case of SU(3). For the interaction of the form $M_1\ M_2\ M_3$ we have to construct the representation contragradient to M_3 from the direct product of M_1 and M_2. This_is obvious from the fact that only the contragradient representations have a scalar representation in the decomposition of their direct product. As in (1.17), there must exist an operator (called isometry) which will allow us to construct, out of the product of M_1 and M_2, a representation that transforms contragradiently to M_3 under SU(3). In fact, there exist as many such operators as the representation contragradient to M_3 occurs in the reduction of the direct product representation of M_1 and M_2. This is exactly how Gell-Mann[8] obtained, in connection with Baryon-meson interaction, two types of vertices namely, the D and F respectively. The D and F types are respectively symmetric and anti-symmetric under the interchange of the two baryons. As the SU(3) symmetry cannot distinguish between D and F type one has to take an arbitrary linear combination of the two. We shall, however, write our interaction Lagrangian between the Baryon and the pseudoscalar mesons in a slightly different way which is convenient if bases of the representations are known in terms of the irreducible tensors.

In our case, both the baryons and the pseudoscalar mesons are given in terms of the mixed tensors (1.14) and (1.12'). As the trace of a tensor is invariant under SU(3) we construct traces of the product $\bar{B}BP$. But there are two ways in which we can obtain this trace as a consequence of which we obtain the following invariants:

$$L_1 = \bar{B}^{\nu}_{\mu}\, B^{\lambda}_{\nu}\, P^{\mu}_{\lambda}$$

$$L_2 = \bar{B}^{\nu}_{\mu}\, P^{\lambda}_{\nu}\, B^{\mu}_{\lambda} \tag{1.18}$$

Instead of (1.18) it is customary to consider the following combinations:

$$L_{int}^{(\pm)} = L_1 \pm L_2 \qquad (1.19)$$

which are symmetric and antisymmetric respectively under the interchange, $\bar{B} \leftrightarrow B$. The general interaction Lagrangian is, therefore, an arbitrary linear combination of $L_{int}^{(\pm)}$ which we write as:

$$L_{int} = (4\pi)^{\frac{1}{2}} \sqrt{2}\, g \left\{ D.L^{(+)} + F L^{(-)} \right\} \qquad (1.20)$$

where we have multiplied with a factor $\sqrt{2}\,(4\pi)^{\frac{1}{2}}$ in order that we obtain $(D + F)g = g_{NN\pi}$, where experimentally $g^2_{NN\pi} \simeq 15$ is known. Without the loss of generality, we may choose $g = (15)^{\frac{1}{2}}$ in which case the experiment restricts the coefficient $D + F = 1$. Thus, (1.20) can be rewritten in the form,

$$L_{int} = (4\pi)^{\frac{1}{2}} \sqrt{2}.g \Big\{ (1 - F)\, tr\, (\bar{B}BP + \bar{B}PB)$$

$$+ F\, tr\, (\bar{B}BP - \bar{B}PB) \Big\} \qquad (1.20')$$

We can calculate the traces in (1.20') and express them in terms of the fields operators (1.12') and (1.14) with appropriate phases (Appendix A). We, then, arrange the different parts in such a way that on comparison with the most general baryon-meson interaction Lagrangian[19] we obtain the following relations between 12 baryon-meson coupling constants,

$$g_{NN\pi} = -g_{\Sigma\ \ K} = g$$

$$g_{\Sigma\Sigma\pi} = 2Fg$$

$$g_{N\Sigma K} = -g_{\ \ \pi} = (1 - 2F)g$$

$$g_{\Sigma\Lambda\pi} = g_{\Sigma\Sigma\eta} = -g_{\Lambda\Lambda\eta} = \frac{2}{\sqrt{3}}(1 - F)g \qquad (1.21)$$

$$g_{NN\eta} = g_{\ \ \Lambda K} = -\frac{1}{\sqrt{3}}(1 - 4F)g$$

$$g_{N\Lambda K} = g_{\ \ \eta} = -\frac{1}{\sqrt{3}}(1 + 2F)g$$

Unfortunately, of all these coupling constants, only one, namely, $g^2_{NN\pi} \approx 15$ is accurately known. However, it is possible to make an indirect estimate of F. Such an investigation has been carried out by Martin and Wali[19] who considered the process $B + P \rightarrow B + P$ taking into account only one baryon exchange. They varied the value of F to get the N* resonance at the appropriate energies. The values of F which gave the best fit was $\approx \frac{1}{4}$, the corresponding F/D ratio, being 1/3.

We can similarly couple the vector mesons to the baryons and obtain relations between the various coupling constants in the same manner as above, involving again the F/D ratio which has to be determined by an indirect method. Therefore, the question arises whether we can write these couplings in terms of just one. Theoretically we would just like to eliminate one on the basis of some other consideration or another postulated invariance. Neeman thus obtained the F-type interaction only for the vector mesons by imposing the gauge invariance principle. Gell-Mann, on the other hand, tried to introduce R-invariance. As D and F couplings are respectively symmetric and antisymmetric under R operator this invariance would demand the existence of D alone. However, in the case of 3 boson couplings, one of the two types is automatically excluded on account of charge conjugation invariance and the fact that the particles and the antiparticles appear simultaneously in the boson octets. Thus, the three vector mesons or one vector and 2 pseudoscalar mesons vertices must be F type while 3 pseudoscalar mesons or one pseudoscalar meson and two vector mesons vertices must necessarily be D type[20].

d. The Symmetry Breaking Interactions and Mass-Formulae:

Exact SU(3) symmetry demands that the particles said to form a supermultiplet must have the same degenerate mass. But this does not happen to be the case in nature and, therefore, one has to consider the breaking of the symmetry. The breaking of the symmetry then allows the mass degeneracy to be removed and we are expected to obtain the correct mass-spectrum from the symmetry breaking. In order to find what is the source which causes this symmetry-breaking, let us go back to the history of the charge independence theory. In the charge-independence theory it was supposed that the particles with very nearly the same mass form the isotopic multiplet and that the correct mass spectrum would be obtained if we did include the effects of the electromagnetic interactions which do not observe charge-independence. In SU(3) symmetry, we likewise assume that so far as the strong interactions are concerned the particles having very nearly the same mass may be grouped into SU(3) supermultiplets and that when we consider the symmetry breaking we shall again be able to remove the mass-degeneracy. In this case, however, the situation is slightly different. Here, neglecting the weak interactions altogether, our hierachy of interactions consist of the very strong, the medium strong and the electromagnetic interactions. The very strong interactions, as we have already assumed, are invariant under the transformations of the full group SU(3). On the other hand, the medium strong interactions are assumed to be invariant under the subgroup $SU(2)_I \otimes U(1)_Y$ of the full group SU(3). The above subgroup is then assumed to contain a further subgroup which leaves the electromagnetic interaction invariant. Thus, the complete removal of the mass degeneracy can be supposed to take effect in two stages. In the first stage, the medium strong interactions are turned on. As the medium strong interactions do not have the symmetry of the very strong ones, the super-multiplets will decompose into various isotopic multiplets. The mass-splitting in this stage is such that although SU(3) is broken, the isotopic spin I and hypercharge Y are still conserved. In the second stage then we switch on the electromagnetic interaction which completely removes the mass-degeneracy resulting in mass-splitting between all the members of the SU(3) super-multiplets. However, the third component of the isotopic spin I_3, and the hypercharge are still conserved in this interaction. The most troublesome interactions are the weak interactions which destroy the strangeness s conservation as well. Finally, only charge and baryon numbers are conserved. Thus, the total interaction Lagrangian of the Hadrons can be written in the form,

$$L_{int} = I_{vs} + I_{ms} + I_{em} + I_w \tag{1.22}$$

where I_{vs}, I_{ms}, I_{em}, I_w are respectively the very strong, the medium strong, the electromagnetic and the weak interactions. We shall, however, discuss the effects only of the medium and electromagnetic interactions.

Let us consider the medium strong interactions which are responsible for the breaking of the SU(3) supermultiplets into the isotopic multiplets. As is well known, this interaction is considered as an operator, T_M, that must commute with the isotopic spin, strangeness and the nucleon number operators I, S, N. This restriction is highly reasonable as we are still in the realm of strong interactions where the strangeness and the baryon number are conserved and any non-commutation with I will result in the mass-splitting between different members of the same isotopic multiplets. Now, the mass splitting due to this interaction can be written in the form,

$$\Delta M = \langle D, \psi | T_M | D, \psi \rangle \tag{1.23}$$

where D is an arbitrary irreducible representation and ψ any vector in its basis. Let us now discuss what more restrictions we can impose on the operator T_M. The conservation of the isotopic spin I requires the operator T_M to be an isospin scalar such that under $SU(2)_I$ it transforms as an isospin singlet. Further, the conservation of hypercharge Y and the condition that it is an additive quantum number demands that the hypercharge carried by the operator must be zero. Thus, summing up we find that the operator T_M must transform as an isosinglet with $Y = 0$. There are quite a few, namely, the irreducible representations $\underset{\sim}{1}$, $\underset{\sim}{8}$, $\underset{\sim}{27}$, $\underset{\sim}{64}$ ... etc. which contain a state having the above quantum numbers. To the first order approximation, only the $\underset{\sim}{1}$ and $\underset{\sim}{8}$ contribute. Therefore, we can express T_M as a linear combination of the operators corresponding to the 1 and 8 dimensional representations,

$$T_M = T^{(1)} (I = Y = 0) + T^{(8)} (I = Y = 0) \qquad (1.24)$$

$T^{(1)}$ just gives a constant term in (1.23) and the traceless octet operator is given by,

$$T^{\mu}_{\nu} = a A^{\mu}_{\nu} + b \left[A^{\mu}_{\lambda} A^{\lambda}_{\nu} - \tfrac{1}{3}\delta^{\mu}_{\nu} A^{\alpha}_{\lambda} A^{\lambda}_{\alpha} \right] \qquad (1.25)$$

where A^{μ}_{ν} are generators of SU(3). From (1.24) it is obvious that we need only the T^3_3 component of the tensor T^{μ}_{ν}. Considering only T^3_3, we can express the quantities in the right hand side of (1.25) in terms of I, Y etc. We then obtain from (1.23) using (1.24) and (1.25) the well known Okubu mass formula[8),]

$$M = m_0 + m_1 Y + m_2 \left[I(I + 1) - \tfrac{1}{4} Y^2 \right] \qquad (1.26)$$

where m_o, m_1, m_2 are arbitrary constants. Let us now consider the application of the formula to the various sets of particles. We first consider the baryon octet N, Λ, Σ, Ξ for which we obtain the well known Gell-Mann-Okubu[8)] relation,

$$\frac{1}{2}(m_N + m_{\Xi}) = \frac{1}{4}(3m_\Lambda + m_\Sigma) \qquad (1.27)$$

Taking $m_N \approx 938.9$, $m_\Sigma \approx 1192.9$, $m_\Lambda \approx 1115.4$, $m_{\Xi} \approx 1317.6$ and the average octet mass $\bar{m} \approx 1152$ we find,

$$\frac{1}{\bar{m}}\left[\frac{1}{2}(m_N + m_{\Xi}) - \frac{1}{4}(3m_\Lambda + m_\Sigma)\right] \leq 1\%$$

The Gell-Mann-Okubu mass formula is, therefore well satisfied for the baryons. For the Decouplet, however, we have $I = 1 + \frac{Y}{2}$ and the formula (1.26) reduces to,

$$M = m_s + m_4 Y \qquad (1.28)$$

where $m_s = (m_o + 2m_2)$; $m_4 = (m_1 + \frac{3}{2}m_2)$. The formula (1.28) then predicts equal spacing between the isomultiplets of the Decouplet. From experiments we have $m_{N^*} = 1238$; $m_{Y^*} \approx 1385$; $m_{\Xi^*} = 1530$; $m_{\Omega^-} = 1675$ and the spacing from (1.28) are the following,

$$m_{N^*} - m_{Y^*} = -147; \quad m_{Y^*} - m_{\Xi^*} = -145,$$

$$m_{\Xi^*} - m_{\Omega^-} = -145$$

At the time when SU(3) was hypothised, the Ω^- particle had not been observed. The above formula predicts its mass to be $m_\Omega \approx m_{\Xi^*} + 146 \approx 1676$ which is in remarkable agreement with the value eventually determined. We now consider the applications of the mass-formula to the pseudoscalar and vector mesons. As the mass of the boson occurs in the Lagrangian in the form μ^2, μ being the mass of the boson, the mass M in (1.26) should be replaced by M^2. Also in order that $m_K^2 = m_{\bar{K}}^2$ we require the coefficient m_1 of Y in the mass formula to be zero. For the pseudoscalar meson we then obtain,

$$m_K^2 = \tfrac{1}{4}(3m_\eta^2 + m_\pi^2) \tag{1.29}$$

which is reasonably satisfied. We now consider the nine vector mesons ρ, K^*, $\bar{K}^*$, ω, ϕ. The masses of these particles are respectively: $m_\rho \approx 763$; $m_{K^*} \approx 891$; $m_\phi \approx 1020$; and $m_\omega \approx 783$. The corresponding mass formula is,

$$m_{K^*}^2 = \tfrac{1}{4}(3m_{\omega_0}^2 + m_\rho^2) \tag{1.30}$$

where m_{ω_0} is taken as the mass of the isosinglet member of the vector octet. From the above we have $m_{\omega_0} \approx 930$ Mev. This value corresponds to neither the ϕ nor the ω. Now considering ϕ_0 as the pure $I = Y = 0$ SU(3) singlet we can write the physical ϕ and ω as mixtures of the ω_0 and ϕ_0 [8] states:

$$\phi = \phi_0 \sin\theta - \omega_0 \cos\theta$$
$$\omega = \psi_0 \cos\theta + \omega_0 \sin\theta \qquad (1.31)$$

The mixing angle θ in (1.31) can be estimated from the ϕ, ω decay width or else by diagonalising the mass matrix. An approximate value of the mixing angle is $\theta \approx \pm 40^o$ [8].

Mass formulae corresponding to the second order $T^s_s T^s_s$ (in fact n^{th} order[21]) have also been worked out. However, we then obtain too many coefficients and we cannot, in fact, derive any relationship between the masses in a representation. We shall, rather, not go into that matter any further.

The mass-splitting among the members of the same isotopic multiplets of the SU(3) multiplets have been discussed in the past by a number of authors[8,22]. We just quote the famous Glashow-Coleman relation,

$$M(\Xi^-) - m(\Xi^0) = m(\Sigma^-) - m(\Sigma^+) + m(p) - m(n) \qquad (1.32)$$

Experimental results give $m(p) - m(n) \approx 1.29$ Mev. $m(\Sigma^-) - m(\Sigma^+) \approx 8.25 \pm .5$ Mev. These yield the mass difference for $m(\Xi^-) - m(\Xi^0) \approx 6.96 \pm .5$ Mev which seems to agree rather well with the experimental values obtained so far.

2. The SU(6) Symmetry for the Hadrons

a. The SU(6) Supermultiplets and the Baryon-meson Yukawa Couplings

The group SU(6) as the symmetry group for the hadrons was first proposed by Gürsey and Radicatti and Sakita[23] independently. This is a group of all the unitary unimodular transformations in some six dimensional complex space and has a subgroup $SU(3) \otimes SU(2)$ which can be identified with the direct product of the SU(3) symmetry group and the ordinary spin group. From this point of view, this theory can be regarded as an extension of the supermultiplet theory of the nucleus of Wigner[24] who classified the Nuclear levels according to the irreducible representations of the group SU(4) which has a subgroup $SU(2)_I \otimes SU(2)_J$ which is the direct product of the isospin and ordinary spin groups respectively. Neglecting the Coloumb and the Non-central forces, Wigner first of all assumed that the forces between two nucleons were invariant under the transformations of the product group $SU(2)_I \otimes SU(2)_J$. He then made the further assumption that the forces responsible for the binding of the Nucleus were spin isospin independent so that it will be invariant under the full group $SU(4)$ which contains the product group $SU(2)_I \otimes SU(2)_J$ and transforms the four objects; $p\uparrow$, $n\uparrow$, $p\downarrow$ and $n\downarrow$ among themselves. Now, with the help of the quark model of the hadrons, the extension of $SU(4)$ to $SU(6)$ is obvious. Here, we consider the transformations on the following six-component objects,

$$\psi = \begin{vmatrix} q_1\uparrow \\ q_2\uparrow \\ q_3\uparrow \\ q_1\downarrow \\ q_2\downarrow \\ q_3\downarrow \end{vmatrix} \qquad (1.33)$$

where the arrows indicate the ordinary spin states of the quarks. We then assume that the strong interactions (usually called the forces binding the quarks to form an elementary particle) are SU_3-spin-spin independent so that they are invariant under the group $SU(6)$ which transforms the components of the fundamental sextet among themselves. The particle states are then classified according to the irreducible representations of $SU(6)$. The irreducible representations of $SU(6)$ are, in general, reducible under the subgroup SU(3) × SU(2) and consequently the SU(3) and spin contents of an irreducible representation of the full group are obtained by reducing it with respect to the product group. For that purpose we specify the irreducible representations of SU(3) ⊗ SU(2) by a set of two numbers $(\alpha\ a)$, denoting the dimension of SU(3) and SU(2) respectively. Now any irreducible representation A of SU(6) can be written as,

$$A = (\alpha\ a) + (\beta\ b) + \ldots \tag{1.34}$$

where A is the dimension of any irreducible representation of SU(6). Thus, (1.34) provides the SU(3) and SU(2) multiplet content of the supermultiplet A. Obviously, the arithmetic equality $A = \alpha a + \beta b + \ldots$ must be satisfied.

As is well known, the sextet (1.33) forms the basis of the fundamental representation of SU(6). The generators of SU(6) for this representation can be taken to consist of the following:

$$|A^{\mu}_{\nu}|_{ij} = \delta_{\mu i}\, \delta_{\nu j} - \frac{1}{6}\, \delta^{\mu}_{\nu}\, \delta_{ij} \tag{1.34'}$$

where $\mu, \nu, i, j = 1 \ldots 6$. The eigenvalues of the sextet corresponding to the operators I_3, Y and J_3 (third component of the ordinary spin) can be expressed in terms of the generators A^{μ}_{ν} as follows:

$$I_3 = \tfrac{1}{2}(A_1^1 - A_2^2 + A_4^4 - A_5^5)$$

$$Y = -(A_3^3 + A_6^6) \qquad (1.35)$$

$$J_3 = \tfrac{1}{2}(A_1^1 + A_2^2 + A_3^3 - A_4^4 - A_5^5 - A_6^6)$$

The basis of the six-dimensional contragradient representation is formed by the following six objects:

$$\bar{\Psi} = \begin{vmatrix} \bar{q}_1\downarrow \\ \bar{q}_2\downarrow \\ \bar{q}_3\downarrow \\ \bar{q}_1\uparrow \\ \bar{q}_2\uparrow \\ \bar{q}_3\uparrow \end{vmatrix} \qquad (1.36)$$

We now construct the states of the mesons from the basic fields of the fundamental sextets. The mesons which are regarded as bound states of quarks and antiquarks in the S-state are assigned to the 35-dimensional representation of SU(6) in the decomposition,

$$\underset{\sim}{6} \otimes \underset{\sim}{\bar{6}} = \underset{\sim}{1} + \underset{\sim}{35} \qquad (1.36)$$

The 35-dimensional representation can be further reduced with respect to $SU(3) \otimes SU(2)$ to obtain the SU_3 and SU_2 content of the 35-plet. The general method for the reduction of an irreducible representation of the group SU_{mn} with respect to the subgroup $SU_m \otimes SU_n$ has been discussed by a number of authors[25]. The above case is very simple and the result is,

$$\underset{\sim}{35} = (1, 3) + (8, 3) + (8, 1) \tag{1.37}$$

From (1.37) it is clear that the 35-plet of SU(6) contains an octet of pseudoscalar mesons (0^-), and a nonet (singlet + Octet) of vector meson (1^-). We now construct the basis tensor of the 35-plet in terms of the corresponding ones of SU(2) and SU(3). As we know already, the basis of the 35-plet is a traceless tensor ϕ^A_B (A,B = 1,6) which in terms of the corresponding ones of SU(2) and SU(3) can be expressed as follows:

$$\phi^A_B = \phi^{i\alpha}_{j\beta} = \tfrac{1}{2}\,\delta^i_j\,\phi^{k\alpha}_{k\beta} + \tfrac{1}{3}\,\delta^\alpha_\beta\,\phi^{i\gamma}_{j\gamma} + \left\{ \phi^{i\alpha}_{j\beta} - \tfrac{1}{2}\,\delta^i_j\,\phi^{k\alpha}_{k\beta} - \tfrac{1}{3}\,\delta^\alpha_\beta\,\phi^{i\gamma}_{j\gamma} \right\} \tag{1.38}$$

where i, j = 1, 2; $\alpha, \beta = 1, 2, 3$; and (1.38) represents the decomposition (1.37). Let us use P^α_β, $\vec{V}(1)$ and $\vec{V}^\alpha_\beta(8)$ for the octet 0^-, singlet 1^- and octet 1^- meson wave function respectively. Then (1.38) can be rewritten as,

$$\phi^{A}_{B} = \phi^{i\alpha}_{j\beta} = \frac{1}{\sqrt{2}}\left\{ \delta^{i}_{j}\, P^{\alpha}_{\beta} + \vec{\sigma}\,^{i}_{j}\, \vec{V}^{\alpha}_{\beta}\,(9) \right\} \tag{1.39a}$$

$$\vec{V}^{\alpha}_{\beta}\,(9) = \frac{1}{\sqrt{3}}\, \delta^{\alpha}_{\beta}\, \vec{V}(1) + \vec{V}^{\alpha}_{\beta}\,(8) \tag{1.39b}$$

where the wave-function has been so normalised that

$$< \phi\, \phi > = \phi^{A}_{B}\, \phi^{B}_{A} = < p\, p > + \vec{V}(1).\vec{V}(1) + < \vec{V}(8)\, \vec{V}(8) > \tag{1.40}$$

Let us now construct the states of the baryons which are to be obtained from the corresponding states of the three quarks as follows:

$$\underline{6} \otimes \underline{6} \otimes \underline{6} = \underline{20} + \underline{56} + \underline{70} + \underline{70} \tag{1.41}$$

where $\underline{20}$ and $\underline{56}$ are respectively completely antisymmetric and symmetric in the interchange of any two quarks states. The 70-plet has the symmetry of the type [2,1]. Again the SU(3) and SU(2) contents of the above irreducible representations are:

$$\begin{aligned} \underline{20} &= (1,4) + (8,2) \\ \underline{56} &= (8,2) + (10,4) \\ \underline{70} &= (1,2) + (8,2) + (8,4) + (10,2) \end{aligned} \tag{1.42}$$

It is clear from (1.42) that 56-plet contains an octet of spin 1/2 and a decouplet of spin 3/2. Therefore we can assign the 8 baryons and 10 baryon resonances to the 56-plet of SU(6). But this assignment goes against the requirement of the Pauli principle if we assume the baryons as bound states of three quarks in the S-state and if the quarks obey Fermi-statistics. However, the ratio of the magnetic moments of the proton and neutron and the mass-splitting obtained from the 56-plet are in much better agreement with the experimental result than those obtained from the 20-plet which would have been in consistence with the Pauli requirement. It is because of these reasons that the baryons are assigned to the 56-plet. Now the wave-function of the 56-dimensional representation can be constructed from the SU(2) and SU(3) wave functions as follows:

$$\psi_{ABC} \equiv \psi_{i\alpha,j\beta,k\gamma} = D_{\alpha\beta\gamma,ijk} + \frac{1}{3\sqrt{2}}\Big\{\epsilon_{\alpha\beta\delta}\,\epsilon_{ij}\,N^{\delta}_{\gamma,k} + \epsilon_{\beta\gamma\delta}\,\epsilon_{jk}\,N^{\delta}_{\alpha,i} + \epsilon_{\gamma\alpha\delta}\,\epsilon_{ki}\,N^{\delta}_{\beta,j}\Big\} \qquad (1.43)$$

where $D_{\alpha\beta\gamma,ijk}$ is completely symmetric with respect to Latin and Greek indices separately so that it represents a decouplet of spin 3/2 wave-function.

Let us now calculate the effective current which transforms like 35-dimensional representation of SU(6). This current $J^{A'}_{A}$ can be obtained from '56' baryons and antibaryons as follows:

$$J_A^{A'} = \bar{\psi}^{A'BC}\psi_{ABC} - \frac{1}{6}\,\delta_A^{A'}\,\bar{\psi}^{DBC}\,\psi_{DBC} \tag{1.44}$$

Substituting the expression of $\bar{\psi}^{ABC}$ and ψ_{ABC} into (1.44), the effective current is given by,

$$\begin{aligned} J_A^{A'} \equiv J_{i\alpha}^{i'\alpha'} &= \bar{D}^{\alpha'\beta\gamma,i'jk}\, D_{\alpha\beta\gamma,ijk} - \frac{1}{6}\,\delta_\alpha^{\alpha'}\delta_i^{i'} < \bar{D}\, D > \\ &+ \frac{\sqrt{2}}{3}\Big\{ \epsilon_{\alpha\beta\delta}\,\epsilon_{ij}\,\bar{D}^{\alpha'\beta\gamma,i'jk}\, N^{\delta}_{\gamma,k} + \epsilon^{\alpha'\beta\delta}\,\epsilon^{i'j}\,\bar{N}^{\gamma,k}_{\delta} \\ &\times\ D_{\alpha\beta\gamma,ijk}\Big\} + \frac{1}{6}\Big\{ \delta_i^{i'}\,(\bar{N}N_F)_\alpha^{\alpha'} + \frac{2}{3}\,(\vec{\sigma}\,)_i^{i'} \\ &\times\Big[\, 3(\bar{N}\vec{\sigma}\,N_D\,)_\alpha^{\alpha'} + 2(\bar{N}\vec{\sigma}\,N_F\,)_\alpha^{\alpha'} - \delta_\alpha^{\alpha'} < \bar{N}\,N > \Big]\Big\} \end{aligned} \tag{1.45}$$

where

$$\begin{aligned} < \bar{D}\, D > &= \bar{D}^{\alpha\beta\gamma,ijk}\, D_{\alpha\beta\gamma,ijk} \\ (\bar{N}N_F)_\alpha^{\alpha'} &= \bar{N}^{\alpha'}_{\beta} N^{\beta}_{\alpha} - \bar{N}^{\beta}_{\alpha}\, N^{\alpha'}_{\beta} \\ (\bar{N}N_D)_\alpha^{\alpha'} &= \bar{N}^{\alpha'}_{\beta} N^{\beta}_{\alpha} + \bar{N}^{\beta}_{\alpha}\, N^{\alpha'}_{\beta} \\ < \bar{N}\, N > &= \bar{N}^{\beta}_{\alpha}\, N^{\alpha}_{\beta} \end{aligned} \tag{1.46}$$

Now the interaction Lagrangian (Yukawa type) between all the baryons and mesons is,

$$L_{int} = J_{A}^{A'} \cdot \phi_{A'}^{A} \tag{1.47}$$

It is well known from the field theory that the pseudoscalar meson-baryon coupling is of the form $\bar{\psi}\gamma_5\psi\phi$ which in the static (as in the case with SU(6)) limit reduces to $(\bar{\psi}\vec{\sigma}\psi)\cdot\vec{\nabla}\phi$; while the vector meson-baryon coupling is of the form $\bar{\psi}\gamma_\mu\psi\phi_\mu$ which reduces to $\bar{\psi}\psi\phi_0$. Hence from (1.45) we find that in SU(6) the vector mesons couple to the baryon in an F type coupling; whereas the pseudoscalar mesons have both D and F types occurring in the ratio F/D = 2/3.

b. The Symmetry Breaking Interactions in SU(6)

The mass-splitting in SU(6) can be considered in the same way as it was done in SU(3). Neglecting the weak interactions altogether as before, we can consider, therefore, the effects of the medium strong and the electromagnetic interactions in the removal of the mass-degeneracy of the SU(6) supermultiplets. We first discuss the effects of the medium strong interactions and shall consider the electromagnetic interactions later in this section.

As we have discussed in relation to SU(3), the medium strong interaction said to be responsible for the symmetry breaking can be considered as an operator T_M. We can now assume, as some authors[26] have done, that this mass operator T_M transforms under SU(6) as an $I = Y = J = 0$ eigenstate. Then the above operator can be expressed as a sum of the operators corresponding to various irreducible representations which contain the eigenstate $I = Y = J = 0$. A few low-dimensional representations which contain the above eigenstate are $\underline{1}$, $\underline{35}$, $\underline{189}$, $\underline{405}$ etc. Therefore, we can write T_M in the form,

$$T_M = T^1(I = Y = J = 0) + T^{35}(I = Y = J = 0)$$
$$+ T^{189}(I = Y = J = 0) + T^{405}(I = Y = J = 0)$$
$$+ \qquad\qquad + \qquad\qquad (1.48)$$

Then the mass-splittings are given by the matrix elements of the operator T_M in (1.48) between the same state of any irreducible representation of SU(6). Kuo and Yao[26], however, assumed that to the first order, the mass-operator T_M transforms as the $I = Y = J = 0$ state of the regular representation of SU(6). The tensor operator they have used, is the SU(6) version of Okubu's operator and is given by,

$$T^{\mu}_{\nu} = a_0 \delta^{\mu}_{\nu} + a_1 A^{\mu}_{\nu} + a_2 (A.A)^{\mu}_{\nu} + a_3 (A.A.A)^{\mu}_{\nu} + a_4 (A.A.A.A)^{\mu}_{\nu}$$

$$+ a_5 (A.A.A.A.A)^{\mu}_{\nu} \qquad (1.49)$$

where the A^{μ}_{ν} are the generators of SU(6) and the ai's depend only on the Casimir operators of the group. The components of the above operator contributing to the mass-splitting is according to the above assumption $T^3_3 + T^6_6$.

Let us now discuss the mass-splitting in 35-plet by using the above operator (1.49). Since $\underline{35}$ occurs twice in the decomposition,

$$\underline{35} \times \underline{35} = \underline{1} + \underline{35} + \underline{35} + \underline{189} + \underline{280} + \underline{280}^* + \underline{405} \qquad (1.50)$$

only the first three terms in (1.49) contribute to the mass-splitting. Thus,

$$\Delta M = < 35|T^{\mu}_{\nu}|35 > = a_0 + a_1 < 35|A^{\mu}_{\nu}|35 > + a_2 < 35|(A.A)^{\mu}_{\nu}|35 >$$

$$(1.51)$$

From (1.51) we can calculate the contribution of $T^3_3 + T^6_6$ to the mass-splitting. The mass-formula obtained by Kuo et al is,

$$M^2 = m_0 + m_1 Y + m_2 \left[2J^{(\lambda)}(J^{(\lambda)} + 1) + \tfrac{1}{4}Y^2 - C_2^{(4)} \right] \qquad (1.52)$$

where $C_2^{(4)}$ is the quadratic Casimir operator of the SU(4) subgroup of SU(6). $\vec{J}^{(\lambda)}$ is the spin of the strangeness bearing quark. The other components of this vector are,

$$J_+^{(\lambda)} = A_6^3$$

$$J_-^{(\lambda)} = A_3^6 \qquad (1.53)$$

$$J_3^{(\lambda)} = 1/2(A_3^3 - A_6^6)$$

Using (1.53) we can calculate the eigenvalues of the operator $\vec{J}^{(\lambda)}$ for the various members of the 35-plet. Using those values we can get the mass-relations which was obtained by Kuo et al. The meson-mass relations are:

$$m_\omega^2 = m_\rho^2 \qquad (1.54a)$$

$$m_\phi^2 + m_\rho^2 = 2m_{K*}^2 \qquad (1.54b)$$

$$m_K^2 = \tfrac{1}{4}(3m_\eta^2 + m_\pi^2) \qquad (1.54c)$$

$$m_{K*}^2 - m_\rho^2 = m_K^2 - m_\pi^2 \qquad (1.54d)$$

The relation (1.54c) is the well known Gell-Mann -Okubu formula we obtained in relation to SU(3). However, the appearance of the relation (1.54a) shows that the mass formula (1.52) requires improvement. For this reason Beg and Singh[27)] considered the contributions from the operators corresponding to the representations 189 and 405 as well. The final formula they obtained is,

$$M = a + bY + c\left[\, 2J^{(\lambda)}(J^{(\lambda)} + 1) + \frac{1}{4}Y^2 - C_2^{(4)} \,\right] + d\,C_2^{(3)}$$

$$+ \, eJ(J+1) + f\left[\, J^{(N)}(J^{(N)} + 1) - J^{(\lambda)}(J^{(\lambda)} + 1) \,\right]$$

$$+ \, g\left[\, I(I+1) - \frac{1}{4}Y^2 \,\right] \qquad (1.55)$$

where $C_2^{(3)}$ is the quadratic Casimir operator of the subgroup SU(3). $\vec{J}^{(N)}$ is the spin of the non-strange quarks. The other components of $\vec{J}^{(N)}$ are,

$$J_+^{(N)} = A_4^1 + A_5^2$$

$$J_-^{(N)} = A_1^4 + A_2^5 \qquad (1.56)$$

$$J_3^{(N)} = \tfrac{1}{2}(A_1^1 - A_4^4 + A_2^2 - A_5^5)$$

Again the operator $\vec{J}$ in (1.55) is defined by

$$\vec{J} = \vec{J}^{(N)} + J^{(\lambda)}$$

such that $J_3 = J_3^{(\lambda)} + J_3^{(N)}$.

The consequence of the mass formula (1.55) is that the relation $m_\omega^2 = m_\rho^2$ disappears and at the same time the other relations in (1.54) which compare well with experimental results are retained. Moreover, for the 56-plet, the mass formula (1.55) reduces to the very simple form given by,

$$M = m_0 + m_1 J(J+1) + m_2 Y + m_3 [I(I+1) - \frac{1}{4} Y^2] \qquad (1.57)$$

This is just the Okubu formula with a spin J part. Using the above formula we can obtain the relation,

$$M_{\Xi} - M_N - \frac{3}{2}(M_\Sigma - M_\Lambda) = M_{\Xi^*} - M_{N^*} \qquad (1.58)$$

The experimental values of the left and right hand sides of (1.58) are 270 Mev and 293 Mev respectively.

Let us now discuss some SU(6) results obtained in the fields of electromagnetic interactions. Chan and Sarkere[28)] have considered the electro-magnetic mass-corrections of the various isomultiplets within the 56-plet. They assumed that the electromagnetic mass operator transforms like a spin singlet in the spin space and like the charge operator Q in the SU(3) space. The relations they obtained for the Baryon octet are:

$$M_{\Sigma^-} - M_{\Sigma^+} = (M_n - M_p) - (\Xi^0 - \Xi^-) \qquad (1.59a)$$

$$M_{\Sigma^-} - M_{\Sigma^0} = \Xi^- - \Xi^0 \qquad (1.59b)$$

The mass relation (1.59a) is the same as in SU(3). This is the famous Coleman-Glashow relation which is in very good agreement with the experimental results. The relation (1.59b) is a new prediction of SU(6). The values of the left and right hand sides of (1.59b) are 4.75 Mev and 6.5 Mev respectively. Thus, the new mass relation is also in good agreement with the experimental results.

Electromagnetic vertex of the baryons have also been discussed in the past. As shown by Sakita[29)] the effective electromagnetic interaction of the baryon (in the static limit) is of the form,

$$H_{eff} = 3 e J_1^1 \phi - 3\mu \vec{J}_1^1 \vec{H} \qquad (1.60)$$

where ϕ and $\vec{H}$ are respectively the external electrostatic potential and the magnetic field, and J_α^β and $\vec{J}_\alpha^\beta$ are respectively defined by,

$$J_\alpha^\beta = J_{i\alpha}^{i\beta} \; ; \; \vec{J}_\alpha^\beta = \vec{\sigma}_i^j \left(J_{j\alpha}^{i\beta} - \frac{1}{3} \delta_\alpha^\beta J_{j\gamma}^{i\gamma} \right) \qquad (1.61a)$$

Again $J_A^B = J_{i\alpha}^{i\beta}$ is given by (1.45). In (1.60) e is the charge and μ is the magnetic moment of the proton. Substituting (1.45) into (1.60) we can obtain the relationships among the total magnetic moments of the baryons. In particular, we can obtain the ratio of the magnetic moment of proton and neutron. The value of this ratio obtained by Sakita is,

$$\mu p/\mu n = -3/2 \tag{1.61b}$$

The above result is one of the important achievements of the SU(6) symmetry.

3. The U(6,6) Symmetry for Hadrons

a. The Supermultiplets of Baryons and Mesons and their Yukawa Coupling.

As we have discussed in the last section, the ordinary spin in the SU(6) theory is considered on the same footing as the isotopic spin and hypercharge. This is possible only in the non-relativistic theory as in the case of Wigner's supermultiplet theory. Then the question arose as to whether the SU(6) theory which has had so much success in relating the internal and spin properties of the observed particle-multiplets, could be extended to the relativistic domain. As a result, many attempts were made in the past to find a larger invariance group which would incorporate the SU(6) in a relativistically covariant manner.

Notwithstanding many difficulties in the relativistic generalisation of SU(6), however, the groups SL(6C)[30] and U(6,6)[12,31] were proposed in early 1965 as two possible approximate dynamical symmetries for a phenomenological description of the hadrons and their interactions. Both SL(6C) and U(6,6) are non-compact groups and, therefore, one has to face the same difficulty in obtaining the finite dimensional unitary representations of either group. The group SL(6C) is a group of 6×6 complex matrices of determinant unity and contains SU(6) as one of its subgroups. Consequently, the little groups of SL(6C), as shown by Rühl are $SU(6)_p$ which determine the multiplet- structure of SL(6C). The inhomogeneous ISL(6C) group is a semi-direct product of the homogeneous SL(6C) and an invariant Abelian subgroup of translations T_{36} in 36-dimensional (generalised) space. On the other hand, the group U(6,6) is the group of 12×12 complex matrices and as shown by many authors[32] the maximal compact subgroup it contains is of the type $U(6) \otimes U(6)$ (Nonchirial[33]). Therefore, it is the group $U(6) \otimes U(6)$ which determines the multiplet-structure of U(6,6). The inhomogeneous IU(6,6) group can, as shown by Charap et al.[34], be obtained by taking a semi-direct product of the homogeneous U(6,6) and the translation group T_{144} in 144-dimensional space. It can be shown[32] that SL(6C) is a subgroup of U(6,6). Further, as the subgroup of translation T_{144} contains T_{36} as a subgroup, the inhomogeneous ISL(6C) is also a subgroup of the inhomogeneous $I\bar{U}(6,6)$. From these points of view, we can conclude as claimed by Rühl[32] that every prediction of SL(6C) model will also be implied by U(6,6) model. We shall, however, consider only $\bar{U}(6,6)$ and discuss some of its consequences.

The group $\bar{U}(6,6)$ appeared in many different names proposed by various authors[12,31]. We shall, however, confine ourselves in our discussion to the theory formulated by Delbourgo, Salam and Strathdee[12]. We start with the basic representation which corresponds to three quarks each having four components as Dirac particles. Let ψ_A be the 12-component wave-function denoting the fields of the three quarks. Then under a transformation of U(6,6) we have,

$$\psi_A \rightarrow \psi_A' = S_A^B \psi_B \tag{1.62}$$

where, A = 1,2 ... 12. S is a 12 × 12 matrix and, as is well known, forms the spinor representation of U(6,6). Expressing in terms of the 144 generators of U(6,6) we have,

$$S_A^B = \exp i [\epsilon^K F_K]_A^B \tag{1.63}$$

where K = 1, ... 144. The ϵ^K's are real and F_K's are the generators and in this spinor representation these are 12 × 12 matrices. Since U(6,6) contains the direct product of U(22) U(3) as the subgroup, the generators F_K can be expressed as the direct product of the generators of these subgroups. Thus, we can write,

$$F_K = \Gamma_R \otimes T_i \tag{1.64}$$

where R = 1 ... 16, i = 0 ... 8. The Γ_R are the generators of U(22) and can be taken to consist of the following 16 Dirac matrices,

$$\Gamma_R = 1, \gamma_\mu, \sigma_{\mu\upsilon} = \frac{i}{2}[\gamma_\mu \gamma_\upsilon], \quad i\gamma_\mu\gamma_5 \quad \gamma_5 \tag{1.65a}$$

where, $\underset{\sim}{\gamma}$ are antihermitian in our representation and γ_0 hermitian with the representation $\gamma_0 = (1,1,-1,-1)$ diag. The metric tensor in the Lorentz space is $g_{\mu\upsilon} = (1,-1,-1,-1)$ diag. The above 4×4 Dirac matrices satisfy the relation,

$$\gamma_0 \Gamma_R^+ \gamma_0 = \Gamma_R \tag{1.65b}$$

From (1.65a) we find that of the sixteen matrices 8 : 1, $\underset{\sim}{\sigma}$, γ_0, $i\underset{\sim}{\gamma}\gamma_5$ are hermitian and the other 8 : γ_5, $-i\,\underset{\sim}{\sigma}_0$, $i\gamma_0\gamma_5$, $\underset{\sim}{\gamma}$ are antihermitian, where $\underset{\sim}{\sigma} = (\sigma_{23}, \sigma_{31}, \sigma_{12})$, $\underset{\sim}{\sigma}_0 = (\sigma_{01}, \sigma_{02}, \sigma_{03})$. Hence the group U(22) generated by the 16 Dirac matrices is non-compact, the defining property being given by (1.65b).

The 9 generators T_i of $U(3)$ have been given by Gell-Mann[8]. The normalisation here is so chosen that the relation between T_i and Gell-Mann's λ_i is,

$$T_o = 1/\sqrt{6}, \quad \mathrm{Tr}\, T_i\, T_j = 1/2\, \delta_{ij}, \quad T_i = \lambda_i/2, \quad i = 1, \dots 8$$

We now consider the transformation property of the field of the antiquarks. As before, we denote the field of the antiquarks by the wave-function $\bar{\psi}^A$ which transforms as,

$$\bar{\psi}^A \rightarrow \bar{\psi}'^A = \bar{\psi}^B (S^{-1})^A_B \qquad (1.66)$$

where $\bar{\psi}^A$ is defined by,

$$\bar{\psi}^A = \psi^+_B\, (\gamma_o)^A_B \qquad (1.67a)$$

where

$$(\gamma_o)^A_B = (\gamma_o)^\alpha_\beta\, \delta^i_j \qquad (1.67b)$$

where $\alpha, \beta = 1, \dots 4$ are the Dirac indices and $i, j = 1, 2, 3$ are the $U(3)$ indices. From (1.62) and (1.66) it is obvious that the quantity $\bar{\psi}^A\ \psi_A$ (sometimes referred to as the mass term) is invariant under $U(6,6)$ and (1.67) shows that the above quantity has six positive and six negative terms. This is one of the ways to define the non-compact group $U(6,6)$. In terms of the matrices S, the other defining property of $U(6,6)$ is,

$$S^{+}(\gamma_0 \otimes 1)S = \gamma_0 \otimes 1 \tag{1.68}$$

where 1 is a 3 x 3 unit matrix of U(3) so that $[\gamma_0 \otimes 1]$ is a 12 x 12 matrix. S^{+} is the hermitian conjugate of S. Now, as before, we construct the states of mesons and baryons from the basic fields of the quarks and the antiquarks. The meson states are constructed from quark - antiquark states and those of baryons from three quarks. These states decompose under U(6,6) in the following way,

$$\begin{aligned} \underline{12} \otimes \underline{12}^{*} &= \underline{1} + \underline{143} \\ \underline{12} \otimes \underline{12} \otimes \underline{12} &= \underline{220} + \underline{364} + \underline{572} + \underline{572} \end{aligned} \tag{1.69}$$

where $\underline{220}$ and $\underline{364}$ correspond respectively to the completely anti-symmetric and symmetric tensor of rank 3 and $\underline{572}$ to the mixed symmetry of the type [21]. Finally, $\underline{143}$ corresponds to the Young tableau $[2,1^{10}]$. Under the subgroup SU(22) x SU(3) these irreducible representations are further reduced and their SU(2,2) and SU(3) contents are given by,

$$\begin{aligned} \underline{143} &= (15,8) + (15,1) + (1,8) \\ \underline{220} &= (20',8) + (20,1) + (4^{*},10) \\ \underline{364} &= (20,10) + (20',8) + (4^{*},1) \\ \underline{572} &= (20',10) + (20',8) + (20,8) + (20',1) \\ &\quad + (4^{*},8) \end{aligned} \tag{1.70}$$

where 15 and 4* of SU(2,2) correspond respectively to the Yong tableaux $[21^2]$ and $[1^3]$ and 20 and 20' of SU(2,2) to the tableaux [3] and [2 1] respectively.

Now the meson fields can be denoted by the mixed tensor ϕ^A_B. For each SU(3) index the meson field ϕ is a 4×4 matrix in the Dirac space. Consequently, it can be expanded in terms of the sixteen independent Dirac matrices. Thus, 144-component meson-field ϕ can be expressed as,

$$\phi^A_B = \left[\phi^i + \gamma_5 \phi^i_5 + i \gamma_\mu \gamma_5 \, \phi^i_{\mu 5} + \gamma_\mu \phi^i_\mu + \tfrac{1}{2} \sigma_{\mu\nu} \, \phi^i_{\mu\nu} \right]^\alpha_\beta (T^i)^p_q \tag{1.71}$$

where $\alpha, \beta = 1, \dots 4$, $p, q = 1, 2, 3$.

The baryons, as it is in SU(6), are assigned to 364 which is completely symmetric with respect to its U(6,6) tensor indices. Let ψ_{ABC} be such a tensor field which is completely symmetric with respect to the interchange of any of the two indices. The tensor of such a symmetry type can be constructed from the corresponding SU(2,2) and SU(3) tensors in three different ways as is obvious from (1.70). Thus, we can express ψ_{ABC} in the form,

$$\psi_{ABC} = \psi_{\alpha p,\beta q,\gamma r} = \frac{\sqrt{3}}{2\sqrt{2}} D_{\alpha\beta\gamma,pqr} + \epsilon_{pqr} V_{[\alpha\beta\gamma]}$$

$$+ \frac{1}{2\sqrt{6}} \Big[\epsilon_{pqs} N_{[\alpha\beta]\gamma,r}^{\;s} + \epsilon_{qrs} N_{[\beta\gamma]\alpha,p}^{\;s} +$$

$$+ \epsilon_{rps} N_{[\gamma\alpha]\beta,q}^{\;s} \Big] \qquad (1.72)$$

where ϵ_{pqr} is SU(3) invariant Levi-Civita tensor. $D_{\alpha\beta\gamma,pqr}$ is completely symmetric with respect to Dirac and SU(3) indices separately. $V_{[\alpha\beta\gamma]}$ is completely antisymmetric with respect to the Dirac indices. $N_{[\alpha\beta]\gamma}$ has the mixed symmetry and satisfy the following relations:

$$N_{[\alpha\beta]\gamma} = -N_{[\beta\alpha]\gamma}$$

$$N_{[\alpha\beta]\gamma} + N_{[\beta\gamma]\alpha} + N_{[\gamma\alpha]\beta} = 0 \qquad (1.73)$$

The normalisation factors in (1.72) have been chosen in the same way as it was done in connection with SU(6) except the coefficient of $D_{\alpha\beta\gamma,pqr}$ which, here is so chosen that it represents the spin 3/2 particle with the correct charge for N^{*++} .

We now construct the meson-baryon Yukawa type strong interaction Lagrangian. The number of ways this Lagrangian can be constructed depend on how many times the representation $\underline{143}$ occurs in the reduction of the product representation $364 \otimes 364^*$. Using the well known method we have,

$$\underline{364} \otimes \underline{364}^* = \underline{1} + \underline{143} + \underline{5940} + \underline{126412} \tag{1.74}$$

From (1.74) it is obvious that the meson-baryon vertex is unique as it was in SU(6). Thus the meson-baryon interaction Lagrangian can be written as,

$$L_{int} = \bar{\psi}^{ABC} \, \phi_A^{A'} \, \psi_{A'BC} \tag{1.75}$$

Now substituting the expressions for $\bar{\psi}^{ABC}$, $\psi_{A'BC}$ from (1.72) and that for $\phi_A^{A'}$ from (1.71) into (1.75) and then using (1.73) we have on simplification,

$$L_{int} = \frac{3}{8} \bar{D}^{\alpha\beta\gamma,pqr} \, (\phi_R^i \, \gamma_R)_\alpha^{\alpha'} \, (T^i)_p^{p'} \, D_{\alpha'\beta\gamma,p'qr}$$

$$+ \frac{1}{4} \Big[\bar{D}^{\alpha\beta\gamma,pqr} \, (\phi^i \, \gamma_R)_\alpha^{\alpha'} \, (T^i)_p^{p'} \, \epsilon_{p'qt} \, N_{[\alpha'\beta]\gamma,r}^{\;t}$$

$$+ \bar{N}^{[\alpha\beta]\gamma,r}{}_s \, \epsilon^{pqs} \, (\phi_R^i \, \gamma_R)_\alpha^{\alpha'} \, (T^i)_p^{p'} \, D_{\alpha'\beta\gamma,p'qr} \Big]$$

$$- \frac{1}{24} \left[\bar{N}^{[\beta\alpha]\gamma} \, (\phi^{i}_{R} \, \gamma_{R})^{\alpha'}_{\alpha} \, N_{[\alpha'\beta]\gamma} \right]^{i}_{3D + 5F + 12S}$$

$$+ \frac{1}{12} \left[\bar{N}^{[\beta\alpha]\gamma} \, (\phi^{i} \, \gamma_{R})^{\alpha'}_{\alpha} \, N_{[\alpha'\gamma]\beta} \right]^{i}_{3D + 2F + 3S}$$

(1.76)

where,

$$(\bar{N}N)^{i}_{F} = \bar{N}^{p}_{r} \, (T^{i})^{q}_{p} \, N^{r}_{q} - \bar{N}^{p}_{r} \, N^{q}_{p} \, (T^{i})^{r}_{q}$$

$$(\bar{N}N)^{i}_{D} = \bar{N}^{p}_{r} \, (T^{i})^{q}_{p} \, N^{r}_{q} + \bar{N}^{p}_{r} \, N^{q}_{p} \, (T^{i})^{r}_{q} \qquad (1.77)$$

$$(\bar{N}N)_{s} = \bar{N}^{p}_{s} \, (T^{o})^{p}_{p} \, N^{s}_{p}$$

In the above expression (1.76) for the meson-baryon interaction we have neglected the terms involving $V_{[\alpha\beta\gamma]}$ which, as we shall discuss in the following section, is identically zero. The above interaction Lagrangian is invariant under U(6,6). We shall simplify it further in the next section to make it much simpler for the application in the S-matrix calculation.

b. Bargmann-Wigner equations and the Final form of the Baryon-Meson Interaction Lagrangian.

We have discussed in the last section the multispinors corresponding to the finite dimensional irreducible representations of U(6,6). But as the group U(6,6) is non-compact, these representations are not unitary and, therefore cannot be associated with physical particles, the reason being that the spaces of the representations have indefinite metric. We can, however, obtain unitary representations by decomposing, as Delbourgo et al. have shown[32], the indefinite spaces of the original representations into a collection of subspaces which are invariant under the subgroup $U(6)\otimes U(6)$ each of the subspaces being definite. Now, if we assume that particles at-rest correspond to the representations of $U(6)\otimes U(6)$, the Bargmann-Wigner equations when applied to the multispinors corresponding to the U(6,6) representations generate the relativistic structure of such $U(6)\otimes U(6)$ multiplets. Thus, the Bargmann-Wigner equations can be looked upon as the relativisitic boosts which generate for a single particle state what we may call the little group structure $(U_6\otimes U_6)_p$. Let $\phi^{B_1B_2\cdots}_{A_1A_2\cdots}$ be a multispinor corresponding to an irreducible representation of U(6,6). Then the Bargmann-Wigner equations corresponding to the lower and upper indices are respectively given by,

$$(\gamma\, p)^{A_1'}_{A_1}\,\Phi^{B_1B_2}_{A_1'A_2}(p) = m\,\phi^{B_1B_2\cdots}_{A_1A_2}(p) \qquad (1.78a)$$

$$(\gamma\, p)^{B_1}_{B_1'}\,\Phi^{B_1'B_2}_{A_1A_2}(p) = -m\,\phi^{B_1B_2}_{A_1A_2}(p) \qquad (1.78b)$$

where p, m are respectively the momentum and the rest-mass of the particle such that $p^2 = m^2$. Corresponding to each of the indices of the irreducible tensor-field there is a Bargmann-Wigner equation of either of the types given above. For identifications with the physical states we are to keep only those vectors whose components vanish outside the space specified by the Bargmann-Wigner equations.

We now consider the reduction of the multispinors of U(2,2) with respect to one of its subgroups. For this prupose, we choose the well known subgroup L_4 , the homogeneous Lorentz group, the generators of which in the spinor representation are the six Dirac matrices $\sigma_{\mu\nu}$. Now, as is well known, for this case we can define an antisymmetric matrix $(C^{-1})^{\alpha\beta}$ within the Dirac algebra such that $C^{-1}\psi^T$ (ψ^T being the transpose of the basic spinor ψ_α) transforms as $\bar{\psi}$. In particular, the quantity $C^{-1}\psi^T\psi$ will be invariant just as $\bar{\psi}\psi$ under an infinitesimal transformation. Thus, the antisymmetric matrix C will play the role for L_4 of the metric tensor such that we may regard $(C^{-1})^{\alpha\beta}$ as the contravariant quantity and its inverse $C_{\alpha\beta}$ the covariant. It can be shown[35] that the matrix C can be realised if the following relation is satisfied,

$$(C^{-1})^{\alpha\xi}(\gamma_\mu)_\xi^\delta C_{\delta\beta} = -(\gamma_\mu)_\beta^\alpha \tag{1.79a}$$

The other properties of the matrix C are,

$$(C^{-1})^{\alpha\beta} = -(C^{-1})^{\beta\alpha} \tag{1.79b}$$

$$C_{\alpha\beta}(C^{-1})^{\beta\gamma} = \delta_{\alpha}^{\gamma} \tag{1.79c}$$

From these relations we can show that the quantities $(\gamma_\mu C)_{\alpha\beta}$, and $(\sigma_{\mu\nu} C)_{\alpha\beta}$ are symmetric. On the other hand, the quantities $(\gamma_5 C)_{\alpha\beta}$ and $(i\gamma_\mu\gamma_5 C)_{\alpha\beta}$ are antisymmetric. It is these quantities which can be used in constructing a multispinor in L_4 having a certain symmetry with respect to the interchange of the Dirac indices. Thus, a completely symmetric tensor of rank 3 can be expressed as,

$$D_{\alpha\beta\gamma} = (\gamma^\mu C)_{\alpha\beta}\, D_{\gamma\mu} + \tfrac{1}{2}(\sigma_{\mu\nu} C)_{\alpha\beta}\, D_{\gamma\mu\nu} \tag{1.80}$$

Considering the symmetry property of $D_{\alpha\beta\gamma}$, we obtain the following

$$\gamma^\mu D_\mu = 0$$

$$\gamma^\mu D_{\mu\nu} + i\, D_\nu = 0 \tag{1.81}$$

The equations (1.81) show that $D_{\alpha\beta\gamma}$ has only 20 independent components. Now the Bargmann-Wigner equations corresponding to each of the three indices of $D_{\alpha\beta\gamma}$ give,

$$P_\mu D_{\mu\upsilon} = -i\,m\,D_\upsilon \tag{1.82a}$$

$$(\not{p} - m)D_\mu = 0 \tag{1.82b}$$

$$p_\mu D_\upsilon - p_\upsilon D_\mu = i\,m\,D_{\mu\upsilon} \tag{1.82c}$$

where p and m are respectively the momentum and the rest-mass of the particle. From (1.82b,c) we can derive the following equation,

$$p^\mu D_\mu = 0 \tag{1.82d}$$

The equations (182b,c,d) are the well known Rarita-Schwinger equations of motion for a spin 3/2 particle. Now using (1.82) the expression (1.80) can be rewritten in the following simple form,

$$D_{\alpha\beta\gamma} = \frac{1}{m}\left[(\not{p} + m)\gamma^\mu C \right]_{\alpha\beta} D_{\gamma\mu} \tag{1.80'}$$

Similarly, the third-rank tensor $N_{[\alpha\beta]\gamma}$ of the mixed symmetry can be expressed as,

$$N_{[\alpha\beta]\gamma} = C_{\alpha\beta}K_\gamma + (\gamma_5 C)_{\alpha\beta}\, N_\gamma + i(\gamma_\mu\gamma_5 C)_{\alpha\beta}\, N_{\gamma\mu} \tag{1.83}$$

where the use of the equation (1.73) gives,

$$\gamma_5 N + i\gamma^{\mu}\gamma_5 N_{\mu} - K = 0 \tag{1.84a}$$

For this tensor-field $N_{[\alpha\beta]\gamma}$ we can also obtain three Bargmann-Wigner equations corresponding to the three spinor indices. From these, the equations we obtain are as follows:

$$(\not{p} - m)N_{\mu} = (\not{p} - m)N = (\not{p} - m)K = 0$$

$$K = 0; \quad p_{\mu} N_{\mu} - p_{\nu} N_{\mu} = 0; \quad p_{\mu} N_{\mu} = -imN \tag{1.84b}$$

$$p_{\mu} N = imN_{\mu}$$

The above equations taken together describe a spin 1/2 particle Using the above relations the expression (1.83) can be rewritten in the form:

$$N_{[\alpha\beta]\gamma} = \frac{1}{m}\left[(\not{p} + m)\gamma_5 C \right]_{\alpha\beta} N_{\gamma} \tag{1.83'}$$

Corresponding to the upper and lower indices of the meson-fields (1.71) we shall obtain two Bargmann-Wigner equations. For any arbitrary SU(3) index we get the following equations,

$$\phi^{j} = 0 \tag{1.85a}$$

$$q_\mu \phi_5^j = i\mu\phi_{\mu 5}^j \; ; \; q^\mu \phi_{\mu 5}^j = -i\mu\phi_5^j \tag{1.85b}$$

$$q_\mu \phi_{\mu\nu}^j = -i\mu\phi_\nu^j \; ; \; q_\mu \phi_\nu^j - q_\nu \phi_\nu^j = i\mu\phi_{\mu\nu}^j \tag{1.85c}$$

where q and μ are respectively momentum and the rest-mass of mesons. From (1.85c) we can obtain,

$$q^\mu \phi_\mu = 0 \tag{1.85d}$$

which shows that the field described by ϕ_μ corresponds to a spin one particle. It is also evident that the equations (1.85b) describe a spin 0^- particle and the equations (1.85c) describe a spin 1^- particle. Thus, the pseudoscalar mesons (nonet) are described by five-component entities and the vector (nonet) mesons by ten-component objects. Now, using the equations (1.85), the mixed tensor $\phi_A^{A'}$ for the meson-fields can be expressed as,

$$\phi_A^{A'} = \frac{1}{\mu}\left[(\not{q} + \mu)(\gamma_5\phi^i + \gamma^\mu\phi_\mu^i \right]_\alpha^{\alpha'} (T^i)_p^{p'} \tag{1.86}$$

We now simplify the expression (1.76) for the Baryon-meson interaction Lagrangian by using the free-field equations we have discussed above. The relevant tensor fields in the expression (1.76) we obtain from (1.80') (1.83') and (1.86). Using the free-field equations of motion (1.82) and (1.84b) for the spin 3/2 and 1/2 particles respectively we simplify the expression and write the interaction Lagrangian in the following form:

$$L(\bar{N}\phi_5 N) = \frac{p^2}{4m^2}\left(1 + \frac{2m}{\mu}\right) (\bar{N}\gamma_5 \phi_5 N)_{S+D+\frac{2}{3}F} \tag{1.87}$$

$$L(\bar{N}\phi_\mu N) = \frac{p^\mu}{2m}\left(1 + \frac{q^2}{2m_\mu}\right) (\bar{N}\phi_\mu N)_{3S+F} + \frac{1}{4m^2}\left(1 + \frac{2m}{\mu}\right)$$

$$(\bar{N} r_\mu \phi_\mu N)_{S+D+\frac{2}{3}F} \tag{1.88}$$

$$L(\bar{N}\phi_5 D) = \frac{1}{m}\left(1 + \frac{2m}{\mu}\right)\left[\bar{N} D_\lambda q_\lambda + q_\lambda \bar{D}_\lambda N\right] \cdot \phi_5 \tag{1.89}$$

$$L(\bar{N}\phi_\mu D) = \frac{1}{2m^2}\left(1 + \frac{2m}{\mu}\right) \epsilon^{\mu\upsilon K\lambda} P_\upsilon q_K \left[\bar{N} D_\lambda + \bar{D}_\lambda N\right] \cdot \phi_\mu \tag{1.90}$$

$$L(\bar{D}\phi_5 D) = \left(1 + \frac{2m}{\mu}\right)\left[\frac{3P^2}{4m^2} \bar{D}^\upsilon \gamma_5 D_\upsilon + \frac{3}{2m^2} q^\lambda D_\lambda \gamma_5 D_\upsilon q_\upsilon\right] \cdot \phi_5$$

(1.91)

$$L(\bar{D}\,\phi_\mu\,D) = \frac{3P^2}{4m^2}\,\bar{D}^\upsilon \left[\left(1 + \frac{2m}{\mu}\right)\gamma^\mu - \frac{P^\mu}{\mu} \right] D_\upsilon\,\phi_\mu + \frac{3}{2m^2}\,q_\lambda\,\bar{D}_\lambda$$

$$\left[\left(1 + \frac{2m}{\mu}\right)\gamma^\mu - \frac{P^\mu}{\mu} \right] D_\upsilon\,q_\upsilon\,\phi_\mu \qquad (1.92)$$

In the above expressions we have $P = p + p'$; $q = p - p'$, where p and p' are respectively the momenta of the ingoing and the outgoing baryons $r_\mu = \epsilon^{\mu\upsilon K\lambda}\,P_\upsilon\,q_K\,\gamma_\lambda\,\gamma_5$, where $\epsilon^{\mu\upsilon K\lambda}$ is the fourth-rank Levi-Civita tensor with $\epsilon^{0123} = 1$. N,D, ϕ_5 and ϕ_μ denote baryon, baryon-resonance, pseudoscalar meson and the vector meson respectively. The expressions $(\bar{N}\,N)_F$, $(\bar{N}\,N)_D$ and $(\bar{N}\,N)_S$ have the usual meaning defined by (1.77).

One of the interesting features of the U(6,6) theory is that the pseudoscalar SU(3) singlet (x^0) coupling is no longer independent of the pseudoscalar octet coupling. These are related through U(6,6) coupling as it is evident from (1.89). Further, the F/D ratio of the meson-baryon coupling is uniquely defined in U(6,6) as it is in SU(6). Salam, Delbourgo and Strathdee have obtained for the proton-neutron magnetic moment ratio, the value which is the same as SU(6) result. The U(6,6) theory not only gives the value of this ratio, but also an expression in terms of Baryon and meson masses for calculating the value of the proton magnetic moment.

Despite many other predictions (discussed by Salam et al[32]) the U(6,6) theory, however, have some shortcomings. As discussed by several authors[36] the requirement of the unitarity of the S-matrix (scattering matrix) is not compatible with the U(6,6) theory. We shall, however, like to elaborate this point when we consider the Baryon-meson scattering in the context of U(6,6) in the later chapter. The incompatibiltiy of the causality relation for baryons with the index invariance theory (as U(6,6) and SL(6 C)) have been discussed by Feldman and Matthews[37]. We shall, rather, not discuss this aspect of the relativistic (higher) symmetry theories any further.

CHAPTER II

The Methods for the Dynamical Calculations

1. The Partial Wave Dispersion Relations and N/D Method

Mandelstam[38)] has given a representation for the invariant amplitude A(s t u) in the form of a double dispersion relation. Here the variables are respectively the c.m. energy squared in each of the three channels that are associated with any two-particle scattering diagram. Mandelstam's representation expresses the scattering amplitude A(s t u) as a function of these variables thus implying that the function A(s t u) is analytically continuable in the different regions of the above variables. Since no convenient method for using the double spectral functions in the Mandelstam's representation is yet available, this representation is used in the form of either the fixed energy dispersion relations in which one of the three variables is kept fixed or the partial wave dispersion relations. It is the latter that we shall be concerned with in this chapter. We shall assume in this chapter that the partial wave amplitudes and the structures of the related singularities corresponding to an arbitrary scattering process are known and then discuss how the dispersion relation techniques are used to solve such a problem. In particular, we shall discuss how the partial wave dispersion relations are used in the self-consistent calculations.

Let $a_\ell(s)$ be a partial wave amplitude corresponding to the orbital angular momentum ℓ with the following properties:

(a) $a_\ell(s)$ is analytically continuable into the entire complex s-plane, where s is the c.m. energy squared.

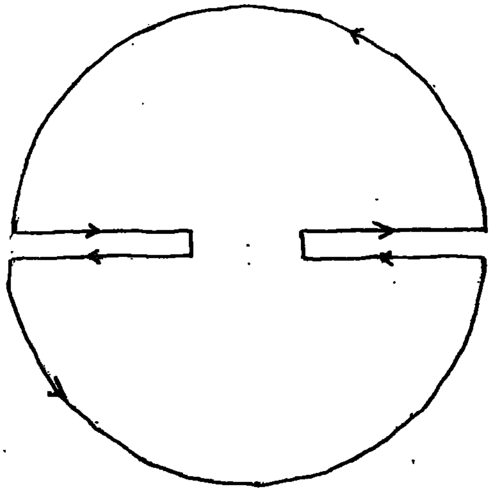
(b) $a_\ell(s)$ is regular (analytic and single valued) everywhere in the entire complex s-plane except for the two branch cuts, namely, the right-hand cut (physical cut) starting from the physical threshold and the left-hand cut that arises due to the singularities in the crossed processes.

(c) $a_\ell(s)$ vanishes at infinity everywhere in the complex s-plane. In otherwords, the partial wave amplitude $a_\ell(s)$ behaves as s^{-n} where $n \geqslant 1$.

(d) the function $a_\ell(s)$ is a real analytic function of the variable under consideration. Mathematically, the reality property of the analytic function is given by,

$$a_\ell^*(s) = a(s^*) \qquad (2.1)$$

We can, then, use the Cauchy theorem in order to obtain the dispersion relation for the partial wave amplitude $a_\ell(s)$. We choose the contour as shown below (Fig. 2.1) in the complex s-plane.



s-plane

Figure 2.1

Now in view of the property (c) of $a_\ell(s)$ we can neglect the contributions from the infinite circle and further, we assume that $a_\ell(s)$ is free from all the kinematic singularities. Then, using (2.1) we obtain the following dispersion relation,

$$a_\ell(s) = \frac{1}{\pi}\int_L ds' \frac{\operatorname{Im} a_\ell(s')}{s'-s} + \frac{1}{\pi}\int_R ds' \frac{\operatorname{Im} a_\ell(s')}{s'-s} \qquad (2.2)$$

where L, R denote the integrals on the left and right cuts respectively. The above dispersion relation is valid provided the condition (c) is satisfied. If $a_\ell(s)$ does not vanish as $|s| \to \infty$ we have to introduce subtractions. Suppose $|a_\ell(s)| \not\to 0$ as $|s| \to \infty$ but $|a_\ell(s)|/|s|^n \to 0$ as $|s| \to \infty$ where $n \geqslant 1$. Then we apply the Cauchy theorem to the function $a_\ell(s)/s^n$ to obtain the dispersion relation. This new function, however, has poles due to the factor s^n in the denominator. The contributions of these poles are always taken into account while deriving the dispersion relation.

Having known the form (2.2) of the dispersion relation for the partial wave amplitude $a_\ell(s)$, the job then is to solve it in order to obtain the informations about a specific scattering process. In order to obtain a solution for the scattering amplitude $a_\ell(s)$ we need to know the imaginary parts Im $a_\ell(s)$ of $a_\ell(s)$ over both the right and left cuts. The imaginary part Im $a_\ell(s)$ over the right-hand cut is known from the unitarity relation above the physical threshold. Then, using the unitarity relation, we can obtain for Im $a_\ell(s)$ over the right-hand cut,

$$\text{Im } a_\ell(s) = \rho(s)|a_\ell(s)|^2 R_\ell(s) \qquad (2.3)$$

where ρ (s) is the kinematic factor which is usually of the form,

$\rho_\ell(s) = \frac{k^{2\ell+1}}{\sqrt{s}}$, where k is the absolute value of the momentum.

The quantity $R_\ell(s)$ in (2.3) is the ratio of the total cross section of all the processes (elastic as well as inelastic, corresponding to the ℓ^{th} wave) to that of the elastic processes. The factor $\rho(s)$ is related with the partial wave amplitude through the relation,

$$a_\ell(s) = \frac{e^{i\delta_\ell} \sin \delta_\ell}{\rho(s)} \qquad (2.4)$$

where δ_ℓ is the phase-shift corresponding to the ℓ^{th} partial wave of the scattering process. In the above we have considered only the elastic scattering for which δ_ℓ is real and we can put $R_\ell(s) = 1$. This approximation is exact only up to the inelastic threshold. For the scattering above the inelastic threshold, the relation (2.4) will have to be modified by introducing the factor $R_\ell(s)$ in order to take into account the inelastic effects. Many calculations have, in fact, been made with the elastic approximation. This approximation, however, would not be bad if the second integral in (2.2) is rapidly convergent enough.

As we have discussed above, the unitarity relation (2.3) gives us some information about the right-hand cut but a difficulty arises as to calculating Im $a_\ell(s)$ over the left-hand cut because the unitarity condition of $a_\ell(s)$ cannot be applied over the left cut which lies in the unphysical region of s-channel. This difficulty is, however, overcome by using the crossing symmetry property of $a_\ell(s)$. It is well known that the left-hand cut is associated with the forces that are responsible for the scattering and these forces arise due to the exchanges of the particles in the crossed channels. The nearby part of the left-hand cut arises from the lightest particles that can be exchanged and so corresponds to the long range forces of the problem. The far-off part of the left-hand cut is associated with the exchange of more massive systems, i.e. multiparticle states and thus corresponds to the short-range forces. If we know the particle that can be exchanged in the crossed channel of the scattering under consideration we can use the crossing symmetry which relates the Im $a_\ell(s)$ on the left-cut with absorptive parts of the crossed processes. These absorptive parts are known from the relevant crossed diagrams and consequently Im $a_\ell(s)$ on the left-cut can be calculated.

Having known the values of Im $a_\ell(s)$ over both the left and the right cuts we may try to solve (2.2) but in that case we have to deal with a non-linear equation which will arise due to the non-linear condition (2.3). Chew and Mandelstam[13)], however, have given a method known as N/D method by which we can convert this non-linear equation into a pair of coupled linear integral equations. Thus, we can write,

$$a_\ell(s) = \frac{N_\ell(s)}{D_\ell(s)} \tag{2.5}$$

where $N_\ell(s)$ contains all the discontinuities on the left-hand cut and is real on the right-hand cut. Thus, it corresponds to the forces.

The function $D_\ell(s)$ on the other hand, is assumed to contain all the discontinuities on the right-hand cut and be real on the left-hand cut. That $a_\ell(s)$ can be expressed in the form (2.5) is essentially due to the fact that the left-hand and right-hand cuts are separated by a gap, i.e. $a_\ell(s)$ is real analytic.

We are now in a position to write dispersion relations for both $N_\ell(s)$ and $D_\ell(s)$. We can again assume that both $N_\ell(s)$ and $D_\ell(s)$ are real analytic functions such that they will satisfy the following relations,

$$N_\ell^*(s) = N_\ell(s^*) \tag{2.6a}$$

$$D_\ell^*(s) = D_\ell(s^*) \tag{2.6b}$$

Now the imaginary or the absorptive parts of $N_\ell(s)$ and $D_\ell(s)$ are respectively given by,

$$\begin{aligned} \mathrm{Im}\, N_\ell(s) &= D_\ell(s)\mathrm{Im}\, a_\ell(s) \quad \text{for } s < s_L \\ &= 0 \quad \text{for } s > s_L \end{aligned} \tag{2.7a}$$

$$\begin{aligned} \mathrm{Im}\, D_\ell(s) &= N_\ell \mathrm{Im}(1/a_\ell) \\ &= -N_\ell(s)\, \rho(s) R_\ell(s') \quad \text{for } s > s_R \\ &= 0 \quad \text{for } s < s_R \end{aligned} \tag{2.7b}$$

where s_L is the beginning of the left-cut and s_R is that of the right-cut. In deriving the relation (2.7b) we have used (2.4) and also taken into account the effect of the inelastic scattering. Assuming further that $N_\ell(s) \to 0$ as $|s| \to \infty$ the dispersion relation for $N_\ell(s)$ is given by,

$$N_\ell(s) = \frac{1}{\pi}\int_{s_L}^{\infty} \frac{D_\ell(s')\text{Im } a_\ell(s')}{(s'-s)}\, ds' \qquad (2.8)$$

In order to obtain the dispersion relation for $D_\ell(s)$ we make further assumption that the partial wave amplitude $a_\ell(s)$ has no CDD[39)] poles. Moreover, since both $N_\ell(s)$ and $D_\ell(s)$ can be multiplied by a constant (matrix in the case of multichannel problem) number without affecting the solution of $a_\ell(s)$ we have the freedom to normalise $D_\ell(s)$ to 1 at a suitable point near the physical region. Thus, normalising $D_\ell(s)$ at a point $s = s_0$, and using (2.7b) we obtain the following dispersion relation,

$$D_\ell(s) = 1 - \frac{(s-s_0)}{\pi}\int_{s_R}^{\infty} ds'\rho(s')R_\ell(s')\,\frac{N_\ell(s')}{(s'-s)(s'-s_0)} \qquad (2.9)$$

Equation (2.8) for $N_\ell(s)$ involves an integral over the left-hand cut. It is often convenient if the integral (2.8) is converted into one over the right-hand cut. In order to achieve that we assume that we know a function $B_\ell(s)$ which has the same left-hand cut as $a_\ell(s)$ but no right-hand so that we can write,

$$B_\ell(s) = \frac{1}{\pi}\int ds' \frac{\text{Im}\, a_\ell(s')}{s' - s} \qquad (2.10)$$

Now considering the above assumption about the property of B_ℓ, we have over the left-hand cut, $\text{Im}[N_\ell - B_\ell D_\ell] = D_\ell \text{ Im } a_\ell - D_\ell \text{ Im } B_\ell = 0$, where we have used (2.71). Thus, the function $N_\ell - B_\ell D_\ell$ has only a right-hand cut. Then, writing dispersion relation for $N_\ell - B_\ell D_\ell$ we have,

$$N_\ell(s) = B_\ell(s) + \frac{1}{\pi}\int_{S_R} ds' \left[B_\ell(s') - \frac{s - s_o}{s' - s_o} B_\ell(s) \right] \times \frac{\rho(s')\, N_\ell(s')}{(s' - s)} \qquad (2.11)$$

where s_o is the same point as it is in (2.9). The equation (2.11) can be used together with (2.9). The advantage of the above method is that the calculation involves integrals only over the physical region so that if we know $B_\ell(s)$ (which we can calculate from the processes in the crossed channels) without having first to evaluate the left-hand cut discontinuity or if we wish to use an approximation for $B_\ell(s)$ which is reasonably good in the physical region, then it is convenient to use (2.11) and (2.9) instead of (2.8) and (2.9) for the solution of the scattering problem. Further the use of (2.11) together with (2.9) gives for the scattering amplitude $a_\ell(s)$ a solution which is independent of the subtraction point s_0 ; and in the case of the multi-channel problem, this solution retains the time reversal property. The above method (2.11) was in fact first used by Uretsky[40] and in the past has been used in many calculations[7,41]. The above method can also be used in the multi-channel problem in which both $N_{ij}(s)$ and $D_{ij}(s)$ become $n \times n$ matrices depending on the number of channels. The integrals of these functions will contain a function $\theta(s' - s_j)$ showing that the integrals start at the appropriate threshold of the channel concerned (see next section for details).

2. The Approximate N/D Methods

a. The Determinantal Method:

In order to solve the integral equations for the partial wave amplitudes $a_\ell(s)$, we have, as mentioned in the last section, to have some information about Im $a_\ell(s)$ on the left-hand cut or about $B_\ell(s)$ on the right-hand cut. As we also mentioned earlier, the values of these functions can be calculated from a set of exchange diagrams in the crossed channels. A difficulty that usually arises in such a procedure is that when we consider exchanges of particles of spin greater than or equal to one, the integrals (2.8) or (2.11) and (2.9) become divergent. These divergences arise from the fact that the contributions from such exchange diagrams involving particles with spin greater than or equal to one, contain terms proportional to $P_\ell(\cos_t)$ or $P_\ell(\cos_u)$ with $\ell \geq 1$ and then since $\cos\theta_t$ and $\cos\theta_u$ are proportional to s, these are proportional to s^ℓ. If this gives rise to, as it usually does, the divergent integrals we are forced to introduce a cut-off. There have been some calculations[7,41] where these integral equations have been solved numerically, but the calculations become rather complicated, and since the results even then involve the cut-off as an arbitrary parameter it has been natural to look for some simplifying approximations.

An approximate N/D method which has been widely used in the past because of its simplicity is the so-called determinantal method[42]. The usual procedure to solve the coupled integral equations (which is the case in N/D method) is to use the iterative method. In the first approximation, we put $D_\ell(s) = 1$ on the left-hand cut to evaluate $N_\ell(s)$ from (2.8) then we use this $N_\ell(s)$ to evaluate $D_\ell(s)$ from (2.9) and so on. In the determinantal method, we go only up to the first approximation then obtain, using (2.7a) $\mathrm{Im}\, N_\ell(s) = \mathrm{Im}\, a_\ell(s) = \mathrm{Im}\, B_\ell(s)$. Consequently, we can put $N_\ell(s) = B_\ell(s)$. The function $B_\ell(s)$ is usually calculated from a set of single-particle exchange diagrams in the crossed channels and that is why this method is, sometimes, referred to as the unitarised Born approximation or the B/D method. In this approximate method, the solution of the partial wave amplitude becomes very simple and it is given by,

$$a_\ell(s) = B_\ell(s) \times \left[1 - \frac{(s - s_o)}{\pi} \int_{S_R} ds' \, \rho(s') \, \frac{R_\ell(s') \, B_\ell(s')}{(s' - s)(s' - s_o)} \right]^{-1} \tag{2.12}$$

The self-consistent calculations become very straight-forward if we use the determinantal approximation. Suppose a bound state occurs at $s = s_B$ and as this corresponds to the zero of the denominator function $D_\ell(s)$, we have $D_\ell(s_B) = 0$. Then the output coupling constant corresponding to this bound state is given by,

$$g'^2 = - \frac{B_\ell(s_B)}{D'(s_B)} \tag{2.13}$$

where $B_\ell(s_B)$ is the value of the Born terms at the pole-position and $D'(s_B)$ is the first-derivative of $D_\ell(s)$ (2.9) with respect to s evaluated at $s = s_B$.

When we are dealing with a multichannel problem, the use of this approximate method makes the calculations very straightforward and much simpler. Suppose we are considering a m-channel scattering then the unitarity relation of the problem can be written in the form,

$$\left[a_{ij}(s) - a^*_{ij}(s) \right] \Big/ 2i = \sum_K a^*_{ik}\, \theta(s - s_K)\rho_{k\gamma}\, a_{\gamma j}(s) \tag{2.14}$$

where s_K is the threshold-energy for the state k and $\theta(s - s_K)$ is the step-function which is one or zero, according to whether s is larger or smaller than s_K. The kinematic factor $\rho_{k\gamma}(s)$ is now a diagonal matrix and is usually of the form,

$$\rho_{ij} = \delta_{ij} \frac{k_i^{2\ell_i+1}}{\sqrt{s}} \tag{2.15}$$

From the unitarity relation (2.14) we can obtain,

$$\mathrm{Im}[a^{-1}(s)]_{ij} = -\,\theta(s - s_j)\, \rho_{ij}(s) \tag{2.16}$$

Now, the scattering amplitude $a_{ij}(s)$ can be written in the form,

$$a_{ij}(s) = N_{ik}(s)\, D^{-1}_{kj}(s) \tag{2.17}$$

Then, using (2.16) and (2.17) and the determinantal approximation we have for the denominator function

$$D_{kj}(s) = \delta_{kj} - \frac{s - s_o}{\pi} \int_{s_k} ds'\, \theta(s' - s_K)\rho_{KK}(s')\, \frac{B_{kj}(s')}{(s' - s)(s' - s_o)} \tag{2.18}$$

The advantage of the determinantal method lies in the fact that the integral (2.18) does not require any iteration and further the scattering amplitude can be given in a very simple form,

$$a_{ij}(s) = \frac{B_{ik}(s)\ \bar{D}_{kj}(s)}{\det D} \tag{2.19}$$

where det D is the determinant of the denominator function. If we are looking for the position of the bound state, then it can be obtained from the condition,

$$\det D = 0 \tag{2.20}$$

If, on the other hand, we are looking for the location of a resonance we have to equate the real part of det D to zero.

The determinantal method discribed above, however, has some shortcomings. Although this method ensures unitarity, it does not give the correct left-hand cut corresponding to the set of diagrams chosen, except at the immediate neighbourhood of the point $s = s_o$ at which the denominator function D is normalised. For this reason, it is expected that the results will, to some extent, be dependent on the choice of the subtraction point. We therefore have to choose the subtraction point somewhere in the nearby part of the left-hand cut from where the maximum contribution to the force is expected. Further, if we use this method in a multi-channel problem described above, it does not give a symmetric scattering matrix as required by the time reversal invariance. Despite these shortcomings, however, this method has been very popular because of its simplicity. Moreover, the problem of divergences mentioned earlier in this section, can be avoided by using this method.

b. The Pole Approximation:

In order to avoid the numerical integrations involved in the N/D method, the pole approximation method has been used in the past in many calculations. In this method, it is assumed that the effects of the left-hand in the physical region can be approximated by means of a set of poles located on the left-hand cut. Then, we can write,

$$B_\ell(s) = \sum_{i=1}^{n} \frac{R_i}{s_i - s} \tag{2.21}$$

where $s_i < s_L$, and $s > s_R$, s_L being the beginning of the left-hand cut and s_R that of the right-hand cut. Now, from the principal value theorem in complex variables we have,

$$\frac{R_i}{\epsilon \to 0,\ s_i - s \mp i\epsilon} = \frac{P}{s_i - s} \pm i\pi\delta(s_i - s) \tag{2.22}$$

Using (2.21) and (2.22), we have from (2.8),

$$N_\ell(s) = \sum_{i=1}^{n} \frac{R_i}{s_i - s} D_\ell(s_i) \tag{2.23}$$

If we now substitute (2.23) into (2.9) the integral of the denominator function can be easily evaluated. Then, the denominator function $D_\ell(s)$ can be expressed in the following simple form:

$$D_\ell(s) = 1 + \sum_{i=1}^{n} F_i(s, s_0 s_i)R_i D_\ell(s_i) \tag{2.24}$$

where F_i's are some functions of the variables appearing in the argument. We can now evaluate $D_\ell(s)$ at, say n points from (2.24) and solve n algebraic equations so obtained for $D_\ell(s)$. The equations (2.23) and (2.24) then give the amplitude in the physical region. This method can also be used in the self-consistent calculations where we can adjust the parameters R_i (residue of the i^{th} pole) and s_i (position of the i^{th} pole on the left-cut) in order to obtain the self-consistent solution. These input parameters can also be determined in different ways. Following the method used by Frautschi and Walecka[43] we can approximate the left-hand cut discontinuities corresponding to a certain set of diagrams by poles in such a way that the Cauchy integral around the poles approximates the Cauchy integral around those discontinuities, when it is evaluated in the physical region. In using this method, however, a difficulty arises when exchanges in the crossed channels of particles with spin greater than or equal to one are considered. In that case the left-hand cut integrals become divergent and consequently we are forced to neglect the far-off part of the cut or introduce a cut-off in the integrals concerned.

c. Pagels Method.

Pagels[44)] has given an approximation scheme for solving the N/D equations and this method has, undoubtedly, much advantage compared to the other approximate methods we have discussed so far. In this scheme the spectral integral over the kinematic factor $\rho_\ell(s)$ on the left is approximated by a set of poles on the right without making any change in the force term $B_\ell(s)$. Following this procedure, we can, getting rid of the integrals, obtain an algebraic expression for $a_\ell(s)$ in terms of $B_\ell(s)$ in such a way that the solution gives the correct discontinuities across both the right- and left-hand cuts. Further, the solution is independent of the subtraction point and has a symmetric $a_\ell(s)$ for a symmetric input $B_\ell(s)$. We shall discuss this method in some detail and show how it can be used in bootstrap calculations. We shall consider only the single-channel problem with only one-pole approximation. Generalisation to the many-channel problems with more than one-pole approximation is, as it will be clear, very straight-forward.

Pagels assumes that the force term $B_\ell(z)$ admits of the Hilbert representation of the form,

$$B_\ell(z) = \frac{1}{\pi} \int_L dz' \frac{\operatorname{Im} B_\ell(z')}{z' - z} \tag{2.25}$$

where the integral extends over the left-hand cut and z is the c.m. energy squared. All through this section we shall take x, y, z as the energy variables for the sake of convenience. Dropping the index ℓ we write the partial wave amplitude in the form (2.5)

$$a(z) = N(z)\, D^{-1}(z) \tag{2.26}$$

Now making the same assumptions as before about the analytic properties of N(z) and D(z) and using the unitarity relations (2.7) and normalising the denominator function at $z = z_o$, we have,

$$D(z, z_o) = 1 - \frac{z - z_o}{\pi} \int_R dx \frac{\rho(x)\, N(x\ \ z_o)}{(x - z_o)(x - z)} \tag{2.27}$$

$$N(z, z_o) = \frac{1}{\pi} \int_L dx \frac{\operatorname{Im} N(x, z_o)}{x - z} \tag{2.28}$$

where,

$$\text{Im } N(x\ z_o) = \text{Im } B(x)\ D(x\ z_o) \tag{2.29}$$

Now substituting (2.28) into (2.27) and interchanging the orders of integration we can easily obtain,

$$D(z\ z_o) = 1 + \frac{z - z_o}{\pi} \int_L dy\ K(z\ y\ z_o)\ \text{Im } N(y\ z_o) \tag{2.30}$$

where, the kernel $K(z\ y\ z_o)$ depends only on the kinematic factor $\rho(z)$ and is given by,

$$K(z\ y\ z_o) = \frac{1}{\pi} \int_R \frac{dx\ \rho(x)}{(x - z_o)(x - z)(x - y)} \tag{2.31}$$

Now using the identity,

$$\frac{1}{(x - z)(x - z_o)} = \frac{1}{z - z_o} \left\{ \frac{z}{x(x - z)} - \frac{z_o}{x(x - z_o)} \right\}$$

we can express the kernel $K(x\ y\ z_o)$ in the following form,

$$K(z\ y\ z_o) = \frac{z\ F(z)}{(z - y)(z - z_o)} + \frac{y\ F(y)}{(y - z)(y - z_o)} + \frac{z_o\ F(z_o)}{(z_o - z)(z_o - y)} \tag{2.32}$$

where the function $F(z)$ (which is a diagonal matrix in a multi-channel problem) is,

$$F(z) = \frac{z}{\pi} \int_R \frac{dx\ \rho(x)}{x^2(x - z)} \tag{2.33}$$

We now substitute (2.32) into (2.30) and on simplification get,

$$D(z\ z_o) = 1 - z F(z) N(z\ z_o) + z_o F(z_o) N(z_o\ z_o)$$

$$+ \frac{1}{\pi} \int_L dy\ F(y)\ \text{Im}\ N(y\ z_o) \left\{ \frac{z}{y-z} - \frac{z_o}{y-z_o} \right\} \qquad (2.34)$$

Now, the function $H_\ell(z) = F_\ell(z)/z$ is a spectral integral over the positive definite kinematical factor $\rho_\ell(z) > 0$ and hence on the left will have all its derivatives positive. Therefore, this function can, quite accurately, be approximated on the left by a pole on the right. Thus, for the ℓ-th partial wave we can write,

$$F_\ell(z)/z = H_\ell(z) \simeq C_\ell/(z - a_\ell) \qquad (2.35)$$

where, C_ℓ and a_ℓ are constants which are chosen to reproduce $H_\ell(z)$, which is known exactly once $\rho_\ell(z)$ is given. The constants C_ℓ and a_ℓ are completely determined once the partial wave is specified. Here, in (2.35) we are making only one-pole approximation. If greater accuracy is desired we can add more pole terms in (2.35), thus more closely approximating the exact $H_\ell(z)$.

Now, with one-pole approximation, we obtain from (2.34), using (2.25), (2.28) and (2.29) the following expression for the denominator function,

$$D(z\ z_o) = g(z_o) - z\,F(z)\,N(z\ z_o) + \frac{Cz}{(z-a)} \times \left[z\,N(z\ z_o) - a\,N(a\ z_o) \right] \qquad (2.36)$$

where,

$$g(z_o) = 1 + z_o\,F(z_o)\,N(z_o\ z_o) - \frac{C\,z_o}{(z_o - a)} \times \left[z_o\,N(z_o\ z_o) - a\,N(a\ z_o) \right] \qquad (2.37a)$$

It is evident from (2.37a) that,

$$g(0) = 1 \qquad (2.37b)$$

Using, now, (2.35) we obtain from (2.36) an accurate expression (within the limit of the approximation) for D(z) along the left-hand cut,

$$D(z\ z_o) = g(z_o) - \frac{C\,a\,z}{z - a}\,N(a\ z_o) \qquad (2.38)$$

Now substituting (2.29) and using (2.38), we obtain from (2.28),

$$N(z\ z_o) = B(z)\,g(z_o) - \left[z\,B(z) - a\,B(a) \right] . \frac{Ca.N(a\ z_o)}{(z-a)} \qquad (2.39)$$

where we have used (2.25) and the following identity

$$\frac{x}{(x - a)(x - z)} = \frac{z}{(z - a)(x - z)} - \frac{a}{(z - a)(x - a)}$$

As we mentioned earlier in this section, the solution of the problem is independent of the subtraction point. Therefore, without the loss of any accuracy we can set $z_o = 0$. From (2.39) we can further obtain,

$$N(a) = \left\{ 1 + Ca \left[B(a) + a B'(a) \right] \right\}^{-1} B(a).g(z_o) \qquad (2.40)$$

where $B'(a) = dB(a)/da$.

We now see that the equations (2.36), (2.38), (2.39) and (2.40) together give the complete solution of the problem. The expressions for the denominator function $D(z)$ on the right and left cut are given by the equations (2.36) and (2.38) respectively.

Let us now discuss how this method is applied in a bootstrap calculation. Setting $z_o = 0$, we have from (2.38) using (2.40),

$$D(z) = 1 - \frac{C\, z a.B(a)}{(z - a)} \left\{ 1 + C.a \left[B(a) + a B'(a) \right] \right\}^{-1} \qquad (2.41)$$

If there is a pole at $z = s_o$, we have $D(s_o) = 0$. Now taking out the coupling constant from the Born term $B(z) \rightarrow g^2 B(z)$ we obtain for the zero of the denominator function $D(s_o) = 0$,

$$1/g^2 = \frac{C\,a^2}{s_o - a}\Big[\,B(a) - (s_o - a)B'(a)\,\Big] \tag{2.42}$$

Now the expression for the output coupling constant is given by,

$$1/g'^2 = -\frac{D'(s_o)}{N(s_o)} \tag{2.43}$$

Using (2.39) and (2.41) we get from (2.43) the expression for the output coupling constant which is given by,

$$1/g'^2 = -\frac{1}{B(a)(s_o - a)} = \frac{1}{g^2 B(a)(s_o - a)} \tag{2.43'}$$

where in (2.43') we take out the input coupling constant from the Born term B(a). For the self-consistency requirement we have $g^2 = g'^2$. Therefore, the condition for the self-consistent result is given by,

$$B(a)(s_o - a) = -1 \tag{2.44}$$

From (2.42) and (2.44) the expression for the self-consistent coupling constant is,

$$1/g'^2 = 1/g^2 = -Ca^2\Big[\,B^2(a) + B'(a)\,\Big] \tag{2.45}$$

The values of the parameters C and a' can be determined by fitting the pole $C/(z - a)$ with the function $H_\ell(z)$ (2.35) on the left hand cut. The Born term $B(a)$ can be obtained from the exchange, in the crossed channel, of the particle for which the self-consistent solution is sought. Thus, using (2.45) we can calculate the self-consistent value of the coupling constant. The divergency problem, unfortunately, also arises in this method. When the Born terms are divergent, the dispersion relation (2.25) is not valid. This difficulty, however, can be overcome by introducing subtraction in (2.25). But this introduces some additional parameters which have to be determined in order to obtain the solution of the problem under consideration.

d. The static Model and Bootstraps in SU(2)

In this section we shall consider the pion-nucleon scattering in the context of the static model in which it is assumed that the nucleon being much heavier than the pion is at rest both during and after the collision. In particular, we shall discuss how the reciprocal bootstrap relationship between nucleon N and nucleon resonance N* (3/2, 3/2) can be explained by static model approach. The idea of such a reciprocal bootstrap was first suggested by Chew[6)] who showed that the static model could, to a reasonable extent, explain the existence of such a relationship. We shall, however, discuss only that aspect of the theory which is relevant to the Quark-bootstrap calculation which will be considered in the next chapter.

A meson-baryon state can be specified by its spin J, isospin I, orbital angular momentum ℓ and the total energy W. For a given orbital angular momentum the scattering amplitude $g_{IJ}(w)$ is related with the phase-shift through the relation given by,

$$g_{IJ}(w) = \frac{e^{i\delta_{IJ}} \sin \delta_{IJ}}{q^{2\ell+1}} \tag{2.46}$$

where δ_{IJ} is the phase-shift corresponding to the amplitude specified by the angular momentum J and isospin I. Here, $q^2 = w^2 - 1$ and $w = W - M$, where w denotes the energy of pion, m the mass of the nucleon and pion-mass is taken as unity for the sake of convenience.

The contribution of the cross-channel can be obtained from the crossing relation

$$g_{IJ}(w) = \sum_{I'J'} \alpha_{II'} \beta_{JJ'} g_{I'J'}(-w) \tag{2.47}$$

where α and β are the crossing matrices for the isospin and total angular momentum respectively. In the static limit, the g's on the right-hand side of the equation (2.47) have the same ℓ as the g's on the left, so that the angular momentum crossing matrix is just like the isospin crossing matrix.

If there occurs a bound state or a resonance in the (I J) state, the corresponding amplitude will have a pole $\gamma_{IJ}/(w_{IJ} - w)$, γ_{IJ} being the residue of the pole. Then the force or the Born term in the ℓ-th partial wave can be obtained from (2.47) and given in the form,

$$B_{IJ}(w) = \sum_{I'J'} \alpha_{II'} \beta_{JJ'} \frac{\gamma_{I'J'}}{(w_{I'J'} + w)} \qquad (2.48)$$

where J, J' = $\ell \pm 1/2$. Usually, the summation in (2.48) is taken over all possible (I'J') states and we put $\gamma_{I'J'} = 0$ whenever there is no particle in any particular state. We can now use $B_{IJ}(w)$ as the input in an N/D calculation.

Let us write, as before,

$$g_{IJ}(w) = N_{IJ}(w)/D_{IJ}(w) \qquad (2.49)$$

Then, using (2.23), (2.46) and (2.48) the numerator and the denominator functions in (2.49) can be written, as is well known, in the following form,

$$N_{IJ}(w) = \sum_{I'J'} \alpha_{II'} \beta_{JJ'} \frac{\gamma_{I'J'} D_{IJ}(-w_{I'J'})}{(w_{I'J'} + w)} \qquad (2.50)$$

$$D_{IJ}(w) = 1 - \frac{w - w_o}{\pi} \int_1^{w_c} dw' \frac{(w'^2 - 1)^{\ell+\frac{1}{2}} N_{IJ}(w')}{(w' - w_o)(w' - w)} \tag{2.51}$$

where w_o is the point at which D is normalised to unity and w_c is the cut-off parameter which parametrises our ignorance of the short range and the high energy effects.

We now suppose that there occurs a bound state or a resonance in the amplitude specified by the quantum numbers I and J. It is well known that the denominator function D(w) behaves, more or less linearly near the position of the pole or the resonance. Therefore, the denominator function, near the pole, can be approximately expressed in the form,

$$\text{Re}\, D_{IJ}(w) = \frac{w_{IJ} - w}{w_{IJ} - w_o} \tag{2.52}$$

Now, the coupling constant corresponding to a pole occurring in the state (I J) is given by

$$\gamma_{IJ} = - \frac{N_{IJ}(w_{IJ})}{D'_{IJ}(w_{IJ})} \tag{2.53}$$

Then, using (2.50) and (2.52) we obtain from (2.53),

$$\gamma_{IJ} = \sum_{I'J'} \alpha_{II'} \beta_{JJ'} \gamma_{I'J'} \tag{2.54}$$

where the γ's on the left-hand side of (2.54) are the 'output' and those on the right-hand side, the 'input' coupling constants respectively. The self-consistency condition requires the input and the output values of the relevant couplings to be equal. Now, (2.54) can be written in the matrix form,

$$\Gamma = C\Gamma \qquad (2.54')$$

where Γ is a column matrix and C is the crossing matrix which is the direct product of the isospin and angular momentum crossing matrices respectively. We now introduce the following quantity,

$$F_{IJ}(w) = \sum_{I'J'} \alpha_{II'} \beta_{JJ'} \gamma_{I'J'} \qquad (2.55)$$

The above quantity F_{IJ} may be regarded as a reasonable measure of force contributed to the state (I J) by the states occuring in the crossed channel. If F_{IJ} is negative for any state then the corresponding output γ_{IJ} will be negative too as a consequence of which such a state cannot exist. On the other hand, if F_{IJ} is positive and large we can expect that a state (I J) may exist. The above conditions we have put forward follows from the very nature of the denominator function (2.51) which requires positive couplings (or forces) for the occurrence of a pole in the amplitude.

Let us now apply the above formalisms to the pion-nucleon problem. As it is known from experiment that the nucleon resonance occurs in p-state, it is natural to consider the p-wave scattering. Corresponding to the p-wave there are two total angular momentum states, namely, the states with $J = \ell \pm \frac{1}{2}$. Again from the isotopic spin analysis we get the channel isotopic spin I = 3/2, 1/2. When the isospin and the ordinary spin states are combined we get, in all, four states. For the p-wave scattering the isospin and spin crossing matrices are the same, i.e. $\alpha = \beta$, where any of the two crossing matrices is given by,

$\alpha_{II'}$ =

I \ I'	3/2	1/2
3/2	- 1/3	4/3
1/2	2/3	1/3

(2.56)

Assuming further that only the states (1/2,1/2) and (3/2,3/2), i.e. N and N* exist, we get from (2.54')

$$\begin{vmatrix} \gamma_{1/2\ 1/2} \\ \gamma_{3/2\ 3/2} \end{vmatrix} = \begin{vmatrix} 1/9 & 16/9 \\ 4/9 & 1/9 \end{vmatrix} \begin{vmatrix} \gamma_{1/2\ 1/2} \\ \gamma_{3/2\ 3/2} \end{vmatrix} \qquad (2.57)$$

The above relation (2.57) gives two equations from which we obtain $\gamma_{1/2\ 1/2} = 2\gamma_{3/2\ 3/2}$. This result is in good agreement with the experimental one, thus showing that a reciprocal bootstrap relationship may exist between N and N*. This is, however, just a preliminary test for the existence of a bootstrap relationship between two particles. In order to see whether such a relationship really exists we have to evaluate the denominator function of the partial wave amplitude.

CHAPTER III

A Reciprocal Bootstrap Mechanism for Quarks

1. Quark-Meson Scattering and the General Kinematics.

It is well known, as we have also discussed in Chapter I (section 1) that the spinor (3-dimensional) representation of SU(3) does not correspond to any known particles. Speculations about the possible existence of these three particles, called 'Quarks' were first made by Gell-Mann and independently by Zweing[11]. According to the scheme of SU_3 (or its relativistic generalisation) symmetry these particles have the baryon number 1/3 and non-integral charges (Other models which give integral charges for the particles belonging to the basic representations of groups concerned, have been considered by a number of authors[45]. Unlike Gell-Mann and Zweing model, these models, however, require the existence of some more quantum numbers, called the super-charge quantum numbers for which there has not yet been found any experimental evidence. We shall, therefore, consider the SU(3) scheme which is the simplest and consistent with the known physical quantum numbers). A considerable search has been made in the past in order to detect the possible existence of the quarks but the experimental results have, so far, been negative[46]. We, therefore, infer that these particles must be very massive and consider a model which is expected to give, at least, a rough estimate about the masses and the coupling constants of these particles.

We consider quark-pseudoscalar meson scattering and adopt the "bootstrap" hypothesis in which all the strongly interacting particles are supposed to be composites of each other, and we ask whether quarks can exist in a self-consistent scheme. In particular, we use the analogy with the well known N-N* bootstrap of Chew[6] and its analogous SU(3) extension considered by a number of authors[9]. In our calculations, we use the determinantal method (discussed in the previous chapter) and assume that the forces that arise due to the fermion exchanges in the crossed channel play a dominant role in this mechanism. Quantitatively, these are unlikely to be good approximations but they appear to be reasonable qualitatively and we, therefore, adopt them for our calculation also. Before, we get deeper into the problem, let us discuss some of the kinematics that we will have to deal with later on.

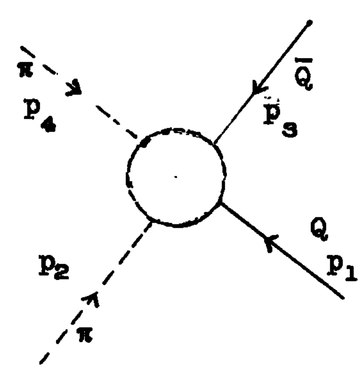


Figure 3.1

The above two-particle scattering diagram (fig. 3.1) represents the quark-meson scattering and all the crossed processes. We follow the conventional rotation of taking the ingoing momenta as positive and define the scalar product of two four-vector as follows: $A.B = A_o B_o - \underset{\sim}{A}.\underset{\sim}{B}$. In other words, we have chosen the metric such that $g_{\mu\upsilon} = (1, -1, -1, -1)$ diagonal. Out of the four-momenta p_1, p_2, p_3, p_4, we can construct the following scalar invariants:

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \tag{3.1a}$$

$$u = (p_1 + p_4)^2 = (p_3 + p_2)^2 \tag{3.1b}$$

$$t = (p_1 + p_3)^2 = (p_4 + p_2)^2 \tag{3.1c}$$

If we consider p_1, p_2 ingoing and p_3, p_4 outgoing then we have quark-meson scattering. The interchange of p_2 and p_4 will mean the interchange of the two pions, so the process described by p_1, p_4 ingoing and p_3,p_2 outgoing will again be quark-meson scattering. If, on the otherhand, we consider p_1, p_3 ingoing and p_2, p_4 outgoing, we obtain the annihilation process $\pi\pi \rightarrow Q\bar{Q}$. Referring to the above processes as channels I, II and III respectively we have,

$$Q(p_1) + \pi(p_2) \rightarrow Q(-p_3) + \pi(-p_4) \qquad \text{I}$$

$$Q(p_1) + \pi(+p_4) \rightarrow Q(-p_3) + \pi(-p_2) \qquad \text{II}$$

$$Q(p_1) + \bar{Q}(p_3) \rightarrow \pi(-p_2) + \pi(-p_4) \qquad \text{III}$$

Figure 3.1 will also describe three anti-particle reactions corresponding to the above three channels. s, u and t will be positive time-like for I, II and III respectively and they are c.m. energies squared in the channel concerned. The other two variables in each channel, are the negative squares of the momentum transfer in that channel. Now, using the four-momenta conservation law $p_1 + p_2 + p_3 + p_4 = 0$ and the mass-energy relations, $p_1^2 = p_3^2 = m^2$; $p_2^2 = p_4^2 = \mu^2$ we have,

$$s + t + u = 2m^2 + 2\mu^2 \qquad (3.2)$$

Thus, out of the three variables s, t, u only two are independent. In each channel, we can, therefore, consider the c.m. energy squared and the cosine of the angle of scattering as the two independent variables. We treat each channel separately and obtain the various kinematic variables in terms of s, t and u.

Channel I:

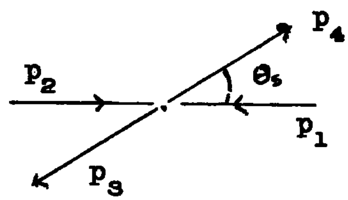


Figure 3.2
s-channel c.m. system

Figure 3.2 shows the process described by channel I in the c.m. system. We have,

$$\underset{\sim}{p}_1 = -\underset{\sim}{p}_2 \; ; \; \underset{\sim}{p}_3 = -\underset{\sim}{p}_4 \; ; \; p_1^2 = p_2^2 = p_3^2 = p_4^2 = k^2 \quad \text{(say)}$$

Then we have,

$$s = (p_{10} + p_{20})^2 \tag{3.3a}$$

where ,

$$p_{10}^2 = m^2 + k^2 \; ; \; p_{20}^2 = k^2 + \mu^2 \tag{3.3b}$$

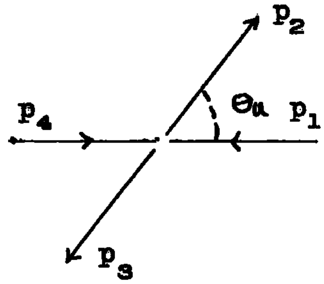
and $|k|$ is the absolute value of the 3-component momentum given by,

$$|k|^2 = \left\{ s - (m + \mu)^2 \right\} \left\{ s - (m - \mu)^2 \right\} \Big/ 4s \tag{3.3c}$$

Now the negative momentum transfer squared between the two mesons or the two quarks is,

$$t = -2k^2(1 - \cos\theta) \tag{3.3d}$$

where $\cos\theta = \underset{\sim}{p}_2 \cdot \underset{\sim}{p}_4 / |p_2 \cdot p_4|^2$, θ being the scattering angle in channel I. The momentum transfer between a pion and a quark can be obtained from (3.2) by using (3.3a) and (3.3d). The physical region in this channel is defined by, $(m + \mu)^2 \leqslant s \leqslant \infty$ and $-4k^2 \leqslant t \leqslant 0$.



u-channel c.m. system

Figure 3.3

Channel II:

In this channel everything is similar to the channel I and is obtained from them by interchanging s and u. Then we have,

$$\underset{\sim}{p}_1 = -\underset{\sim}{p}_4 \; ; \; \underset{\sim}{p}_3 = -\underset{\sim}{p}_2 \; ; \; p_1^2 = p_2^2 = p_3^2 = p_4^2 = \bar{k}^2 \quad \text{(say)}$$

The c.m. energy squared is then

$$u = (p_{10} + p_{40})^2 \qquad (3.4a)$$

where,

$$p_{10}^2 = p_{30}^2 = \bar{k}^2 + m^2 \; ; \; p_{20}^2 = p_{40}^2 = \bar{k}^2 + \mu^2$$

and

$$\bar{k}^2 = \left\{ u - (m+\mu)^2 \right\}\left\{ u - (m-\mu)^2 \right\} \Big/ 4u \qquad (3.4b)$$

Again the negative momentum transfer squared is

$$t = -2\bar{k}^2(1 - \cos\bar{\theta}) \qquad (3.4c)$$

where $\cos\bar{\theta} = \underset{\sim}{p}_2 \cdot \underset{\sim}{p}_4 / |p_2 \cdot p_4|^2$, θ being the scattering angle in channel II. The physical region in this channel is defined by

$(m + \mu)^2 \leq u \leq \infty$, $-4k^2 \leq t \leq 0$. The kinematic variables in channel III can be similarly obtained. We shall not, however, need them in our calculation.

2. The Crossing Matrices

In order to solve a scattering problem, we require, as we have discussed in the previous chapter, the information about the imaginary part of the partial wave amplitude $a_\ell(s)$ over the left-hand cut which is associated with the forces that arise due to the exchanges of the various particles in the crossed channels II and III. This information is conveniently obtained by using the crossing relations which are provided by the crossing matrices. Since we shall mostly be dealing with SU(3) crossing matrices, in this and the next chapter, it will be convenient if we derive the general expressions for the crossing matrices. With that end in view, we consider a general two-particle scattering process with the following channels:

$$a_1 + a_2 \rightarrow a_3 + a_4 \qquad \text{I}$$

$$a_1 + \bar{a}_4 \rightarrow a_3 + \bar{a}_2 \qquad \text{II}$$

$$a_1 + \bar{a}_3 \rightarrow \bar{a}_2 + a_4 \qquad \text{III}$$

The SU(3)-invariant scattering amplitude in channel I can be given by[47],

$$\left\langle \begin{matrix} N_3 & N_4 \\ n_3 & n_4 \end{matrix} |F| \begin{matrix} N_1 & N_2 \\ n_1 & n_2 \end{matrix} \right\rangle = \sum_{N,n\beta\gamma} \begin{pmatrix} N_3 & N_4 & N & \\ & & & \beta \\ n_3 & n_4 & n & \end{pmatrix} < N\,n\,\beta|F^I|N\,n\,\gamma >$$

$$\begin{pmatrix} N_1 & N_2 & N & \\ & & & \gamma \\ n_1 & n_2 & n & \end{pmatrix} \qquad (3.5)$$

where N_i, $i = 1, 2, 3, 4$ denote the dimensions of the irreducible representations to which the above four particles belong. $n = (I, I_3, Y)$, the quantum numbers assigned to the particles that form the basis of the representation concerned. In (3.5) N denotes the dimension of the irreducible representation that is obtained from the direct product of two representations and γ, β denote how many times a particular representation occurs in the initial and final states of the scattering process. The quantities $\begin{pmatrix} N_1 & N_2 & N & \\ & & & \gamma \\ n_1 & n_2 & n & \end{pmatrix}$ are the SU(3) Clebsch-Gordon coefficients.

The expressions for the SU(3)-invariant scattering amplitudes in channels II and II can be similarly obtained and these are as follows:

$$\left\langle \begin{matrix} N_3 & N_2^* \\ n_3 & -n_2 \end{matrix} |F| \begin{matrix} N_1 & N_4^* \\ n_1 & -n_4 \end{matrix} \right\rangle = \sum_{Nn,\beta\gamma} \begin{pmatrix} N_3 & N_2^* & N & \\ & & & \beta \\ n_3 & -n_2 & n & \end{pmatrix} < N\,n\,\beta|F^{II}|N\,n\,\gamma >$$

$$\begin{pmatrix} N_1 & N_4^* & N & \\ & & & \gamma \\ n_1 & -n_4 & n & \end{pmatrix} \qquad (3.6)$$

$$\left\langle \begin{matrix} N_2^* & N_4 \\ -n_2 & n_4 \end{matrix} \middle| F \middle| \begin{matrix} N_1 & N_3^* \\ n_1 & -n_3 \end{matrix} \right\rangle = \sum_{Nn\beta\gamma} \left(\begin{matrix} N_2^* & N_4 & N \\ -n_2 & n_4 & n \end{matrix} \; \beta \right) \langle N \beta n | F^{III} | N n \gamma \rangle$$

$$\left(\begin{matrix} N_1 & N_3^* & N \\ n_1 & -n_3 & n \end{matrix} \; \gamma \right) \qquad (3.7)$$

Now, the relation between a particle-state and the corresponding antiparticle state is,

$$|N^*; I, I_3\, Y \rangle = (-1)^{\bar{n}} \left\{ |N; I, -I_3, -Y \rangle \right\}^* \qquad (3.8)$$

where N^* denotes the contragradient representation and $\bar{n} = I_3 + \frac{Y}{2}$. Using (3.8) we can derive the following relations.

$$\left\langle \begin{matrix} N_3 & N_4 \\ n_3 & n_4 \end{matrix} \middle| F \middle| \begin{matrix} N_1 & N_2 \\ n_1 & n_2 \end{matrix} \right\rangle = (-1)^{\bar{n}_4 - \bar{n}_2} \left\langle \begin{matrix} N_3 & N_2^* \\ n_3 & -n_2 \end{matrix} \middle| F \middle| \begin{matrix} N_1 & N_4^* \\ n_1 & -n_4 \end{matrix} \right\rangle \qquad (3.9a)$$

$$\left\langle \begin{matrix} N_3 & N_4 \\ n_3 & n_4 \end{matrix} \middle| F \middle| \begin{matrix} N_1 & N_2 \\ n_1 & n_2 \end{matrix} \right\rangle = (-1)^{\bar{n}_3 - \bar{n}_2} \left\langle \begin{matrix} N_2^* & N_4 \\ -n_2 & N_4 \end{matrix} \middle| F \middle| \begin{matrix} N_1 & N_3^* \\ n_1 & -n_3 \end{matrix} \right\rangle \qquad (3.9b)$$

Now, the orthogonality relations the Clebsch-Gordon coefficients satisfy are as follows:

$$\sum_{n_1 n_2} \begin{pmatrix} N_1 N_2 N & \\ & \gamma \\ n_1 n_2 n & \end{pmatrix} \begin{pmatrix} N_1 & N_2 & N' & \\ & & & \gamma' \\ n_1 & n_2 & n' & \end{pmatrix} = \delta_{NN'}\, \delta_{nn'}\, \delta_{\gamma\gamma'} \qquad (3.10a)$$

$$\sum_{N\, n\, \gamma} \begin{pmatrix} N_1 N_2 N & \\ & \gamma \\ n_1 n_2 n & \end{pmatrix} \begin{pmatrix} N_1 & N_2 & N & \\ & & & \gamma \\ n_1' & n_2' & n & \end{pmatrix} = \delta_{n_1 n_1'}\, \delta_{n_2 n_2'} \qquad (3.10b)$$

Using (3.9a) and (3.10a) we can obtain from (3.5) and (3.6) the relation between the amplitudes in the channel I with those in II and this is as follows:

$$< N\, n\, \beta | F^{I} | N\, n\, \gamma > = \sum_{N' \beta' \gamma} < N \beta \gamma | A_I (N_1 N_2 N_3 N_4) | N' \beta' \gamma' >$$

$$\times \quad < N'\, n'\, \beta | F^{II} | N'\, n'\, \gamma' > \qquad (3.11)$$

where the elements of the crossing matrix $< N \beta \gamma | A_I | N' \beta' \gamma' >$ are given by,

$$< N \beta \gamma | A_{II} | N' \beta' \gamma' > = \sum_{\substack{n_1 n_2 n_3 n_4 \\ n'}} (-1)^{\bar{n}_4 - \bar{n}_2} \begin{pmatrix} N_1 & N_2 & N & \\ & & & \gamma \\ n_1 & n_2 & n & \end{pmatrix} \begin{pmatrix} N_3 & N_4 & N & \\ & & & \beta \\ n_3 & n_4 & n & \end{pmatrix}$$

$$\times \begin{pmatrix} N_1 & N_4^* & N' & \\ & & & \gamma' \\ n_1 & -n_4 & n' & \end{pmatrix} \begin{pmatrix} N_3 & N_2^* & N' & \\ & & & \beta' \\ n_3 & -n_2 & n' & \end{pmatrix}$$

(3.12)

The relation between the amplitudes in channel I and those in III, can similarly be obtained. We obtain the following:

$$< N\, n\, \beta | F^{I} | N\, n\, \gamma > \; = \sum_{N'\beta'\gamma'} < N\, \beta\gamma | A_{III}(N_1\; N_2\; N_3\; N_4) | N'\beta'\gamma > \times < N' n' \beta' | F^{III} | N' n' \gamma' > \qquad (3.13)$$

where the elements of the crossing matrix $< N\,\beta\gamma | A_{III} | N'\beta'\gamma' >$ are given by,

$$< N\,\beta\gamma | A_{III} | N'\beta'\gamma' > \; = \sum_{\substack{n_1 n_2 n_3 n_4 \\ n'}} (-1)^{\bar{n}_3 - \bar{n}_2} \begin{pmatrix} N_1 & N_2 & N & \\ n_1 & n_2 & n & \gamma \end{pmatrix} \times \begin{pmatrix} N_3 & N_4 & N & \\ n_3 & n_4 & n & \beta \end{pmatrix} \begin{pmatrix} N_1 & N_3^* & N' & \\ n_1 & -n_3 & n' & \gamma' \end{pmatrix} \begin{pmatrix} N_2^* & N_4 & N' & \\ -n_2 & n_4 & n' & \beta' \end{pmatrix} \qquad (3.14)$$

We now calculate the relevant crossing matrices for the quark-meson scattering. First, we consider the channels I and II. In both the initial and the final states of these channels, we have a quark-pion system which, being a product of an octet and a triplet, representations of SU(3), can be resolved into the following irreducible representations:

$$\underset{\sim}{3} \otimes \underset{\sim}{8} \; = \; \underline{3} \oplus \underline{6}^* \oplus \underline{15} \qquad (3.15)$$

If space spin is neglected, there are in each of the above channels three SU(3) invariant amplitudes corresponding respectively to the transitions $\underset{\sim}{3} \rightleftarrows \underset{\sim}{3}$, $\underset{\sim}{6}^* \rightleftarrows \underset{\sim}{6}^*$ and $\underset{\sim}{15} \rightleftarrows \underset{\sim}{15}$. The amplitudes in the channels I and II are related to each other through (3.11). Using the Clebsch-Gordon coefficients obtained from the isoscalar factors calculated by Edmonds[48] we have obtained the following crossing matrix:

A_{II} =

N \ N'	3	6*	15
3	$-\frac{1}{8}$	$-\frac{3}{4}$	$\frac{15}{8}$
6*	$-\frac{3}{8}$	$\frac{3}{4}$	$\frac{5}{8}$
15	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{3}{8}$

(3.16)

Let us now find the scattering amplitudes in the channel III. The initial state consists of a quark and an antiquark system which on decomposition gives the following

$$3 \otimes \bar{3} = \underset{\sim}{1} \oplus \underset{\sim}{8} \tag{3.17}$$

In the final state we have a two-meson system which being the direct product of two octets decomposes into the following:

$$\underset{\sim}{8} \otimes \underset{\sim}{8} = \underset{\sim}{1} \oplus 8_1 \oplus 8_2 \oplus \underset{\sim}{10} \oplus \underset{\sim}{10^*} \oplus \underset{\sim}{27} \qquad (3.18)$$

As there cannot be any transition between two different representations, there are altogether three SU(3) invariant amplitudes in the channel III, corresponding respectively to the transitions, $\underset{\sim}{1} \rightleftharpoons 1$; $8 \rightleftharpoons 8_1$ and $8 \rightleftharpoons 8_2$. These amplitudes are related with those in channel I, through the following crossing matrix[49]:

A_{III} = (3.18)

N \ N', β'	$\underset{\sim}{1}$	8_1	8_2
$\underset{\sim}{3}$	$\frac{\sqrt{6}}{4}$	$-\frac{\sqrt{30}}{16}$	$\frac{3\sqrt{6}}{16}$
$\underset{\sim}{6^*}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{30}}{8}$	$\frac{\sqrt{6}}{8}$
$\underset{\sim}{15}$	$\frac{5\sqrt{6}}{4}$	$-\frac{\sqrt{30}}{16}$	$-\frac{5\sqrt{6}}{16}$

Let us now assume that the forces that arise due to fermion exchanges in the crossed channel II play a very important role in quark-meson scattering and consider only the contribution from p-wave in analogy with πN-scattering. Then, each of the amplitudes in the channels I and II will have two components corresponding respectively to the transitions, (N, J = 1/2) $\rightleftharpoons$ (N, J = 1/2) and (N, J = 3/2) $\rightleftharpoons$ (N, J = 3/2). The spin crossing matrix B can be easily calculated and is,

B =

	J = 1/2	J = 3/2
J = 1/2	$-\frac{1}{3}$	$\frac{4}{3}$
J = 3/2	$\frac{2}{3}$	$\frac{1}{3}$

(3.19)

Now, the total crossing matrix relating the channel II amplitudes with those of the channel I is the direct product of A_{II} (3.16) and B (3.19). Calling this total crossing matrix C we have from (3.16) and (3.19)

C =

	N'	3		6*		15	
N	J \ J	1/2	3/2	1/3	3/2	1/2	3/2
3	$\frac{1}{2}$	$\frac{1}{24}$	$-\frac{1}{6}$	$\frac{1}{4}$	-1	$-\frac{5}{8}$	$\frac{5}{2}$
	$\frac{3}{2}$	$-\frac{1}{12}$	$-\frac{1}{24}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{5}{4}$	$\frac{5}{8}$
6*	$\frac{1}{2}$	$\frac{1}{8}$	$-\frac{1}{2}$	$-\frac{1}{4}$	1	$-\frac{5}{24}$	$\frac{5}{6}$
	$\frac{3}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{5}{12}$	$\frac{5}{24}$

$C =$

	N'	3		6*		15	
N	J \ J'	1/2	3/2	1/2	3/2	1/2	3/2
15	$\frac{1}{2}$	$-\frac{1}{8}$	$\frac{1}{2}$	$-\frac{1}{12}$	$\frac{1}{3}$	$-\frac{1}{8}$	$\frac{1}{2}$
	$\frac{3}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{8}$

(3.20)

It is evident from (3.20) that the exchange of 15 with J = 3/2 gives rise to the most dominant force for 3 with J = 1/2; it also gives attractive force for itself in the direct channel. Likewise the exchange of 3 with J = 1/2 gives an attractive force for both 15 with J = 3/2 and 3 with J = 1/2. This is analogous to the relation between N and N* in the SU_2 bootstrap of Chew. Let us also calculate the ratio of the couplings of these two states by using static model. Denoting the 3 with J = 1/2 and 15 with J = 3/2 couplings by $\Gamma_{3,1/2}$ and $\Gamma_{15,3/2}$ respectively we find using (2.54) that from (3, 1/2) strap $\Gamma_{3,1/2}/\Gamma_{15,3/2} \approx 2.6$ and that from (15, 3/2) strap $\Gamma_{3,1/2} / \Gamma_{15,3/2} \approx 3.5$. Thus, it seems, within the static approximations, that there may be a self-consistent reciprocal bootstrap relationship between quarks, Q and some other particle Q* having baryonic number B = 1/3, spin 3/2 and belonging to 15 dimensional representation of SU(3).

3. The N/D Method and the Self-Consistent Solution for Quarks

We have shown in the last section that there exists a reciprocal bootstrap relationship between quarks Q and some other particles Q* having baryonic number B = 1/3, spin 3/2 and belonging to the $\underset{\sim}{15}$ dimensional representation of SU(3) symmetry. We now use the well known N/D method[13] to calculate the consistent masses and the coupling constants of these particles. In particular, we use the so called determinantal method[42] which is expected to provide us at least with a rough estimate about the masses and couplings of these particles.

We denote, as usual, by s, u the c.m. energies squared in the channels I and II respectively, and assume that the Q and Q* poles in s-variable occur only in the amplitudes $F^{3}_{1/2}$ and $F^{15}_{3/2}$ respectively with the residues given by,

$$f_{1/2} = \frac{4\pi g_1^2}{m^2 - s} \tag{3.21}$$

$$f_{3/2} = \frac{g_2^2}{M^2 - s - \frac{1}{2}\Gamma} \tag{3.22}$$

where we define $g_r^2 = 4\pi g_1^2$ as the $QQ\pi$ coupling, m, the average mass of the quark-triplet and M, that of Q*. The residue g_2^2 is related to the width by unitarity at $s = M^2$ as $\Gamma = 2q^* g_2^2$. The poles (3.21) and (3.22) however are to be multiplied by the respective factors given in the appendix B (B.4) in order to take into account the effects of the SU(3) couplings.

The above pole terms in the various amplitudes in the u-variable can be calculated by using the crossing matrix C (3.20) so that we have,

$$F^C(u\ s) = C\,F(s\ u) \tag{3.23}$$

where F(s u) is a column matrix with the elements $F^{s}_{1/2}$, $F^{s}_{3/2}$, $F^{6*}_{1/2}$, $F^{6*}_{3/2}$, $F^{15}_{1/2}$, $F^{15}_{3/2}$. Then, the desired partial wave amplitudes are obtained from the following relations :

$$f_{\ell\pm}(s) = \frac{1}{2}\int_{-1}^{+1} dz \left[f_1(u,s)\, P_\ell(z) + f_2(u,s)\, P_{\ell\pm1}(z) \right] \tag{3.24}$$

where

$$f_1(u,s) = \sum_{\ell=0}^{\infty} f_{\ell+}(u)\, P'_{\ell+1}(x) - \sum_{\ell=2}^{\infty} f_{\ell-}\, P'_{\ell-1}(x) \tag{3.25a}$$

$$f_2(u,s) = \sum_{\ell=1}^{\infty} f_{\ell-}(u)\, P'_\ell(x) - \sum_{\ell=1}^{\infty} f_{\ell+}(u)\, P'_\ell(x) \tag{3.25b}$$

$$x = \cos\theta \tag{3.25c}$$

$$P'_{\ell}(x) = \frac{d}{dx} P_{\ell}(x) \tag{3.25d}$$

$$f_{\ell\pm} = \exp[i\,\delta_{\ell\pm}] \sin \delta_{\ell\pm} / q \tag{3.25e}$$

Here $f_{\ell\pm}$ has orbital angular momentum ℓ and the total angular momentum $J = \ell \pm 1/2$. $\delta_{\ell\pm}$ is the phase-shift corresponding to the ℓ-th partial wave and θ is the angle of scattering in the c.m. system of the crossed channel (section 1).

In order to obtain the force term for the $J = 1/2$ state we consider the contribution from only the Q* exchange (since Q exchange gives a very small effect) in the crossed channel. Using (3.22) and (3.23) we obtain from (3.24) and (3.25) the following for the p-wave with $J = 1/2$ state,

$$f_{1-}(s) = \frac{5g_2^2}{8q^2}\left[3x\left(2 - a\log\frac{a+1}{a-1}\right) - \log\frac{a+1}{a-1}\right] \tag{3.26a}$$

For the $J = 3/2$ state we consider the contribution from both Q and Q* exchanges (since both have almost equally significant crossing matrix elements) in the crossed channel. Using (3.21), (3.22) and (3.23) we obtain from (3.24) and (3.25) for the p-wave with $J = 3/2$ state the following,

$$f_{1+}(s) = \frac{g_2^2}{32q^2}\left[3x\left(2 - a \log \frac{a+1}{a-1} \right) + 3a - \frac{3a^2 - 1}{2} \log \frac{a+1}{a-1} \right] + \frac{2\pi g_1^2}{3}\left[-3b + \frac{3b^2 - 1}{2} \log \frac{b+1}{b-1} \right]$$

(3.26b)

where in both (3.26a) and (3.26b) we have,

$$a = \frac{s + 2s\,M^2 - 2(m^2 + \mu^2)}{2q^2} - 1 \qquad (3.27a)$$

$$b = \frac{s + m^2 - 2\mu^2}{2q^2} - 1 \qquad (3.27b)$$

$$x = 1 + \frac{2M^2[2(m^2 + \mu^2) - M^2 - s]}{[M^4 - 2M^2(m^2 + \mu^2) + (m^2 - \mu^2)^2]} \qquad (3.27c)$$

$$q^2 = \left\{ s - (m + \mu)^2 \right\}\left\{ s - (m - \mu)^2 \right\} \Big/ 4s \qquad (3.27d)$$

where μ is taken as the average mass of the meson octet.

The Born terms (3.26a) and (3.26b) behave as q^2 at the threshold and so we can divide them by the factor q^2 without introducing any additional singularities. Further, we have to multiply these Born terms by s in order to remove the kinematic singularities associated with them. Thus, we define the following kinematic singularity free Born terms:

$$B_{1\mp}(s) = \frac{s}{q^2} f_{1\mp}(s) \tag{3.28}$$

For the same reason described above we work with the following kinematic singularity free partial wave amplitudes:

$$h_{1\mp}(s) = \frac{s}{q^2} f_{1\mp}(s) \tag{3.29}$$

Here the multiplication of the partial wave amplitudes $f_{1\mp}(s)$ by the factor s/q^2 ensures that $h_{1\mp}(s)$ is free from all the kinematic singularities and further that the final results will maintain the correct threshold behaviour as $q^2 \to 0$. We now write,

$$h_{1\mp}(s) = \frac{N_{1\mp}(s)}{D_{1\mp}(s)} \tag{3.30}$$

where, as is well known, $N_{1\mp}(s)$ are analytic on the force cut (left cut) and real on the physical cut and $D_{1\mp}(s)$ are analytic on the physical cut and real on the force cut. Now, the one-subtraction dispersion relations of $D_{1\mp}(s)$ can be written in the form,

$$D_{1-}(s) = 1 + \frac{(s - s_1)}{\pi} \int_{(m+1)^2}^{\infty} \frac{\text{Im } D_{1-}(s')}{(s' - s)(s' - s_1)} ds' \tag{3.31a}$$

$$D_{1+}(s) = 1 + \frac{(s - s_2)}{\pi} \int_{(m+1)^2}^{\infty} \frac{\text{Im } D_{1+}(s')}{(s' - s)(s' - s_2)} ds' \tag{3.31b}$$

where we have normalised $D_{1-}(s)$ at $s = s_1$ and $D_{1+}(s)$ at $s = s_2$ and set the meson mass $\mu = 1$. Now, over the left-hand cut we get from (3.30) the following:

$$\begin{aligned} \text{Im}\, N_{1\mp}(s) &= D_{1\mp}(s)\ \text{Im}\, h_{1\mp}(s) && s < s_L \\ &= 0 && \text{for} \quad s > s_L \end{aligned} \tag{3.32}$$

where s_L is the beginning of the left-hand cut. Now, the determinantal approximation allows us to write,

$$N_{1\mp}(s) = B_{1\mp}(s) \tag{3.33}$$

Again, the unitarity relations over the physical cut give us the following information:

$$\begin{aligned} \text{Im}\, D_{1\mp}(s) &= N_{1\mp}(s)\ \text{Im}(1/h_{1\mp}(s)) \\ &= -\, B_{1\mp}(s)\ \rho(s) \end{aligned} \tag{3.34}$$

where $\rho(s)$ is the kinematic factor given by,

$$\rho(s) = \frac{q^3}{s} \tag{3.35}$$

where q is given by (3.27d).

Using the equations (3.32) - (3.35), we obtain from (3.31a) and (3.31b) the following dispersion relations for $D_{1\mp}(s)$:

$$D_{1-}(s) = 1 - \frac{(s - s_1)}{\pi} \int_{(m+1)^2}^{\infty} \rho(s') \frac{B_{1-}(s')}{(s' - s)(s' - s_1)} ds' \tag{3.36}$$

$$D_{1+}(s) = 1 - \frac{(s - s_2)}{\pi} \int_{(m+1)^2}^{\infty} \rho(s') \frac{B_{1+}(s')}{(s' - s)(s' - s_2)} ds' \tag{3.37}$$

If $D_{1-}(m^2) = 0$ with m^2 below the physical threshold there is a $P_{1/2}$ bound state near which we can write $h_{1-}(s)$ in the form:

$$h_{1-}(s) = \frac{N_{1-}(m^2)/D'_{1-}(m^2)}{(s - m^2)} \tag{3.38}$$

Comparing (3.38) with (3.21) the output coupling constant for the quarks is given by,

$$g_1'^2 = - \frac{N_{1-}(m^2)}{D'_{1-}(m^2)} \tag{3.39}$$

Similarly if there is a resonance at $s = M^2$ we have Re $D_{1+}(M^2) = 0$ and near the resonance position we have

$$h_{1+}(s) = \frac{N_{1+}(M^2)/\mathrm{Re}\ D'(M^2)}{(s - M^2) - i\,\rho\,(M^2)\ N_{1+}(M^2)/\mathrm{Re}\ D'(M^2)} \tag{3.40}$$

Comparing (3.40) with (3.22) we have,

$$\Gamma' = -2\rho(M^2)\,\frac{N_{1+}(M^2)}{\mathrm{Re}\ D'(M^2)} \qquad (3.41)$$

We can define a coupling constant for Q* as proportional to the width Γ. Thus, we can have,

$$g_2'^2 = -\frac{\Gamma'}{\rho(M^2)} = -\frac{N_{1+}(M^2)}{\mathrm{Re}\ D'(M^2)} \qquad (3.42)$$

where $\rho(M^2)$ is given by (3.35).

For a given input value of m and M the values of g_1^2, $g_1'^2$ and g_2^2, $g_2'^2$ can be calculated from the equations (3.36), (3.37), (3.39) and (3.42). Taking $s_1 = (m-1)^2$ and $s_2 = m^2$, the relevant integrations have been solved numerically, the investigation being carried out for the quark-mass upto 20μ, where μ is the average mass of the meson octet. If there exists a complete reciprocal bootstrap relationship between Q and Q*, then g_1^2, $g_1'^2$ and g_2^2, $g_2'^2$ have to be consistent simultaneously. For the computations, we have used the "Optimisation Method" which minimises the sums of the squares of the differences of the above two sets of the coupling constants for the two given ranges of the masses m and M. The self-consistent solutions we have obtained correspond to m $\simeq$ 2429 Mev, M $\simeq$ 5251 Mev, with the corresponding values of the couplings 22 and 32 respectively. In the above calculation we have taken 360 Mev as the average mass of the pseudoscalar meson octet. The above value of the $QQ^*\pi$ coupling corresponds to the full-width $\Gamma \simeq$ 360 Mev, where Γ is the average full-width of Q^*.

An investigation very similar to ours has been carried out by Nieto[51)] who have considered the contributions of the vector meson exchange as well. The reason for their obtaining the negative result may be attributed to the fact that the vector meson exchange forces, as is evident from the crossing matrix (3.18), are repulsive, being strongly repulsive in the Q^* channel. It is, therefore, very likely that the vector meson exchange has a very insignificant or rather opposite effect in the quark-bootstraps. The method we have used in our calculations, however, suffers from some shortcomings. The determinantal method employed in the calculations is valid very approximately. In addition, the inelastic effects which we have neglected in the calculations may have some significant influence on the quark-meson scattering if the quarks are supposed to be very massive. Considering the above limitations of the calculational method employed, our results are, therefore, to be taken with that spirit.

CHAPTER IV

An N/D Calculation for the Mass-Splitting of Baryons in Broken U(6,6) Symmetry

1. Baryon-Meson Scattering and Related Processes:

Scatterings between two U(6,6) supermultiplets have been considered by a number of authors[52)] who, by studying only the elastic collinear processes, have tried to obtain results which, within the limitations of the approximations used, would be comparable to the experimental ones. Consideration only of the elastic forward scatterings, no doubt, allows one to use the optical theorem for the comparison of the total cross-sections, but a question naturally arises as to the validity of these results because of the approximations that have been used in these calculations. In these investigations, only the homogeneous U(6,6) invariant part of the amplitudes has been considered. This, however, is not sufficient for the scattering amplitudes to be compatible with the unitarity[36)] conditions in the physical regions. In order to be consistent with the unitarity, the scattering amplitudes, in addition to the homogeneous U(6,6) invariant terms, must have two more terms corresponding respectively to the irregular (involving the derivative couplings) and the higher order spurion[53] terms. It is these spurion terms that are supposed to cause the mass-splittings within a U(6,6) supermultiplet.

Some calculations using the irregular coupling terms have been carried out by a number of authors[54] who also introduce ad hoc mass-splittings between the SU(3) multiplets within the U(6,6) supermultiplets of the external particles involved in a scattering process. Besides these mass-splittings Rivers[55], on the other hand, also introduces the similar mass-splittings within the U(6,6) supermultiplet occuring in the intermediate state of a second order Feynman diagram corresponding to a two-particle scattering process. This is done by expressing the propagator of the U(6,6) multiplet with U(6,6) degenerate mass as a sum of the propagators of the constituent SU(3) multiplets with SU(3) degenerate masses. It has been shown by Rivers that such a procedure introduces in the scattering amplitudes the same higher order spurion terms as we have discussed above. Since this procedure has no group-theoretical basis, one may look at the problem from the opposite point of view. Instead of introducing the mass-splittings right at the very outset of the calculation, one may start the calculation with the U(6,6) degenerate masses and expect the mass-splittings result from the spurion terms associated with the scattering amplitudes. As it is extremely difficult to deal with the irregular couplings and the higher order spurion terms in the scattering amplitudes, one would naturally look for a simpler method. A convenient method is the so-called N/D method by which one can get round the difficulties that usually arise when one is dealing with the full U(6,6) invariant amplitudes. Such a method has been used by Gatto and Venziano[56)] in connection with the calculation of the mass of N_{33} resonance by using the $SU(6)_W$[57)] invariant vertices in the calculations. They consider the pion-nucleon scattering and the exchanges of a nucleon, a nucleon resonance and a rho-meson in the respective crossed channels. The result they have obtained for the mass of N_{33} resonance is so encouraging that one would feel very much tempted to use the above proceduere in order to calculate the mass of the nucleon as well. That is what, in short, we propose to do, our purpose and the problem we shall be dealing with, however, being different from theirs.

We consider the meson-baryon scattering from the point of view of U(6,6) theory and investigate the mass-splitting between the baryon octet and decouplet by using the N/D method. We assume that the SU(3) symmetry is exact so that the masses of the octet and decouplet will correspond respectively to the average mass of the eight baryons and that of the ten baryon resonances. As these

SU(3) multiplets belong to the same irreducible 364-dimensional representation of U(6,6) symmetry, they are supposed to have the same mass from the view-point of U(6,6) theory. Now, if we use the U(6,6) vertices in our calculations, then it is expected that by using the N/D method we shall, to a reasonable extent, get the SU(3) degenerate masses of these SU(3) multiplets. In our calculations we shall, however, follow the hypothesis of N-N* bootstrap of Chew and consequently consider that the forces responsible for the binding of the baryon octet and decouplet come predominantly from the exchanges of these SU(3) multiplets themselves in the crossed channels. The problem we are going to solve is, in fact, a multi-channel problem and therefore the matter will be clear if we discuss how these processes arise.

It is well known that all the mesons belong to the 143-dimensional irreducible representation of U(6,6). But not all of the states of 143 correspond to the physical particles. The states which correspond to the physical particles are, as we have discussed in Chapter I (section 3), obtained by applying on the irreducible tensor corresponding to 143 representation, the Bergmann-Wigner equations under which the trivial components vanish identically. We have shown (1.70) the $U(22) \otimes SU(3)$ contents of 143 representation. The physical states which are obtained by applying the Bergmann-Wigner equations correspond to the contents of the subgroup $IL_4 \otimes SU(3)$, where IL_4 is the inhomogeneous Lorentz group. Thus, the physical states of $\underline{143}$ representation can be expressed as follows:

$$\underline{143} \equiv \underline{P} \oplus \underline{V} \oplus P_o \oplus V_o \qquad (4.1)$$

where P, V, P_o, V_o denote respectively the pseudoscalar meson, octet, the vector meson octet, the pseudoscalar meson singlet and the vector meson singlet.

Following a similar analysis discussed above, the $\underline{364}$ dimensional irreducible representation of U(6,6) can be expressed in terms of the physical baryons as follows,

$$\underline{364} \equiv B + D \qquad (4.2)$$

where, B, D are the baryon octet and decouplet respectively.

If we now consider the meson ($\underline{143}$) and baryon ($\underline{364}$) scattering, all the SU(3) multiplets in (4.1) and (4.2) will take part independently of each other in the scattering phenomena and thereby give rise to the multichannel processes. From the point of view of SU(3) symmetry along, this is, in fact, an eight-channel problem; each of the processes, however, being elastic in view of the U(6,6) symmetry. Let us denote an arbitrary SU(3) multiplet in (4.1) by M_α, and that in (4.2) by N_i. Then the above eight-channel scattering phenomena can be expressed as,

$$M_\alpha + N_i \leftrightarrow M_\beta + N_k \qquad (4.3)$$

where α, β = 1, 2, 3, 4, i, k = 1, 2.

If we are interested in the baryon octet poles in the direct channels, two pairs of the above SU(3) multiplets will not contribute to such poles. These two pairs correspond respectively to the occurrence of a pseudoscalar (vector) singlet and a decouplet at either the initial or the final states of the scattering phenomena. Thus, for the baryon octet pole calculation, the problem reduces to a six-channel one. It can, similarly, be shown that the two pairs consisting of a pseudoscalar (vector) singlet and a baryon octet occurring at either the initial or the final states will not contribute to the decouplet poles. Thus, in both the octet and decouplet pole calculations, the problem reduces to two six-channel ones. These have been explained in detail by the second order Feynman diagrams, Fig. B.4 - Fig. B.15 in the appendix B.

We assume, as we have mentioned earlier in this section, that the forces responsible for the binding of the baryon octet with $J = 1/2^+$ and baryon decouplet with $J = 3/2^+$ arise predominantly from the exchanges of these particles in the respective crossed channels. The left-hand cuts (sometimes referred to as force-cuts) are associated with these forces. As it is well known that the long-range part of the forces play a very dominant role in any scattering phenomena, we shall confine ourselves to the consideration of single-particle exchanges only. For each of the processes discussed above, there will occur two crossed diagrams corresponding respectively to the exchanges of the baryon octet with $J = 1/2$ and baryon decouplet with $J = 3/2$. The space-time parts of these processes are associated with the so-called exchange Born terms which we discuss in section 3 of this chapter and the SU(3) symmetry coefficients connected with these exchange Born terms are to be obtained by using the direct pole-coefficients given in tables B.1 and B.2 (appendix B) and the related SU(3) crossing matrices which we discuss below.

For all the processes we shall be dealing with, there are only three independent SU(3) crossing matrices. We shall discuss these crossing matrices one by one. First, we consider the process of a baryon octet and a meson (pseudoscalar vector) octet going to a baryon and a meson octet. The two related channels (chapter III) we are interested in are as follows:

$$a_1(8) + a_2(8) \rightarrow a_3(8) + a_4(8) \qquad \text{I}$$

$$(4.4)$$

$$a_1(8) + \bar{a}_4(8) \rightarrow a_3(8) + \bar{a}_2(8) \qquad \text{II}$$

where the number 8 denotes the dimension of the SU(3) irreducible representation to which the respective particle belongs. The system at either the initial or final state of the above processes being a direct product of two octets can be decomposed into the following irreducible representations:

$$\underline{8} \otimes \underline{8} = \underline{1} + \underline{8}_1 + \underline{8}_2 + \underline{10} + \underline{10}^* + \underline{27} \qquad (4.5)$$

where 8_1 and 8_2 are respectively the well known symmetric and anti-symmetric combinations.

In each channel (4.4) there are seven SU(3) invariant amplitudes (we are not considering spins of the particles in this section) corresponding respectively to the transitions, $1 \to 1$, $8_1 \to 8_1$, $8_2 \to 8_2$, $8_1 \leftrightarrow 8_2$, $10 \to 10$, $10^* \to 10^*$ and $\underline{27} \to \underline{27}$. The amplitudes in the channel II are related with those in channel I through the equation (3.11), the expression for the elements of the corresponding crossing-matrix being given by (3.12). The related crossing matrix is the following:

C_1 =

N,β,γ \ N',β',γ'	1	27	10*	10	8_{11}	8_{12}	8_{21}	8_{22}
1	$\frac{1}{8}$	$\frac{27}{8}$	$-\frac{5}{4}$	$-\frac{5}{4}$	1	0	0	−1
27	$\frac{1}{8}$	$\frac{7}{40}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{5}$	0	0	$\frac{1}{3}$
10*	$-\frac{1}{8}$	$\frac{9}{40}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{2}{5}$	$-\frac{1}{\sqrt{5}}$	$-\frac{1}{\sqrt{5}}$	0

N, β, γ \ N', β', γ'	1	27	10*	10	8_{11}	8_{12}	8_{21}	8_{22}
10	$-\frac{1}{8}$	$\frac{9}{40}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{2}{5}$	$\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{5}}$	0
8_{11}	$\frac{1}{8}$	$\frac{27}{40}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{10}$	0	0	$-\frac{1}{2}$
8_{12}	0	0	$-\frac{\sqrt{5}}{4}$	$\frac{\sqrt{5}}{4}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	0
8_{21}	0	0	$-\frac{\sqrt{5}}{4}$	$\frac{\sqrt{5}}{4}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0
8_{22}	$-\frac{1}{8}$	$\frac{9}{8}$	0	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$

(4.6)

Let us now consider the process of a baryon decouplet and a meson (pseudoscalar or vector) octet going to a baryon octet and a meson octet. The related two channels, as before, are given by,

$$a_1(10) + a_2(8) \rightarrow a_3(8) + a_4(8) \qquad \text{I}$$

(4.7)

$$a_1(10) + \bar{a}_4(8) \rightarrow a_3(8) + \bar{a}_2(8) \qquad \text{II}$$

The initial system being a direct product of a decouplet and an octet can be decomposed as follows:

$$\underline{10} \otimes \underline{8} = \underline{8} \oplus \underline{10} \oplus \oplus \underline{27} \oplus \underline{35} \qquad (4.8)$$

In each of the above channels (4.7) there are four SU(3) invariant amplitudes corresponding respectively to the transitions $8 \rightarrow 8_1$, $8 \rightarrow 8_2$, $10 \rightarrow 10$, and $27 \rightarrow 27$. The related crossing matrix can be calculated as before and is given by,

$C_2 =$

N,β \ N',β'	27	10	8_1	8_2
27	$\frac{1}{10}$	$\frac{1}{3\sqrt{2}}$	$\frac{2}{5}$	$\frac{2}{3\sqrt{5}}$
10	$\frac{9}{10\sqrt{2}}$	$-\frac{1}{2}$	$-\frac{2}{5\sqrt{2}}$	$\frac{2}{\sqrt{10}}$
8_1	$\frac{27}{20}$	$-\frac{1}{2\sqrt{2}}$	$\frac{2}{5}$	$-\frac{1}{\sqrt{5}}$
8_2	$\frac{-9}{4\sqrt{5}}$	$\frac{5}{2\sqrt{10}}$	$-\frac{1}{\sqrt{5}}$	0

(4.9)

We now consider the process in which a bayron decouplet and a meson (vector or pseudoscalar)octet occur at both the initial and final states and in both the channels discussed in this connection above. The irreducible SU(3) representations at both the initial and final states in either channels are those given by (4.8). There are in either channels four SU(3) invariant amplitudes corresponding respectively to the transitions, $8 \rightarrow 8$, $10 \rightarrow 10$, $27 \rightarrow 27$ and and $35 \rightarrow 35$. The following is the related SU(3) crossing matrix:

$$C_s = \begin{array}{c|cccc} N \backslash N' & 35 & 27 & 10 & 8 \\ \hline 35 & \frac{1}{8} & -\frac{9}{40} & \frac{1}{4} & \frac{2}{5} \\ 27 & \frac{7}{24} & \frac{37}{40} & -\frac{1}{12} & -\frac{2}{15} \\ 10 & \frac{7}{8} & -\frac{9}{40} & \frac{3}{4} & -\frac{2}{5} \\ 8 & \frac{7}{4} & -\frac{9}{20} & -\frac{1}{2} & \frac{1}{5} \end{array} \qquad (4.10)$$

As mentioned before, these crossing matrices are to be used in conjunction with the octet and decouplet pole coefficients given in tables B.1 and B.2 (appendix B) in order to obtain the related SU(3) coefficients connected with the exchange Born terms.

Let us now consider the processes involving the pseudo-scalar (vector) singlets. There are in these channels six SU(3) crossing matrices which are required in order to calculate the SU(3) exchange coefficients. We consider them one by one.

I. Process $P_o(V_o)B \longleftrightarrow P(V)B$

In the above process there are in s-channel two SU(3)-invariant amplitudes corresponding to the transitions $\underset{\sim}{8} \rightarrow \underset{\sim}{8}_1$ and $\underset{\sim}{8} \rightarrow \underset{\sim}{8}_2$ respectively. In the crossed (u) channel, there are also two amplitudes corresponding to the transitions $\underset{\sim}{8}_1 \rightarrow \underset{\sim}{8}$ and $\underset{\sim}{8}_2 \rightarrow \underset{\sim}{8}$ respectively. Consequently, the SU(3) crossing matrix, is as follows:

C_4 =

N', γ' / N, β	$8 - 8_1$	$8 - 8_2$
$8_1 - 8$	1	0
$8_2 - 8$	0	1

(4.10a)

II. Process $P_o(V_o)B \leftrightarrow P(V)D$

In this process there is one SU(3) invariant ampltitude in the direct (s-channel) corresponding to the transition $\underset{\sim}{8} \rightarrow \underset{\sim}{8}$. In the crossed (u) channel there also occurs one amplitude corresponding to the transition $\underset{\sim}{10} \leftrightarrow \underset{\sim}{10}$. Hence the one-dimensional crossing matrix is,

C_5 =

N' / N	$10 - 10$
$8 - 8$	$-\frac{\sqrt{5}}{2}$

(4.10b)

III. Process $P_o(V_o)D \leftrightarrow P(V)B$

In the above process, there is one amplitude in the s-channel corresponding to the transition $\underset{\sim}{10} \longleftrightarrow \underset{\sim}{10}$. There also occurs one amplitude in the u-channel corresponding to the transition $\underset{\sim}{8} \longleftrightarrow \underset{\sim}{8}$. Hence the one-dimensional crossing matrix is given by,

$C_8 =$

N \ N'	8 - 8
10 - 10	$-\frac{2}{\sqrt{5}}$

(4.10c)

IV. Process $P_o(V_o)D \longleftrightarrow P(V)D$

There is in the above process one amplitude corresponding to the transition $\underset{\sim}{10} \longleftrightarrow 10$ in both s- and u-channel. Consequently the crossing matrix which is one-dimensional is as follows:

$C_7 =$

N \ N'	10 - 10
10 - 10	1

(4.10d)

V. Process $P_o(V_o) \longleftrightarrow P_o(V_o)B$

There occurs in the above process one amplitude corresponding to the transitions $8 \longleftrightarrow 8$ in both the s- and u-channel and the relevant crossing matrix is,

C_s =

N \ N'	8 - 8
8 - 8	1

(4.10e)

VI. Process $P_o(V_o) \longleftrightarrow P_o(V_o)D$

In the above process there is in both the s- and u-channel one SU(3) invariant amplitude corresponding to the transition $\underset{\sim}{10} - \underset{\sim}{10}$ and the crossing matrix is given by,

C_s =

N \ N'	10 - 10
10 - 10	1

(4.10f)

As mentioned before, these SU(3) crossing matrices corresponding to the processes involving the pseudoscalar (vector) singlets are to be used in conjunction with the relevant octet and decouplet SU(3)-pole coefficients given in the table B.1 and table B.2 (appendix B) in order to obtain the related SU(3) coefficients associated with the relevant exchange Born terms.

2. Helicity formalisms and partial wave amplitudes:

In the previous section, we have discussed how, when the U(6,6) symmetry is broken by Bergmann-Wigner equations, the meson-baryon scattering decomposes into a number of independent processes involving two SU(3) multiplets at both the initial and the final states. If we now take into account the spins of the scattering particles, then the scattering amplitude corresponding to each of the above processes will be a n $\times$ n matrix, the dimension of the matrix, of course, being dependent on the spins of the particles involved in the processes concerned. We shall discuss, in this section, this particular aspect of the scattering amplitudes from the view-point of the helicity formalism discussed by Jacob and Wick[58)]. As we shall be solving the N/D equations only for the octet and decouplet poles, we need to construct the parity amplitudes which contribute to $J = 1/2$ and $J = 3/2$. states. Finally, we shall discuss how these parity amplitudes are expressed in terms of the partial wave amplitudes. In fact, one deals with the partial wave amplitudes when one is using the N/D method in order to solve a scattering problem. It is these partial wave amplitudes or the parity amplitudes discussed above that give rise to a number of independent channels in any of the reactions we have discussed in the previous section. Since the space-time properties of the pseudoscalar (the vector) singlet are the same as those of the pseudoscalar (the vector) octet, it will be sufficient if we consider the processes involving only the pseudoscalar (the vector) meson octets.

Let us consider a general scattering process (Fig. 4.1) involving a baryon (octet or decouplet) and a meson (pseudoscalar or vector) at both the initial and final states.

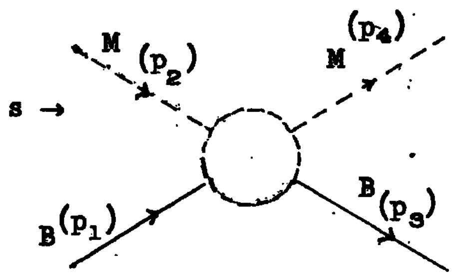


Figure 4.1a

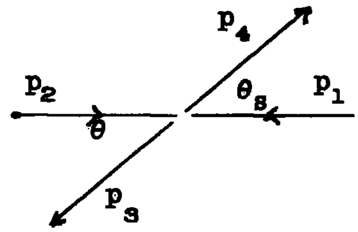


Figure 4.1b

s-channel C.M. system

The s-matrix corresponding to the above process (Fig. 3.1) of the baryon-meson scattering can be written in the form:

$$< f|s|i > = \delta_{fi} - i(2\pi)^4 \, \delta_4(p_f - p_i) < f|R|i > \qquad (4.11)$$

where

$$< f|R|i > = \frac{1}{(2\pi)^6} \sqrt{\frac{m^2}{4E_1E_2E_3E_4}} < f|F|i > \qquad (4.12)$$

where, F is the well known Feynman amplitudes. In (4.12), m is the mass of the baryon and, as we have already mentioned in the previous section, is to be taken as the U(6,6) degenerate mass. E_1, E_2 are the energies of the baryon and meson respectively in the initial state and E_3, E_4 the corresponding energies in the final states. The three invariant quantities that can be formed out of the four momenta, $p_1\ p_2\ p_3\ p_4$ are as follows:

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 = W^2$$

$$t = (p_1 - p_3)^2 = -2k^2(1 - \cos\theta) \qquad (4.13)$$

$$u = (p_1 - p_4)^2 = 2m^2 + 2\mu^2 - s + 2k^2(1 - \cos\theta)$$

where, $\underset{\sim}{p}_1^2 = \underset{\sim}{p}_2^2 = \underset{\sim}{p}_3^2 = \underset{\sim}{p}_4^2 = k^2$

$$k^2 = \left\{ s - (m + \mu)^2 \right\}\left\{ s - (m - \mu)^2 \right\} \Big/ 4s \qquad (4.14a)$$

$$E_1 = E_3 = \sqrt{k^2 + m^2}\ ; \quad E_2 = E_4 = \sqrt{k^2 + \mu^2} \qquad (4.14b)$$

where μ is the U(6,6) degenerate mass of the mesons and θ is the angle of scattering in the c.m. system and is given by,

$$\cos\theta = \underset{\sim}{p}_2 \cdot \underset{\sim}{p}_4 \Big/ |p_2 \cdot p_4|^2 \qquad (4.14c)$$

Now, using equation (C.23) we have,

$$< p_3\ p_4\ \lambda_3\ \lambda_4 | R | p_1\ p_2\ \lambda_1\ \lambda_2 > = \frac{W}{\sqrt{p_1\ p_3}} \ \frac{1}{\sqrt{E_1\ E_2\ E_3\ E_4}}$$

$$< \theta_f\ \phi_f\ \lambda_3\ \lambda_4 | R | \theta_i\ \phi_i, \lambda_1\ \lambda_2 > \qquad (4.15)$$

The, using (4.15) the helicity representation of the Feynman amplitude, F, obtained from (4.11) and (4.12) is as follows:

$$< p_3\, p_4\, \lambda_3\, \lambda_4 | F | p_1\, p_2\, \lambda_1\, \lambda_2 > \;=\; - \frac{(4\pi)^2}{\sqrt{m_1\, m_3}} \sum_{JJ',MM'}$$

$$< \theta_f\, \phi_f \lambda_3\, \lambda_4 \mid J'M'\, \lambda_3\, \lambda_4 > < J'M'\, \lambda_3\, \lambda_4\, | s-1 | JM\, \lambda_1\, \lambda_2 >$$

$$< JM\, \lambda_1\, \lambda_2 \mid \theta_i\, \phi_i\, \lambda_1\, \lambda_2 > \frac{W}{2i\sqrt{p_1\, p_3}} \qquad (4.16)$$

Now the helicity amplitude between the same total angular momentum states is defined as,

$$h^J_{\lambda_3\lambda_4;\lambda_1\lambda_2} = < JM\, \lambda_3\, \lambda_4 | s-1 | JM\, \lambda_1\, \lambda_2 > \cdot \frac{1}{2i\sqrt{p_1\, p_3}}$$

$$= \frac{W}{k} \cdot \frac{s^J - 1}{2i} = \frac{W}{k} \cdot \frac{e^{2i\delta_J} - 1}{2i} \qquad (4.17)$$

where δ_J is the phase-shift corresponding to the J-th partial wave.

Then substituting (4.17) into (4.16) we have,

$$< p_3\, p_4\, \lambda_3\, \lambda_4 | F | p_1\, p_2\, \lambda_1\, \lambda_2 > \;=\; - \frac{(4\pi)^2}{\sqrt{m_1\, m_3}} \sum_{JM}$$

$$< \theta_f \; \phi_f \; \lambda_3 \; \lambda_4 | JM \; \lambda_3 \; \lambda_4 > < JM \; \lambda_1 \; \lambda_2 | \theta_i \; \phi_i \; \lambda_1 \; \lambda_2 > \; h^J_{\lambda_3\lambda_4;\lambda_1\lambda_2} \tag{4.18}$$

Let $\lambda = \lambda_3 - \lambda_4$; $\mu = \lambda_1 - \lambda_2$. Now taking the three-component momenta of the initial particles along the z-axis and in the xz-plane such that $\theta_i = \phi_i = 0$, we have from (C.29)

$$< JM \; \lambda_1 \; \lambda_2 \; | \; \theta_i \; \phi_i \; \lambda_1 \; \lambda_2 > \; = \; N_J \; \delta_{M\mu} \tag{4.19}$$

Now substituting (4.19) and (C.29) into (4.18) we have,

$$< p_3 \; p_4 \; \lambda_3 \; \lambda_4 | F | p_1 \; p_2 \; \lambda_1 \; \lambda_2 > \; = \; - \frac{(4\pi)^2}{\sqrt{m_1 \; m_2}} \sum_{JM} \; h^J_{\lambda_3\lambda_4;\lambda_1\lambda_2}$$

$$e^{iM\phi_f} \; d^J_{M\lambda}(\theta) \; e^{-i\lambda\phi_f} \; N_J^2 \; \delta_{M\mu}$$

We also take the two momenta of the final-state particles in the xz-plane. Then multiplying the above equation by $d^J_{\mu\lambda}(\theta) \sin\theta$ and integrating over θ and using (C.26) we get,

$$< \lambda_3 \; \lambda_4 | h^J | \lambda_1 \; \lambda_2 > \; = \; - \frac{m}{8\pi} \int_{-1}^{+1} < \lambda_3 \; \lambda_4 | F | \lambda_1 \; \lambda_2 > \; d^J_{\mu\lambda} \; (\theta) \; d(\cos\theta) \tag{4.20}$$

For each total angular momentum J there are $(2s_1 + 1)(2s_2 + 1)$ $(2s_3 + 1)(2s_4 + 1)$ helicity amplitudes, where s_i's are the spins of the scattering particles. In fact, one has to evaluate the independent helicity amplitudes by invoking the law of invariance of the scattering matrix s under parity and time reversal. To see how it is done let us define the helicity of a free particle as follows:

$$\lambda = \mathbf{J}.\mathbf{P}/|\mathbf{P}| \tag{4.21}$$

where J is the total angular momentum, P is the three-component momentum.

a) Invariance under parity:

Under space-inversion J does not change sign but P does change sign. Consequently, a state with helicity λ is transformed into a state with helicity $-\lambda$ under parity operation. It can be shown[7),]

$$P|J\ M\ \lambda_1\ \lambda_2 > \; = \; \eta_1\ \eta_2(-1)^{J-s_1-s_2}\ |J\ M;\ -\lambda_1\ -\lambda_2 > \tag{4.22}$$

where η_1 and η_2 are the intrinsic parities of the two particles of spin s_1 and s_2 and P stands for the parity operator. Now, the invariance of the s-matrix under parity implies,

$$P\ S\ P^{-1} = S \tag{4.23}$$

Using (4.22) and (4.23) we obtain from (4.17) the following relation between matrix elements with states of opposite parities:

$$< -\lambda_3, -\lambda_4 | h^J | -\lambda_1, -\lambda_2 > = \frac{\eta_3 \; \eta_4}{\eta_1 \; \eta_2} (-1)^{s_3+s_4-s_1-s_2} < \lambda_3, \lambda_4 | h^J | \lambda_1, \lambda_2 > \tag{4.24}$$

b) Time reversal invariance:

Under time reversal both J and P change sign, so the helicity λ (4.21) does not change. By applying time reversal, T, to a state we obtain a new state with the same angular momentum and helicities but with an opposite eigenvalue for J_z. Thus, we obtain

$$T \, | J \; M; \; \lambda_1 \; \lambda_2 > = (-1)^{J-M} \, | \, J, -M; \; \lambda_1 \; \lambda_2 > \tag{4.25}$$

Now, the invariance of the s-matrix under time reversal implies,

$$T \; S \; T^{-1} = S^{+} \tag{4.26}$$

Use of (4.25) and (4.26) in (4.17) then gives the following relation of the helicity amplitudes:

$$< \lambda_3 \; \lambda_4 | h^J | \lambda_1 \; \lambda_2 > = < \lambda_1 \; \lambda_2 | h^J | \lambda_3 \; \lambda_4 > \tag{4.27}$$

In this connection, we mention that when we consider a process like B + P $\longleftrightarrow$ B + V, with a pseudoscalar meson at one state and a vector meson at the other, the helicity amplitudes corresponding to the time reversed process, are to be multiplied by (-1). This factor arises due to the spin of the vector meson[59].

Let us now construct the parity states which contribute to $J = 1/2$ and $J = 3/2$ states. These are obtained by taking a linear combination of the helicity states. The parity of such a combination can be given by,

$$P\left\{ |J\, M;\, \lambda_1\, \lambda_2 > \pm\, |J,M;\, -\lambda_1,-\lambda_2 > \right\} = \eta_1\, \eta_2 (-1)^{J-S_1-S_2} (\pm) \left\{ |J\, M;\, \lambda_1\, \lambda_2 > \pm\, |J,M;\, -\lambda_1,-\lambda_2 > \right\} \quad (4.28)$$

Considering the spins of the scattering particles concerned, it can be shown[7] that the above combination with one sign contributes to $J^P = 1/2^+$ and the other to the state $J^P = 3/2^+$. We shall, however, express these parity states in terms of the states having a definite orbital angular momentum and total channel spins. For that purpose we make use of the following relation[7],

$$< JM;\, LS|JM;\lambda_1\, \lambda_2 > = \left(\frac{2L+1}{2J+1}\right)^{\frac{1}{2}} C(L\,S\,J;\, 0\lambda)C(s_1 s_2 s;\, \lambda_1,\, -\lambda_2) \quad (4.29)$$

where $\lambda = \lambda_1 - \lambda_2$ and C's are the related Clebsch-Gordon coefficients. From (4.29) we can obtain,

$$|JM;\, \lambda_1\, \lambda_2 > = \sum_{L,S} \left(\frac{2L+1}{2J+1}\right)^{\frac{1}{2}} C^{LSJ}_{0\lambda\lambda} \cdot C^{s_1 s_2 s}_{\lambda_1,-\lambda_2}\, \lambda|JM;\, LS > \quad (4.30)$$

In (4.30) we can drop the symbol M because the conservation of M is automatically taken care of. In what follows we express the above mentioned parity states in terms of the states with a definite orbital angular momentum and channel spins for the relevant vertices from which all the processes we consider can be obtained. The related orbital angular momenta are obtained by considering the $J^P = 1/2^+$ or $J^P = 3/2^+$ states in the intermediate states in the direct channels (Fig. B.4 - B.15). We shall write $J^P = 1/2^+$ state as B's and $J^P = 3/2^+$ as D's. Using (4.28) and (4.30) we obtain the following[60]:

I. For the external particles PB the states are:

$$B_1 = \frac{1}{\sqrt{2}}\left[\,|1/2;\,1/2> - \,|\,1/2;\,-1/2>\right] = -\,|\,1/2;\,P_{1/2}>$$

$$D_1 = \frac{1}{\sqrt{2}}\left[\,|3/2;\,1/2> + \,|\,3/2;\,-1/2>\right] = \,|\,3/2;\,P_{1/2}> \tag{4.31}$$

II. For the external particles PD the states are:

$$B_1 = \frac{1}{\sqrt{2}}\left[\,|\,1/2;\,1/2> + \,|\,1/2;\,-1/2>\right] = -\,|\,1/2;\,P_{3/2}>$$

$$D_1 = \frac{1}{\sqrt{2}}\left[\,|\,3/2;\,1/2> - \,|\,3/2;\,-1/2>\right] = -\frac{1}{\sqrt{10}}\,|3/2;P_{3/2}> + \frac{3}{\sqrt{10}}\,|\,3/2;\,F_{3/2}>$$

$$D_2 = \frac{1}{\sqrt{2}}\left[\ |\ 3/2;\ 3/2 > - \ |\ 3/2;\ -3/2 > \right] = -\frac{3}{\sqrt{10}}\ |3/2;\ P_{3/2} >$$

$$-\frac{1}{\sqrt{10}}\ |\ 3/2;\ F_{3/2} > \qquad (4.32)$$

III. For the external particles VB the states are:

$$B_1 = \frac{1}{\sqrt{2}}\left[\ |\ 1/2;\ 1/2,1 > + \ |\ 1/2;\ -1/2,\ -1 > \right] = \frac{\sqrt{2}}{\sqrt{3}}\ |1/2;\ P_{1/2} >$$

$$-\frac{1}{\sqrt{3}}\ |\ 1/2;\ P_{3/2} >$$

$$B_2 = \frac{1}{\sqrt{2}}\left[\ |\ 1/2;\ 1/2,0 > + \ |\ 1/2;\ -1/2,\ 0 > \right] = -\frac{1}{\sqrt{3}}\ |\ 1/2;\ P_{1/2} >$$

$$-\frac{\sqrt{2}}{\sqrt{3}}\ |\ 1/2;\ P_{3/2} >$$

$$D_1 = \frac{1}{\sqrt{2}}\left[\ |\ 3/2;\ 1/2,1 > - \ |\ 3/2;\ -1/2,-1 > \right] = \frac{\sqrt{2}}{\sqrt{3}}\ |\ 3/2;\ P_{1/2} >$$

$$+\frac{1}{\sqrt{30}}\ |\ 3/2;\ P_{3/2} > - \frac{3}{\sqrt{30}}\ |\ 3/2;\ F_{3/2} >$$

$$D_2 = \frac{1}{\sqrt{2}}\left[\ |\ 3/2;\ 1/2,0 > - \ |\ 3/2;\ -1/2,0 > \right] = \frac{1}{\sqrt{3}}\ |\ 3/2;\ P_{1/2} >$$

$$-\frac{2}{\sqrt{60}}\ |3/2;\ P_{3/2} > + \frac{6}{\sqrt{60}}\ |\ 3/2;\ F_{3/2} >$$

$$D_8 = \frac{1}{\sqrt{2}}\left[\,|\,3/2;\ 1/2,-1> - \,|\,3/2;\ -1/2,1>\right] = -\frac{3}{\sqrt{10}}\,|3/2;\ P_{3/2}>$$

$$-\frac{1}{\sqrt{10}}\,|\,3/2;\ F_{3/2}>$$

(4.33)

IV. For the external particles V D the states are:

$$B_1 = \frac{1}{\sqrt{2}}\left[\,|\,1/2;\ 1/2,1> - \,|\,1/2;\ -1/2,-1>\right] = \frac{1}{\sqrt{6}}\,|\,1/2;\ P_{1/2}>$$

$$-\frac{4}{\sqrt{30}}\,|\,1/2;\ P_{3/2}> + \frac{3}{\sqrt{30}}\,|\,1/2;\ F_{5/2}>$$

$$B_2 = \frac{1}{\sqrt{2}}\left[\,|\,1/2;\ 1/2,0> - \,|\,1/2;\ -1/2,0>\right] = \frac{1}{\sqrt{3}}\,|\,1/2;\ P_{1/2}>$$

$$-\frac{2}{\sqrt{60}}\,|\,1/2;\ P_{3/2}> - \frac{6}{\sqrt{60}}\,|\,1/2;\ F_{5/2}>$$

$$B_4 = \frac{1}{\sqrt{2}}\left[\,|\,1/2;\ 3/2,1> - \,|\,1/2;\ -3/2,-1>\right] = \frac{1}{\sqrt{2}}\,|\,1/2;\ P_{1/2}>$$

$$-\frac{2}{\sqrt{10}}\,|\,1/2;\ P_{3/2}> - \frac{1}{\sqrt{10}}\,|\,1/2;\ F_{5/2}>$$

$$D_1 = \frac{1}{\sqrt{2}}\left[\,|\,3/2;\ 1/2,1> + \,|\,3/2;\ -1/2,-1>\right] = \frac{1}{\sqrt{6}}\,|\,3/2;\ P_{1/2}>$$

$$+\frac{2}{5\sqrt{3}}\,|\,3/2;\ P_{3/2}> - \frac{3}{5\sqrt{2}}\,|\,3/2;\ P_{5/2}>$$

$$- \frac{2\sqrt{3}}{5} \mid 3/2;\ F_{3/2} \rangle + \frac{\sqrt{3}}{5} \mid 3/2;\ F_{5/2} \rangle$$

$$D_2 = \frac{1}{\sqrt{2}} \Big[\mid 3/2;\ 1/2,0 \rangle + \mid 3/2;\ -1/2,0 \rangle \Big] = - \frac{1}{\sqrt{3}} \mid 3/2;\ P_{1/2} \rangle$$

$$- \frac{1}{5\sqrt{6}} \mid 3/2;\ P_{3/2} \rangle - \frac{3}{5} \mid 3/2;\ P_{5/2} \rangle + \frac{3}{5\sqrt{6}} \mid 3/2;\ F_{3/2} \rangle$$

$$+ \frac{6}{5\sqrt{6}} \mid 3/2;\ F_{5/2} \rangle$$

$$D_3 = \frac{1}{\sqrt{2}} \Big[\mid 3/2;\ 1/2,-1 \rangle + \mid 3/2;\ -1/2,1 \rangle \Big] = \frac{3}{5} \mid 3/2;\ P_{3/2} \rangle$$

$$- \frac{6}{5\sqrt{6}} \mid 3/2;\ P_{5/2} \rangle + \frac{1}{5} \mid 3/2;\ F_{3/2} \rangle - \frac{3}{5} \mid 3/2;\ F_{5/2} \rangle$$

$$D_4 = \frac{1}{\sqrt{2}} \Big[\mid 3/2;\ 3/2,1 \rangle + \mid 3/2;\ -3/2,-1 \rangle \Big] = \frac{1}{\sqrt{2}} \mid 3/2;\ P_{1/2} \rangle$$

$$- \frac{1}{5} \mid 3/2;\ P_{3/2} \rangle - \frac{3}{5\sqrt{6}} \mid 3/2;\ P_{5/2} \rangle + \frac{3}{5} \mid 3/2;\ F_{3/2} \rangle$$

$$+ \frac{1}{5} \mid 3/2;\ F_{5/2} \rangle$$

$$D_5 = \frac{1}{\sqrt{2}} \Big[\mid 3/2;\ 3/2,0 \rangle + \mid 3/2;\ -3/2,0 \rangle \Big] = - \frac{9}{5\sqrt{6}} \mid 3/2;\ P_{3/2} \rangle$$

$$- \frac{2}{5} \mid 3/2;\ P_{5/2} > \ - \ \frac{3}{5\sqrt{6}} \mid 3/2;\ F_{3/2} >$$

$$- \frac{6}{5\sqrt{6}} \mid 3/2;\ F_{5/2} > \qquad (4.34)$$

In the above, P_s, F_s denote the orbital angular momentum in accordance with the convention of the atomic theory and the associated suffices denote the corresponding channel spins. Inverting the above relations we obtain the following:

i. For the external particles PB we have,

$$\mid 1/2;\ P_{1/2} > \ = \ - B_1$$

$$\mid 3/2;\ P_{1/2} > \ = \ D_1 \qquad (4.31')$$

ii. For the external particles PD we have,

$$\mid 1/2;\ P_{3/2} > \ = \ - B_1$$

$$\mid 3/2;\ P_{3/2} > \ = \ - \frac{1}{\sqrt{10}} (D_1 + 3D_2) \qquad (4.32')$$

$$\mid 3/2;\ F_{3/2} > \ = \ \frac{1}{\sqrt{10}} (3D_1 - D_2)$$

iii. For the external particles VB we have,

$$| 1/2;\ P_{1/2} > \ = \ \frac{1}{\sqrt{3}} (\sqrt{2}\ B_1 - B_2)$$

$$| 1/2;\ P_{3/2} > \ = \ -\frac{1}{\sqrt{3}} (B_1 + \sqrt{2}\ B_2)$$

$$| 3/2;\ P_{1/2} > \ = \ \frac{1}{\sqrt{3}} (\sqrt{2}\ D_1 + D_2) \qquad (4.33')$$

$$| 3/2;\ P_{3/2} > \ = \ \frac{1}{\sqrt{30}} (D_1 - \sqrt{2}\ D_2 - 3\sqrt{3}\ D_3)$$

$$| 3/2;\ F_{3/2} > \ = \ -\frac{1}{\sqrt{10}} (\sqrt{3}\ D_1 - \sqrt{6}\ D_2 + D_3)$$

iv. For the external particles VD we have,

$$| 1/2;\ P_{1/2} > \ = \ \frac{1}{\sqrt{6}} (B_1 + \sqrt{2}\ B_2 - \sqrt{3}\ B_4)$$

$$| 1/2;\ P_{3/2} > \ = \ -\frac{1}{\sqrt{15}} (2\sqrt{2}\ B_1 + B_2 + \sqrt{6}\ B_4)$$

$$| 1/2;\ F_{5/2} > \ = \ \frac{1}{\sqrt{10}} (\sqrt{3}\ B_1 - \sqrt{6}\ B_2 - B_4)$$

$$| 3/2;\ P_{1/2} > \ = \ \frac{1}{\sqrt{6}} (D_1 + \sqrt{2}\ D_2 + \sqrt{3}\ D_4)$$

$$| 3/2; P_{3/2} > = \frac{1}{10\sqrt{3}} (4D_1 - \sqrt{2}\, D_2 + 6\sqrt{3}\, D_3 - 2\sqrt{3}\, D_4 - 9\sqrt{2}\, D_5)$$

$$| 3/2; P_{5/2} > = -\frac{1}{5\sqrt{2}} (3D_1 + 3\sqrt{2}\, D_2 + 2\sqrt{3}\, D_3 + \sqrt{3}\, D_4 + 2\sqrt{2}\, D_5)$$

$$| 3/2; F_{3/2} > = -\frac{1}{10} (4\sqrt{3}\, D_1 - \sqrt{6}\, D_2 - 2\, D_3 - 6\, D_4 + \sqrt{6}\, D_5)$$

$$| 3/2; F_{5/2} > = \frac{1}{5} (\sqrt{3}\, D_1 + \sqrt{6}\, D_2 - 3\, D_3 + D_4 - \sqrt{6}\, D_5)$$

(4.34')

From these relations one can express the partial wave amplitudes in terms of the desired parity amplitudes. From the four-types of vertices considered above, we can obtain all the partial wave amplitudes corresponding to sixteen processes by taking appropriate combinations of these vertices. There are, however, only ten independent processes; others being related through time reversal invariance property of the scattering matrix s.

3. Calculations of the Born Terms

a. The Direct Pole Terms:

We have shown in the last section that both P and F waves occur in some of the scattering processes we consider. It is clear from the relations (4.31') - (4.34') that F-waves occur in those processes which involve the baryon decouplets and vector mesons as the scattering particles. But it is known from the experiments that N^*_{33} resonance occurs as πN resonance in P-wave. Therefore, we make an investigation to see how much contribution these F-waves have to the $J^P = 1/2^+$ and $J^P = 3/2^+$ poles with which we are concerned. With this end in view, we calculate the direct $J = 1/2$ and $J = 3/2$ Born terms using the U(6,6) vertices (1.87) - (1.92). We evaluate the Feynman amplitudes $< \lambda_3 \, \lambda_4 | F | \lambda_1 \, \lambda_2 >$ which we use in order to obtain the helicity amplitudes from the equation (4.20). The processes which we have to consider have been shown (appendix B) by the Fig. B.4 - Fig. B.15. We shall, however, ignore the SU(3) symmetry properties of the multiplets and consequently the processes we consider are shown in Fig. 4.2

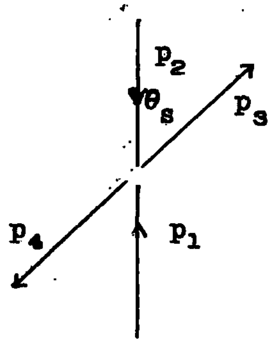


Figure 4.2a

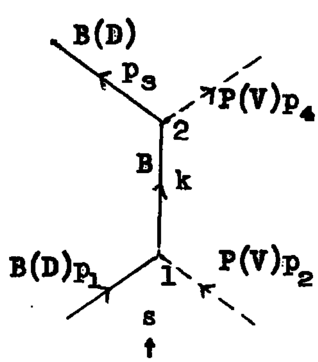


Figure 4.2b

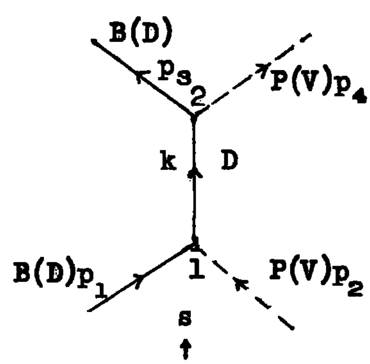


Figure 4.2c

In order to obtain the pole contributions we evaluate the related vertex functions by putting all the particles on the respective mass shells. As the three-component momentum of the intermediate particle is zero in the c.m. system, the helicity states of the intermediate particles can be taken as being normalised in the positive z-direction. Further, we take the momenta of all the particles associated with the diagrams fig. 4.2b and Fig. 4.2c in the xz-plane. In particular, we take the momenta of the particles in the initial state along the z-axis and those of the particles in the final state along a direction making an angle θ with the z-axis, θ being the angle of scattering. Now, since the propagator of the intermediate particles (spin 1/2 or spin 3/2 particles in this case) can be expressed as a sum over the helicity states of the particles concerned, the Feynman amplitude corresponding to any of the processes shown in the above can be expressed in the form[61)]

$$< \lambda_3 \, \lambda_4 \, \underline{p}_3 \, \underline{p}_4 | F | \underline{p}_1 \, \underline{p}_2 \, \lambda_1 \, \lambda_2 > \; = \; F^2_{\lambda\upsilon}(\theta) \, \frac{1}{s - m^2} \, F^1_{\upsilon\mu}(0) \tag{4.35}$$

where $F^2_{\lambda\upsilon}(\theta)$, $F^1_{\upsilon\mu}(0)$ are the vertex functions obtained by putting the particles at the vertices 2 and 1 respectively on the respective mass shells. These can be expressed in the following forms:

$$F^2_{\lambda\upsilon}(\theta) \; = \; \sqrt{2m} < p_3 \, p_4 \, \lambda_3 \, \lambda_4 | J_\ell | k\upsilon > \tag{4.36a}$$

$$F^{1}_{\upsilon\mu}(0) = \sqrt{2}m < k\upsilon|J_{\eta}|p_1\ p_2\ \lambda_1\ \lambda_2 > \qquad (4.36b)$$

where $\lambda = \lambda_3 - \lambda_4$; $\mu = \lambda_1 - \lambda$ and υ is the helicity of the intermediate particle with U(6,6) degenerate mass m. Thus, the four-component momentum k of the intermediate particles can be given in the c.m. system by,

$$k = (m, 0) \qquad (4.36c)$$

In (4.36) J_{ℓ} is the U(6,6) form factors. Taking various terms from (1.87) - (1.92) we can obtain all the pole terms corresponding to the sixteen processes discussed in the last section. We shall, however, discuss two examples and write down the results for the rest. For that we need the helicity states of the external particles as well as those of the intermediate J = 1/2 or J = 3/2 particles. The helicity states of the external particles have been given in appendix C so we have to obtain the helicity states of the intermediate J = 1/2 and J = 3/2 particles.

Corresponding to the spin 1/2 particle in the intermediate state the helicity states, using (4.36c), are as follows:

$$u(k) = \frac{1}{\sqrt{2m(E+m)}} \begin{vmatrix} E+m \\ \sigma \cdot p \end{vmatrix} \chi_{\upsilon} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \chi_{\upsilon} \qquad (4.37)$$

For the spin 3/2 particle in the intermediate, the helicity states are taken (given by C.15) as the vector sums of (4.37) and the following:

$$\epsilon^{(\pm)} = \left\{ 0, \xi^{(\pm)} \right\}; \qquad \epsilon^{(0)} = \left\{ 0, \xi^{(0)} \right\} \qquad (4.38)$$

$$\text{where,} \quad \xi^{(+)} = \frac{1}{\sqrt{2}} \begin{vmatrix} -1 \\ -i \\ 0 \end{vmatrix} \qquad \xi^{(0)} = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} \qquad \xi^{(-)} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ -i \\ 0 \end{vmatrix}$$

In order to show how these pole terms are obtained we consider the process $P + B \rightarrow P + B$ with $J = 1/2$ and $J = 3/2$ in the intermediate state.

I. For the $J = 1/2$ particle particle in the intermediate state:

In this case we consider the $U(6,6)$ vertices given by equation (1.87). The two vertex functions corresponding to all the particles on the respective mass-shells are obtained as follows:

$$F^{2}_{\lambda\upsilon}(\theta) = \frac{P^2}{4m^2} \left(1 + \frac{2m}{\mu} \right) \sqrt{2m}\, \bar{u}_{\lambda}\, (p_3)\, r_5\, u_{\upsilon}(k) \qquad (4.39)$$

$$F^{1}_{\upsilon\mu}(0) = \frac{P^2}{4m^2} \left(1 + \frac{2m}{\mu} \right) \sqrt{2m}\, \bar{u}_{\upsilon\mu}(k)\, r_5\, u(p_1)$$

Using (4.37), (C.2) and (C.5) we evaluate (4.39) by putting all the particles on the mass-shells then substitute them into (4.35) which gives the following pole terms:

$$\langle \lambda | F | \mu \rangle = C \times \begin{array}{c|cc} {}_{\lambda} \diagdown {}^{\mu} & 1/2 & -1/2 \\ \hline \frac{1}{2} & \cos\theta/2 & -\sin\theta/2 \\ -\frac{1}{2} & \sin\theta/2 & \cos\theta/2 \end{array} \tag{4.40}$$

$$\text{where, } C = -\left(1 + \frac{2m}{\mu}\right)^2 \frac{(4m^2 - \mu^2)^2}{32\, m^5} \frac{1}{(s - m^2)} .$$

II The J = 3/2 particle in the intermediate state:

In this case, the U(6,6) vertices we have to consider are given by equation (1.89). Then the two related vertex functions are as follows:

$$F^{2}_{\lambda\upsilon}(\theta) = \frac{1}{m}\left(1 + \frac{2m}{\mu}\right) \sqrt{2m}\, \bar{u}_{\lambda}(p_s)\, u(k) \otimes \epsilon_k \cdot q^k$$

$$F^{1}_{\upsilon\mu}(0) = \frac{1}{m}\left(1 + \frac{2m}{\mu}\right) \sqrt{2m}\, q^k\, \epsilon^{+}_{k} \otimes \bar{u}(k) \cdot u(p_1) \tag{4.41}$$

The above functions are evaluated by using (C.1), (C.5) and (C.15) together with (4.37) and (4.38). After having evaluated these functions by putting all the particles on the appropriate mass shells we substitute them into (4.35) which gives the following pole terms for $J = 3/2$ in the intermediate state:

$< \lambda|F|\mu > =$

$C \times$

$\lambda \backslash \mu$	$1/2$	$-1/2$
$\frac{1}{2}$	$\frac{4}{3}(1 - 3\cos\theta)\cos\theta/2;$	$-\frac{4}{3}(1 + 3\cos\theta)\sin\theta/2$
$-\frac{1}{2}$	$\frac{4}{3}(1 + 3\cos\theta)\sin\theta/2;$	$\frac{4}{3}(1 - 3\cos\theta)\cos\theta/2$

(4.42)

Following the above procedures and using the various U(6,6) vertices given by equations (1.87) - (1.92) we can evaluate all the pole terms corresponding to $J = 1/2$ and $J = 3/2$ states for the sixteen processes discussed earlier in this section. Apart from some kinematic factors, it should be noted, these pole terms contain functions of $\cos\theta$ or $\sin\theta$ in such a manner that these functions are the same as the appropriate elements of $d^J(\theta)$ matrices given by equations (C.4) and (C.16). We can then obtain the related helicity amplitudes after performing the integral on the right hand side of (4.20) by using the orthogonality relation (C.26) of the reduced matrices $d^J(\theta)$. Having obtained the helicity amplitudes for the total angular momentum $J = 1/2$ and $J = 3/2$ we obtain the parity amplitudes discussed in the last section, by evaluating the appropriate combinations, given by the equations (4.31) - (4.34). We present these values of the parity amplitudes in tables separately for J = 1/2 and J = 3/2 poles. We mention only those processes which involve pseudoscalar (vector) octets at either the initial or final states. As the space-time properties of the pseudoscalar (vector) singlet are the same as those of the corresponding octet, we need not have to consider them separately.

Table 4.1 : Direct pole-terms for J = 3/2 states,

		PB	PD		VB			VD
		D_1	D_1	D_2	D_1	D_2	D_3	D_1 D_2 D_3 D_4 D_5
PB	D_1	$-\frac{4}{3}$	$\frac{\sqrt{2}}{\sqrt{3}}\,i$	$\sqrt{6}\,i$	$\frac{2\sqrt{2}}{3}\,i$	0	$-\frac{2\sqrt{2}}{\sqrt{3}}\,i,$	$\frac{4}{\sqrt{3}}, \sqrt{6}, 2, 2, \sqrt{6}$
PD	D_1	$\frac{\sqrt{2}}{\sqrt{3}}\,i$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{\sqrt{3}}$	0	-1	$-\sqrt{2}\,i, -\frac{3}{2}\,i, -\frac{\sqrt{3}}{\sqrt{2}}\,i, -\frac{\sqrt{3}}{\sqrt{2}}\,i, -\frac{3}{2}\,i$
	D_2	$\sqrt{6}\,i$	$\frac{3}{2}$	$\frac{9}{2}$	$\sqrt{3}$	0	-3	$-3\sqrt{2}\,i, -\frac{9}{2}\,i, -\frac{3\sqrt{3}}{\sqrt{2}}\,i, -\frac{3\sqrt{3}}{\sqrt{2}}\,i, -\frac{9}{2}\,i$

Table 4.1 (contd.)

		PB	PD		VB			VD
VB	D_1	$-\frac{2\sqrt{2}}{3}\, i$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$-\frac{2}{3}$	0	$\frac{2}{\sqrt{3}}$	$\frac{2\sqrt{2}}{\sqrt{3}}\, i,\ \sqrt{3}\, i,\ \sqrt{2}\, i,\ \sqrt{2}\, i,\ \sqrt{3}\, i$
	D_2	0	0	0	0	0	0	0 0 0 0 0
	D_3	$\frac{2\sqrt{2}}{\sqrt{3}}\, i$	1	3	$\frac{2}{\sqrt{3}}$	0	-2	$-2\sqrt{2}\, i,\ -3\, i,\ -\sqrt{6}\, i,\ -\sqrt{6}\, i,\ -3\, i$
VD	D_1	$-\frac{4}{\sqrt{3}}$	$\sqrt{2}\, i$	$3\sqrt{2}\, i$	$\frac{2\sqrt{3}}{\sqrt{3}}\, i$	0	$-2\sqrt{2}\, i$	$4,\ 3\sqrt{2},\ 2\sqrt{3},\ 2\sqrt{3},\ 3\sqrt{2}$
	D_2	$-\sqrt{6}$	$\frac{3}{2}\, i$	$\frac{9}{2}\, i$	$\sqrt{3}\, i$	0	$-3\, i$	$3\sqrt{2},\ \frac{9}{2},\ \frac{3\sqrt{3}}{\sqrt{2}},\ \frac{3\sqrt{3}}{\sqrt{2}},\ \frac{9}{2}$
	D_3	-2	$\frac{\sqrt{3}}{\sqrt{2}}\, i$	$\frac{3\sqrt{3}}{\sqrt{2}}\, i$	$\sqrt{2}\, i$	0	$-\sqrt{6}\, i$	$2\sqrt{3},\ \frac{3\sqrt{3}}{\sqrt{2}},\ 3,\ 3,\ \frac{3\sqrt{3}}{\sqrt{2}}$
	D_4	-2	$\frac{\sqrt{3}}{\sqrt{2}}\, i$	$\frac{3\sqrt{3}}{\sqrt{2}}\, i$	$\sqrt{2}\, i$	0	$-\sqrt{6}\, i$	$2\sqrt{3},\ \frac{3\sqrt{3}}{\sqrt{2}},\ 3,\ 3,\ \frac{3\sqrt{3}}{\sqrt{2}}$
	D_5	$-\sqrt{6}$	$\frac{3}{2}\, i$	$\frac{9}{2}\, i$	$\sqrt{3}\, i$	0	$-3\, i$	$3\sqrt{2},\ \frac{9}{2},\ \frac{3\sqrt{3}}{\sqrt{2}},\ \frac{3\sqrt{3}}{\sqrt{2}},\ \frac{9}{2}$

The elements in both table 4.1 and table 4.2 are to be multiplied by the factor,

$$A = \frac{1}{32m^4}\left(4m^2 - \mu^2\right)^2\left(1 + \frac{2m}{\mu}\right)^2 \frac{4\pi g^2}{s - m^2}.$$

Table 4.2: Direct pole terms for J = 1/2 states,

		PB	PD	VB		VD		
		B_1	B_1	B_1	B_2	B_1	B_2	B_4
PB	B_1	1	$-\frac{2\sqrt{2}}{\sqrt{3}}\, i$	$-\sqrt{2}\, i$	$-i$	$\frac{2}{\sqrt{3}}$	0	2
PD	B_1	$-\frac{2\sqrt{2}}{\sqrt{3}}\, i$	$-\frac{8}{3}$	$-\frac{4}{\sqrt{3}}$	$-\frac{2\sqrt{2}}{\sqrt{3}}$	$-\frac{4\sqrt{2}}{3}\, i$	0	$-\frac{4\sqrt{2}}{\sqrt{3}}\, i$
VB	B_1	$\sqrt{2}\, i$	$\frac{4}{\sqrt{3}}$	2	$\sqrt{2}$	$\frac{2\sqrt{2}}{\sqrt{3}}\, i$	0	$2\sqrt{2}\, i$
	B_2	1	$\frac{2\sqrt{2}}{\sqrt{3}}$	$\sqrt{2}$	2	$\frac{2}{\sqrt{3}}\, i$	0	$2\, i$
VD	B_1	$-\frac{2}{\sqrt{3}}$	$\frac{4\sqrt{2}}{3}\, i$	$\frac{2\sqrt{2}}{\sqrt{3}}\, i$	$\frac{2}{\sqrt{3}}\, i$	$-\frac{4}{3}$	0	$-\frac{4}{\sqrt{3}}$
	B_2	0	0	0	0	0	0	0
	B_4	-2	$\frac{4\sqrt{2}}{\sqrt{3}}\, i$	$2\sqrt{2}\, i$	$2\, i$	$\frac{4}{\sqrt{3}}$	0	-4

Taking the elements from either the table 4.1 or the table 4.2, the contributions of the F-waves corresponding to any of the related processes discussed earlier in this section, to either $J^P = 3/2^+$ or $J^P = 1/2^+$ state can be evaluated by using the equations (4.31') - (4.34'). It has been found that the F-wave contributions corresponding to the above pole terms vanish. It is, therefore, very likely that the effects of the F-waves in the mass-splittings between baryon octet and decouplet may be negligible. If the approximations that have been used are justified, we may neglect the F-waves in the N/D calculations.

b. The Baryon Octet Exchange Born Terms:

We have already mentioned in the first section of this chapter that we shall assume that the forces responsible for the binding of either octet with $J^P = 1/2^+$ or decouplet with $J^P = 3/2^+$ arise predominantly from the exchanges of both baryon octet and decouplet in the relevant crossed channels. The relevant SU(3) coefficients of these exchange poles are to be obtained from the corresponding SU(3) direct-channel-pole-coefficients given in tables B.1 and B.2 (appendix B) using the related SU(3) crossing matrices given by equations (4.6), (4.9) and (4.10). In this and the next section, the methods for the evaluations of the space-time parts of these exchange Born terms will be discussed. In this connection, it is assumed further that only the long range forces are important in the mass-splittings between baryon octet and decouplet. In other words, we assume that only that part of the left-hand cut which is very near to the physical cut is important and consequently confine ourselves to the evaluations only of the single-particle exchange diagrams. The exchange Born terms required to be evaluated correspond to the exchange diagrams given by Fig. 4.3 - Fig. 4.9.

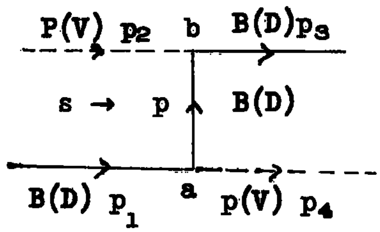


Fig. 4.3

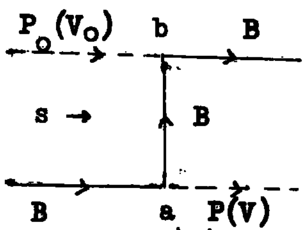


Fig. 4.4

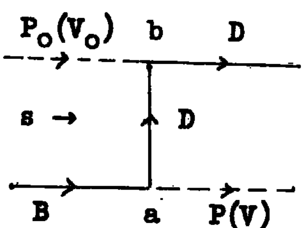


Fig. 4.5

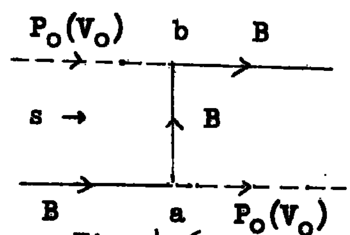


Fig. 4.6

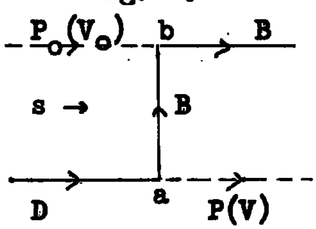


Fig. 4.7

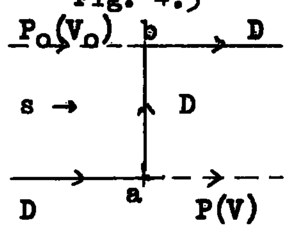


Fig. 4.8

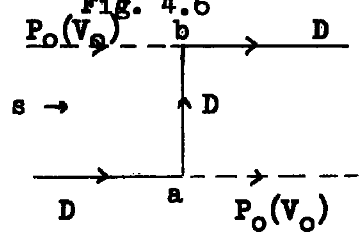


Fig. 4.9.

It is well known that the exchange Born terms can be expressed in terms of the direct Born terms in the crossed channel (u-channel) using the crossing relations. In the 'pole approximation' the direct Born terms in the u-channel are calculated with all the energy-dependence (u-dependence) at the pole position $u = m^2$, where m is the mass of the exchanged baryon and in our particular case is taken to be the U(6,6) degenerate mass. It turns out that the threshold conditions for the contributions of the exchange diagrams calculated at the 'pole approximation' to the partial wave amplitudes are not satisfied, so one must, in fact, keep some explicit dependence on the energy of the crossed channel. In order to do this in a unique way it is necessary to make some distinction between 'kinematic factors' - in which the energy dependence should be maintained - and 'dynamic factors' in which the energy dependence should be put on the mass-shell.

In order to make clear the above prescription for how the energy dependence factors should be put on the mass-shell let us consider a general scattering process for which the invariant scattering amplitudes can be written in the form,

$$T = \sum_{i=1}^{n} Y_i A_i \qquad (4.43)$$

where A_i are the scalar invariants and Y_i are the factors that arise due to the spins of the external particles and we shall call them the spin kinematic factors. It is these kinematic factors whose energy-dependence should be maintained according to our prescription[62]. On the other hand, the functions associated with the invariant amplitudes A_i can be put on the mass-shell when one is interested in the pole approximation. If the above procedure is followed, then it has been found that the partial wave amplitudes obtained from these Born terms will have correct threshold behaviours. To follow the above technique of calculating the exchange Born terms, one first has to find out the decomposition of the scattering amplitude into the scalar invariant amplitudes. Then the direct pole term obtained by applying Feynman rules has to be analysed in order to find out its contribution to each of the scalar invariant amplitudes A_i (s t u). Further, one has to sort out all the crossing relations for the scalar invariants A_i (s t u) and the 'spin kinematics' Y_i. This method, however, becomes very complicated when one has to consider a process like VD → VD, where both the scattering particles have spin greater than 1/2. In this particular case, there are 30 scalar invariant amplitudes A_i (s t u) and consequently the decomposition (4.43) is itself a major problem.

The problem of decomposition into scalar amplitudes may be avoided by calculating the Born terms between states of definite helicities. A typical Feynman amplitude calculated between states of total final helicity μ and total initial helicity λ would look like $g^2F(s,u)d^s_{\lambda\mu}/(u - m^2)$, where $d^s_{\lambda\mu}$ is the relevant rotation matrix with spin, $s = \max (|\lambda|, |\mu|)$. The function F(s u) is the sum of the terms, each of which has a part from 'spin kinematics' and a part from the dynamic factors. If we are able to separate out the spin and the dynamic parts, our mass-shell prescription can be applied. Assuming that this separation is possible one may proceed in two different ways. Firstly, the helicity amplitudes may be taken as direct pole terms in the u-channel. Then, making use of the helicity crossing matrices[63)] one can obtain the exchange Born terms in the s-channel. This, however, involves some tedious manipulations with the elements of the helicity crossing matrices. Secondly, one can follow the more direct procedure. That is, we calculate the exchange Born terms in the s-channel directly with our mass-shell prescription. That is what, in short, we propose to do. In this section, we discuss how we obtain the Born terms for the exchange of octet with $J^P = 1/2^+$ in the u-channel and the next section will be devoted to the evaluations of the Born terms corresponding to the exchange of decouplet with $J^P = 3/2^+$. Since nothing has to be put on the mass-shell for the baryon octet exchange, the method of vertex functions which we have used in evaluating the direct-channel poles in the last section can also be used in this case. The evaluations of the Born terms corresponding to the exchange decouplet with $J^P = 3/2^+$, on the other hand, require many energy dependence factors to be put on the mass-shell and consequently we have to tackle the problem in a slightly different manner.

The processes for which we require the baryon octet, B, exchange Born terms are shown in Fig. 4.3, where the incoming baryon and meson four-momenta are p_1 and p_2 respectively and the outgoing baryon and meson momenta are p_3 and p_4 respectively. The related three Lorentz invariant quantities s, t and u have been given by equation (4.13) in the c.m. system. Let p denote the four-component momentum of the internal baryon. We assume that the three-component momenta of all the external as well as the internal particles are in the xz-plane so that the azimuthal angle ϕ is zero. Now, we take three-component momentum of the incoming baryon in the positive z-direction and that of the incoming meson along the negative z-direction. Considering θ as the scattering angle (shown by Fig. 4.1b) we take the three-component momentum of the outgoing baryon in a direction making an angle θ with the positive z-direction and that of the outgoing meson in a direction making an angle $\pi + \theta$ with the positive z-direction. As a consequence of the above convention, the helicity states of the incoming and outgoing baryon octet are respectively given by equations (C.2) and (C.5) and those of the incoming and outgoing vector mesons are given by equations (C.13) and (C.14). For the incoming J = 3/2 particle, the corresponding helicity states are obtained from the equation (C.15) by using (C.2) and (C.10) along with (C.9) and for J = 3/2 outgoing baryon we use (C.5) and (C.12) to obtain the corresponding helicity states.

From Fig. 4.1b we then obtain the following:

$$p_1 = (E, 0, 0, k) \; ; \quad p_2 = (w, 0, 0, -k)$$

$$p_3 = (E, k \sin\theta, 0, k \cos\theta); \quad P_4 = (w, -k \sin\theta, 0, -k \cos\theta)$$

$$E = \sqrt{k^2 + m^2} \; ; \quad w = \sqrt{k^2 + \mu^2} \qquad (4.45)$$

From (4.45) we readily obtain,

$$p = (E', k' \sin\alpha, 0, k' \cos\alpha)$$

$$E' = E - w \; ; \quad k' = 2k \cos\alpha, \quad \text{with } \alpha = \theta/2 \qquad (4.46a)$$

We also have, $u = p^2 = E'^2 - k'^2$ so that when we put the exchanged baryon on the mass-shell we have,

$$k'^2 = E'^2 - m^2 \qquad (4.46b)$$

Now, as in the calculations of the direct poles, the Feynman amplitude in the helicity representation with the baryon octet as the internal particle can be expressed in the form:

$$< \lambda_3\lambda_4; \underline{p}_3\underline{p}_4 | F | \underline{p}_1\underline{p}_2; \lambda_1\lambda_2 > = F^b_{\mu\upsilon}(\theta) \frac{1}{u - m^2} F^a_{\upsilon\lambda}(\theta) \qquad (4.47)$$

where the vertex functions $F^b_{\mu\upsilon}(\theta)$ and $F^a_{\upsilon\lambda}(\theta)$ are calculated at the vertices 'b' and 'a' respectively (Fig. 4.3) and they are given by,

$$< \mu, k|F^b|k' \ \upsilon > \ = \ \sqrt{2m} < \lambda_3 \ \lambda_2; \ p_3 \ p_2|J_\ell|k' \ \upsilon \ >$$

(4.48)

$$< \upsilon; \ k'|F^a|k; \ \lambda > \ = \sqrt{2m} < \upsilon; \ k'|J_\eta|p_1 \ p_4; \ \lambda_1 \ \lambda_4 >$$

where, $\mu = \lambda_3 - \lambda_2$; $\lambda = \lambda_1 - \lambda_4$ and υ is the helicity of the internal baryon octet with the U(6,6) degenerate baryon mass m. The helicity states of the internal baryon octet can be written in the form :

$$u_\upsilon(k') = \frac{1}{\sqrt{2m(E' + m)}} \begin{vmatrix} E' + m \\ 2k' \cdot \upsilon \end{vmatrix} \chi_\upsilon \qquad (4.49a)$$

where we have using (4.46a),

$$\chi_{1/2} = \begin{vmatrix} \cos \alpha/2 \\ \sin \alpha/2 \end{vmatrix} \qquad \chi_{-1/2} = \begin{vmatrix} -\sin \alpha/2 \\ \cos \alpha/2 \end{vmatrix} \qquad (4.49b)$$

In equation (4.48), J_α are the related U(6,6) form factors which have been given by equations (1.87) - (1.92). It is evident from these equations that we need to evaluate only five different vertex functions of the types given by equation (4.48) in order to calculate the exchange Born terms corresponding to sixteen processes shown by Fig. 4.3 with only baryon octet exchange. Using the various terms given in the equations (1.87) - (1.92) we evaluate the relevant vertex functions and write them down one by one. Throughout these calculations we shall omit the factor

$4\pi g^2/(u - m^2)$ which will be taken into account when the total angular momentum J projections of the Feynman amplitudes are obtained by using the equation (4.20). The representation of the Dirac matrices that will come up in the calculations is given by,

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \quad \gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

$$\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = -i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad (4.50a)$$

where, $\gamma_0^+ = \gamma_0$; $\gamma_k^+ = -\gamma_k$; $\gamma_5^+ = -\gamma_5$.

Having defined above all the relevant quantities we write down the different vertex functions as follows:

I. This type of vertex functions correspond to a pseudoscalar meson and baryon octet at either the vertex b or the vertex a. Taking the relevant factors from the equation (1.87) the two vertex functions are given, by using equation (4.48), in the form:

$$< \lambda_3; p_3 p_2 | F_1^b | k' \upsilon > = \frac{P^2}{4m^2} \left(1 + \frac{2m}{\mu}\right) \bar{u}_{\lambda_3}(p_3)\, \gamma_5\, u_\upsilon\, (k') \sqrt{2m} \qquad (4.50b)$$

$$< \upsilon\, k' | F_1^a | p_1 p_4\, \lambda_1 > = \frac{P^2}{4m^2} \left(1 + \frac{2m}{\mu}\right) \bar{u}_\upsilon(k')\, \gamma_5\, u_{\lambda_1}(p_1) \sqrt{2m}$$

Considering the representations of the helicity states of all the particles involved and, in particular, using the relations (4.45), (4.46) and (4.50) these vertex functions are evaluated. The factors which will be common to all the elements of the vertex functions are the following :

$$c_1 = \frac{p^2}{4m^2}\left(1 + \frac{2m}{\mu}\right)\frac{i}{\sqrt{2m(E + m)(E' + m)}} \qquad (4.51)$$

$$a_1 = (E' + m)k - (E + m)k' ; \quad a_2 = (E' + m)k + (E + m)k'$$

Then the two relevant vertex functions are as follows :

$$< \lambda_s ; |F_1^b| \upsilon > = c_1 \begin{vmatrix} a_1 \cos\alpha/2 ; & a_2 \sin\alpha/2 \\ a_2 \sin\alpha/2 ; & -a_1 \cos\alpha/2 \end{vmatrix} \qquad (4.52a)$$

$$< \upsilon \mid F_1^a \mid \lambda_1 > = c_1 \begin{vmatrix} -a_1 \cos\alpha/2 ; & a_2 \sin\alpha/2 \\ a_2 \sin\alpha/2 & a_1 \cos\alpha/2 \end{vmatrix} \qquad (4.52b)$$

II. This type of vertex functions correspond to a pseudoscalar meson and a baryon decouplet, D, at either the vertex 'b' or the vertex 'a' and are obtained by using the U(6,6) vertices given by equation (1.89). We write down the two vertex functions one by one.

$$< \lambda_s; \underset{\sim}{p}_s | F_s^b | k' \upsilon > = \frac{1}{m}\left(1 + \frac{2m}{\mu}\right) q^\lambda \left[u_{\lambda_s}(p_s) \otimes \epsilon_\lambda^+ \right] u_\upsilon(k') \sqrt{2m}$$

(4.53a)

Here, the overall common factors are as follows:

$$c_2 = \frac{1}{m}\left(1 + \frac{2m}{\mu}\right) \frac{|k|}{\sqrt{2m(E + m)(E' + m)}}$$

$$a_3 = (E + m)(E' + m) - kk' \qquad (4.53b)$$

$$a_4 = (E + m)(E' + m) + kk'$$

Then, the elements of the above vertex function are as follows:

$$< 3/2|1/2 > = -\frac{a_3}{\sqrt{2}} \sin\theta \cos\alpha/2$$

$$< 3/2|-1/2 > = -\frac{a_4}{\sqrt{2}} \sin\theta \sin\alpha/2$$

$$< 1/2|1/2 > = -\frac{1}{m\sqrt{6}}\left[2\, a_3(w + E\cos\theta)\cos\alpha/2 - a_4\, m \sin\theta \sin\alpha/2\right]$$

$$< 1/2|-1/2 > = -\frac{1}{m\sqrt{6}}\left[2\, a_4(w + E\cos\theta)\sin\alpha/2 + a_3\, m \sin\theta \cos\alpha/2\right]$$

$$< -1/2\ 1/2 > = \frac{1}{m\sqrt{6}}\left[2\, a_4(w + E\cos\theta)\sin\alpha/2 + a_3 m \sin\theta \cos\alpha/2\right]$$

$$< -1/2|-1/2 > = -\frac{1}{m\sqrt{6}}\left[2\,a_3(w + E\cos\theta)\cos\alpha/2 - a_4 m\sin\theta\sin\alpha/2\right]$$

$$< -3/2|1/2 > = -\frac{a_4}{\sqrt{2}}\sin\theta\sin\alpha/2$$

$$< -3/2|-1/2 > = \frac{a_3}{\sqrt{2}}\sin\theta\cos\alpha/2 \qquad (4.53c)$$

The vertex function at the vertex 'a' is given by,

$$< \upsilon;\, k'|F_2^a|p_1\,\lambda_1 > = \frac{1}{m}\left(1 + \frac{2m}{\mu}\right)\bar{u}_\upsilon\,(k')\left[u_{\lambda_1}(p_1)\otimes\epsilon_\lambda\right]\cdot q^\lambda\sqrt{2m}$$

(4.53d)

Here, the overall common factors are the same as given by (4.53b).

Now, the elements of this vertex function are as follows:

$$< 1/2|3/2 > = -\frac{a_3}{\sqrt{2}}\sin\theta\cos\alpha/2$$

$$< -1/2|3/2 > = \frac{a_4}{\sqrt{2}}\sin\theta\sin\alpha/2$$

$$< 1/2|1/2 > = \frac{1}{m\sqrt{6}}\left[2\,a_3(w + E\cos\theta)\cos\alpha/2 - a_4 m\sin\theta\sin\alpha/2\right]$$

$$< -1/2|1/2 > = -\frac{1}{m\sqrt{6}}\left[2\,a_4(w + E\cos\theta)\sin\alpha/2 + a_3 m\sin\theta\cos\alpha/2\right]$$

$$< 1/2|-1/2 > = \frac{1}{m\sqrt{6}}\left[2\, a_4(w + E \cos\theta) \sin\alpha/2 + a_8 m \sin\theta \cos\alpha/2\right]$$

$$< -1/2|-1/2 > = \frac{1}{m\sqrt{6}}\left[2\, a_8(w + E \cos\theta) \cos\alpha/2 - a_4 m \sin\theta \sin\alpha/2\right]$$

$$< 1/2|-3/2 = \frac{a_4}{\sqrt{2}} \sin\theta \sin\alpha/2$$

$$< -1/2|-3/2 > = \frac{a_8}{\sqrt{2}} \sin\theta \cos\alpha/2 \qquad (4.53e)$$

III. This type of vertex functions correspond to the occurrence of a vector meson and a baryon octet at either the vertex 'b' or the vertex 'a'. On account of the two types of couplings corresponding respectively to the charge and magnetic couplings of the vector meson, we shall split them into two parts for the sake of convenience. First, we evaluate the two vertex functions which correspond to the charge couplings of the vector meson. Taking the first term from the equation (1.88) we have for the vertex function at the vertex 'b',

$$< \lambda_8\, \lambda_2 ; p_8\, p_2 |F_8^b| k' \upsilon > = \frac{\mu}{4m^2}\left(1 + \frac{2m}{\mu}\right) P_\mu\, \epsilon_\mu^{(\lambda_2)}\, \bar{u}_{\lambda_8}(p_8) u_\upsilon(k')\sqrt{2m} \qquad (4.54a)$$

Here, the overall common factor is,

$$c_s = \frac{\mu}{4m^2}\left(1 + \frac{2m}{\mu}\right) \frac{|k|}{\sqrt{2m(E + m)(E' + m)}} \qquad (4.54b)$$

For the sake of convenience, we write down the vertex functions separately for each of the vector meson helicity states. Then there results three different vertex functions of the above type. Considering the factors a_s, a_4 as given by the equation (4.53b) these vertex functions are as follows:

$$< \lambda_s, +1; \underset{\sim}{p}_s\, \underset{\sim}{p}_2 | F_s^b | k' \upsilon > = -\sqrt{2}\sin\theta \begin{vmatrix} a_s \cos\alpha/2, & a_4 \sin\alpha/2 \\ -a_4 \sin\alpha/2, & a_s \cos\alpha/2 \end{vmatrix} \qquad (4.54c)$$

$$< \lambda_s, 0; \underset{\sim}{p}_s\, \underset{\sim}{p}_2 | F_s^b | k' \upsilon > = -\frac{2}{\mu}(E + w\cos\theta) \begin{vmatrix} a_s \cos\alpha/2, & a_4 \sin\alpha/2 \\ -a_4 \sin\alpha/2, & a_s \cos\alpha/2 \end{vmatrix} \qquad (4.54d)$$

$$< \lambda_s, -1; \underset{\sim}{p}_s\, \underset{\sim}{p}_2 | F_s^b | k' \upsilon > = \sqrt{2}\sin\theta \begin{vmatrix} a_s \cos\alpha/2, & a_4 \sin\alpha/2 \\ -a_4 \sin\alpha/2, & a_s \cos\alpha/2 \end{vmatrix} \qquad (4.54e)$$

Now, the vertex function of the above type at the vertex 'a' is given by,

$$< \upsilon\, k' | F_s^a | p_1\, p_4 ; \lambda_1\, \lambda_4 > = \frac{\mu}{4m^2}\left(1 + \frac{2m}{\mu}\right) \bar{u}_\upsilon(k')\, u_{\lambda_1}(p_1)\, \overset{*}{\epsilon}{}_\mu^{(\lambda_4)}\, P_\mu \sqrt{2m} \tag{4.54f}$$

Here, the overall common factors are the same as in the above case. Then, the three different vertex functions are given, as before,

$$< \upsilon\, k' | F_s^a | p_1\, p_4 ; \lambda_1, +1 > = - < \lambda_s, +1 ; p_s\, p_2 | F_s^b | k' \upsilon >$$

$$< \upsilon\, k' | F_s^a | p_1\, p_4 ; \lambda_1, 0 > = < \lambda_s, 0 ; p_s\, p_2 | F_s^b | k' \upsilon > \tag{4.54g}$$

$$< \upsilon\, k' | F_s^a | p_1\, p_4 ; \lambda_1, -1 > = - < \lambda_s, -1 ; p_s\, p_2 | F_s^b | k' \upsilon >$$

We now evaluate the vertex functions that arise due to the magnetic couplings of the vector meson. Taking the second term from (1.88) this type of vertex function at the vertex 'b' is,

$$< \lambda_s\, \lambda_2 ; p_s\, p_2 | F_4^b | k' \upsilon > = \frac{1}{4m^2}\left(1 + \frac{2m}{\mu}\right) \epsilon_\mu^{(\lambda_2)}\, \bar{u}_{\lambda_s}(p_s) r_\mu\, u_\upsilon(k') \sqrt{2m} \tag{4.55a}$$

where, $r_\mu = \epsilon_{\mu\upsilon k\lambda}\, P_\upsilon\, q_k\, \gamma_\lambda\, \gamma_5$

Taking out the overall common factor,

$$c_4 = \frac{|k|}{2m^2}\left(1 + \frac{2m}{\mu}\right) \frac{1}{\sqrt{2m(E + m)(E' + m)}} \tag{4.55b}$$

the elements of the above function are as follows:

$$< 1/2,1|1/2 > = \frac{1}{\sqrt{2}}\Big[(a_2 k - a_4 E)\sin\theta\cos\alpha/2 - a_4(E - w)(1-\cos\theta)\sin\alpha/2\Big]$$

$$< 1/2,1|-1/2 > = \frac{1}{\sqrt{2}}\Big[(a_1 k - a_3 E)\sin\theta\sin\alpha/2 + a_3(E - w)(1-\cos\theta)\cos\alpha/2\Big]$$

$$< -1/2,1|1/2 > = \frac{1}{\sqrt{2}}\Big[(a_1 k - a_3 E)\sin\theta\sin\alpha/2 - a_3(E + w)(1+\cos\theta)\cos\alpha/2\Big]$$

$$< -1/2,1|-1/2 > = -\frac{1}{\sqrt{2}}\Big[(a_2 k - a_4 E)\sin\theta\cos\alpha/2 + a_4(E + w)(1+\cos\theta)\sin\alpha/2\Big]$$

$$< 1/2,0|1/2 > = a_4\,\mu\sin\theta\sin\alpha/2; \quad < 1/2,0|-1/2 > = -a_3\,\mu\sin\theta\cos\alpha/2$$

$$< -1/2,0|1/2 > = a_3\,\mu\sin\theta\cos\alpha/2; \quad < -1/2,0|-1/2 > = a_4\,\mu\sin\theta\sin\alpha/2$$

$$< 1/2,-1|1/2 > = \frac{1}{\sqrt{2}}\Big[a_2 k - a_4 E)\sin\theta\cos\alpha/2 + a_4(E + w)(1+\cos\theta)\sin\alpha/2\Big]$$

$$< 1/2,-1|-1/2 > = \frac{1}{\sqrt{2}}\left[(a_1k - a_3E) \sin\theta \sin\alpha/2 - a_3(E + w)\right.$$

$$\left.(1 + \cos\theta)\cos\alpha/2\right]$$

$$< -1/2,-1|1/2 > = \frac{1}{\sqrt{2}}\left[(a_1k - a_3E)\sin\theta \sin\alpha/2 + a_3(E - w)\right.$$

$$\left.(1 - \cos\theta)\cos\alpha/2\right]$$

$$< -1/2,-1|-1/2 > = -\frac{1}{\sqrt{2}}\left[(a_2k - a_4E)\sin\theta \cos\alpha/2 - a_4(E - w)\right.$$

$$\left.(1 - \cos\theta)\sin\alpha/2\right]$$

(4.55c)

Using (1.88) again, the vertex function of the above type at the vertex 'a' is given by,

$$< \upsilon\, k'|F_4^a|p_1\, p_4\, \lambda_1,\lambda_4 > = \frac{1}{4m^2}\left(1 + \frac{2m}{\mu}\right)\bar{u}_\upsilon(k')r_\mu\, u_{\lambda_1}(p_1)\, \epsilon_\mu^{\pm(\lambda_4)}\sqrt{2m}$$

(4.56a)

Here, the overall common factor is the same as given by (4.55b). Then the elements of the above vertex function are as follows:

$$< 1/2|1/2,1 > = -\frac{1}{\sqrt{2}}\left[(a_2k - a_4E)\sin\theta \cos\alpha/2 - a_4(E - w)\right.$$

$$\left.(1 - \cos\theta)\sin\alpha/2\right]$$

$$< -1/2|1/2,1 > = \frac{1}{\sqrt{2}}\left[a_1k - a_3E)\sin\theta \sin\alpha/2 + a_3(E - w)\right.$$

$$\left.(1 - \cos\theta)\cos\alpha/2\right]$$

$$\langle 1/2 | -1/2, 1 \rangle = \frac{1}{\sqrt{2}} \Big[a_1 k - a_3 E) \sin\theta \sin\alpha/2 - a_3 (E + w) (1 + \cos\theta) \cos\alpha/2 \Big]$$

$$\langle -1/2 | -1/2, 1 \rangle = \frac{1}{\sqrt{2}} \Big[(a_2 k - a_4 E) \sin\theta \cos\alpha/2 + a_4 (E + w) (1 + \cos\theta) \sin\alpha/2 \Big]$$

$$\langle 1/2 | 1/2, 0 \rangle = a_4 \mu \sin\theta \sin\alpha/2; \quad \langle -1/2 | 1/2, 0 \rangle = a_3 \mu \sin\theta \cos\alpha/2$$

$$\langle 1/2 | -1/2, 0 \rangle = -a_3 \mu \sin\theta \cos\alpha/2; \quad \langle -1/2 | -1/2, 0 \rangle = a_4 \mu \sin\theta \sin\alpha/2$$

$$\langle 1/2 | 1/2, -1 \rangle = -\frac{1}{\sqrt{2}} \Big[(a_2 k - a_4 E) \sin\theta \cos\alpha/2 + a_4 (E + w) (1 + \cos\theta) \sin\alpha/2 \Big]$$

$$\langle -1/2 | 1/2, -1 \rangle = \frac{1}{\sqrt{2}} \Big[(a_1 k - a_3 E) \sin\theta \sin\alpha/2 - a_3 (E + w) (1 + \cos\theta) \cos\alpha/2$$

$$\langle 1/2 | -1/2, -1 \rangle = \frac{1}{\sqrt{2}} \Big[(a_1 k - a_3 E) \sin\theta \sin\alpha/2 + a_3 (E - w) (1 - \cos\theta) \cos\alpha/2 \Big]$$

$$< -1/2|-1/2,-1 > = \frac{1}{\sqrt{2}}\left[(a_2k - a_4E)\sin\theta\cos\alpha/2 - a_4(E - w)\right.$$

$$\left.(1 - \cos\theta)\sin\alpha/2\right]$$

(4.56b)

IV. This type of vertex functions correspond to the occurrence of a vector meson and a spin 3/2 particle at either the vertex 'b' or the vertex 'a'. Using the U(6,6) vertex given by (1.90), the vertex function at the vertex 'b' is given by,

$$< \lambda_s\ \lambda_2; \underset{\sim}{p}_s\ \underset{\sim}{p}_2|F_5^b|k'\upsilon > = \frac{1}{2m^2}\left(1 + \frac{2m}{\mu}\right)\epsilon_{\mu\upsilon k\lambda}\ P_\upsilon\ q_k\left[\bar{u}_{\lambda_s}(p_s)\otimes\epsilon^+_\lambda\right].\epsilon_\mu$$

$$\times\ u_\upsilon(k')\sqrt{2m} \qquad (4.57a)$$

The overall common factor is,

$$c_s = \frac{i|k|}{m^2}\left(1 + \frac{2m}{\mu}\right)\frac{1}{\sqrt{2m(E + m)(E' + m)}} \qquad (4.57b)$$

Then the elements of the above vertex function are as follows:

$$< 3/2,1|1/2 > = -\frac{1}{2}\,a_s(E - w)(1 - \cos\theta)\cos\alpha/2$$

$$< 3/2,1|-1/2 > = -\frac{1}{2}\,a_4(E - w)(1 - \cos\theta)\sin\alpha/2$$

$$< 1/2,1|1/2 > = -\frac{1}{2\sqrt{3}}\left[2\,a_s\,m\sin\theta\cos\alpha/2 - a_4(E - w)\right.$$

$$\left.(1 - \cos\theta)\sin\alpha/2\right]$$

$$< 1/2,1|-1/2 > = -\frac{1}{2\sqrt{3}}\Big[2\ a_4\ m \sin\theta \sin\alpha/2 + a_3(E - w)$$

$$(1 - \cos\theta)\cos\alpha/2\Big]$$

$$< -1/2,1|1/2 > = \frac{1}{2\sqrt{3}}\Big[2\ a_4\ m \sin\theta \sin\alpha/2 - a_3(E + w)$$

$$(1 + \cos\theta)\cos\alpha/2\Big]$$

$$< -1/2,1|-1/2 = -\frac{1}{2\sqrt{3}}\Big[2\ a_3\ m \sin\theta \cos\alpha/2 + a_4(E + w)$$

$$(1 + \cos\theta)\sin\alpha/2\Big]$$

$$< -3/2,1|1/2 > = \frac{1}{2}\ a_4(E + w)(1 + \cos\theta)\sin\alpha/2$$

$$< -3/2,1|-1/2 = -\frac{1}{2}\ a_3(E + w)(1 + \cos\theta)\cos\alpha/2$$

$$< 3/2,0|1/2 > = \frac{1}{\sqrt{2}}\ a_3\ \mu \sin\theta \cos\alpha/2; \quad < 3/2,0|-1/2 >$$

$$= \frac{1}{\sqrt{2}}\ a_4\ \mu \sin\theta \sin\alpha/2$$

$$< 1/2,0|1/2 > = -\frac{1}{\sqrt{6}}\ a_4\ \mu \sin\theta \sin\alpha/2; \quad < 1/2,0|-1/2 >$$

$$= \frac{1}{\sqrt{6}}\ a_3\ \mu \sin\theta \cos\alpha/2$$

$$< -1/2,0|1/2 > = \frac{1}{\sqrt{6}}\ a_3\ \mu \sin\theta \cos\alpha/2; \quad < -1/2,0|-1/2 >$$

$$= \frac{1}{\sqrt{6}}\ a_4\ \mu \sin\theta \sin\alpha/2$$

$$< -3/2,0|1/2 > = -\frac{1}{\sqrt{2}}\, a_4\, \mu \sin\theta \sin\alpha/2; \quad < -3/2,0|-1/2 > = \frac{1}{\sqrt{2}}\, a_3\, \mu \sin\theta \cos\alpha/2$$

$$< 3/2,-1|1/2 > = \frac{1}{2}\, a_3(E + w)(1 + \cos\theta)\cos\alpha/2$$

$$< 3/2,-1|-1/2 > = \frac{1}{2}\, a_4(E + w)(1 + \cos\theta)\sin\alpha/2$$

$$< 1/2,-1|1/2 > = -\frac{1}{2\sqrt{3}}\left[2\, a_3\, m \sin\theta \cos\alpha/2 + a_4(E + w)(1 + \cos\theta)\sin\alpha/2\right]$$

$$< 1/2,-1|-1/2 > = -\frac{1}{2\sqrt{3}}\left[2\, a_4\, m \sin\theta \sin\alpha/2 - a_3(E + w)(1 + \cos\theta)\cos\alpha/2\right]$$

$$< -1/2,-1|1/2 > = \frac{1}{2\sqrt{3}}\left[2\, a_4\, m \sin\theta \sin\alpha/2 + a_3(E - w)(1 - \cos\theta)\cos\alpha/2\right]$$

$$< -1/2,-1|-1/2 > = -\frac{1}{2\sqrt{3}}\left[2\, a_3\, m \sin\theta \cos\alpha/2 - a_4(E - w)(1 - \cos\theta)\sin\alpha/2\right]$$

$$< -3/2,-1|1/2 > = -\frac{1}{2}\, a_4(E - w)(1 - \cos\theta)\sin\alpha/2$$

$$< -3/2,-1|-1/2 > = \frac{1}{2}\, a_3(E - w)(1 - \cos\theta)\cos\alpha/2 \qquad (4.57c)$$

The other vertex function of the above type is,

$$< \upsilon\, k' | F_5^a | p_1\, p_4 ; \lambda_1\, \lambda_4 > = \frac{1}{2m^2}\left(1 + \frac{2m}{\mu}\right) \bar{u}_\upsilon(k') \left[u_{\lambda_1}(p_1) \times \epsilon_\lambda \right] \epsilon_{\mu\upsilon k\lambda}$$

$$\times\ P_\upsilon\, q_k\, \overset{+}{\epsilon}{}_\mu^{(\lambda_4)} \sqrt{2m} \qquad (4.59a)$$

Here, the overall common factor is the same as in (4.57b). Now, the elements of the above vertex function are as follows:

$$< 1/2 | 3/2,1 > = -\frac{1}{2}\, a_3 (E - w)(1 - \cos\theta) \cos\alpha/2$$

$$< -1/2 | 3/2,1 > = \frac{1}{2}\, a_4 (E - w)(1 - \cos\theta) \sin\alpha/2$$

$$< 1/2 | 1/2,1 > = \frac{1}{2\sqrt{3}} \Big[2\, a_3\, m \sin\theta \cos\alpha/2 - a_4 (E - w)(1 - \cos\theta) \sin\alpha/2 \Big]$$

$$< -1/2 | 1/2,1 > = -\frac{1}{2\sqrt{3}} \Big[2\, a_4\, m \sin\theta \sin\alpha/2 + a_3 (E - w)(1 - \cos\theta) \cos\alpha/2 \Big]$$

$$< 1/2 | -1/2,1 > = \frac{1}{2\sqrt{3}} \Big[2\, a_4\, m \sin\theta \sin\alpha/2 - a_3 (E + w)(1 + \cos\theta) \cos\alpha/2 \Big]$$

$$< -1/2 | -1/2,1 > = \frac{1}{2\sqrt{3}} \Big[2\, a_3\, m \sin\theta \cos\alpha/2 + a_4 (E + w)(1 + \cos\theta) \sin\alpha/2 \Big]$$

$$< 1/2|-3/2,1 > = -\frac{1}{2} a_4(E + w)(1 + \cos\theta)\sin\alpha/2$$

$$< -1/2|-3/2,1 > = -\frac{1}{2} a_3(E + w)(1 + \cos\theta)\cos\alpha/2$$

$$< 1/2|3/2,0 > = -\frac{1}{\sqrt{2}} a_3 \mu \sin\theta \cos\alpha/2; \quad < -1/2|3/2,0 > = \frac{1}{\sqrt{2}} a_4 \mu \sin\theta \sin\alpha/2$$

$$< 1/2|1/2,0 > = -\frac{1}{\sqrt{6}} a_4 \mu \sin\theta \sin\alpha/2; \quad < -1/2|1/2,0 > = -\frac{1}{\sqrt{6}} a_3 \mu \sin\theta \cos\alpha/2$$

$$< 1/2|-1/2,0 > = -\frac{1}{\sqrt{6}} a_3 \mu \sin\theta \cos\alpha/2; \quad < -1/2|-1/2,0 > = \frac{1}{\sqrt{6}} a_4 \mu \sin\theta \sin\alpha/2$$

$$< 1/2|-3/2,0 > = -\frac{1}{\sqrt{2}} a_4 \mu \sin\theta \sin\alpha/2; \quad < -1/2|-3/2,0 > = -\frac{1}{\sqrt{2}} a_3 \mu \sin\theta \cos\alpha/2$$

$$< 1/2|3/2,-1 > = \frac{1}{2} a_3(E + w)(1 + \cos\theta)\cos\alpha/2$$

$$< -1/2|3/2,-1 > = -\frac{1}{2} a_4(E + w)(1 + \cos\theta)\sin\alpha/2$$

$$< 1/2|1/2,-1 > = \frac{1}{2\sqrt{3}} \Big[2 a_3 m \sin\theta \cos\alpha/2 + a_4(E + w)(1 + \cos\theta)\sin\alpha/2\Big]$$

$$< -1/2|1/2,-1 > = -\frac{1}{2\sqrt{3}}\Big[2\, a_4\, m \sin\theta \sin\alpha/2 - a_s(E + w)$$
$$(1 + \cos\theta)\cos\alpha/2\Big]$$

$$< 1/2|-1/2,-1 > = \frac{1}{2\sqrt{3}}\Big[2\, a_4\, m \sin\theta \sin\alpha/2 + a_s(E - w)$$
$$(1 - \cos\theta)\cos\alpha/2\Big]$$

$$< -1/2|-1/2,-1 > = \frac{1}{2\sqrt{3}}\Big[2\, a_s\, m \sin\theta \cos\alpha/2 - a_4(E - w)$$
$$(1 - \cos\theta)\sin\alpha/2\Big]$$

$$< 1/2|-3/2,-1 > = \frac{1}{2}\, a_4(E - w(1 - \cos\theta)\sin\alpha/2$$

$$< -1/2|-3/2,-1 > = \frac{1}{2}\, a_s(E - w)(1 - \cos\theta)\cos\alpha/2$$

(4.59b)

In the above, we have given all the relevant vertex functions which are sufficient for calculating all the required baryon octet exchange Born terms corresponding to the sixteen processes described by Fig. 4.3. In the appendix D, we have given some useful relations, the use of which makes the calculations much easier. Knowing these Feynman amplitudes, the $J = 1/2$ and $J = 3/2$ projections of the helicity amplitudes are obtained from the equation (4.20), where the matrices $d^J_{\mu\lambda}(\theta)$ expressed in terms of Legendre polynomials are given by the equation (C.31) (appendix C).

In this connection, it should be noted that when F is calculated in definite helicity states it takes the form of a function of $\cos\theta$ multiplied by factors which exactly cancel the denominator of $d^J(\theta)$ in the integral of the equation (4.20). The projection then involves simple integrals over the products of P_ℓ's. These products arise due to the factor $4\pi g^2/(u - m^2)$ which we have omitted so far. The factor 4π cancels with the corresponding factor in equation (4.20) and the denominator of the above factor can be expressed, by using equation (4.13), in the form:

$$\frac{1}{u - m^2} = \frac{1}{2k^2}\,\frac{1}{a - \cos\theta} = \sum \frac{2\ell + 1}{2k^2}\, Q_\ell(a)\, P_\ell(\cos\theta) \tag{4.60a}$$

where,

$$a = 1 - \frac{s - m^2 - 2u^2}{2k^2} \tag{4.60b}$$

In equation (4.60a) $Q_\ell(a)$ is the well known Legendre polynomial of the second kind. The right hand side of the equation (4.20) then involves integrals over some polynomials in $\cos\theta$ and the products of P_ℓ's. The relevant integrations are then easily performed by using the following properties of the Legendre polynomials:

$$(\ell + 1)P_{\ell+1}(\cos\theta) + \ell\, P_{\ell-1}(\cos\theta) = (2\ell + 1)\cos\theta\, P_\ell(\cos\theta) \tag{4.61a}$$

$$\frac{1}{2}\int_{-1}^{+1} P_\ell(\cos\,\theta)\, P_m(\cos\,\theta)\, d(\cos\,\theta) = \frac{\delta_{\ell m}}{2 + 1} \qquad (4.61b)$$

Following the procedures discussed above, the contributions of octet exchanges, corresponding to all the processes we are concerned with, to the helicity amplitudes with $J = 1/2$ and $J = 3/2$ are obtained. Knowing these helicity amplitudes of the exchange Born terms, the corresponding contributions to the relevant partial wave amplitudes are obtained by following the method discussed in section 2 of this chapter. The partial wave amplitudes so obtained for $J = 1/2$ and $J = 3/2$ then constitute the octet exchange Born terms contributing to the states with $J^P = 1/2^+$ and $J^P = 3/2^+$ respectively. These are what we shall use as the respective force terms in the N/D calculations.

c. The Baryon Decouplet Exchange Born Terms

In the subsection 'b' of this section we have discussed how the Born terms corresponding to the exchange of baryon octet with $J^P = 1/2^+$ in the crossed (u) channel are evaluated by using the vertex function method discussed in the above mentioned subsection. Following the same procedure as before, we shall discuss here in this subsection how the Born terms corresponding to the exchange of the decouplet with $J^P = 3/2^+$ are calculated. For all the relevant processes (as shown by Fig. 4.3) we are required to calculate only four vertex functions corresponding to each of the two vertices of the second order Feynman diagrams. We discuss them one by one for both the final and initial vertices.

Now, as in the case of the baryon octet exchange Born terms, the Feyynman amplitude in the helicity representation with the baryon decouplet as the internal particle can be expressed in the form:

$$< \lambda_3\ \lambda_4;\ p_3\ p_4 | F | \lambda_1\ \lambda_2;\ p_1\ p_2 > \; = \; F^b_{\mu\upsilon}(\theta)\, \frac{1}{u - m^2}\, F^a_{\upsilon\lambda}(\theta) \qquad (4.62)$$

where the vertex functions $F^b(\theta)$ and $F^a(\theta)$ are evaluated at the vertices 'b' and 'a' respectively (Fig. 4.3) and are given by,

$$< \mu,\ \underset{\sim}{k} | F^b | \underset{\sim}{k}' \upsilon > \; = \; \sqrt{2m} < \lambda_3\ \lambda_2;\ \underset{\sim}{p}_3\ \underset{\sim}{p}_2 | J_\ell | \underset{\sim}{k}' \upsilon >$$

$$(4.63)$$

$$< \upsilon;\ \underset{\sim}{k}' | F^a | \underset{\sim}{k}\ \lambda > \; = \; \sqrt{2m} < \upsilon;\ \underset{\sim}{k}' | J_\eta | \underset{\sim}{p}_1\ \underset{\sim}{p}_4;\ \lambda_1\ \lambda_4 >$$

where $\mu = \lambda_3 - \lambda_2$; $\lambda = \lambda_1 - \lambda_4$ and $\upsilon = 3/2, 1/2, -1/2, -3/2$, is the helicity of the internal baryon decouplet. The helicity states of the spin 3/2 particle expressed as a vector sum of the helicity states of the spin 1/2 and spin 1 particles have been given in the appendix C by equation (C.15). The spinor part of the spin 3/2 internal particle helicity states has been given by equation (4.49) and the vector part of corresponding states is given by,

$$\zeta^{(\pm)} = (0, \underset{\sim}{\zeta}^{(\pm)}) \qquad \zeta^{(0)} = \frac{1}{m} (|k'|, E' \underset{\sim}{\zeta}^{(0)}) \tag{4.64a}$$

where

$$\underset{\sim}{\zeta}^{(\pm)} = \frac{1}{\sqrt{2}} \begin{vmatrix} \mp \cos\alpha \\ -i \\ \pm \sin\alpha \end{vmatrix} \qquad \underset{\sim}{\zeta}^{(0)} = \begin{vmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{vmatrix} \tag{4.64b}$$

In the equation (4.64a) E' and $\underline{k}'$ are respectively the energy and three-component momentum of the internal baryon and have been given by equation (4.46). Let ϵ and $\epsilon^{\dagger}$ be the polarization vectors associated with the incoming and the outgoing decouplet respectively and ξ and $\xi^{\dagger}$ the polarization vectors of the incoming and outgoing vector mesons respectively. The representations of the helicity states of the incoming and outgoing vector mesons have been given by equations (C.13) and (C.14) respectively. For the incoming spin 3/2 particle the corresponding helicity states are obtained from the equation (C.15) by using (C.2) and (C.10) along with (C.9) and for the spin 3/2 outgoing baryon we use (C.5) and (C.12) in order to obtain the corresponding helicity states. We now write down the required vertex functions one by one.

I. This type of vertex functions correspond to the occurrence of a pseudoscalar meson and a baryon octet at either the vertex b or the vertex a. Taking the relevant factors from the equation (1.89) the two vertex functions are given, by using the equation (4.63), in the form:

$$< \lambda_3;\ p_3\ p_2 | F_1^b | k' \upsilon > = \frac{1}{m}\left(1 + \frac{2m}{\mu}\right) \bar{u}_{\lambda_3}(p_3)\ u_\upsilon(k') \otimes \zeta_\mu\ q^\mu$$

$$< \upsilon; k' | F_1^a | p_1\ p_4;\ \lambda_1 > = \frac{1}{m}\left(1 + \frac{2m}{\mu}\right) q^\mu\ \zeta_\mu^\dagger \otimes \bar{u}_\upsilon(k')\ u_{\lambda_1}(p_1) \tag{4.65}$$

II. This type of vertex functions result from the occurrence of a pseudoscalar meson and baryon decouplet at either the vertex b or the vertex a as external particles. Taking the relevant factors from the equation (1.91) these are given as follows:

$$< \lambda_3;\ p_3\ p_2 | F_2^b | k' \upsilon > = \left(1 + \frac{2m}{\mu}\right)\left\{ \frac{3P^2}{4m^2}\, \epsilon^\dagger . \zeta + \frac{3}{2m^2}\, \epsilon^\dagger . q\ \zeta^\mu\ q_\mu \right\}$$

$$\times\ \bar{u}_{\lambda_3}(p_3)\ \gamma_5\ u_\upsilon(k')$$

$$< \upsilon;\ k' | F_2^a | p_1\ p_4;\ \lambda_1 > = \left(1 + \frac{2m}{\mu}\right)\left\{ \frac{3P^2}{4m^2}\, \zeta^\dagger . \epsilon + \frac{3}{2m^2}\, q^\mu\ \zeta_\mu^\dagger\ \epsilon . q \right\} \tag{4.66}$$

$$\times\ \bar{u}_\upsilon(k')\ \gamma_5\ u_{\lambda_1}(p_1)$$

III. This type of the vertex functions arise from the presence of a vector meson and a baryon octet at either the vertex b or the vertex a as external particles. Taking the relevant factors from the equation (1.90) these are given by,

$$< \lambda_s \lambda_2 ; p_s p_2 | F_s^b | k' ; \upsilon > = \frac{1}{2m^2}\left(1 + \frac{2m}{\mu}\right) \epsilon^{\mu\lambda\alpha k} \xi_\mu P_\lambda q_\alpha \zeta_k \times \bar{u}_{\lambda_s}(p_s) u_\upsilon(k') \tag{4.67}$$

$$< \upsilon ; k' | F_s^a | p_1 p_4 \lambda_1 \lambda_4 > = \frac{1}{2m^2}\left(1 + \frac{2m}{\mu}\right) \epsilon^{\mu\lambda\alpha k} \xi^\dagger_\mu P_\lambda q_\alpha \zeta^\dagger_k \times \otimes \bar{u}_\upsilon(k') u_{\lambda_1}(p_1)$$

IV. This type of vertex functions arise from the occurrence of a vector meson and a baryon decouplet at either the vertex b or the vertex a as external particles. Taking the relevant factors from the equation (1.92) these are given by,

$$< \lambda_s \lambda_2 ; p_s p_2 | F_4^b | k' ; \upsilon > = \left\{ \frac{3P^2}{4m^2} \epsilon^\dagger . \zeta + \frac{3}{2m^2} \epsilon^\dagger . q \, \zeta^\mu . q_\mu \right\} \bar{u}_{\lambda_s}(p_s) \times \left\{ \left(1 + \frac{2m}{\mu}\right) \gamma . \xi - \frac{P . \xi}{\mu} \right\} u_\upsilon(k')$$

$$< \upsilon;k' | F_4^a | p_1\ p_4;\ \lambda_1\ \lambda_4 > = \left\{ \frac{3P^2}{4m^2} \zeta^{\dagger}.\epsilon + \frac{3}{2m^2} \zeta_\mu^{\dagger}\ q^\mu\ \epsilon.q \right\} \bar{u}_\upsilon(k')$$

$$\times \left\{ \left(1 + \frac{2m}{\mu}\right) \gamma.\xi^{\dagger} - \frac{P.\xi^{\dagger}}{\mu} \right\} u_{\lambda_1}(p_1)$$

(4.68)

Let us now write,

$$a_1 = (E' + m)k - (E + m)k' ; \quad a_2 = (E' + m)k + (E + m)k'$$

(4.69)

$$a_3 = (E' + m)(E + m) - kk' ; \quad (E' + m)(E + m) + kk' = a_4$$

$$x_1 = \sqrt{2} \left\{ \left(1 + \frac{2m}{\mu}\right) a_2 + \frac{2k}{\mu} \cos\alpha \,.\, a_3 \right\} \sin\alpha \cos\alpha/2$$

$$x_2 = \sqrt{2} \left\{ -\left(1 + \frac{2m}{\mu}\right) a_2 + \frac{2k}{\mu} (1 + \cos\alpha) a_3 \right\} \cos\alpha \sin\alpha/2$$

$$x_3 = \sqrt{2} \left\{ \left(1 + \frac{2m}{\mu}\right) a_1 + \frac{2k}{\mu} (1 - \cos\alpha) a_4 \right\} \cos\alpha \cos\alpha/2$$

$$x_4 = \sqrt{2} \left\{ \left(1 + \frac{2m}{\mu}\right) a_1 + \frac{2k \cos\alpha}{\mu} a_4 \right\} \sin\alpha \sin\alpha/2$$

$$x_5 = \frac{1}{\mu} \left\{ -\left(1 + \frac{2m}{\mu}\right) \left[k a_4 - w(1 - 2\cos\alpha) a_2 \right] + \frac{2k}{\mu} (E + w\cos\theta) a_3 \right\}$$

$$\times \quad \cos\alpha/2$$

$$x_6 = \frac{1}{\mu}\left\{-\left(1+\frac{2m}{\mu}\right)\left[k a_3 - w(1+2\cos\alpha)a_1\right] + \frac{2k}{\mu}\right.$$

$$\left.\times \quad (E + w\cos\theta)a_4\right\} \quad \sin\alpha/2$$

(4.70)

$$c = \frac{1}{\sqrt{2m(E+m)(E'+m)}}$$

Then for the calculation of the vertex functions discussed above we require to evaluate them at the vertex b the following results:

$$S_1^b = \bar{u}u = c \times \begin{vmatrix} a_3\cos\alpha/2 & a_4\sin\alpha/2 \\ -a_4\sin\alpha/2 & a_3\cos\alpha/2 \end{vmatrix} \qquad (4.71a)$$

$$S_2^b = \bar{u}\,\gamma_5\, u = c.i \times \begin{vmatrix} a_1\cos\alpha/2 & a_2\sin\alpha/2 \\ a_2\sin\alpha/2 & -a_1\cos\alpha/2 \end{vmatrix} \qquad (4.71b)$$

$$S_3^b = \bar{u}\left\{\left(1+\frac{2m}{\mu}\right)\gamma\cdot\xi - \frac{1}{\mu}P\cdot\xi\right\}u$$

where,

$$S_3^{(+)b} = c \times \begin{vmatrix} -x_1 & x_4 \\ x_3 & x_2 \end{vmatrix} \;;\quad S_3^{(-)b} = c \times \begin{vmatrix} -x_2 & x_3 \\ x_4 & x_1 \end{vmatrix}$$

$$S_3^{(0)b} = c \times \begin{vmatrix} x_5 & x_6 \\ -x_6 & x_5 \end{vmatrix}$$

(4.71c)

Similarly, in order to evaluate the vertex functions at the vertex a we require the following results:

$$S_1^a = \bar{u}u = c \times \begin{vmatrix} a_3 \cos \alpha/2 & a_4 \sin \alpha/2 \\ -a_4 \sin \alpha/2 & a_3 \cos \alpha/2 \end{vmatrix} \qquad (4.72a)$$

$$S_2^a = \bar{u}\,\gamma_5\, u = c.i \times \begin{vmatrix} -a_1 \cos \alpha/2 & a_2 \sin \alpha/2 \\ a_2 \sin \alpha/2 & a_1 \cos \alpha/2 \end{vmatrix} \qquad (4.72b)$$

$$S_3^a = \bar{u}\left\{\left(1 + \frac{2m}{\mu}\right)\gamma.\xi^{\dagger} - \frac{1}{\mu}p.\xi^{\dagger}\right\} u$$

$$S_3^{(+)a} = c \times \begin{vmatrix} x_1 & x_3 \\ x_4 & -x_2 \end{vmatrix} ; \quad S_3^{(-)a} = c \times \begin{vmatrix} x_2 & x_4 \\ x_3 & -x_1 \end{vmatrix} ;$$

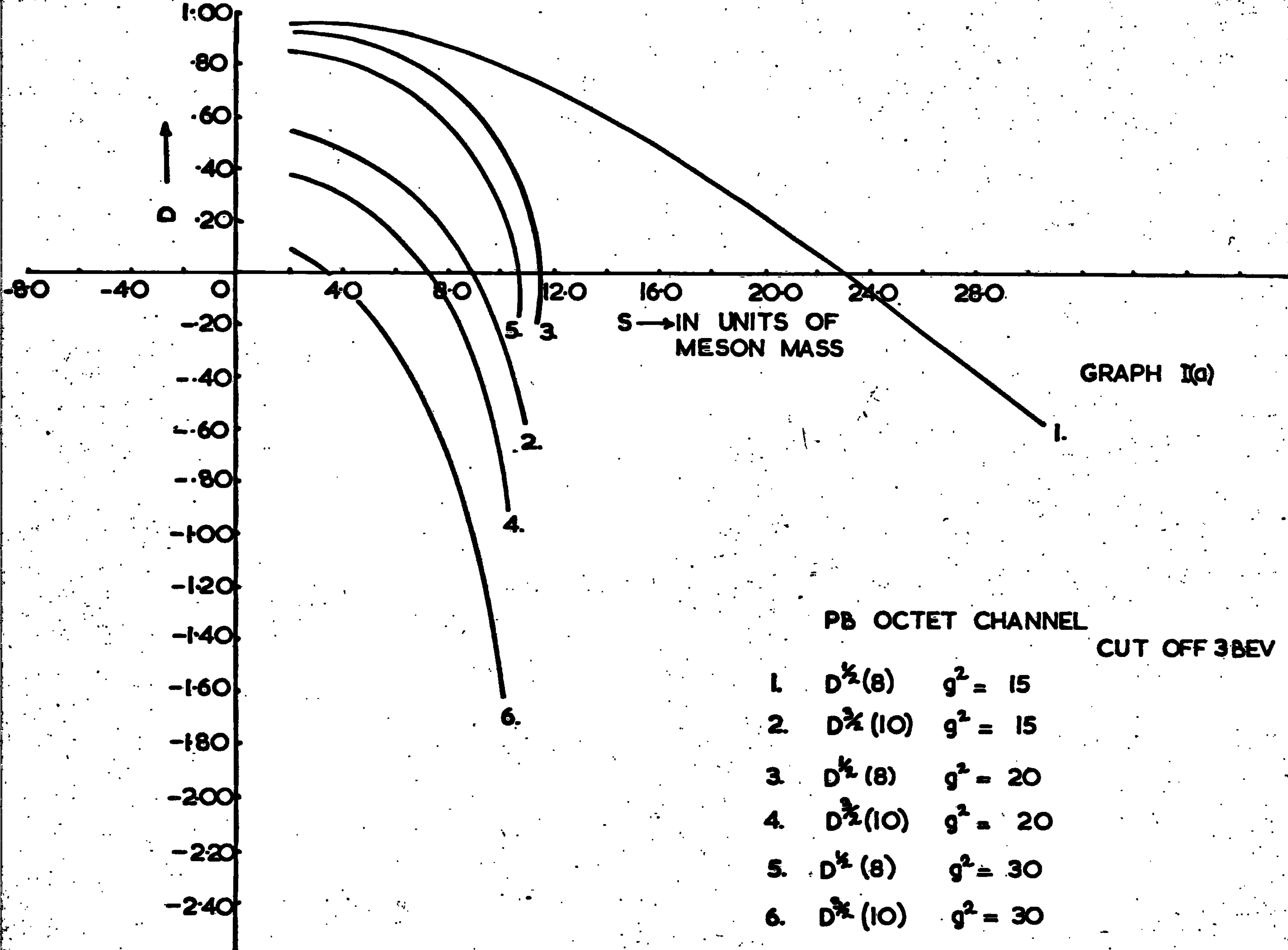
$$S_3^{(0)a} = c \times \begin{vmatrix} x_5 & x_6 \\ -x_6 & x_5 \end{vmatrix} \qquad (4.72c)$$

Having known the vertex functions given by the equations (4.65) - (4.68) the Feynman amplitudes corresponding to the sixteen processes shown by Fig. 4.3 with the spin 3/2 particle in the crossed channel can be evaluated by taking the appropriate combinations of the above mentioned vertex functions. The contributions of these spin 3/2 exchange Born terms to the helicity amplitudes with $J = 1/2$ and

$J = 3/2$ are then obtained by using the equations (4.20), (4.60) and (4.61). The contributions of these helicity states to the partial wave amplitudes are obtained by taking the appropriate combinations of these helicity amplitudes which have definite total angular momentum. The total Born terms for the states with $J^P = 1/2^+$ and $J^P = 3/2$ are then the sum of the contributions of the spin 1/2 exchange Born terms discussed in the last subsection and the spin 3/2 exchange Born terms discussed above. These Born terms corresponding to the relevant processes we are concerned with have been used as input forces in the calculations which we discuss in the following section.

4. The N/D Methods and the Results

As has been mentioned in the section 1 of this chapter, our objective is to investigate the mass splitting between the baryon Octet and Decouplet by using the N/D method. In U(6,6) theory the baryon Octet and Decouplet being the member of the same irreducible representation of U(6,6) symmetry are supposed to have the same degenerate mass. If we use the same degenerate mass by using the U(6,6) vertices in the calculations, then it is expected that the N/D method will give masses for the baryon Octet and Decouplet which will be different from the input degenerate mass and thereby will provide us with an idea as to the mass-splitting of these two SU(3) multiplets. In the calculation, it is assumed that the SU(3)



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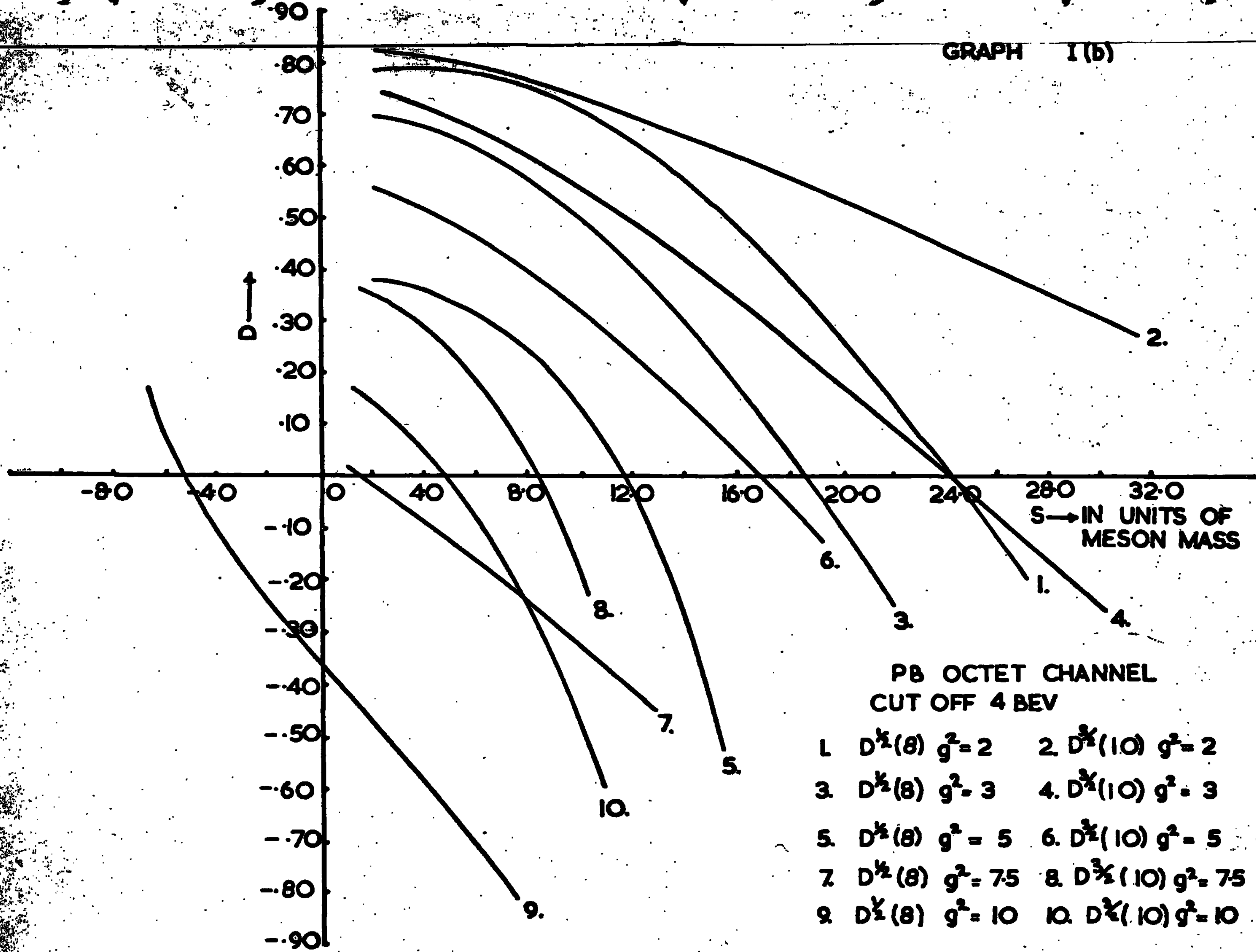
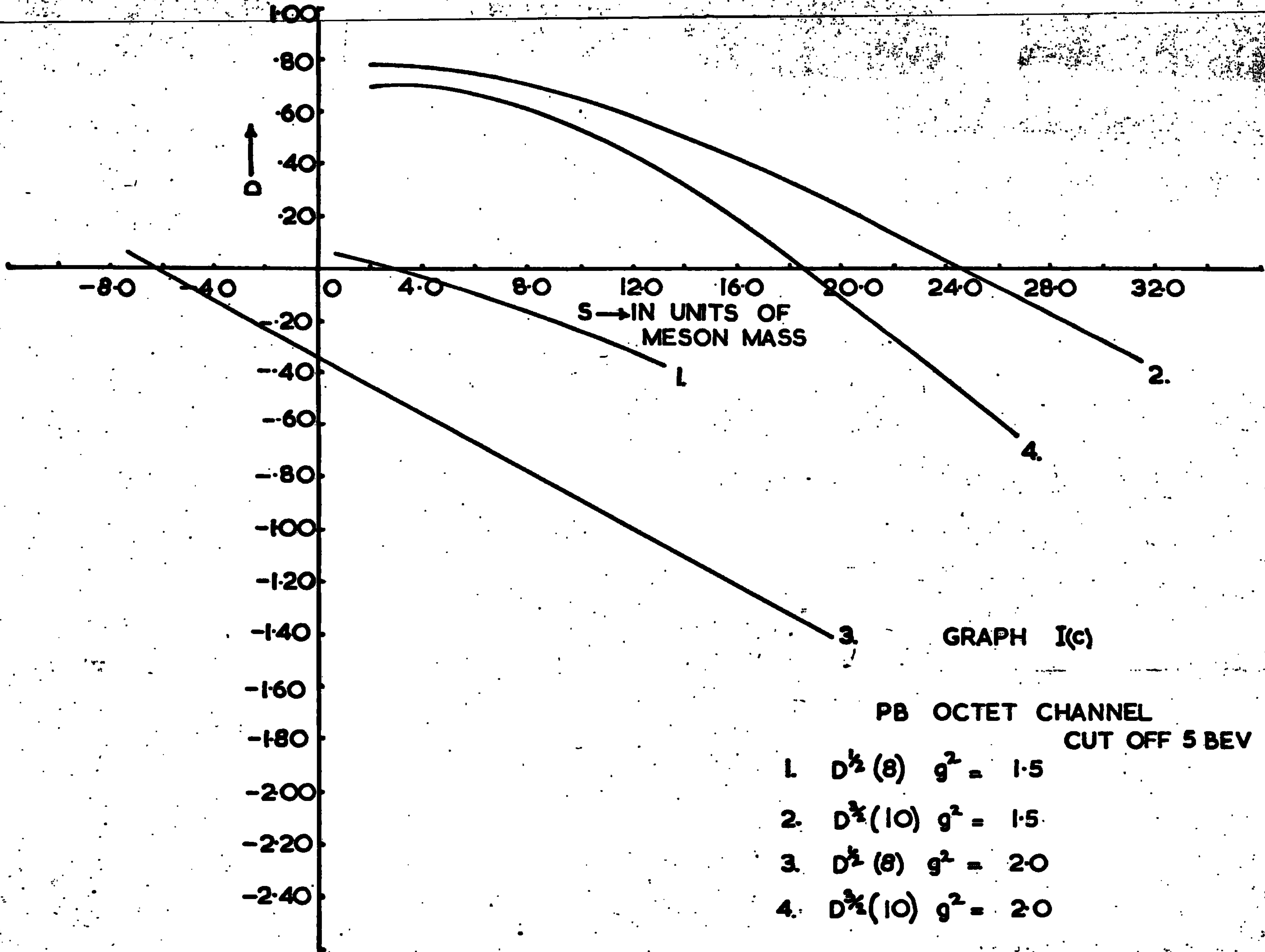
GRAPH I(a)
PB OCTET CHANNEL
CUT OFF 3BEV
1. $D^{1/2}(8)$ $g^2 = 15$
2. $D^{3/2}(10)$ $g^2 = 15$
3. $D^{1/2}(8)$ $g^2 = 20$
4. $D^{3/2}(10)$ $g^2 = 20$
5. $D^{1/2}(8)$ $g^2 = 30$
6. $D^{3/2}(10)$ $g^2 = 30$

GRAPH I (b)



D
S → IN UNITS OF MESON MASS
PB OCTET CHANNEL
CUT OFF 4 BEV
1. $D^{1/2}(8)$ $g^2 = 2$
2. $D^{3/2}(10)$ $g^2 = 2$
3. $D^{1/2}(8)$ $g^2 = 3$
4. $D^{3/2}(10)$ $g^2 = 3$
5. $D^{1/2}(8)$ $g^2 = 5$
6. $D^{3/2}(10)$ $g^2 = 5$
7. $D^{1/2}(8)$ $g^2 = 7.5$
8. $D^{3/2}(10)$ $g^2 = 7.5$
9. $D^{1/2}(8)$ $g^2 = 10$
10. $D^{3/2}(10)$ $g^2 = 10$



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S → IN UNITS OF MESON MASS
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PB OCTET CHANNEL
CUT OFF 5 BEV
1. $D^{1/2}$ (8) g^2 = 1.5
2. $D^{3/2}$ (10) g^2 = 1.5
3. $D^{1/2}$ (8) g^2 = 2.0
4. $D^{3/2}$ (10) g^2 = 2.0

GRAPH I(c)

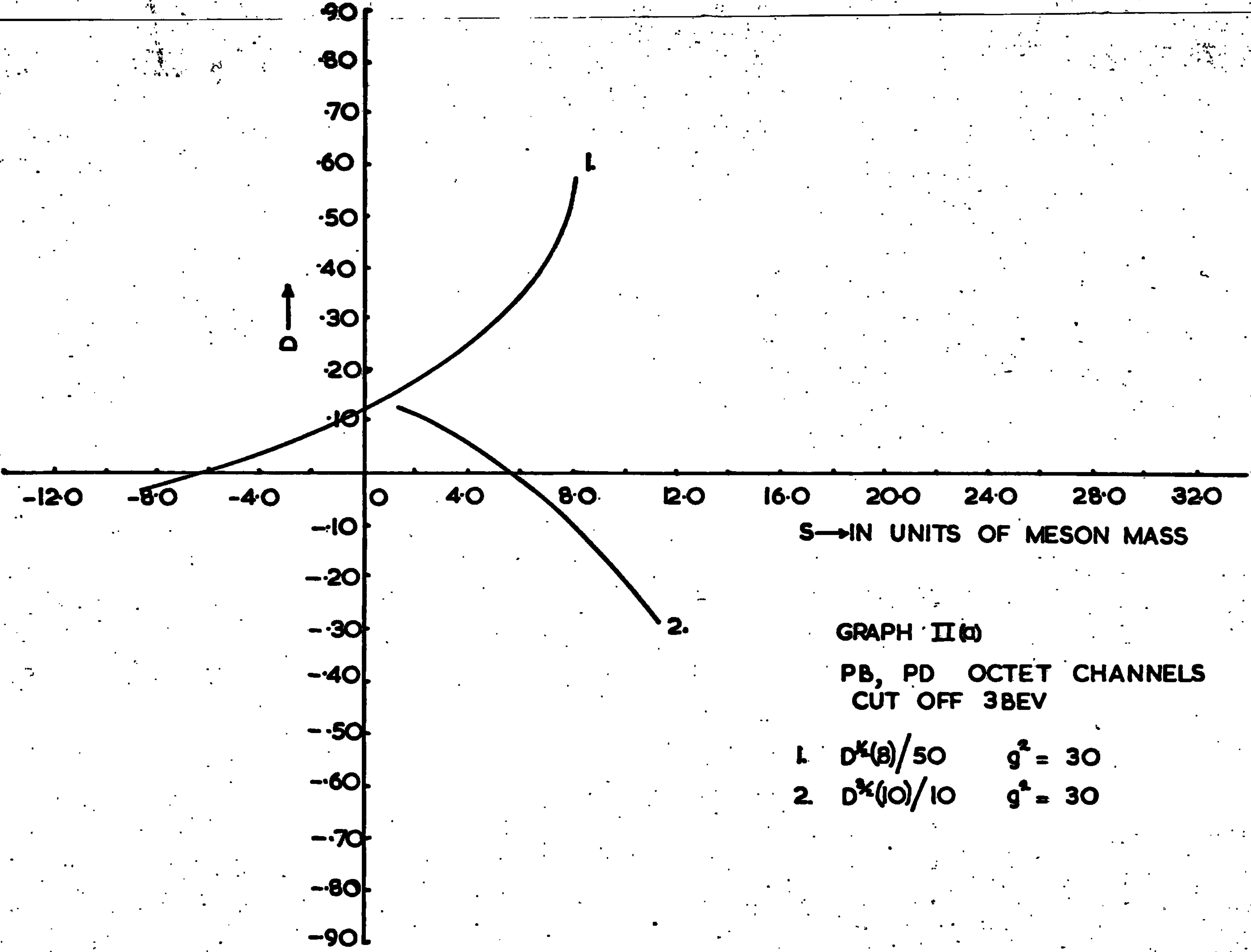
symmetry is approximately exact so that the masses which we will obtain by N/D method for the baryon Octet and Decouplet should, to a reasonable extent, correspond to the SU(3) degenerate masses of these multiplets. In the calculation we have invoked the bootstrap hypothesis that the forces responsible for the binding of the baryon octet and decouplet come predominantly from the exchanges of the multiplets themselves in the crossed (u) channel for each of the processes that arise from the consideration of meson-baryon scattering in the context of U(6,6) symmetry.

The states which contribute to the state $J^P = 1/2^+$ are as follows:

$$\begin{aligned}
&PB : \underline{8}_s(P_{1/2}),\ \underline{8}_A(P_{1/2});\quad P_oB : \underline{8}(P_{1/2});\quad PD : \underline{8}(P_{3/2})\\
&VB : \underline{8}_s(P_{1/2},\ P_{3/2}),\ \underline{8}_A(P_{1/2},\ P_{3/2});\quad V_oB : \underline{8}(P_{1/2},\ P_{3/2})\\
&VD : \underline{8}(P_{1/2},\ P_{3/2},\ F_{5/2})
\end{aligned} \qquad (4.73)$$

where $P_{1/2}$, $P_{3/2}$ and $F_{5/2}$ denote respectively the P and F partial wave amplitudes with the channel spin as mentioned by the respective subscript. It is thus evident from (4.73) that the calculation for the $J^P = 1/2^+$ state involves a thirteen-channel problem. Since the contribution of the F-wave, as discussed in the subsection 'a' of the previous section, is negligible, the F-wave is discarded in the calculation so that the number of channels reduces to twelve.

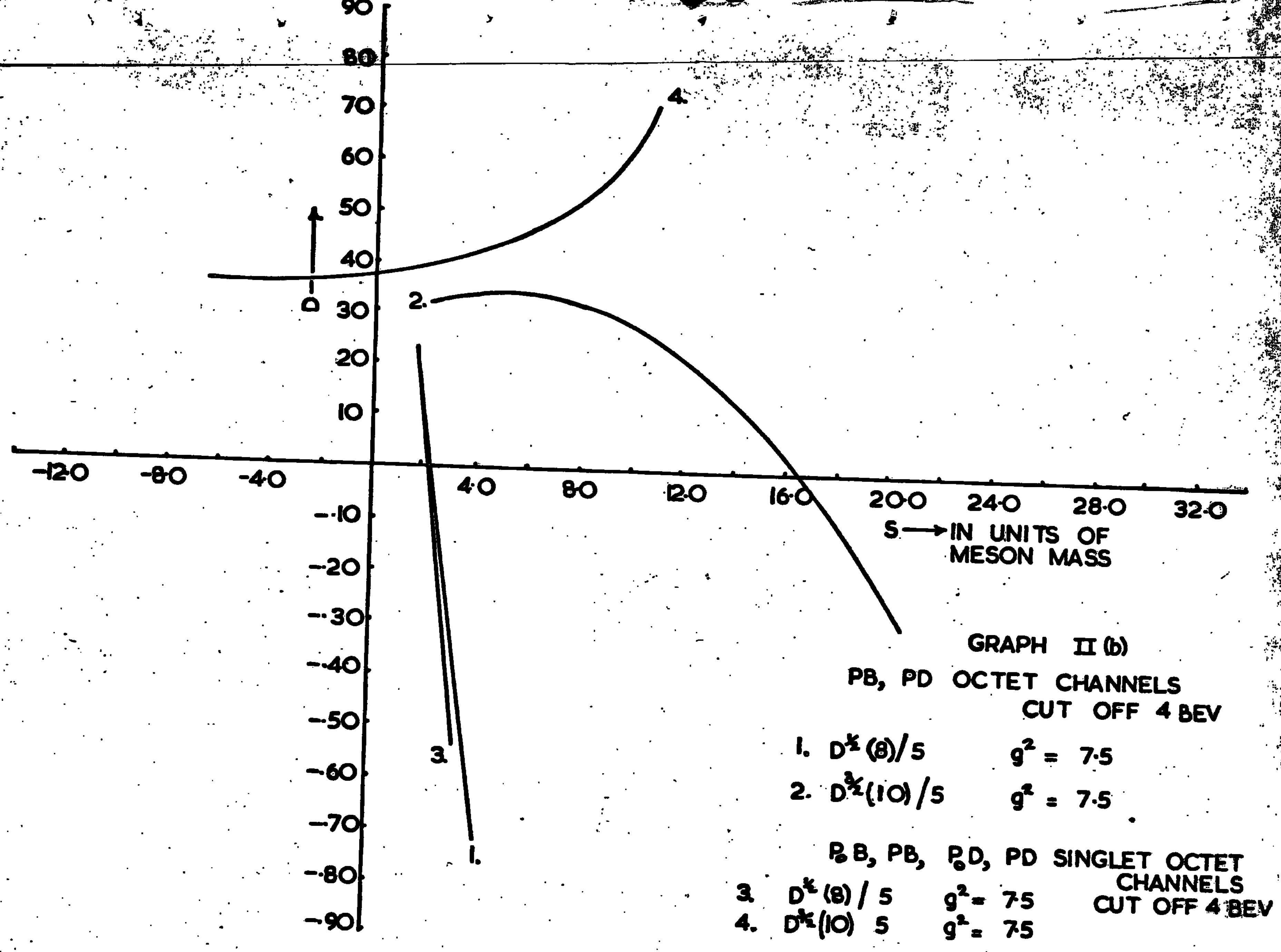
Let us consider the states which contribute to $J^P = 3/2^+$ state. These are as follows:



GRAPH II(a)

PB, PD OCTET CHANNELS
CUT OFF 3BEV

1. $D^{1/2}(8)/50 \quad g^2 = 30$
2. $D^{3/2}(10)/10 \quad g^2 = 30$


D
S → IN UNITS OF MESON MASS
GRAPH II (b)
PB, PD OCTET CHANNELS
CUT OFF 4 BEV
1. $D^{1/2}(8)/5$ $g^2 = 7.5$
2. $D^{3/2}(10)/5$ $g^2 = 7.5$
P_0B, PB, P_0D, PD SINGLET OCTET CHANNELS
CUT OFF 4 BEV
3. $D^{1/2}(8)/5$ $g^2 = 7.5$
4. $D^{1/2}(10)$ 5 $g^2 = 7.5$

$$PB : \underline{10}(P_{1/2}); \quad P_oD : \underline{10}(P_{3/2}, F_{3/2}); \quad PD : \underline{10}(P_{3/2}, F_{3/2});$$

$$VB : \underline{10}(P_{1/2}, P_{3/2}, F_{3/2}); \quad V_oD : \underline{10}(P_{1/2}, P_{3/2}, P_{5/2}, F_{3/2}, F_{5/2});$$

$$VD : \underline{10}(P_{1/2}, P_{3/2}, P_{5/2}, F_{3/2}, F_{5/2}) \tag{4.74}$$

It is thus evident from (4.74) that the number of channels involved in this calculation is eighteen. The neglect of the F-waves reduces the number of channels to eleven.

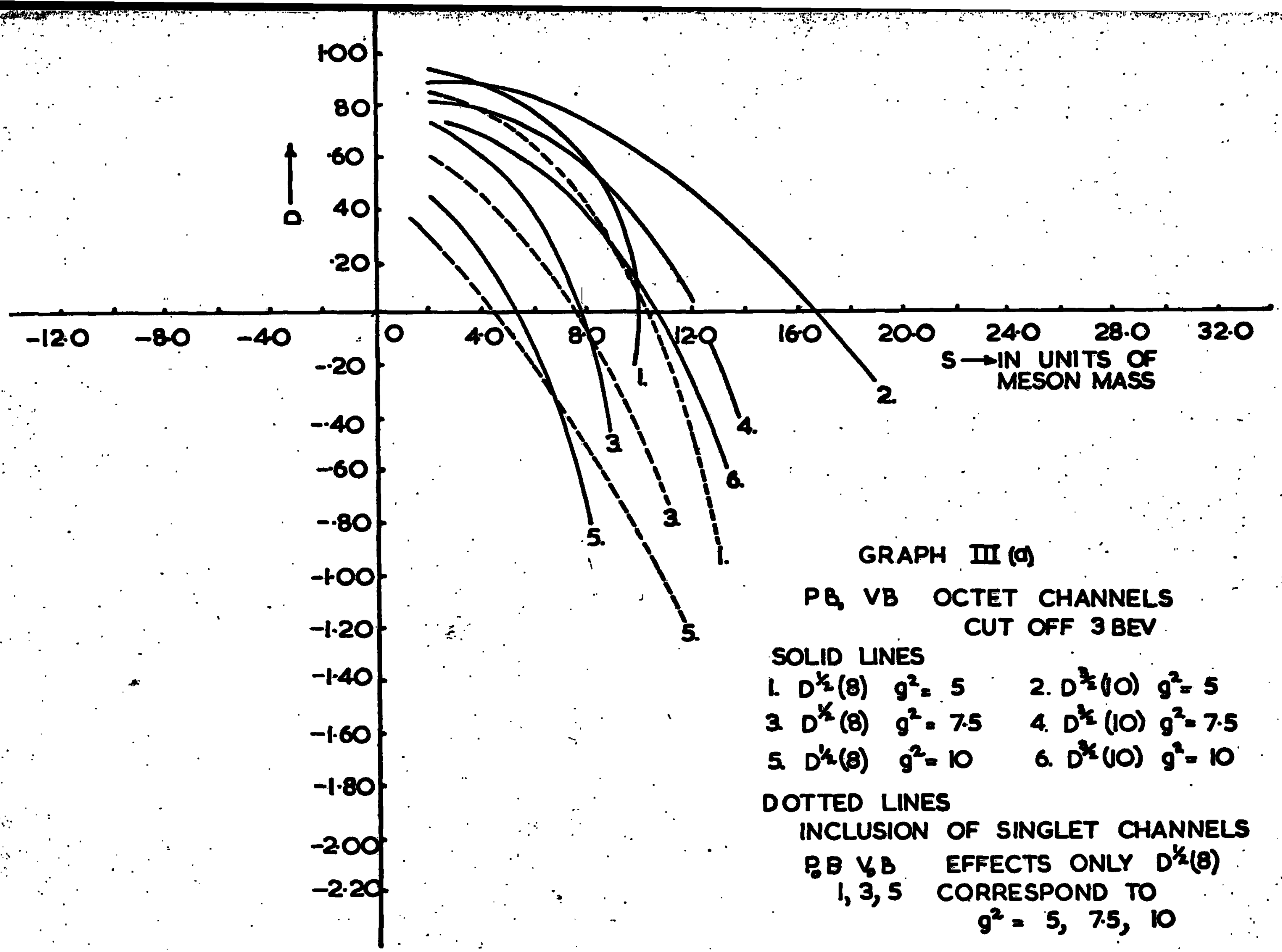
Now, using the determinantal approximation discussed in Chapter II, the dispersion relation for the denominator function of both $J^P = 1/2^+$ and $J^P = 3/2^+$ states is given, by using the equation (2.18), in the form:

$$D^J_{kj}(s) = \delta_{kj} - \frac{1}{\pi} \int_{(m+\mu)^2}^{\infty} ds' \; \rho_{kk}(s') \; \frac{B_{kj}(s')}{(s' - s)} \tag{4.75}$$

where,

$$\rho_{ij}(s) = \delta_{ij} \frac{k^{2\ell+1}}{\sqrt{s}} \tag{4.76}$$

In deriving (4.75) we have normalised the function $D^J(s)$ at $s = -\infty$ and $B_{kj}(s)$ in the same equation denote the Born terms that are obtained, as has been discussed in detail in the previous section, from the exchanges of the baryon Octet with $J = 1/2$ and baryon decouplet with $J = 3/2$ in the crossed (u) channel of the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S → IN UNITS OF MESON MASS
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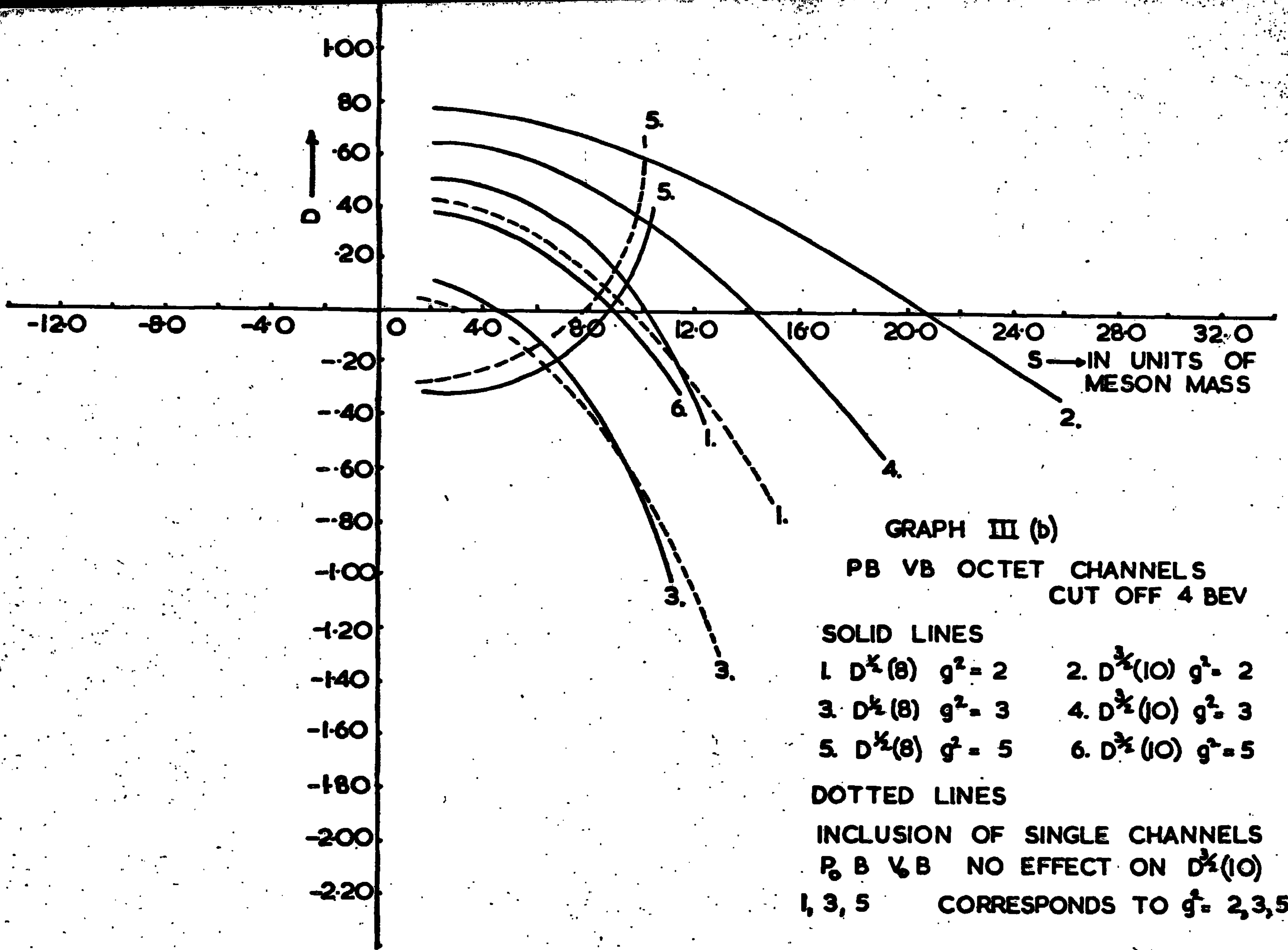
GRAPH III (a)
PB, VB OCTET CHANNELS
CUT OFF 3 BEV
SOLID LINES
1. $D^{1/2}(8)$ $g^2 = 5$
2. $D^{3/2}(10)$ $g^2 = 5$
3. $D^{1/2}(8)$ $g^2 = 7.5$
4. $D^{3/2}(10)$ $g^2 = 7.5$
5. $D^{1/2}(8)$ $g^2 = 10$
6. $D^{3/2}(10)$ $g^2 = 10$
DOTTED LINES
INCLUSION OF SINGLET CHANNELS
P_0B V_0B EFFECTS ONLY $D^{1/2}(8)$
1, 3, 5 CORRESPOND TO
$g^2 = 5, 7.5, 10$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-1·20
-1·40
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-1·80
-2·00
-2·20
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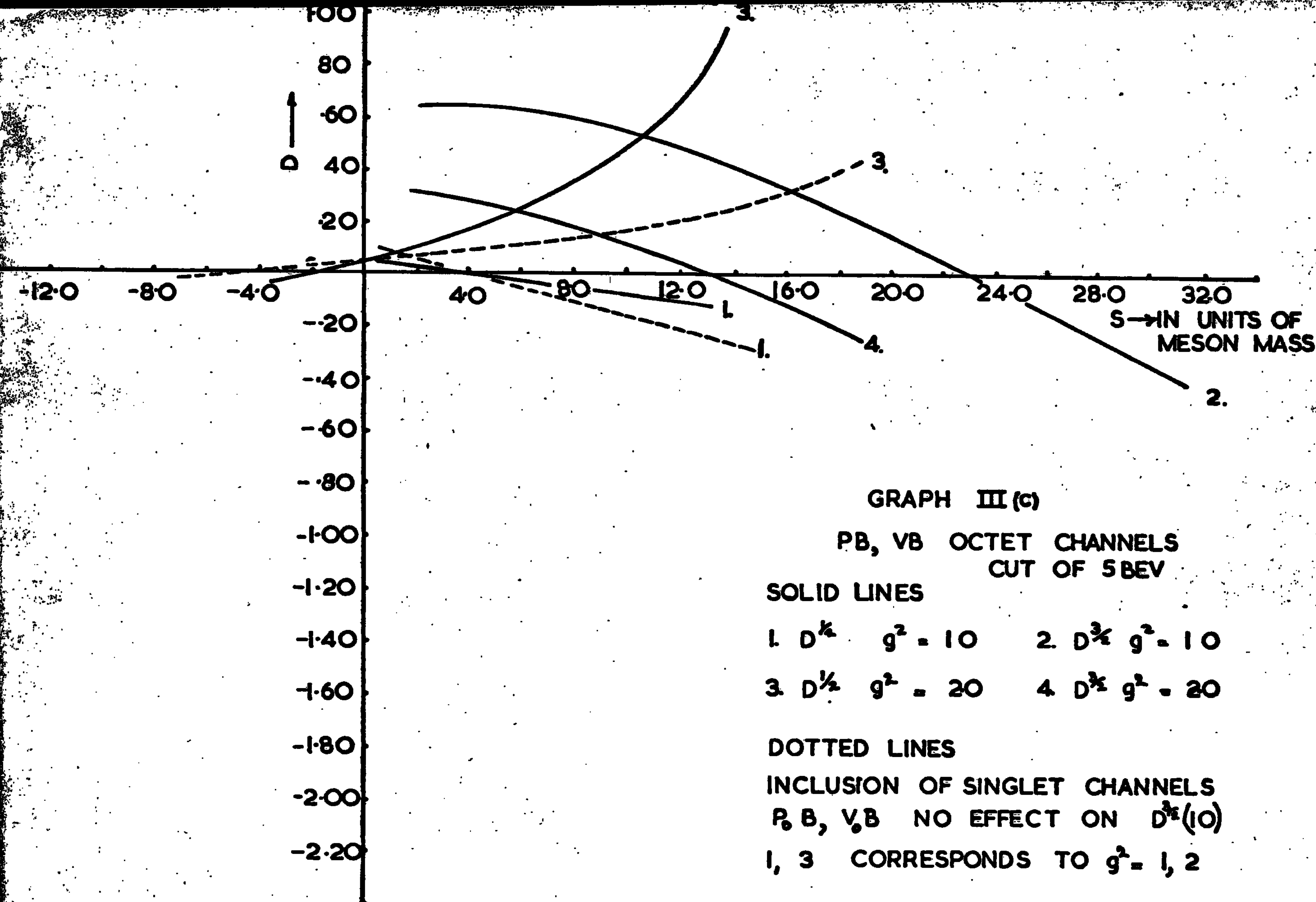
2.
3.
4.
5.
6.
GRAPH III (b)
PB VB OCTET CHANNELS
CUT OFF 4 BEV
SOLID LINES
1. $D^{1/2}(8)$ $g^2 = 2$
2. $D^{3/2}(10)$ $g^2 = 2$
3. $D^{1/2}(8)$ $g^2 = 3$
4. $D^{3/2}(10)$ $g^2 = 3$
5. $D^{1/2}(8)$ $g^2 = 5$
6. $D^{3/2}(10)$ $g^2 = 5$
DOTTED LINES
INCLUSION OF SINGLE CHANNELS
P_0 B V_0 B NO EFFECT ON $D^{3/2}(10)$
1, 3, 5 CORRESPONDS TO $g^2 = 2, 3, 5$



GRAPH III (c)

PB, VB OCTET CHANNELS
CUT OF 5 BEV

SOLID LINES

1. $D^{1/2}$ $g^2 = 10$ 2. $D^{3/2}$ $g^2 = 10$

3. $D^{1/2}$ $g^2 = 20$ 4. $D^{3/2}$ $g^2 = 20$

DOTTED LINES

INCLUSION OF SINGLET CHANNELS

P_0B, V_0B NO EFFECT ON $D^{3/2}(10)$

1, 3 CORRESPONDS TO $g^2 = 1, 2$

relevant processes concerned. Using, now, the determinantal approximation the partial wave amplitude corresponding to the ijth channel is given by,

$$a^J_{ij}(s) = \frac{B^J_{ik}(s)\,\bar{D}^J_{kj}(s)}{\det D^J(s)} \qquad (4.77)$$

where $\bar{D}(s)$ is the cofactor of the determinant of the denominator function D(s). If there occurs a bound state at $s = s_B$ we have

$$\det D^J(s_B) = 0 \qquad (4.78a)$$

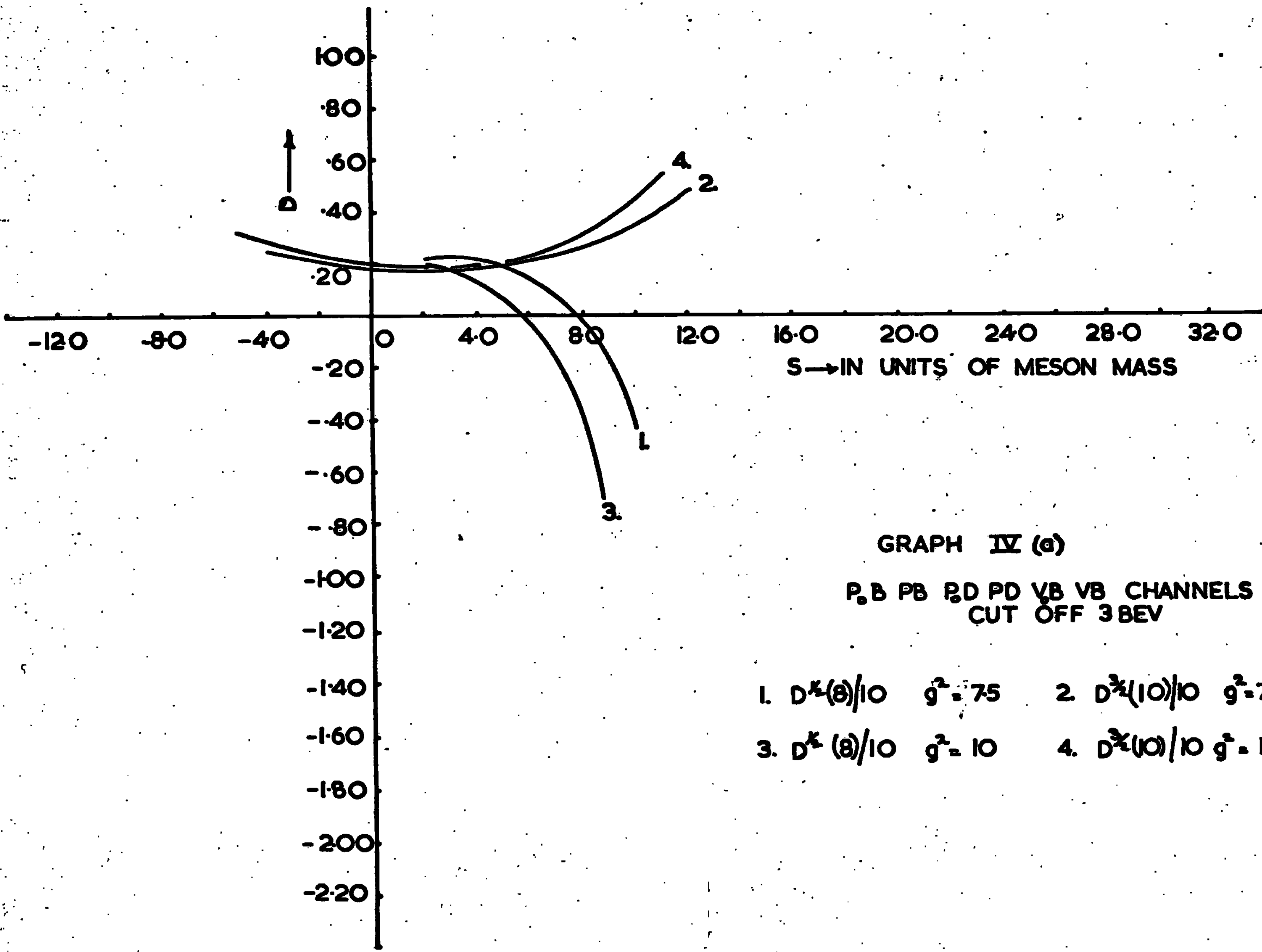
On the other hand, if there occurs a resonance at $s = s_k$ we have,

$$\mathrm{Re}\,\det D^J(s_k) = 0 \qquad (4.78b)$$

Using the equation (4.75) together with the equations (4.78a) and (4.78b) we evaluate the masses of the $J^P = 1/2^+$ and $J^P = 3/2^+$ states. Since the Born terms in the integral (4.75) are very divergent we had to use cut-off in performing the integrations which have been carried out by numerical computation. The relevant determinants have also been evaluated by numerical computation, the zeros of the determinant having been obtained by graphical method. In what follows we discuss the results in detail.

The experimental values of the masses of the baryon Octet and Decouplet are as follows:

$$M_B = 1146.3 \text{ Mev.} \qquad M_D = 1384.6 \text{ Mev.} \qquad (4.79)$$

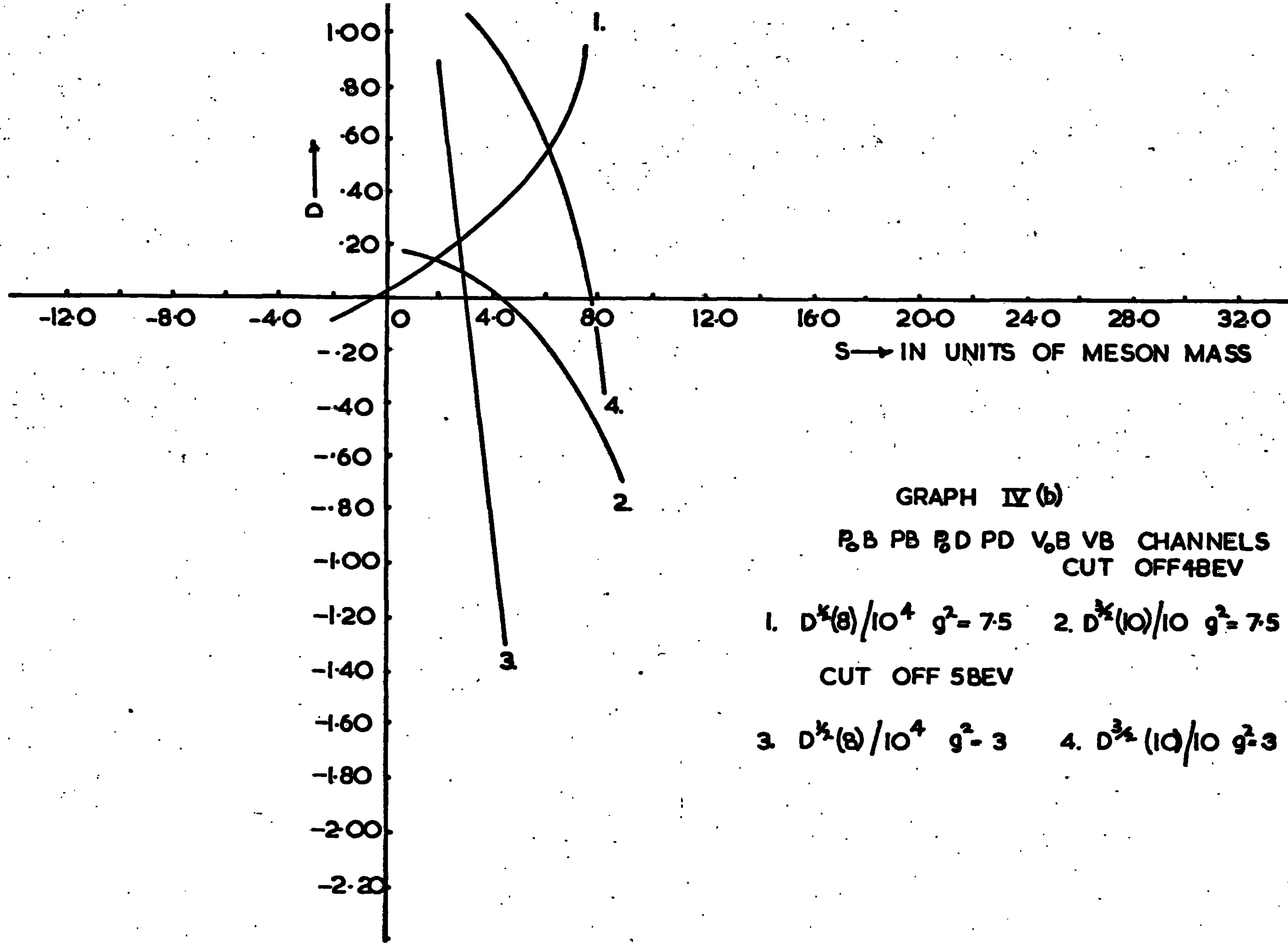


GRAPH IV (a)

P_0B PB P_0D PD VB VB CHANNELS
CUT OFF 3 BEV

1. $D^{1/2}(8)/10$ $g^2 = 7.5$ 2. $D^{3/2}(10)/10$ $g^2 = 7.5$

3. $D^{1/2}(8)/10$ $g^2 = 10$ 4. $D^{3/2}(10)/10$ $g^2 = 10$



GRAPH IV (b)

P_0B PB P_0D PD V_0B VB CHANNELS CUT OFF 4BEV

1. $D^{1/2}(8)/10^4$ $g^2 = 7{\cdot}5$ 2. $D^{3/2}(10)/10$ $g^2 = 7{\cdot}5$

CUT OFF 5BEV

3. $D^{1/2}(8)/10^4$ $g^2 = 3$ 4. $D^{3/2}(10)/10$ $g^2 = 3$

The theoretical results for the masses of the baryon Octet and Decouplet are presented in the following table:

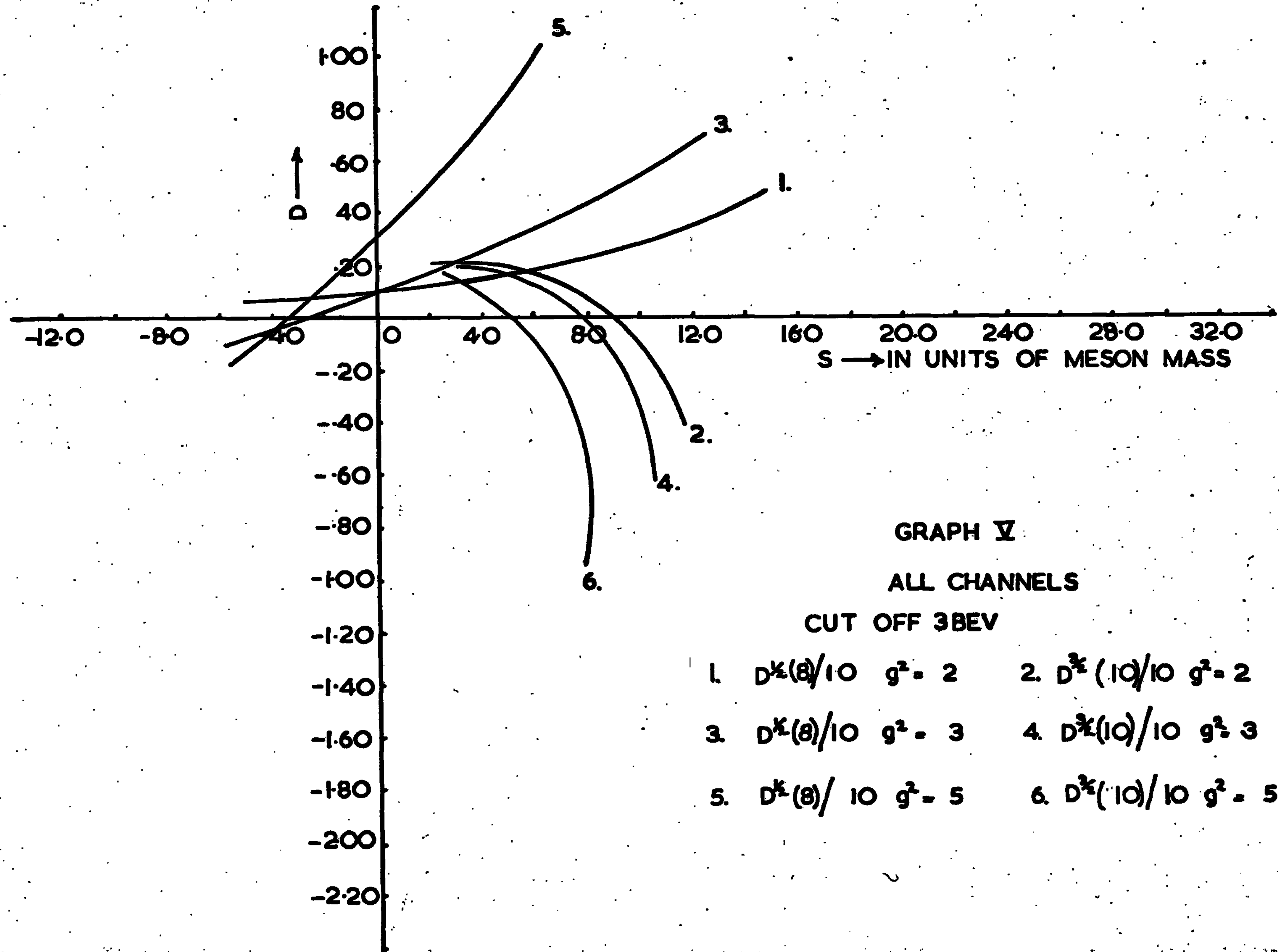
Table 4.1

Channels Considered	S_c Cut off in Mev	U(6,6) Coupling g^2	M_B Mass of the Baryon Octet in Mev.	M_D Mass of the Baryon Decouplet in Mev	$\Delta M = M_D - M_B$ in Mev
PB	3	15	3091.2	1932.0	- 1159.2
		20	2183.2	1725.9	- 457.3
		30	2118.8	1217.2	- 901.6
	4	2	3174.9	-	
		3	2769.2	3149.2	380.0
		5	2196.1	2640.4	444.3
		7.5	817.9	1835.4	1017.5
		10	-1468.3	1410.4	
	5	1.5	1114.1	3187.8	2073.7
		2	-	2691.9	
PB,PD	3	30	-	3604.6	
	4	7.5	1011.0	2614.6	1603.6
PB,VB	3	5	2035.0	2627.5	592.5
		7.5	1822.5	2260.4	437.9
		10	1500.5	2099.4	598.9

Table 4.1

Channels Considered	S_c Cut off in Bev	U(6,6) Coupling g^2	M_B Mass of the Baryon Octet in Mev	M_D Mass of the Baryon Decouplet in Mev	$\Delta M = M_D - M_B$ in Mev
PB,VB	4	2.0	2015.7	2923.8	908.1
		3.0	1352.4	2408.6	1056.2
		5.0	1912.7	1912.7	0
	5	1.0	1288.0	3091.2	1803.2
		2.0	-	2324.8	
P_oB, PB, P_oD, PD, V_oB, VB	3	7.5	1822.5	-	
		10	1526.3	-	
	4	7.5	-	1320.2	
	5	3.0	1114.1	1764.6	650.5
All Channels	3	2	-	1932.0	
		3	-	1796.8	
		5	-	1442.6	

The masses of the baryon Octet and Decouplet obtained from the N/D method by using the determinantal approximation have been presented in the table 4.1. As the Born terms, in particular, the baryon Decouplet exchange Born terms are very divergent, we had to introduce cut off in performing the integrations associated with the denominator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S → IN UNITS OF MESON MASS
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GRAPH V
ALL CHANNELS
CUT OFF 3BEV
1. $D^{1/2}(8)/10$ $g^2 = 2$
2. $D^{3/2}(10)/10$ $g^2 = 2$
3. $D^{1/2}(8)/10$ $g^2 = 3$
4. $D^{3/2}(10)/10$ $g^2 = 3$
5. $D^{1/2}(8)/10$ $g^2 = 5$
6. $D^{3/2}(10)/10$ $g^2 = 5$

function D(s) (4.75) and consequently the results presented in the above mentioned table are extremely dependent on the cut off. In order to obtain an idea as to the trend of the mass-splitting of the baryons we have carried out investigations taking into account the various channels, the results having been obtained by the graphical methods. For each of the calculations involving the various channels considered we have varied the cut off parameter S_c as well as the U(6,6) coupling g^2. Corresponding to each of the combinations of the parameters S_c and g^2 the values of the determinant of the denominator function $D^J(s)$ have been plotted against the c.m. energy squared. The results for all combinations of the channels that have been considered, have been depicted by the graphs I - V. The positions of the zeros of the determinant of the denominator functions $D^J(s)$ then give the values of the masses squared of the corresponding states and in our case the states concerned are respectively the baryon Octet with $J^P = 1/2^+$ and baryon Decouplet with $J^P = 3/2^+$.

It is fairly evident from the table 4.1 that when only the PB channel is considered the mass-splittings for the smaller values of the cut off S_c and higher values of the parameters g^2 occur in the wrong direction. Higher values of the cut off and smaller values of g^2, on the other hand, however cause mass-splittings in the right direction but the values of the masses of the baryon Octet and Decouplet are much higher than the corresponding experimental results.

Inclusion of the VB channel with 4 bev as the value of the cut off parameter and the smaller values of the coupling parameter g^2 does indeed give reasonably good value for the mass of the baryon Octet but the value of that of the Decouplet is much higher than the corresponding experimental results.

It is also evident from the table 4.1 that reasonably good results can be obtained if one considers the channels P_oB, PB, P_oD, PD, V_oB and VB with cut off around 5 bev and the value of coupling g^2 around 3. This combination of the cut off parameter and the coupling g^2 gives reasonably good value for the mass of the baryon Octet although the value of that of the Decouplet is little higher than the experimental value. The most striking feature of this result is that the corresponding value of the U(6,6) coupling g^2 gives for the pion-nucleon coupling $g^2_{\pi NN}$ the value $g^2_{\pi NN} \approx 20$. Considering the approximation we have used and the very much involved nature of the calculation we may conclude that the above result is fairly good.

We have also carried out the calculation taking into account all the channels that arise from the consideration of the meson-baryon scattering in the context of U(6,6) symmetry. The results have been presented at the bottom of the table 4.1. It is clear from the above results that the forces resulting from the inclusion of the VD channel may be so repulsive for the J = 1/2 state that the determinant of the denominator function corresponding to the J = 1/2 state does not give any zero at all. As the inclusion of the VD channel gives only one F-wave channel to the $J_- = 1/2$ state, one cannot conclude that the above non-decreasing nature of the relevant determinant is due to the exclusion of the contribution of the F-waves in the calculation.

Considering the very much involved nature of the calculation described in this chapter, the results we have obtained for the masses of the baryon Octet and Decouplet are reasonably comparable to the experimental ones. In conclusion, we may add that the bootstrap hypothesis, in particular, the N-N* bootstrap of Chew not only works in SU(2) symmetry but it also appears to be reasonably true in higher symmetries as well. Further, we are led to believe that the U(6,6) theory combining the internal symmetry with the space-time symmetries of the strong Interactions of the Hadrons has provided us with a reasonably good basis for carrying out the s-matrix calculations like the one that has been described in this chapter.

APPENDIX A

Phase-conventions and SU(3) vertices

In writing SU(3) invariant Yukawa type strong interaction Lagrangians one has to fix the relative phases of the eigenstates of the multiplets. Determination of the relative phases has been discussed by de Swart[64] who has used Condon-Shortley[65] phase convention in SU(3). Following de Swart, we discuss in this appendix how we have fixed the relative phases of the states of some SU(3)-multiplets which we have considered in our calculations. The procedures we have followed are as follows:

(1) We first fix the phase (real) of an eigenstate (usually the state having the highest weight) then determine the relative phases of the other members of that isomultiplet by the actions of the isospin raising or the isospin lowering $I^{\pm}$, operators. Following Condon-Shortley phase convention we have,

$$I_{\pm}\,\Phi(I,I_3,Y) = \left[(I \mp I_3)(I \pm I_3 + 1)\right]^{\frac{1}{2}} \Phi(I,\, I_3 \pm 1,\, Y) \tag{A.1}$$

(2) The actions of the operators $U^{\pm}$ and $V^{\pm}$ (1.9) take us to different isospin multiplets. Thus, using these operators, we can determine the phases of the other iso-multiplets relative to that of the one we started with. As it is in (A.1), the action of $V+$ on an eigenstate $\Phi(I,I_3\, Y)$ is given by,

$$V_+ \phi(I, I_s, Y) = a_+ \phi(I+\tfrac{1}{2}, I_s+\tfrac{1}{2}, Y+1) + a_- \phi(I-\tfrac{1}{2}, I_s+\tfrac{1}{2}, Y+1) \tag{A.2}$$

where

$$a_+ = \Big\{ (I+I_s+1)[\tfrac{1}{3}(p-q) + I + \tfrac{Y}{2} + 1][\tfrac{1}{3}(p+2q) + I + \tfrac{Y}{2} + 2]$$

$$[\tfrac{1}{3}(2p+q) - I - \tfrac{Y}{2}\,] \Big/ 2(I+1)(2I+1) \Big\}^{\frac{1}{2}} \tag{A.3}$$

$$a_- = \Big\{ (I-I_s)[\tfrac{1}{3}(q-p) + I - \tfrac{Y}{2}\,][\tfrac{1}{3}(p+2q) - I + \tfrac{Y}{2} + 1]$$

$$[\tfrac{1}{3}(2q+p) + I - \tfrac{Y}{2} + 1] \Big/ 2I(2I+1) \Big\}^{\frac{1}{2}} \tag{A.4}$$

where p, q are respectively the numbers of lower and upper indices of the irreducible tensors symmetric in either indices. The effects of the other operators V_-, $U^{\pm}$ can be obtained by using the commutation relations (1.10).

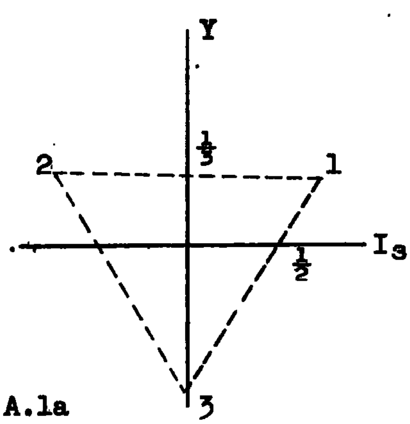

Fig. A.1a

Weight diagram for $D^3(10)$

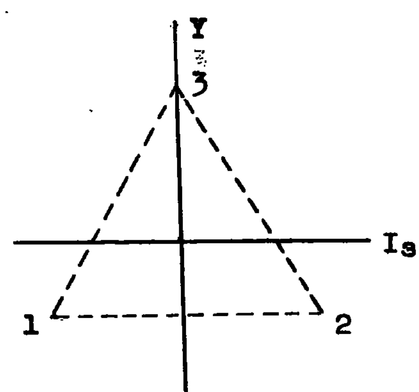

Fig. A.1b

Weight diagram for $D^{3*}(01)$

The operators A^k_ℓ (1.7) can be written in the form:

$$A^k_\ell = x_\ell \frac{\partial}{\partial x^k} - \frac{1}{3}\delta^k_\ell x_n \frac{\partial}{\partial x^n} \tag{A.5}$$

Now, for the three quarks we have the basis vectors x_1, x_2, x_3. Phases are so chosen that we make the following identifications.

$$Q_1 = x_1; \quad Q_2 = x_2; \quad Q_3 = x_3 \tag{A.6}$$

The action of A^k_ℓ on x_k is given by,

$$A^k_\ell x_m = \delta^k_m x_\ell - \frac{1}{3}\delta^k_\ell x_m \tag{A.7}$$

For the three antiquarks we have the basis vectors x^1, x^2, x^3. Phases are to be so chosen that these form the basis of the 3-dimensional contragradient representation. Consequently, we have to choose the following phases:

$$\bar{Q}_1 = -x^1; \quad \bar{Q}_2 = x^2; \quad \bar{Q}_3 = x^3 \tag{A.8}$$

With the above phases, the action of A^k_ℓ on x^m is given by,

$$A^k_\ell x^m = -(\delta^m_\ell x^k + \frac{1}{3}\delta^k_\ell x^m) \tag{A.9}$$

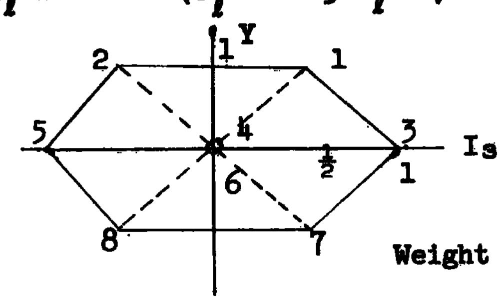

Fig. A.2 Weight diagram for $D^8(11)$

The irreducible tensor for the octet is,

$$\phi^i_j = x^i y_j - \tfrac{1}{3}\,\delta^i_j\,(x^\ell y_\ell) \qquad \text{(A.10)}$$

Using (A.7) and (A.9) we have, from (A.10),

$$A^k_\ell\,\phi^i_j = \delta^k_j\,\phi^i_\ell - \delta^i_\ell\,\phi^k_j \qquad \text{(A.11)}$$

For the pseudoscalar octet we write,

$$P_1 = K_+; \quad P_2 = K_0; \quad P_3 = \pi_+; \quad P_4 = \pi_0;$$

$$P_5 = \pi_-; \quad P_6 = \eta; \quad P_7 = \bar{K}_0; \quad P_8 = K_- \qquad \text{(A.12)}$$

Calculating the eigenvalues, the identifications of the components of the mixed tensor ϕ^i_j with the physical particles with arbitrary phases can be given by,

$$\phi^3_1 = \eta_1 P_1; \quad \phi^3_2 = \eta_2 P_2; \quad \phi^2_1 = \eta_3 P_3;$$

$$\phi^1_2 = \eta_5 P_5; \quad \phi^2_3 = \eta_7 P_7; \quad \phi^1_3 = \eta_8 P_8 \qquad \text{(A.13)}$$

For convenience we set $\eta_3 = -1$, i.e. $P_3 = -\phi^2_1$. The other phases can then be fixed by making use of the actions of $I_\pm$, $U_\pm$, $V_\pm$ on the various states. In order to obtain the phases of P_4 and P_6 we make use of,

$$I - P_3 = \sqrt{2}\, P_4 \; ; \qquad A^1_2\, \phi^2_1 = \phi^2_2 - \phi^1_1$$

$$K - P_1 = \frac{1}{\sqrt{2}} P_4 + \frac{\sqrt{6}}{2} P_6 \; ; \qquad A^1_3\, \phi^3_1 = \phi^3_3 - \phi^1_1$$

and the traceless condition $\phi^1_1 + \phi^2_2 + \phi^3_3 = 0$. Then solving for P_4 and P_6

$$P_4 = \frac{\phi^1_1 - \phi^2_2}{\sqrt{2}} \; ; \quad P_6 = -\frac{\sqrt{3}}{\sqrt{2}}\, \phi^3_3 \, .$$

Now, determining all the relative phases, the pseudoscalar meson octet can be written in the matrix form:

$$P^k_i = \begin{vmatrix} \frac{P_4}{\sqrt{2}} + \frac{P_6}{\sqrt{6}} & -P_3 & -P_1 \\ P_5 & -\frac{P_4}{\sqrt{2}} + \frac{P_6}{\sqrt{6}} & -P_2 \\ P_8 & -P_7 & -\frac{\sqrt{2}}{\sqrt{3}} P_6 \end{vmatrix} \qquad \text{(A.14)}$$

where the lower and upper indices denote rows and columns respectively. The vector octet can also be written in the above form:

$$V_i^k = \begin{vmatrix} \frac{V_4}{\sqrt{2}} + \frac{V_6}{\sqrt{6}} & - V_3 & - V_1 \\ V_5 & -\frac{V_4}{\sqrt{2}} + \frac{V_6}{\sqrt{6}} & - V_2 \\ V_8 & - V_7 & -\sqrt{\frac{2}{3}}\, V_6 \end{vmatrix} \qquad (A.15)$$

where, the physical vector meson-fields have been replaced by V's in (A.15). These relations are as follows:

$$V_1 = K_+^* \; ; \quad V_2 = K_0^* \; ; \quad V_3 = \rho_+ \; ; \quad V_4 = \rho_0 \; ; \quad V_5 = \rho_-$$

$$V_6 = \omega_0 \; ; \quad V_7 = \bar{K}_0^* \; ; \quad V_8 = K_-^* \qquad (A.15')$$

For the baryon octet we make the following assignments

$$B_1 = p \; ; \quad B_2 = n \; ; \quad B_3 = \Sigma_+ \; ; \quad B_4 = \Sigma_0 \; ; \quad B_5 = \Sigma_-$$

$$B_6 = \Lambda_0 \; ; \quad B_7 = \Xi_0 \; ; \quad B_8 = \Xi_- \qquad (A.16)$$

Calculating the eigenvalues of the components of the mixed tensor ϕ_k^i (A.10) we can make similar association as in (A.13) with the physical particles (A.16). As in the case of meson octets we set $B_3 = - \phi_1^2$ and then calculate the relative phases of the other states. Finally, we also write the Baryon octet in the matrix form,

$$B_i^k = \begin{vmatrix} \frac{B_4}{\sqrt{2}} + \frac{B_6}{\sqrt{6}} & -B_3 & -B_1 \\ B_5 & -\frac{B_4}{\sqrt{2}} + \frac{B_6}{\sqrt{6}} & -B_2 \\ B_8 & -B_7 & -\sqrt{\frac{2}{3}}B_6 \end{vmatrix} \qquad (A.17)$$

For the assignments of the antiparticle states we make use of the following relation:

$$|N^* \,;\, I\, I_3\, Y > \;=\; (-1)^{I_3 + \frac{Y}{2}} \{|N \,;\, I,\, -I_3,\, -Y > \}^* \qquad (A.18)$$

where N denotes the dimension of the irreducible representation and N^* that of the contragradient representation. Making use of (A.18) we obtain from (A.16) and (A.17) for the anti-baryon octet the following assignments:

$$\bar{B}_i^k = \begin{vmatrix} \frac{\bar{B}_4}{\sqrt{2}} + \frac{\bar{B}_6}{\sqrt{6}}, & -\bar{B}_5, & -\bar{B}_8 \\ \bar{B}_3 & -\frac{\bar{B}_4}{\sqrt{2}} + \frac{\bar{B}_6}{\sqrt{6}} & -\bar{B}_7 \\ \bar{B}_1 & -\bar{B}_2 & -\sqrt{\frac{2}{3}}\bar{B}_6 \end{vmatrix} \qquad (A.19)$$

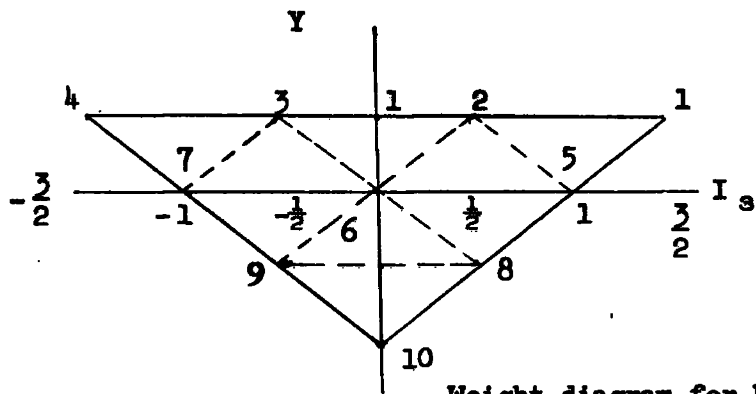

Fig. A.3 Weight diagram for $D^{10}(3,0)$

The fully symmetrised tensor (1.15) of rank 3 and with all the lower indices is,

$$D_{pq\gamma} = \frac{1}{6}[x_p y_q z_\gamma + x_q y_\gamma z_p + x_\gamma y_p z_q + x_\gamma y_q z_p + x_q y_p z_\gamma + x_p y_\gamma z_q] \tag{A.20}$$

Now, the actions of the operator A^k_ℓ on the quantities x_p, y_q, z_γ are given by (A.5), so that we have,

$$A^k_\ell \, x_p y_q z_\gamma = \delta^k_p \, x_\ell y_q z_\gamma + \delta^k_q \, x_p y_\ell z_\gamma + \delta^k_\gamma \, x_p y_q z_\ell - \delta^k_\ell \, x_p y_q z_\gamma \tag{A.21}$$

For the states in the decouplet we first make the following assignments:

$D_1 = N^*_{++}$; $D_2 = N^*_{+}$; $D_3 = N^*_0$; $D_4 = N^*_-$; $D_5 = Y^*_+$

$$D_6 = Y_0^* \; ; \quad D_7 = Y_-^* \; ; \quad D_8 = \Xi_-^* \; ; \quad D_9 = \Xi_0^* \; ; \quad D_{10} = \Omega_- \tag{A.22}$$

Choosing arbitrary phases, the physical particles (A.22) can be identified with the independent components of the tensor $D_{pq\gamma}$ (A.20) as follows:

$$D_{111} = \eta_1 D_1 \; ; \quad D_{112} = \eta_2 D_2 \; ; \quad D_{221} = \eta_3 D_3 \; ;$$

$$D_{222} = \eta_4 D_4 \; ; \quad D_{113} = \eta_5 D_5 \; ; \quad D_{123} = \eta_6 D_6 \; ;$$

$$D_{223} = \eta_7 D_7 \; ; \quad D_{331} = \eta_8 D_8 \; ; \quad D_{332} = \eta_9 D_9 \; ; \tag{A.23}$$

$$D_{333} = \eta_{10} D_{10}$$

We choose $\eta_1 = 1$, i.e. $D_{111} = D_1$, for the sake of convenience. The other phases are fixed by the actions of A_ℓ^k on $D_{pq\gamma}$ and using the phase conventions (1) and (2). Thus, we obtain the following assignments:

$$D_1 = D_{111} \; ; \quad D_2 = \sqrt{3}\, D_{112} \; ; \quad D_3 = \sqrt{3}\, D_{122} \; ;$$

$$D_4 = D_{222} \; ; \quad D_5 = \sqrt{3}\, D_{113} \; ; \quad D_6 = \sqrt{6}\, D_{123} \; ;$$

$$D_7 = \sqrt{3}\, D_{223} \; ; \quad D_8 = \sqrt{3}\, D_{133} \; ; \quad D_9 = \sqrt{3}\, D_{233} \; ;$$

$$D_{10} = D_{333} \tag{A.24}$$

For the anti-decimet states we use (A.18) and obtain the following assignments:

$$\bar{D}_1 = \bar{D}^{111} ; \quad \bar{D}_2 = -\sqrt{3}\,\bar{D}^{112} ; \quad \bar{D}_3 = \sqrt{3}\,\bar{D}^{122} ;$$

$$\bar{D}_4 = -\bar{D}^{222} ; \quad \bar{D}_5 = -\sqrt{3}\,\bar{D}^{113} ; \quad \bar{D}_6 = \sqrt{6}\,\bar{D}^{123} ;$$

$$\bar{D}_7 = -\sqrt{3}\,\bar{D}^{223} ; \quad \bar{D}_8 = \sqrt{3}\,\bar{D}^{133} ; \quad \bar{D}_9 = -\sqrt{3}\,\bar{D}^{233} ;$$

$$\bar{D}_{10} = -\bar{D}^{333} \qquad (A.25)$$

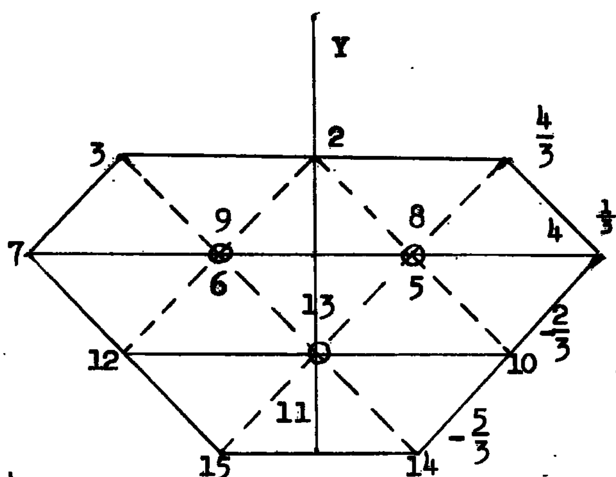

Fig. A.4a
Weight diagram for $D^{15}(21)$

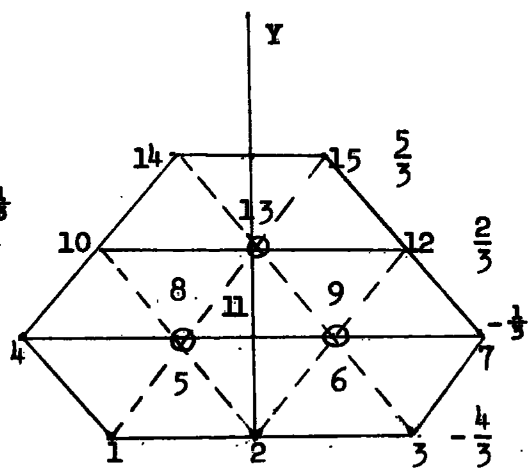

Fig. A.4b
Weight diagram for $D^{15*}(1,2)$

A traceless tensor of rank 3 with one upper and two lower indices (symmetric in the lower indices) form the basis of 15-plet of

SU_3. Such a tensor is given by[66],

$$D^k_{ij} = \tfrac{1}{2}(x_i y_j + x_j y_i)z^k - \tfrac{1}{8}[\delta^k_i(x_\ell y_j + x_j y_\ell)z^\ell$$

$$+ \delta^k_j(x_i y_\ell + x_\ell y_i)z^\ell] \qquad (A.26)$$

Now, the actions of the operators A^k_ℓ on the quantities $x_i y_j z^k$ are given by,

$$A^i_j\, x_k y_\ell z^m = \delta^i_k\, x_j y_\ell z^m + \delta^i_\ell\, x_k y_j z^m - \delta^m_j\, x_k y_\ell z^i - \tfrac{1}{3}\delta^i_j\, x_k y_\ell z^m \qquad (A.27)$$

As explained in the weight diagram (Figure A.4a) we can readily make the following identifications:

$$D^3_{11} = \eta_1 D_1 \; ; \quad D^3_{12} = \eta_2 D_2 \; ; \quad D^3_{22} = \eta_3 D_3 \; ; \quad D^2_{11} = \eta_4 D_4$$

$$D^1_{22} = \eta_7 D_7 \; ; \quad D^2_{13} = \eta_{10} D_{10} \; ; \quad D^1_{23} = \eta_{12} D_{12} \; ;$$

$$D^2_{33} = \eta_{14} D_{14} \; ; \quad D^1_{33} = \eta_{15} D_{15} \qquad (A.28)$$

We have chosen $\eta_4 = 1$, i.e. $D_4 = D^2_{11}$ for the sake of convenience. The other phases in (A.28) are determined by the same procedure as we have used before. The phases of the other six states are obtained by considering the following three trace conditions:

$$D^1_{11} + D^2_{21} + D^3_{31} = 0 \; ; \qquad D^1_{12} + D^2_{22} + D^3_{32} = 0$$

$$D^1_{13} + D^2_{23} + D^3_{33} = 0 \qquad (A.29)$$

Then all the states of the 15-plet are expressed in terms of the independent components of the tensor D^k_{ij} (A.26) as follows:

$$D_1 = D^3_{11} \; ; \quad D_2 = \sqrt{2}\, D^3_{12} \; ; \quad D_3 = D^3_{22} \; ; \quad D_4 = D^2_{11}$$

$$D_5 = -\frac{1}{\sqrt{3}}(D^1_{11} - 2D^2_{21}) \; ; \quad D_6 = \frac{1}{\sqrt{3}}(D^2_{22} - 2D^1_{12})$$

$$D_7 = -D^1_{22} \; ; \quad D_8 = -\frac{4}{\sqrt{6}}(D^1_{11} + D^2_{21}) \; ;$$

$$D_9 = -\frac{4}{\sqrt{6}}(D^2_{22} + D^1_{12}) \; ; \quad D_{10} = \sqrt{2}\, D^2_{13} \; ;$$

$$D_{11} = D^2_{23} - D^1_{13} \; ; \quad D_{12} = -\sqrt{2}\, D^1_{23} \; ; \quad D_{13} = -\sqrt{2}(D^1_{13} + D^2_{23})$$

$$D_{14} = D^2_{33} \; ; \quad D_{15} = -D^1_{33} \qquad (A.30)$$

Since the states belonging to 15-plet are associated with fractional charge, we cannot make use of (A.18) to determine the phases of the states of the contragradient representation. Therefore, we have to follow the same procedure as we have used for the 15-plet and the results are as follows:

$$\bar{D}_1 = \bar{D}^{11}_3 \; ; \quad \bar{D}_2 = -\sqrt{2}\,\bar{D}^{12}_3 ; \quad \bar{D}_3 = \bar{D}^{22}_3 \; ; \quad \bar{D}_4 = \bar{D}^{11}_2$$

$$\bar{D}_5 = \frac{1}{\sqrt{3}}\left(\bar{D}^{11}_1 - 2\bar{D}^{21}_2\right) ; \quad \bar{D}_6 = \frac{1}{\sqrt{3}}\left(\bar{D}^{22}_2 - 2\bar{D}^{12}_1\right)$$

$$\bar{D}_7 = \bar{D}^{22}_1 \; ; \quad \bar{D}_8 = \frac{4}{\sqrt{6}}\left(\bar{D}^{11}_1 + \bar{D}^{21}_2\right) ; \quad \bar{D}_9 = -\frac{4}{\sqrt{6}}\left(\bar{D}^{22}_2 + \bar{D}^{12}_1\right)$$

$$\bar{D}_{10} = -\sqrt{2}\,\bar{D}^{13}_2 \; ; \quad \bar{D}_{11} = \bar{D}^{23}_2 - \bar{D}^{13}_1 \; ; \quad \bar{D}_{12} = \sqrt{2}\,\bar{D}^{23}_1$$

$$\bar{D}_{13} = -\sqrt{2}\left(\bar{D}^{13}_1 + \bar{D}^{23}_2\right) ; \quad \bar{D}_{14} = \bar{D}^{33}_2 \; ; \quad \bar{D}_{15} = \bar{D}^{33}_1 \qquad (A.31)$$

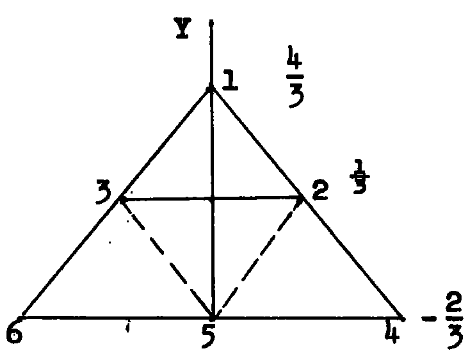

Fig. A.5a

Weight diagram for $D^{6*}(02)$

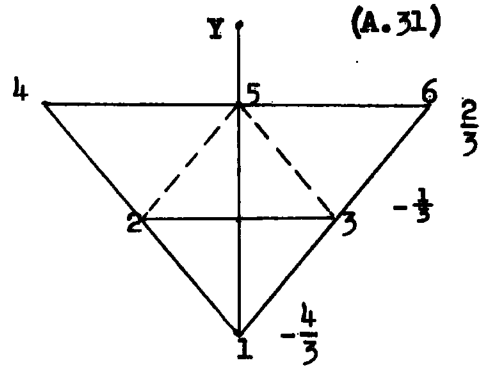

Fig. A.5b

Weight diagram for $D^{6}(2,0)$

A traceless tensor of rank 3 with one upper and two lower indices (antisymmetric in the lower indices) form the basis of 6*-plet of SU_3. Such a tensor is given by[66],

$$S^k_{ij} = \tfrac{1}{2}(x_i y_j - x_j y_i)x^k - \tfrac{1}{4}[\delta^k_i (x_\ell y_j - x_j y_\ell)z^\ell$$

$$+ \delta^k_j (x_i y_\ell - x_\ell y_i)z^\ell] \qquad (A.32)$$

Let $S_k (k = 1, \dots 6)$ denote the six states (Figure A.5a) of the 6*-plet. Then following the same procedure as we have used in the case of 15-plet, we obtain the following identifications:

$$S_1 = \sqrt{2}\, S^3_{12} \; ; \quad S_2 = 2S^2_{12} \; ; \quad S_3 = -2S^1_{12} \; ;$$

$$S_4 = -\sqrt{2}\, S^2_{13} \; ; \quad S_5 = 2S^1_{13} \; ; \quad S_6 = \sqrt{2}\, S^1_{23} \qquad (A.33)$$

$$S^3_{32} = -S^1_{12} \; ; \quad S^2_{12} = -S^3_{13} \; ; \quad S^3_{23} = -S^1_{13}$$

$$S^k_{11} = S^k_{22} = S^k_{33} = 0$$

For the states belonging to the 6-plet we follow the same procedure as we used in the case of 6*-plet above. The results are as follows:

$$\bar{S}_1 = -\sqrt{2}\,\bar{S}_3^{12} \;;\quad \bar{S}_2 = -2\bar{S}_2^{12} \;;\quad \bar{S}_3 = -2\bar{S}_1^{12}$$

$$\bar{S}_4 = \sqrt{2}\,\bar{S}_2^{13} \;;\quad \bar{S}_5 = 2\bar{S}_1^{13} \;;\quad \bar{S}_6 = -\sqrt{2}\,\bar{S}_1^{23}$$

(A.34)

$$\bar{S}_1^{12} = -\bar{S}_3^{32} \;;\quad \bar{S}_2^{12} = -\bar{S}_3^{13} \;;\quad \bar{S}_1^{13} = -\bar{S}_2^{23}$$

$$\bar{S}_k^{11} = \bar{S}_k^{22} = \bar{S}_k^{33} = 0$$

Having determined in the above the fields of the particles belonging to the irreducible representations of SU_3, we can now easily write down the SU_3 - invariant Yukawa type strong interaction Lagrangians. In what follows, we write down those SU(3) vertices which we need in our calculations.

For meson-quark scattering with a Quark in the intermediate state the following interaction Lagrangian which we obtain by using (A.6), (A.8) and (A.14), occurs at both the vertices

$$\mathcal{L}(\bar{Q}Q\phi) = \bar{\psi}^{\alpha}\psi_{\beta}\phi_{\alpha}^{\beta} = \{ P_1\bar{Q}_1Q_3 - P_2\bar{Q}_2Q_3 + P_3\bar{Q}_1Q_2$$

$$- \frac{1}{\sqrt{2}} P_4(\bar{Q}_1Q_1 + \bar{Q}_2Q_2) + P_5\,\bar{Q}_2Q_1 + \frac{1}{\sqrt{6}} P_6(-\bar{Q}_1Q_1$$

$$+ \bar{Q}_2Q_2 - 2\bar{Q}_3Q_3) - P_7\,\bar{Q}_3Q_2 + P_8\,\bar{Q}_3Q_1 \} \qquad (A.35)$$

The vertices involving one Quark, one meson and a 15-plet can be obtained by using (A.6), (A.8), (A.14), (A.30) and (A.31) and these are as follows:

$$L_1(\bar{D}\phi Q) = \bar{D}_\gamma^{\alpha\beta}\,\psi_\beta\,\phi_\alpha^\gamma = \{ P_1(-\bar{D}_1 Q_1 + \frac{1}{\sqrt{2}}\bar{D}_2 Q_2 + \frac{\sqrt{3}}{2\sqrt{2}}\bar{D}_8\,Q_3)$$

$$+ P_2(\frac{1}{\sqrt{2}}\bar{D}_2 Q_1 - \bar{D}_3 Q_2 - \frac{\sqrt{3}}{2\sqrt{2}}\bar{D}_9 Q_3) + P_3(-\bar{D}_4\,Q_1$$

$$+ \frac{1}{\sqrt{3}}\bar{D}_5\,Q_2 - \frac{1}{2\sqrt{6}}\bar{D}_8\,Q_2 + \frac{1}{\sqrt{2}}\bar{D}_{10}\,Q_3) +$$

$$+ P_4\,(\frac{\sqrt{2}}{\sqrt{3}}\bar{D}_5\,Q_1 + \frac{1}{4\sqrt{3}}\bar{D}_8 Q_1 - \frac{\sqrt{2}}{\sqrt{3}}\bar{D}_6 Q_2$$

$$+ \frac{1}{4\sqrt{3}}\bar{D}_9 Q_2 + \frac{1}{\sqrt{2}}\bar{D}_{11} Q_3) + P_5(-\frac{1}{\sqrt{3}}\bar{D}_6 Q_1 - \frac{1}{2\sqrt{6}}\bar{D}_9 Q_1$$

$$+ \bar{D}_7\,Q_2 + \frac{1}{\sqrt{2}}\bar{D}_{12}\,Q_3) + P_6(\frac{3}{4}\bar{D}_8\,Q_1 - \frac{3}{4}\bar{D}_9\,Q_2$$

$$- \frac{\sqrt{3}}{2}\bar{D}_{13}\,Q_3) + P_7(\frac{1}{\sqrt{2}}\bar{D}_{10}\,Q_1 - \frac{1}{2}\bar{D}_{11}\,Q_2$$

$$+ \frac{1}{2\sqrt{2}}\bar{D}_{13}\,Q_2 - \bar{D}_{14}\,Q_3) + P_8(-\frac{1}{2}\bar{D}_{11}\,Q_1$$

$$- \frac{1}{2\sqrt{2}} \bar{D}_{13} Q_1 + \frac{1}{\sqrt{2}} \bar{D}_{12} Q_2 + \bar{D}_{15} Q_3) \} \quad .$$

(A36a)

$$L_2(\bar{Q}_2 \phi D) = \bar{\psi}^{\alpha} D^{\gamma}_{\alpha\beta} \phi^{\beta}_{\gamma} = \Big\{ P_1 (-\tfrac{1}{2} \bar{Q}_1 D_{11} - \frac{1}{2\sqrt{2}} \bar{Q}_1 D_{13}$$

$$+ \frac{1}{\sqrt{2}} \bar{Q}_2 D_{12} + \bar{Q}_3 D_{15}) + P_2 (\frac{1}{\sqrt{2}} \bar{Q}_1 D_{10} - \tfrac{1}{2} \bar{Q}_2 D_{11}$$

$$+ \frac{1}{2\sqrt{2}} \bar{Q}_2 D_{13} - \bar{Q}_3 D_{14}) + P_3 (- \frac{1}{\sqrt{3}} \bar{Q}_1 D_6$$

$$- \frac{1}{2\sqrt{6}} \bar{Q}_1 D_9 + \bar{Q}_2 D_7 + \frac{1}{\sqrt{2}} \bar{Q}_3 D_{12}) + P_4 (\frac{\sqrt{2}}{\sqrt{3}} \bar{Q}_1 D_5$$

$$+ \frac{1}{4\sqrt{3}} \bar{Q}_1 D_8 - \frac{\sqrt{2}}{\sqrt{3}} \bar{Q}_2 D_6 + \frac{1}{4\sqrt{3}} \bar{Q}_2 D_9 - \frac{1}{\sqrt{2}} \bar{Q}_3 D_{11})$$

$$+ P_5 (-\bar{Q}_1 D_4 + \frac{1}{\sqrt{3}} \bar{Q}_2 D_5 - \frac{1}{2\sqrt{6}} \bar{Q}_2 D_8 + \frac{1}{\sqrt{2}} \bar{Q}_3 D_{10})$$

$$+ P_6 (\frac{3}{4} \bar{Q}_1 D_8 - \frac{3}{4} \bar{Q}_2 D_9 - \frac{\sqrt{3}}{2} \bar{Q}_3 D_{13}) + P_7 (\frac{1}{\sqrt{2}} \bar{Q}_1 D_2$$

$$- \bar{Q}_2 D_s - \frac{\sqrt{3}}{2\sqrt{2}} \bar{Q}_s D_8) + P_8(-\bar{Q}_1 D_1 + \frac{1}{\sqrt{2}} \bar{Q}_2 D_2$$

$$+ \frac{\sqrt{3}}{2\sqrt{2}} \bar{Q}_s D_8) \Big\} \qquad (A.36b)$$

The vertices involving one quark, one meson and a 6*-plet can be obtained by using (A.6), (A.8), (A.14), (A.33) and (A.34) and these are as follows:

$$L_1(\bar{S}\phi Q) = \bar{S}^{\alpha\beta}_{\gamma} \psi_\alpha \phi^{\gamma}_{\beta} = \Big\{ P_1(-\frac{1}{\sqrt{2}} \bar{S}_1 Q_2 + \tfrac{1}{2} \bar{S}_2 Q_s)$$

$$+ P_2(\frac{1}{\sqrt{2}} \bar{S}_1 Q_1 + \tfrac{1}{2} \bar{S}_3 Q_s) + P_3(-\tfrac{1}{2} \bar{S}_2 Q_2 + \frac{1}{\sqrt{2}} \bar{S}_4 Q_s)$$

$$+ P_4(\frac{1}{2\sqrt{2}} \bar{S}_2 Q_1 + \frac{1}{2\sqrt{2}} \bar{S}_3 Q_2 - \frac{1}{\sqrt{2}} \bar{S}_5 Q_s) + P_5(-\tfrac{1}{2} \bar{S}_3 Q_1$$

$$+ \frac{1}{\sqrt{2}} \bar{S}_6 Q_s) + P_6(-\frac{\sqrt{3}}{2\sqrt{2}} \bar{S}_2 Q_1 + \frac{\sqrt{3}}{2\sqrt{2}} \bar{S}_3 Q_2)$$

$$+ P_7(-\frac{1}{\sqrt{2}} \bar{S}_4 Q_1 + \tfrac{1}{2} \bar{S}_5 Q_2) + P_8(\tfrac{1}{2}\bar{S}_5 Q_1 - \frac{1}{\sqrt{2}} \bar{S}_6 Q_2) \Big\}$$

(A.37a)

$$L_2(\bar{Q}\phi S) = \bar{\psi}^{\alpha} S^{\gamma}_{\alpha\beta}\, \phi^{\gamma}_{\beta} = \Big\{ P_1(\tfrac{1}{2}\bar{Q}_1 S_5 - \frac{1}{\sqrt{2}}\bar{Q}_2 S_6) + P_2(\tfrac{1}{2}\bar{Q}_2 S_5$$

$$- \frac{1}{\sqrt{2}}\bar{Q}_1 S_4) + P_3(-\tfrac{1}{2}\bar{Q}_1 S_3 + \frac{1}{\sqrt{2}}\bar{Q}_3 S_6) + P_4(\frac{1}{2\sqrt{2}}\bar{Q}_1 S_2$$

$$+ \frac{1}{2\sqrt{2}}\bar{Q}_2 S_3 - \frac{1}{\sqrt{2}}\bar{Q}_3 S_5) + P_5(-\tfrac{1}{2}\bar{Q}_2 S_2 + \frac{1}{\sqrt{2}}\bar{Q}_3 S_4)$$

$$+ P_6(-\frac{\sqrt{3}}{2\sqrt{2}}\bar{Q}_1 S_2 + \frac{\sqrt{3}}{2\sqrt{2}}\bar{Q}_2 S_3) + P_7(\frac{1}{\sqrt{2}}\bar{Q}_1 S_1$$

$$+ \tfrac{1}{2}\bar{Q}_3 S_3) + P_8(-\frac{1}{\sqrt{2}}\bar{Q}_2 S_1 + \tfrac{1}{2}\bar{Q}_3 S_2) \Big\} \qquad \text{(A.37b)}$$

The vertices involving two nucleons and one meson are obtained by using (A.14), (A.17) and (A.19). In our calculations we require the following:

$$\mathcal{L}(\bar{B}\phi B)F = \Big\{ P_1(-\frac{1}{\sqrt{2}}\bar{B}_4 B_8 + \frac{1}{\sqrt{2}}\bar{B}_1 B_4 + \bar{B}_3 B_7 - \bar{B}_2 B_5$$

$$- \frac{\sqrt{3}}{\sqrt{2}}\bar{B}_6 B_8 + \frac{\sqrt{3}}{\sqrt{2}}\bar{B}_1 B_6) + P_2(\bar{B}_5 B_8 - \bar{B}_1 B_3$$

$$- \frac{1}{\sqrt{2}}\bar{B}_4 B_7 + \frac{1}{\sqrt{2}}\bar{B}_2 B_4 + \frac{\sqrt{3}}{\sqrt{2}}\bar{B}_6 B_7 - \frac{\sqrt{3}}{\sqrt{2}}\bar{B}_2 B_6)$$

$$+ P_3 (- \sqrt{2}\, \bar{B}_4 B_5 + \sqrt{2}\, \bar{B}_3 B_4 + \bar{B}_1 B_2 - \bar{B}_7 B_8)$$

$$+ P_4 (\sqrt{2}\, \bar{B}_5 B_5 - \sqrt{2}\, \bar{B}_3 B_3 - \frac{1}{\sqrt{2}} \bar{B}_1 B_1 - \frac{1}{\sqrt{2}} \bar{B}_2 B_2$$

$$+ \frac{1}{\sqrt{2}} \bar{B}_7 B_7 + \frac{1}{\sqrt{2}} \bar{B}_8 B_8) + P_5 (- \sqrt{2}\, \bar{B}_5 B_4$$

$$+ \sqrt{2}\, \bar{B}_4 B_3 + \bar{B}_2 B_1 - \bar{B}_8 B_7) + P_6 (- \frac{\sqrt{3}}{\sqrt{2}} \bar{B}_1 B_1$$

$$+ \frac{\sqrt{3}}{\sqrt{2}} \bar{B}_2 B_2 - \frac{\sqrt{3}}{\sqrt{2}} \bar{B}_7 B_7 + \frac{\sqrt{3}}{\sqrt{2}} \bar{B}_8 B_8) + P_7 (\bar{B}_8 B_5$$

$$- \bar{B}_3 B_1 - \frac{1}{\sqrt{2}} \bar{B}_7 B_4 + \frac{1}{\sqrt{2}} \bar{B}_4 B_2 + \frac{\sqrt{3}}{\sqrt{2}} \bar{B}_7 B_6$$

$$- \frac{\sqrt{3}}{\sqrt{2}} \bar{B}_6 B_2) + P_8 (- \frac{1}{\sqrt{2}} \bar{B}_8 B_4 + \frac{1}{\sqrt{2}} \bar{B}_4 B_1 + \bar{B}_7 B_3$$

$$- \bar{B}_5 B_2 - \frac{\sqrt{3}}{\sqrt{2}} \bar{B}_8 B_6 + \frac{\sqrt{3}}{\sqrt{2}} \bar{B}_6 B_1) \Big\}$$

(A.38a)

$$\mathcal{L}(\bar{B}\,\phi\,B)D + \tfrac{2}{3}F = \Big\{ P_1\Big(-\frac{5}{3\sqrt{2}}\bar{B}_4 B_8 - \frac{1}{3\sqrt{2}}\bar{B}_1 B_4 + \frac{5}{3}\bar{B}_3 B_7$$

$$+ \frac{1}{3}\bar{B}_2 B_5 + \frac{\sqrt{3}}{\sqrt{2}}\bar{B}_1 B_6 - \frac{1}{\sqrt{6}}\bar{B}_6 B_8\Big) + P_2\Big(\frac{5}{3}\bar{B}_5 B_8$$

$$+ \frac{1}{3}\bar{B}_1 B_3 - \frac{5}{3\sqrt{2}}\bar{B}_4 B_7 - \frac{1}{3\sqrt{2}}\bar{B}_2 B_4 - \frac{\sqrt{3}}{\sqrt{2}}\bar{B}_2 B_6$$

$$+ \frac{1}{\sqrt{6}}\bar{B}_6 B_7\Big) + P_3\Big(\frac{5}{3}\bar{B}_1 B_2 + \frac{1}{3}\bar{B}_7 B_8 - \frac{\sqrt{2}}{\sqrt{3}}\bar{B}_6 B_5$$

$$- \frac{\sqrt{2}}{\sqrt{3}}\bar{B}_3 B_6 - \frac{2\sqrt{2}}{3}\bar{B}_4 B_5 + \frac{2\sqrt{2}}{3}\bar{B}_3 B_4\Big)$$

$$+ P_4\Big(\frac{\sqrt{2}}{\sqrt{3}}\bar{B}_4 B_6 + \frac{\sqrt{2}}{\sqrt{3}}\bar{B}_6 B_4 - \frac{5}{3\sqrt{2}}\bar{B}_1 B_1 - \frac{5}{3\sqrt{2}}\bar{B}_2 B_2$$

$$- \frac{1}{3\sqrt{2}}\bar{B}_7 B_7 - \frac{1}{3\sqrt{2}}\bar{B}_8 B_8 + \frac{2\sqrt{2}}{3}\bar{B}_5 B_5 - \frac{2\sqrt{2}}{3}\bar{B}_3 B_3\Big)$$

$$+ P_5\Big(\frac{5}{3}\bar{B}_2 B_1 + \frac{1}{3}\bar{B}_8 B_7 - \frac{\sqrt{2}}{\sqrt{3}}\bar{B}_5 B_6 - \frac{\sqrt{2}}{\sqrt{3}}\bar{B}_6 B_3$$

$$- \frac{2\sqrt{2}}{3} \bar{B}_5 B_4 + \frac{2\sqrt{2}}{3} \bar{B}_4 B_8) + P_6 (- \frac{\sqrt{2}}{\sqrt{3}} \bar{B}_8 B_8$$

$$+ \frac{\sqrt{2}}{\sqrt{3}} \bar{B}_4 B_4 - \frac{\sqrt{2}}{\sqrt{3}} \bar{B}_5 B_5 - \frac{\sqrt{2}}{\sqrt{3}} \bar{B}_6 B_6 - \frac{1}{\sqrt{6}} \bar{B}_1 B_1$$

$$+ \frac{1}{\sqrt{6}} \bar{B}_2 B_2 - \frac{\sqrt{3}}{\sqrt{2}} \bar{B}_7 B_7 + \frac{\sqrt{3}}{\sqrt{2}} \bar{B}_8 B_8) + P_7 (\frac{5}{3} \bar{B}_8 B_5$$

$$+ \frac{1}{3} \bar{B}_8 B_1 - \frac{5}{3\sqrt{2}} \bar{B}_7 B_4 - \frac{1}{3\sqrt{2}} \bar{B}_4 B_2 + \frac{1}{\sqrt{6}} \bar{B}_7 B_6$$

$$- \frac{\sqrt{3}}{\sqrt{2}} \bar{B}_6 B_2) + P_8 (- \frac{5}{3\sqrt{2}} \bar{B}_8 B_4 - \frac{1}{3\sqrt{2}} \bar{B}_4 B_1$$

$$+ \frac{5}{3} \bar{B}_7 B_8 + \frac{1}{3} \bar{B}_5 B_2 - \frac{1}{\sqrt{6}} \bar{B}_8 B_6 + \frac{\sqrt{3}}{\sqrt{2}} \bar{B}_6 B_1) \Big\}$$

(A.38b)

The vertex involving one meson and two decouplets can be obtained from (A.14), (A.24) and (A.25) and is as follows:

$$\mathcal{L} (\bar{D} \phi D) = \Big\{ P_1 (- \frac{1}{\sqrt{3}} \bar{D}_1 D_5 + \frac{\sqrt{2}}{3} \bar{D}_2 D_6 + \frac{1}{3} \bar{D}_8 D_7 + \frac{2}{3} \bar{D}_5 D_8$$

$$- \frac{\sqrt{2}}{3}\bar{D}_6 D_9 - \frac{1}{\sqrt{3}}\bar{D}_8 D_{10}) + P_2\left(\frac{1}{3}\bar{D}_2 D_5 - \frac{\sqrt{2}}{3}\bar{D}_3 D_6\right.$$

$$+ \frac{1}{\sqrt{3}}\bar{D}_4 D_7 - \frac{\sqrt{2}}{3}\bar{D}_6 D_8 + \frac{2}{3}\bar{D}_7 D_9 + \frac{1}{\sqrt{3}}\bar{D}_9 D_{10})$$

$$+ P_3\left(- \frac{1}{\sqrt{3}}\bar{D}_1 D_2 + \frac{2}{3}\bar{D}_2 D_3 - \frac{1}{\sqrt{3}}\bar{D}_3 D_4 + \frac{\sqrt{2}}{3}\bar{D}_5 D_6\right.$$

$$- \frac{\sqrt{2}}{3}\bar{D}_6 D_7 - \frac{1}{3}\bar{D}_8 D_9) + P_4\left(\frac{1}{\sqrt{2}}\bar{D}_1 D_1 - \frac{1}{3\sqrt{2}}\bar{D}_2 D_2\right.$$

$$- \frac{1}{3\sqrt{2}}\bar{D}_3 D_3 - \frac{\sqrt{2}}{3}\bar{D}_5 D_5 + \frac{1}{3\sqrt{2}}\bar{D}_8 D_8 + \frac{1}{\sqrt{2}}\bar{D}_4 D_4$$

$$+ \frac{\sqrt{2}}{3}\bar{D}_7 D_7 + \frac{1}{3\sqrt{2}}\bar{D}_9 D_9) + P_5\left(- \frac{1}{\sqrt{3}}\bar{D}_2 D_1 + \frac{2}{3}\bar{D}_3 D_2\right.$$

$$- \frac{1}{\sqrt{3}}\bar{D}_4 D_3 + \frac{\sqrt{2}}{3}\bar{D}_6 D_5 - \frac{\sqrt{2}}{3}\bar{D}_7 D_6 - \frac{1}{3}\bar{D}_9 D_8)$$

$$+ P_6\left(\frac{1}{\sqrt{6}}\bar{D}_1 D_1 - \frac{1}{\sqrt{6}}\bar{D}_2 D_2 + \frac{1}{\sqrt{6}}\bar{D}_3 D_3 - \frac{1}{\sqrt{6}}\bar{D}_4 D_4\right.$$

$$- \frac{1}{\sqrt{6}} \bar{D}_8 D_8 + \frac{1}{\sqrt{6}} \bar{D}_9 D_9 + \frac{\sqrt{2}}{\sqrt{3}} \bar{D}_{10} D_{10})$$

$$+ P_7(\frac{1}{3} \bar{D}_5 D_2 - \frac{\sqrt{2}}{3} \bar{D}_6 D_3 - \frac{\sqrt{2}}{3} \bar{D}_8 D_6 + \frac{1}{\sqrt{3}} \bar{D}_7 D_4$$

$$+ \frac{2}{3} \bar{D}_9 D_7 + \frac{1}{\sqrt{3}} \bar{D}_{10} D_9) + P_8(- \frac{1}{\sqrt{3}} \bar{D}_5 D_1$$

$$+ \frac{\sqrt{2}}{3} \bar{D}_6 D_2 - \frac{1}{3} \bar{D}_7 D_3 + \frac{2}{3} \bar{D}_8 D_5 - \frac{\sqrt{2}}{3} \bar{D}_9 D_6$$

$$- \frac{1}{\sqrt{3}} \bar{D}_{10} D_8) \Big\} \qquad (A.39)$$

The vertices involving one meson, one baryon and one decouplet are obtained by using (A.14), (A.17), (A.19), (A.24) and (A.25) and these are as follows:

$$L_1(\bar{D} \phi N) = \Big\{ P_1(- \frac{\sqrt{2}}{\sqrt{3}} \bar{D}_2 B_4 + \frac{1}{\sqrt{3}} \bar{D}_3 B_5 + \bar{D}_1 B_3 - \frac{1}{\sqrt{3}} \bar{D}_5 B_7$$

$$+ \frac{1}{\sqrt{6}} \bar{D}_6 B_8) + P_2(- \frac{1}{\sqrt{3}} \bar{D}_2 B_3 + \frac{\sqrt{2}}{\sqrt{3}} \bar{D}_3 B_4 - \bar{D}_4 B_5$$

$$+ \frac{1}{\sqrt{6}} \bar{D}_6 B_7 - \frac{1}{\sqrt{3}} \bar{D}_7 B_8) + P_3(- \bar{D}_1 B_1 + \frac{1}{\sqrt{3}} \bar{D}_2 B_2$$

$$+ \frac{1}{\sqrt{6}} \bar{D}_5 B_4 + \frac{1}{\sqrt{2}} \bar{D}_5 B_6 - \frac{1}{\sqrt{6}} \bar{D}_6 B_5 - \frac{1}{\sqrt{3}} \bar{D}_8 B_8)$$

$$+ P_4(\frac{\sqrt{2}}{\sqrt{3}} \bar{D}_2 B_1 - \frac{\sqrt{2}}{\sqrt{3}} \bar{D}_3 B_2 - \frac{1}{\sqrt{6}} \bar{D}_5 B_3 - \frac{1}{\sqrt{2}} \bar{D}_6 B_6$$

$$+ \frac{1}{\sqrt{6}} \bar{D}_7 B_5 + \frac{1}{\sqrt{6}} \bar{D}_8 B_7 + \frac{1}{\sqrt{6}} \bar{D}_9 B_8)$$

$$+ P_5(- \frac{1}{\sqrt{3}} \bar{D}_3 B_1 + \bar{D}_4 B_2 + \frac{1}{\sqrt{6}} \bar{D}_6 B_3 - \frac{1}{\sqrt{6}} \bar{D}_7 B_4$$

$$+ \frac{1}{\sqrt{2}} \bar{D}_7 B_6 - \frac{1}{\sqrt{3}} \bar{D}_9 B_7) + P_6(- \frac{1}{\sqrt{2}} \bar{D}_5 B_3$$

$$+ \frac{1}{\sqrt{2}} \bar{D}_6 B_4 - \frac{1}{\sqrt{2}} \bar{D}_7 B_5 + \frac{1}{\sqrt{2}} \bar{D}_8 B_7 - \frac{1}{\sqrt{2}} \bar{D}_9 B_8)$$

$$+ P_7(\frac{1}{\sqrt{3}} \bar{D}_5 B_1 - \frac{1}{\sqrt{6}} \bar{D}_6 B_2 - \frac{1}{\sqrt{6}} \bar{D}_8 B_4$$

$$- \frac{1}{\sqrt{2}} \bar{D}_8 B_6 + \frac{1}{\sqrt{3}} \bar{D}_9 B_5 + \bar{D}_{10} B_8) + P_8(- \frac{1}{\sqrt{6}} \bar{D}_6 B_1$$

$$+ \frac{1}{\sqrt{3}} \bar{D}_7 B_2 + \frac{1}{\sqrt{3}} \bar{D}_8 B_3 - \frac{1}{\sqrt{6}} \bar{D}_9 B_4 + \frac{1}{\sqrt{2}} \bar{D}_9 B_6$$

$$- \bar{D}_{10} B_7) \Big\} \qquad \text{(A.40a)}$$

$$L_2 (\bar{B} \phi D) = \Big\{ P_1 (- \frac{1}{\sqrt{6}} \bar{B}_1 D_6 + \frac{1}{\sqrt{3}} \bar{B}_2 D_7 + \frac{1}{\sqrt{3}} \bar{B}_3 D_8$$

$$- \frac{1}{\sqrt{6}} \bar{B}_4 D_9 + \frac{1}{\sqrt{2}} \bar{B}_6 D_9 - \bar{B}_7 D_{10}) + P_2 (\frac{1}{\sqrt{3}} \bar{B}_1 D_5$$

$$- \frac{1}{\sqrt{6}} \bar{B}_2 D_6 - \frac{1}{\sqrt{6}} \bar{B}_4 D_8 + \frac{1}{\sqrt{3}} \bar{B}_5 D_9 - \frac{1}{\sqrt{2}} \bar{B}_6 D_8$$

$$+ \bar{B}_8 D_{10}) + P_3 (- \frac{1}{\sqrt{3}} \bar{B}_1 D_3 + \bar{B}_2 D_4 + \frac{1}{\sqrt{6}} \bar{B}_3 D_6$$

$$- \frac{1}{\sqrt{6}} \bar{B}_4 D_7 + \frac{1}{\sqrt{2}} \bar{B}_6 D_7 - \frac{1}{\sqrt{3}} \bar{B}_7 D_9) + P_4 (\frac{\sqrt{2}}{\sqrt{3}} \bar{B}_1 D_2$$

$$- \frac{\sqrt{2}}{\sqrt{3}} \bar{B}_2 D_3 - \frac{1}{\sqrt{6}} \bar{B}_3 D_5 + \frac{1}{\sqrt{6}} \bar{B}_5 D_7 - \frac{1}{\sqrt{2}} \bar{B}_6 D_6$$

$$+ \frac{1}{\sqrt{6}} \bar{B}_7 D_8 + \frac{1}{\sqrt{6}} \bar{B}_8 D_9) + P_5 (- \bar{B}_1 D_1 + \frac{1}{\sqrt{3}} \bar{B}_2 D_2$$

$$+ \frac{1}{\sqrt{6}} \bar{B}_4 D_5 - \frac{1}{\sqrt{6}} \bar{B}_5 D_6 + \frac{1}{\sqrt{2}} \bar{B}_6 D_5 - \frac{1}{\sqrt{3}} \bar{B}_8 D_8)$$

$$+ P_6(- \frac{1}{\sqrt{2}} \bar{B}_3 D_5 + \frac{1}{\sqrt{2}} \bar{B}_4 \ D_6 - \frac{1}{\sqrt{2}} \bar{B}_5 D_7$$

$$+ \frac{1}{\sqrt{2}} \bar{B}_7 D_8 - \frac{1}{\sqrt{2}} \bar{B}_8 D_9) + P_7(- \frac{1}{\sqrt{3}} \bar{B}_3 D_2$$

$$+ \qquad + \frac{\sqrt{2}}{\sqrt{3}} \bar{B}_4 D_3 - \bar{B}_5 D_4 + \frac{1}{\sqrt{6}} \bar{B}_7 D_6 - \frac{1}{\sqrt{3}} \bar{B}_8 D_7)$$

$$+ P_8(\bar{B}_3 D_1 - \frac{\sqrt{2}}{\sqrt{3}} \bar{B}_4 D_2 + \frac{1}{\sqrt{3}} \bar{B}_5 D_3 - \frac{1}{\sqrt{3}} \bar{B}_7 D_5$$

$$+ \frac{1}{\sqrt{6}} \bar{B}_8 D_6) \Big\} \qquad \text{(A.40b)}$$

Again the vertex involving a meson-singlet and two baryon octets is given by,

$$\mathcal{L}(\bar{B}\phi^{\circ}B)s = \bar{B}^{\beta}_{\alpha} B^{\alpha}_{\beta} \phi^{\alpha}_{\alpha} = \frac{1}{3} P_0(-\bar{B}_1 B_1 + \bar{B}_2 B_2 - \bar{B}_3 B_3 + \bar{B}_4 B_4$$

$$- \bar{B}_5 B_5 + \bar{B}_6 B_6 + \bar{B}_7 B_7 - \bar{B}_8 B_8) \qquad \text{(A.41)}$$

Similarly the vertex involving a meson singlet and two decouplets is,

$$\mathcal{L}(\bar{D}\phi^{o}D) = \bar{D}^{\alpha\beta\gamma} D_{\alpha\beta\gamma}\phi^{\gamma}_{\gamma} = \frac{1}{3} P_0 (\bar{D}_1 D_1 - \bar{D}_2 D_2 + \bar{D}_3 D_3 - \bar{D}_4 D_4 - \bar{D}_5 D_5 + \bar{D}_6 D_6 - \bar{D}_7 D_7 + \bar{D}_8 D_8 - \bar{D}_9 D_9 - \bar{D}_{10} D_{10}) \qquad (A.42)$$

In the derivations of the above Lagrangians we have, for convenience, omitted the space-time factors which are required to make these vertices Lorentz invariant. These factors are to be taken into account when we use these vertices in the related calculations.

APPENDIX B

Calculations of SU(3) Coupling Coefficients

In almost all the dynamical calculations involving SU(3) symmetry, we require SU(3)-coupling coefficients. Here, in this appendix, we discuss a method which makes use of the SU(3) invariant vertices (appendix A) for calculating the pole coefficients corresponding to a direct channel scattering diagram. This method can also be used for the exchange diagram; but it is always convenient to use the crossing matrix when direct channel pole coefficients are known.

Let us consider the case of quark-meson scattering. There can occur, in this case, three poles in the direct-channel as shown in the following diagrams:

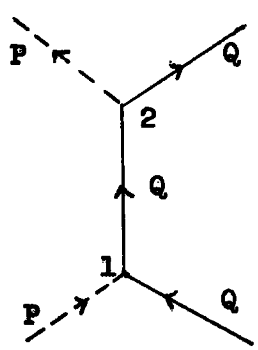

Fig. B.1

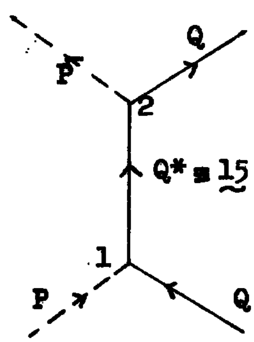

Fig. B.2

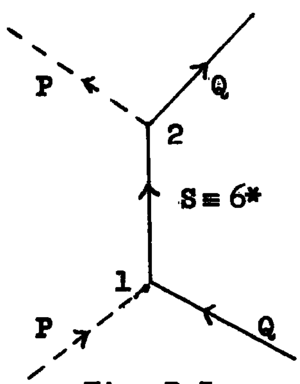

Fig. B.3

The method we have used can be described as follows: we construct a state $|N; I\ I_3\ Y>$ in terms of the product states $|N_1; I_1\ I_{1z}\ Y_1>|N_2; I_2\ I_{2z}\ Y_2>$ of the two particles involved in the scattering process. As I, I_3 and Y are conserved in the strong interactions, the state $|N; I, I_3, Y>$ must occur in the intermediate state of the above diagrams (Fig. B.1,2,3). We are then to pick out those coefficients from the two vertices associated with the above diagrams such that we get, by contraction, the relevant state $|N\ I\ I_3\ Y>$ as the intermediate state as well as the related external particles at the initial and theffinal states. We shall follow this method to calculate all the SU(3)-coefficients we require in our calculations.

For the $\underset{\sim}{3}$ $\underset{\sim}{15}$ and 6* pole coefficients we construct the following states:

$$|3; 0\ 0 - {}^2/_3> = |\underset{\sim}{3}; Q_3> = \frac{1}{2\sqrt{2}}\Big[-\sqrt{2}|P_6\ Q_3> - \sqrt{3}|P_7\ Q_2> + \sqrt{3}|P_8\ Q_1)\Big] \qquad (B.1)$$

$$|15; 0\ 0 - {}^2/_3> = |\underset{\sim}{15}; D_{13}> = \frac{1}{2\sqrt{2}}\Big[\sqrt{6}|P_6\ Q_3> - |P_7\ Q_2> + |P_8\ Q_1>\Big] \qquad (B.2)$$

$$|\underline{6}^*;\, 0\, 0\, {}^4\!/_3 > \;=\; |6^*;\, s_1 > \;=\; \frac{1}{\sqrt{2}}\left[|Q_1 P_2 > - |Q_2\, P_1 >\right] \quad (B.3)$$

Considering the diagram (Fig. B.1) and using the interaction Lagrangian (A.35) we obtain for 3-pole coefficient,

$$< \underline{3}|\underline{3} > \;=\; \frac{1}{8}\left[\frac{2}{\sqrt{3}} + \sqrt{3} + \sqrt{3}\right]\left[\frac{2}{\sqrt{3}} + \sqrt{3} + \sqrt{3}\right]$$

$$= \frac{1}{8} \cdot \frac{8}{\sqrt{3}} \cdot \frac{8}{\sqrt{3}} = \frac{8}{3}$$

For the 15-pole coefficient we consider the diagram (Fig. B.2) and use (A.36a) and (A.3) for vertices 1 and 2 respectively. Thus, we obtain,

$$< \underset{\sim}{15}|\underset{\sim}{15} > \;=\; \frac{1}{8}\left[-\frac{3}{\sqrt{2}} - \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\right]\left[-\frac{3}{\sqrt{2}} - \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\right] = 1.$$

For 6* we use Fig. B.3 and consider for the vertices 1 and 2 the Lagrangians (A.37a) and (A.37b) and obtain the following:

$$< \underline{6}^*|\underline{6}^* > \;=\; \frac{1}{2}\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right]\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] = 1$$

Thus, collecting all the above pole-coefficients we have

$$\langle \underset{\sim}{3} | \underset{\sim}{3} \rangle = \frac{8}{3}\,; \qquad \langle \underset{\sim}{15} | 15 \rangle = 1; \qquad \langle \underline{6}^* | \underline{6}^* \rangle = 1 \tag{B.4}$$

In the Baryon mass-splitting calculations, we consider the scattering of Baryons (Octet as well as Decouplet) and Mesons (Pseudoscalar Nonet as well as vector Nonet). The SU(3) coupling coefficients which are required correspond to the Baryong Octet and Decouplet poles in the direct channel. Following the method discussed earlier in this appendix we calculate all the SU(3) coupling coefficients that are required in the above mentioned calculations. For this purpose, we have to consider the following uncrossed Feynman diagrams:

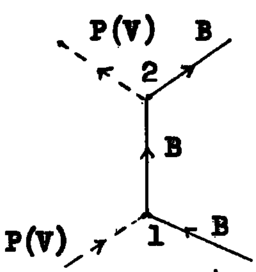

Fig. B.4

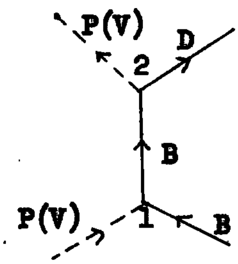

Fig. B.5

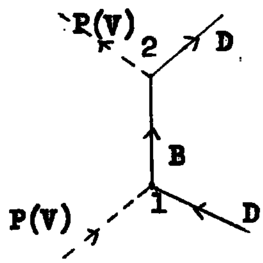

Fig. B.6

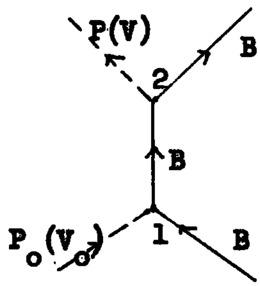

Fig. B.7

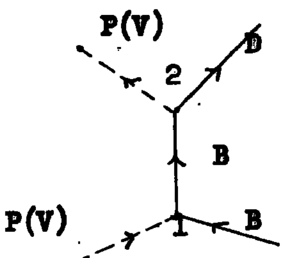

Fig. B.8

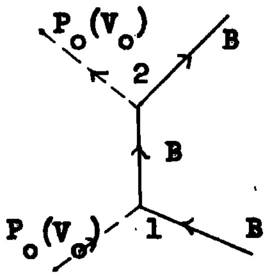

Fig. B.9

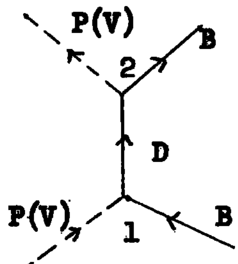

Fig. B.10

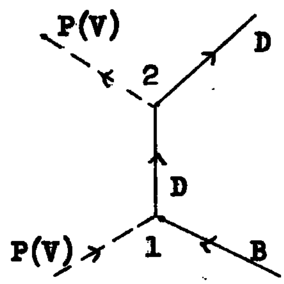

Fig. B.11

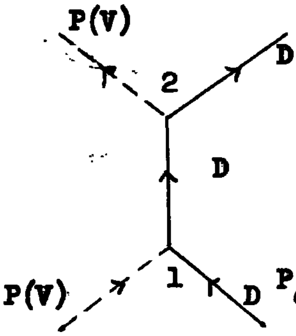

Fig. B.12

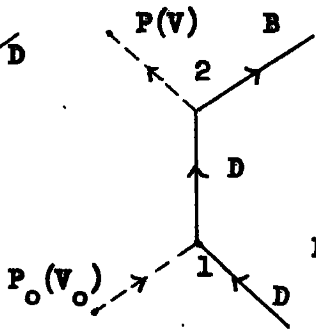

Fig. B.13

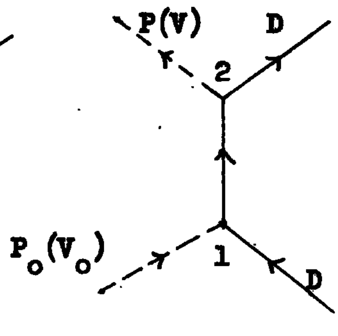

Fig. B.14

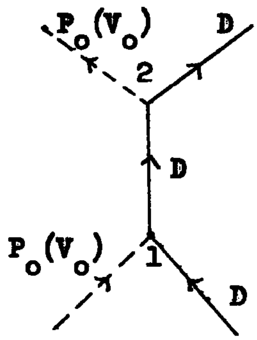

Fig. B.15

Where, P, V denote the pseudoscalar and vector octet respectively and B, D the baryon octet and decouplet respectively. P_o, V_o are respectively the pseudoscalar and vector singlet.

To calculate the Octet and Decouplet pole coefficients for the processes invdving pseudoscalar mesons (Octet or Singlet) we can follow the same procedure as we used in the case of quark-meson scattering process. As an example, let us calculate the baryon Octet pole coefficient in the process PB → PB (Fig.B.4). For that purpose, we construct the following states[67] :

$$|8;\,000> = \frac{1}{\sqrt{5}}\Big[\tfrac{1}{2}|P_1\,B_8> - \tfrac{1}{2}|P_2\,B_7> - |P_3\,B_5> + |P_4\,B_4>$$

$$- |P_5\,B_3> - |P_6\,B_6> - \tfrac{1}{2}|P_7\,B_2> + \tfrac{1}{2}|P_8\,B_1>\Big] \tag{B.5}$$

$$|8;\ 0\ 0\ 0> = \tfrac{1}{2}\Big[\ |P_1\ B_8> - |P_2\ B_7> + |P_7\ B_2> - |P_8\ B_1>\ \Big] \tag{B.6}$$

where 8, 8' denote the symmetric and antisymmetric octet representation respectively.

Using (B.5) and (B.6) we obtain from (A.38b) the following Baryon Octet pole coefficients:

$$<8|8> = \frac{1}{5}\left\{-\frac{1}{2\sqrt{6}}-\frac{1}{2\sqrt{6}}+\frac{\sqrt{2}}{\sqrt{3}}+\frac{\sqrt{2}}{\sqrt{3}}+\frac{\sqrt{2}}{\sqrt{3}}+\frac{\sqrt{2}}{\sqrt{3}}+\frac{\sqrt{3}}{2\sqrt{2}}\right.$$

$$\left.+\frac{\sqrt{3}}{2\sqrt{2}}\right\}\left\{-\frac{1}{2\sqrt{6}}-\frac{1}{2\sqrt{6}}+\frac{\sqrt{2}}{\sqrt{3}}+\frac{\sqrt{2}}{\sqrt{3}}+\frac{\sqrt{2}}{\sqrt{3}}+\frac{\sqrt{2}}{\sqrt{3}}\right.$$

$$\left.+\frac{\sqrt{3}}{2\sqrt{2}}+\frac{\sqrt{3}}{2\sqrt{2}}\right\}$$

$$= \frac{1}{5}\cdot\frac{10}{\sqrt{6}}\cdot\frac{10}{\sqrt{6}} = \frac{10}{3}$$

$$<8'|8'> = \frac{1}{4}\left\{-\frac{1}{\sqrt{6}}-\frac{1}{\sqrt{6}}-\frac{\sqrt{3}}{\sqrt{2}}-\frac{\sqrt{3}}{\sqrt{2}}\right\}\left\{-\frac{1}{\sqrt{6}}-\frac{1}{\sqrt{6}}-\frac{\sqrt{3}}{\sqrt{2}}-\frac{\sqrt{3}}{\sqrt{2}}\right\}$$

$$= \frac{1}{4}\cdot-\frac{8}{\sqrt{6}}\cdot-\frac{8}{\sqrt{6}} = \frac{8}{3}$$

$$< 8'|8 > = \frac{1}{2} \cdot \frac{1}{\sqrt{5}} \cdot \left(-\frac{8}{\sqrt{6}}\right) \cdot \frac{10}{\sqrt{6}} = -\frac{4\sqrt{5}}{3}$$

For the Decouplet pole coefficient corresponding to the same process (Fig. B.10) we construct the following state:

$$|10;\ 0\ 0\ -2 > = \frac{1}{\sqrt{2}} [\ |P_7\ B_8 > - |P_8\ B_7 >] \qquad (B.7)$$

Then, using (B.7) we have from (A.40b) and (A.40a)

$$< 10|10 > = \tfrac{1}{2} [1 + 1]\ [1 + 1] = 2$$

The above results have been checked against those calculated by a number of authors[19,68]. Following the above procedure we can calculate all the relevant SU(3) coefficients corresponding to the processes involving only the pseudoscalar mesons at both the vertices. For the processes involving vector meson (Octet or Singlet) at both the vertices or vector meson at one vertex and pseudoscalar meson at the other the situation is, however, slightly different owing to the occurrence of two types of couplings (Chapter I section 3). For this specific purpose, we write the relevant vertices in the following form:

For the vector (Octet) at the final vertex we write:

$$\mathcal{L}_2(\bar{B}\ \phi^*_\mu\ B) = \alpha' \mathcal{L}(\bar{B}\ \phi^*_\mu\ B)_F + \beta' \mathcal{L}(\bar{B}\ \phi^*_\mu\ B)_D + \tfrac{2}{3}F \qquad (B.8a)$$

Similarly for the vector octet at the initial vertex we have,

$$\mathcal{L}_1(\bar{B}\,\phi_\mu\,B) = \alpha\mathcal{L}(\bar{B}\,\phi_\mu B)_F + \beta\mathcal{L}(\bar{B}\,\phi_\mu B)_D + \frac{2}{3}F \qquad (B.8b)$$

Again, for the vector singlet at the final and initial vertices we write respectively,

$$\mathcal{L}_2(\bar{B}\,\phi_\mu^{*0}B) = \alpha'\mathcal{L}(\bar{B}\,\phi_\mu^{*0}B)_{38} + \beta'.\mathcal{L}(\bar{B}\,\phi_\mu^{*0}\,B)_8 \qquad (B.9a)$$

$$\mathcal{L}_1(\bar{B}\,\phi_\mu^{0}B) = \alpha.\mathcal{L}(\bar{B}\,\phi_\mu^{0}B) + \beta'.\mathcal{L}(\bar{B}\,\phi_\mu^{0}B) \qquad (B.9b)$$

In the above, α, β are the kinematic factors of the vertices concerned. For the Decouplet, on the other hand, the vertices are of the same form as with the pseudoscalar mesons. So, we need to calculate the coefficients corresponding to only one type of meson. Having discussed the relevant procedures and the notations, we present the results in the following tables:

Table B 1: B-pole coefficients in the direct channel:

PROCESS	POLES	POLE COEFFICIENTS
	$8 \leftrightarrow 8$	$10/3$
$PB \rightarrow PB$	$8' \leftrightarrow 8$	$+4\sqrt{5}/3$
	$8' \leftrightarrow 8'$	$8/3$

Table B 1: B-pole coefficients in the direct channels:

PROCESS	POLES	POLE COEFFICIENTS
$PB \rightarrow PD$	$8 \leftrightarrow 8$	$-5/\sqrt{3}$
	$8 \leftrightarrow 8'$	$-2\sqrt{5}/\sqrt{3}$
$PB \leftrightarrow P_oB$	$8 \leftrightarrow 8$	$2\sqrt{5}/3\sqrt{6}$
	$8 \leftrightarrow 8'$	$+4/3\sqrt{6}$
$PD \leftrightarrow PD$	$8 \leftrightarrow 8$	$5/2$
$P_oB \leftrightarrow P_oB$	$88 \leftrightarrow 8$	$1/9$
$P_oB \leftrightarrow PD$	$8 \leftrightarrow 8$	$-\sqrt{5}/3\sqrt{2}$
$PB \leftrightarrow VB$	$8 \leftrightarrow 8$	$10\beta'/3$
	$8' \leftrightarrow 8$	$+2\sqrt{5}\alpha' + 4\sqrt{5}\beta'/3$
	$8' \leftrightarrow 8'$	$4\alpha' + 8\beta'/3$
$PB \leftrightarrow V_oB$	$8 \leftrightarrow 8$	$2\sqrt{5}\alpha'/\sqrt{6} + 2\sqrt{5}\beta'/3\sqrt{6}$
	$8 \leftrightarrow 8'$	$+4\alpha'/\sqrt{6} + 4\beta'/3\sqrt{6}$
$PB \leftrightarrow VD$	$8 \leftrightarrow 8$	$-5/\sqrt{3}$
	$8 \leftrightarrow 8'$	$-2\sqrt{5}/\sqrt{3}$
$P_oB \leftrightarrow VB$	$8 \leftrightarrow 8$	$2\sqrt{5}\beta'/3\sqrt{6}$
	$8' \leftrightarrow 8$	$+\sqrt{6}\alpha'/3 + 2\sqrt{2}\beta'/3\sqrt{3}$
$P_oB \leftrightarrow V_oB$	$8 \leftrightarrow 8$	$\alpha'/3 + \beta'/9$
$P_oB \leftrightarrow VD$	$8 \leftrightarrow 8$	$-\sqrt{5}/3\sqrt{2}$

PROCESS	POLES	POLE COEFFICIENTS
$PD \leftrightarrow VB$	$8 \leftrightarrow 8$	$-5\beta'/\sqrt{3}$
	$8' \leftrightarrow 8$	$-\sqrt{15}\,\alpha' + 2\sqrt{5}\,\beta'/\sqrt{3}$
$PD \leftrightarrow V_0B$	$8 \leftrightarrow 8$	$-\sqrt{5}\,\alpha'/\sqrt{2} - \sqrt{5}\,\beta'/3\sqrt{2}$
$PD \leftrightarrow VD$	$8 \leftrightarrow 8$	$5/2$
$VB \leftrightarrow VB$	$8 \leftrightarrow 8$	$10\beta\beta'/3$
	$8' \leftrightarrow 8$	$+2\sqrt{5}\alpha'\beta + 4\sqrt{5}\beta'\beta/3$
	$8' \leftrightarrow 8'$	$6\alpha'\alpha + 4\alpha'\beta + 4\alpha\beta' + 8\beta'\beta/3$
$VB \leftrightarrow V_0B$	$8 \leftrightarrow 8$	$2\sqrt{5}\alpha'\beta/\sqrt{6} + 2\sqrt{5}\beta'\beta/3\sqrt{6}$
	$8 \leftrightarrow 8'$	$+\sqrt{6}\,\alpha'\alpha + 4\alpha'\beta/\sqrt{6} + 2\beta'\alpha/\sqrt{6} + 4\beta'\beta/3\sqrt{6}$
$VB \leftrightarrow VD$	$8 \leftrightarrow 8$	$-5\beta/\sqrt{3}$
	$8 \leftrightarrow 8'$	$-\sqrt{15}\,\alpha + 2\sqrt{5}\,\beta/\sqrt{3}$
$V_0B \leftrightarrow V_0B$	$8 \leftrightarrow 8$	$\alpha'\alpha + \alpha'\beta/3 + \beta'\alpha/3 + \beta'\beta/9$
$V_0B \leftrightarrow VD$	$8 \leftrightarrow 8$	$-\sqrt{5}\,\alpha/\sqrt{2} - \sqrt{5}\,\beta/3\sqrt{2}$
$VD \leftrightarrow VD$		$5/2$

Table B 2: D-pole Coefficients in the direct channel:

PROCESS	POLES	POLE COEFFICIENTS
$PB \leftrightarrow PB$	$10 \leftrightarrow 10$	2
$PB \leftrightarrow PD$	$10 \leftrightarrow 10$	$4/\sqrt{6}$
$PD \leftrightarrow PD$	$10 \leftrightarrow 10$	$4/3$
$P_oD \leftrightarrow PB$	$10 \leftrightarrow 10$	$+\sqrt{2}/3$
$P_oD \leftrightarrow PD$	$10 \leftrightarrow 10$	$+2/3\sqrt{3}$
$P_oD \leftrightarrow P_oD$	$10 \leftrightarrow 10$	$1/9$

APPENDIX C

Helicity Formalism and Rotation Matrices

When dealing with the scattering of particles having spins it is always convenient to work with the helicity representations of the scattering amplitudes. In order to obtain that one has to construct the helicity states of free particles and this appendix will be devoted to that purpose. In our calculations, we are concerned with the two-particle scatterings and consequently we require the helicity states of the two-particle systems. First, we shall discuss the helicity states of those single free-particles which we encounter in our calculations and then construct the helicity states of the two-particle systems in the form which can be conveniently used in obtaining the helicity representations of the scattering amplitudes. While discussing the helicity states of the free particles we assume that the three-component momenta of the particles are in the xz-plane so that we can set the azimuthal angle $\phi = 0$ in our calculations. The inclusion of ϕ in the representation, in fact, does not make any difference in the final results of the calculation when one is working in the centre of mass systems. As regards the phases and the normalisations of the helicity states we shall mostly follow the conventions used by Jacob and Wick[58).

1. **Helicity states of spin $\frac{1}{2}$ particles :**

The four-component positive-energy spinor u with momentum $\underset{\sim}{p}$ is,

$$u_\lambda(p) = \frac{1}{\sqrt{2m(E+m)}} \begin{vmatrix} E+m \\ \underset{\sim}{\sigma} \cdot \underset{\sim}{p} \end{vmatrix} \chi_\lambda \qquad (C.1)$$

where $E = \sqrt{p^2 + m^2}$; $(\sigma \cdot p)\chi_\lambda = 2|p|\lambda . \chi_\lambda$, where λ is called the helicity of the particle. For spin $\frac{1}{2}$ particle we have $\lambda = \pm\frac{1}{2}$. The normalisation constant is so chosen that $\bar{u}u = u^+\gamma_0 u = 1$. In (C.1) m is the mass of the spin $\frac{1}{2}$ particle and $\underset{\sim}{\sigma}$'s are the familiar Pauli matrices. For the momentum $\underset{\sim}{p}$ in the z-direction, the helicity states χ_λ are well known and given by,

$$\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (C.2)$$

The helicity states corresponding to the momentum $\underset{\sim}{p}$ in a direction making an angle θ with the positive z-direction can be obtained by applying rotation around the y-axis. This is given by the following relation,

$$\chi_\lambda = \sum_{\lambda'} d^{\frac{1}{2}}_{\lambda'\lambda}(\theta)\, \chi_\lambda \qquad (C.3)$$

where $d^{\frac{1}{2}}_{\lambda'\lambda}(\theta)$ is the matrix representation of a rotation by an angle θ around the Y-axis. This is given by[60],

$$d^{\frac{1}{2}}_{\lambda'\lambda}(\theta) =$$ (C.4)

λ' \ λ	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\cos\theta/2$	$-\sin\theta/2$
$-\frac{1}{2}$	$\sin\theta/2$	$\cos\theta/2$

Using (C.2) and (C.4), the representation of the helicity states of a spin $\frac{1}{2}$ particle with momentum in a direction making an angle θ with the z-axis are obtained from (C.3).

$$\chi_{\frac{1}{2}} = \begin{vmatrix} \cos\theta/2 \\ \sin\theta/2 \end{vmatrix} \qquad \chi_{-\frac{1}{2}} = \begin{vmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{vmatrix} \qquad \text{(C.5)}$$

While working in the c.m. system, one requires the representations of the helicity states corresponding to the momentum $\underset{\sim}{p}$ in a direction making an angle θ' with the positive z-direction, where $\theta' = \pi$ or $\pi + \theta$. In such a case the corresponding helicity states are obtained from the following relation

$$\chi_\lambda = \sum_{\lambda'} (-1)^{s-\lambda}\, d^{s}_{\lambda'\lambda}(\theta)\chi_\lambda \qquad \text{(C.6)}$$

where s is the spin of the particle. Here, the phase factor $(-1)^{s-\lambda}$ is introduced for convenience in such a way that for the particle at rest $|p| = 0$, the helicity states corresponding to the momentum in the negative $z(\pi+\theta)$ direction reduce to those corresponding to the momentum in the positive $z(\theta)$ direction.

Thus, using (C.2) and (C.4) the helicity states corresponding to the momentum $\underset{\sim}{p}$ in the negative z-direction are obtained from (C.6) given by

$$\chi_{\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \chi_{-\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{C.7}$$

The helicity states corresponding to the momentum $\underset{\sim}{p}$ in $\pi + \theta$ direction can be, similarly, obtained and are,

$$\chi_{\frac{1}{2}} = \begin{vmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{vmatrix}, \qquad \chi_{-\frac{1}{2}} = \begin{vmatrix} \cos\theta/2 \\ \sin\theta/2 \end{vmatrix} \tag{C.8}$$

2. Helicity states of spin 1 particles:

Let us now discuss the representations of the helicity states of a spin 1 particle corresponding to the different situations discussed just above. Here, there are three helicity states which are given by,

$$\epsilon^{(\pm)} = \left\{ 0, \underset{\sim}{\eta}^{(\pm)} \right\}; \quad \epsilon^{(0)} = \frac{1}{\mu} \left\{ |k|, \omega\, \underset{\sim}{\eta}^{(0)} \right\} \tag{C.9}$$

where μ is the mass of the particle and ω is the energy such that $\omega = \sqrt{|k|^2 + \mu^2}$. For the momentum $\underset{\sim}{k}$ in the positive z-direction, the representations of the three-component polarization vectors

$\underset{\sim}{\eta}^{(+)}$, $\underset{\sim}{\eta}^{(0)}$ are given by,

$$\underset{\sim}{\eta}^{(+)} = \frac{1}{\sqrt{2}}\begin{vmatrix}-1\\-i\\0\end{vmatrix}; \quad \underset{\sim}{\eta}^{(0)} = \begin{vmatrix}0\\0\\1\end{vmatrix}; \quad \underset{\sim}{\eta}^{(-)} = \frac{1}{\sqrt{2}}\begin{vmatrix}1\\-i\\0\end{vmatrix}$$

(C.10)

The helicity states corresponding to other three situations discussed in connection with spin $\frac{1}{2}$ particles can be obtained by using (C.9) and (C.10) from the relations similar to (C.3) and (C.6), the rotation matrix $d^{1}_{\lambda',\lambda}(\theta)$ in this case being,

$d^{1}_{\lambda',\lambda}(\theta)$ = (C.11)

$\lambda' \backslash \lambda$	1	0	-1
1	$\frac{1+\cos\theta}{2}$	$-\frac{\sin\theta}{\sqrt{2}}$	$\frac{(1-\cos\theta)}{2}$
0	$\frac{\sin\theta}{\sqrt{2}}$	$\cos\theta$	$-\frac{\sin\theta}{\sqrt{2}}$
-1	$\frac{(1-\cos\theta)}{2}$	$\frac{\sin\theta}{\sqrt{2}}$	$\frac{1+\cos\theta}{2}$

Thus, for the momentum $\underset{\sim}{k}$ in a direction making an angle θ with the positive z-direction, the representations of the polarization vectors $\underset{\sim}{\eta}^{(\pm)}$ $\underset{\sim}{\eta}^{(0)}$ are,

$$\eta^{(+)} = \frac{1}{\sqrt{2}} \begin{vmatrix} -\cos\theta \\ -i \\ \sin\theta \end{vmatrix} \qquad \eta^{(0)} = \begin{vmatrix} \sin\theta \\ 0 \\ \cos\theta \end{vmatrix}$$

$$\eta^{(-)} = \frac{1}{\sqrt{2}} \begin{vmatrix} \cos\theta \\ -i \\ -\sin\theta \end{vmatrix} \qquad (C.12)$$

Taking into account the phase-factor discussed above, the representations of $\eta^{(\pm)}$, $\eta^{(0)}$ corresponding to the momentum $\underset{\sim}{k}$ in the negative z-direction are,

$$\eta^{(+)} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 \\ -i \\ 0 \end{vmatrix} ; \quad \eta^{0} = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} \quad \eta^{(-)} = \frac{1}{\sqrt{2}} \begin{vmatrix} -1 \\ -i \\ 0 \end{vmatrix}$$

(C.13)

where, $\epsilon^{(0)} = \frac{-1}{\mu} \left\{ |k|, -\omega\, \eta^{(0)} \right\}$

Following the similar procedure corresponding to the momentum $\underset{\sim}{k}$ in the direction $\theta + \pi$ with the z-axis we have,

$$\eta^{(+)} = \frac{1}{\sqrt{2}} \begin{vmatrix} \cos\theta \\ -i \\ -\sin\theta \end{vmatrix} \qquad \eta^{(0)} = \begin{vmatrix} \sin\theta \\ 0 \\ \cos\theta \end{vmatrix}$$

$$\eta^{(-)} = \frac{1}{\sqrt{2}} \begin{vmatrix} -\cos\theta \\ -i \\ \sin\theta \end{vmatrix} \qquad (C.14)$$

With, $\epsilon^{(0)} = \frac{-1}{\mu}\left\{ |k|, -\omega\, \eta^{(0)} \right\}$

3. **Helicity states of spin 3/2 particles:**

The four helicity states corresponding to the spin 3/2 particles are obtained by taking the vector addition of the helicity states of the spin $\frac{1}{2}$ and spin 1 particles. Corresponding to the four situations described above, there will be four sets of such states. If ξ_λ denotes a helicity state of spin 3/2 particles, then corresponding to any of the four-combinations that we can obtain we have,

$$\xi_{3/2} = u_{\frac{1}{2}} \cdot \epsilon^{(+)}$$

$$\xi_{\frac{1}{2}} = \frac{1}{\sqrt{3}}\left\{ u_{-\frac{1}{2}}\, \epsilon^{(+)} + \sqrt{2}\, u_{\frac{1}{2}}\, \epsilon^{(0)} \right\}$$

(C.15)

$$\xi_{-\frac{1}{2}} = \frac{1}{\sqrt{3}}\left\{ u_{\frac{1}{2}}\, \epsilon^{(-)} + \sqrt{2}\, u_{-\frac{1}{2}}\, \epsilon^{(0)} \right\}$$

$$\xi_{-3/2} = u_{-\frac{1}{2}}\, \epsilon^{(-)}$$

We shall also require the matrix $d^{3/2}_{\lambda'\lambda}(\theta)$, the elements of which are obtained from (C.4) and (C.11) with the help of the related Clebsch-Gordon coefficients[60]. Thus, we obtain the following:

$d^{3/2}_{\lambda'\lambda}(\theta)$ =

3/2	1/2	-1/2	-3/2
$\frac{1}{2}(1+\cos\theta)\cos\frac{\theta}{2}$,	$-\frac{\sqrt{3}}{2}(1+\cos\theta)\sin\frac{\theta}{2}$,	$\frac{\sqrt{3}}{2}(1-\cos\theta)\cos\frac{\theta}{2}$,	$-\frac{1}{2}(1-\cos\theta)\sin\frac{\theta}{2}$
$\frac{\sqrt{3}}{2}(1+\cos\theta)\sin\frac{\theta}{2}$,	$-\frac{1}{2}(1-3\cos\theta)\cos\frac{\theta}{2}$,	$-\frac{1}{2}(1+3\cos\theta)\sin\frac{\theta}{2}$,	$\frac{\sqrt{3}}{2}(1-\cos\theta)\cos\frac{\theta}{2}$
$\frac{\sqrt{3}}{2}(1-\cos\theta)\cos\frac{\theta}{2}$,	$\frac{1}{2}(1+3\cos\theta)\sin\frac{\theta}{2}$,	$-\frac{1}{2}(1-3\cos\theta)\cos\frac{\theta}{2}$,	$-\frac{\sqrt{3}}{2}(1+\cos\theta)\sin\frac{\theta}{2}$
$\frac{1}{2}(1-\cos\theta)\sin\frac{\theta}{2}$,	$\frac{\sqrt{3}}{2}(1-\cos\theta)\cos\frac{\theta}{2}$,	$\frac{\sqrt{3}}{2}(1+\cos\theta)\sin\frac{\theta}{2}$,	$\frac{1}{2}(1+\cos\theta)\cos\frac{\theta}{2}$

(C.16)

4. Two-particle states and helicity representation of the scattering amplitudes:

Let us construct the helicity states of two free particles 1 and 2 with masses m_1 and m_2, and spins s_1 and s_2. These states may, of course, be constructed as direct product $R^{(1)}_{\phi_1,\theta_1,-\phi_1}\,\psi_{p_1\lambda_1}$

$R^{(2)}_{\phi_2,\theta_2,-\phi_2}\ \psi_{p_2\lambda_2}$ of the individual states of the above particles, where $R^{(i)}_{\phi_i,\theta_i,-\phi_i}$ are the rotation operators corresponding to the rotations by the Euler's angles ϕ_i, θ_i, $-\phi_i$, operating on the i-th particle. When one is dealing with a scattering problem one works in the C.M. system of the two particles. In C.M. system, we have $p_1 = -p_2 = p$ with zero total linear momentum. If the linear momentum $\underset{\sim}{p}$ is directed along a direction defined by the angles θ, ϕ, then $\theta_1 = \theta$, $\phi_1 = \phi$ and $\theta_2 = \pi - \theta$, $\phi_2 = \phi \pm \pi$ and the two rotations $R^{(1)}$ and $R^{(2)}$ can be replaced by a single rotation R involving the total angular momentum $\underset{\sim}{J} = \underset{\sim}{J}_1 + \underset{\sim}{J}_2$ of the two particles. In order to obtain that one usually takes the relative momentum $\underset{\sim}{p}$ of the two-particles in the C. M. system along the z-axis. Taking into account for one of the two particles the phase-factors discussed earlier in this appendix we can define a product state for the two particles,

$$\psi_{p\lambda_1\lambda_2} = \psi_{p\lambda_1}(1)\chi_{p\lambda_2}(2) \qquad (C.17)$$

where the relative momentum $\underset{\sim}{p}$ is directed along the positive z-direction. States with the direction of p defined by the angles $\theta\ \phi$ are obtained by applying a rotation,

$$|p\ \theta\ \phi, \lambda_1 \lambda_2 > \;=\; R_{\phi,\theta,-\phi}\, \psi_{p\lambda_1\lambda_2} \;=\; e^{i\lambda\phi}\, R_{\phi\theta 0}\, \psi_{p\lambda_1\lambda_2} \qquad (C.18)$$

where, $\lambda = \lambda_1 - \lambda_2$ is the resultant angular momentum of the two-particle system, the particle numbered 2 having been taken initially with momentum in the negative z-direction.

Now, the method of constructing a state of two particles having definite helicities, relative momentum $\underset{\sim}{p}$; total angular momentum J with $J_z = M$ is well known. Such a state is constructed as follows:

$$|p;\ J\ M\ \lambda_1\lambda_2 > \;=\; \frac{N}{2\pi} \int \underset{\sim}{du}\, D^{*}_{M\lambda}(\alpha\beta\gamma)\, R_{\alpha\beta\gamma}\, \psi_{p\lambda_1\lambda_2} \qquad (C.19)$$

where $R_{\alpha\beta\gamma}$ is a rotation operator corresponding to a rotation through arbitrary Euler's angles α, β, γ and $D_{M\lambda}(\alpha\beta\gamma)$ is the corresponding representation and is given by,

$$D_{M\lambda}(\alpha\beta\gamma) \;=\; e^{-iM\alpha}\, d^{J}_{M\lambda}(\beta)\, e^{-i\lambda\gamma} \qquad (C.20)$$

The integral in (C.19) extends to the region $0 < \alpha < 2\pi$, $0 < \beta < \pi$, $0 < \gamma < 2\pi$ and $\underset{\sim}{du} = \sin\beta\, d\alpha\, d\beta\, d\gamma$.

Now, substituting (C.18) into (C.19) and performing the integration over one ϕ's we have,

$$|p; J\,M\,\lambda_1\lambda_2 > \; = \; N \int D^{*}_{M\lambda}(\phi,\theta,-\phi)\,|p\,\theta\,\phi;\; \lambda_1\lambda_2 > d\,\Omega \qquad (C.21)$$

where $d\Omega = \sin\theta\, d\theta\, d\phi$.

Now for the product state $|p_1 p_2 \lambda_1 \lambda_2 >$ with the independent momenta for the two particles we assume the following normalisation,

$$< \underset{\sim}{p}'_1 \underset{\sim}{p}'_2 \lambda'_1 \lambda'_2 | \underset{\sim}{p}_1 \underset{\sim}{p}_2 \lambda_1 \lambda_2 > \; = \; \delta(p_1 - p'_1)\,\delta(p_2 - p'_2)\; \delta_{\lambda_1 \lambda'_1}\, \delta_{\lambda_2 \lambda'_2} \qquad (C.22)$$

Then introducing the new variables $P_\mu = (P_o, \underset{\sim}{P})$ with $P_o = E_1 + E_2$ and $\underset{\sim}{P} = \underset{\sim}{p}_1 + \underset{\sim}{p}_2$ and the two polar angles θ, ϕ to specify the direction of the relative momentum $\underset{\sim}{p} = \underset{\sim}{p}_1 - \underset{\sim}{p}_2$, the scattering matrix s corresponding to the two-particle scattering can be written in the form:

$$< \underset{\sim}{p}'_1 \underset{\sim}{p}'_2 \lambda'_1 \lambda'_2 | s | \underset{\sim}{p}_1 \underset{\sim}{p}_2 \lambda_1 \lambda_2 > \; = \; \delta_4(P_\mu - P'_\mu)(vv')^{\frac{1}{2}}\, (\underset{\sim}{p}\, \underset{\sim}{p}')^{-1}$$

$$\times \; < \theta' \phi' \lambda'_1 \lambda'_2 | s | \theta\, \phi\, \lambda_1 \lambda_2 > \qquad (C.23)$$

where, v p and v', p' are the magnitudes of relative velocity and the momentum in the C.M. system of the initial and the final states respectively.

Equation (C.22) together with (C.23) then fixes the following normalisation:

$$< \theta' \phi' \lambda_1' \lambda_2' | \theta \phi \lambda_1 \lambda_2 > = \delta(\cos\theta - \cos\theta')\, \delta(\phi - \phi')\, \delta_{\lambda_1 \lambda_1'}\, \delta_{\lambda_2 \lambda_2'}$$

(C.24)

The orthogonality relation of the rotation $D(\alpha\beta\gamma)$ is,

$$\int d\Omega\, D^{j_2 *}_{\mu_1 m_1}(\alpha\beta\gamma)\, D^{j_2}_{\mu_2 m_2}(\alpha\beta\gamma) = \frac{8\pi^2}{2j_1 + 1}\, \delta_{\mu_1 \mu_2}\, \delta_{m_1 m_2}\, \delta_{j_1 j_2}$$

(C.25)

From (C.25) we obtain,

$$\int_0^\pi d^j_{m\mu}(\beta) d^{j'}_{m\mu}(\beta) \sin\beta\, d\beta = \frac{2}{2j + 1}\, \delta_{jj'} \qquad (C.26)$$

Also, we have,

$$\frac{1}{2} \sum_j (2j + 1) d^j_{m\mu}(\beta) d^j_{m\mu}(\beta') = \delta(\cos\beta - \cos\beta') \qquad (C.27)$$

Now, using (C.24) - (C.26) we obtain from (C.21) the following normalisation:

$$< J' M' \lambda_1' \lambda_2' | J\, M\, \lambda_1 \lambda_2 > = \delta\, J\, J'\, \delta\, M\, M'\, \delta_{\lambda_1 \lambda_1'}\, \delta_{\lambda_2 \lambda_2'}$$

(C.28)

where we have chosen $N_J = \left\{ (2J+1) / 4\pi \right\}^{\frac{1}{2}}$.

Again, using (C.24) the transformation matrix obtained from (C.21) is,

$$< \theta \phi \lambda_1' \lambda_2' | J M \lambda_1 \lambda_2 > = N_J \, \delta_{\lambda_1 \lambda_1'} \, \delta_{\lambda_2 \lambda_2'} \, D^{J*}_{M\lambda} (\phi, \theta, -\phi) \qquad (C.29)$$

where $\lambda = \lambda_1 - \lambda_2$ and N_J is given as in (C.28).

The above (C.29) transformation matrix satisfies the following unitarity conditions:

$$\int d\Omega < \theta \phi \lambda_1 \lambda_2 | J M \lambda_1 \lambda_2 > < \theta \phi \lambda_1 \lambda_2 | J' M' \lambda_1 \lambda_2 >^* = \delta_{JJ'} \, \delta_{MM'} \qquad (C.30a)$$

$$\sum_{JM} < \theta \phi \lambda_1 \lambda_2 | J M \lambda_1 \lambda_2 > < \theta' \phi' \lambda_1 \lambda_2 | J M \lambda_1 \lambda_2 >^* = \delta(\cos\theta - \cos\theta') \times \delta(\phi - \phi') \qquad (C.30b)$$

5. Representations of $d^J(\theta)$ matrices in terms of the Legendre polynomials:

In order to obtain the partial wave amplitude one needs to express the matrices $d^J(\theta)$ in terms of the Legendre polynomials $P_\ell(\cos\theta)$. The following matrices are required for our purpose:

$$d^J_{\frac{1}{2}\frac{1}{2}}(\theta) = \frac{1}{2\cos\theta/2}\left[P_{J+\frac{1}{2}} + P_{J-\frac{1}{2}} \right]$$

$$d^J_{3/2\ \frac{1}{2}}(\theta) = \frac{[(J-\frac{1}{2})(J+3/2)]^{\frac{1}{2}}}{4(1+\cos\theta)\sin\theta/2}\left[\frac{1}{J+1} \cdot P_{J+3/2} + \frac{1}{J} P_{J+\frac{1}{2}} - \frac{1}{J+1} P_{J-\frac{1}{2}} - \frac{1}{J} P_{J-3/2} \right] \quad \text{(C.31)}$$

$$d^J_{3/2\ 3/2}(\theta) = \frac{1}{4(1+\cos\theta)\cos\theta/2}\left[\frac{J-\frac{1}{2}}{J+1} P_{J+3/2} + \frac{3(J-\frac{1}{2})}{J} P_{J+\frac{1}{2}} + \frac{3(J+3/2)}{J+1} P_{J-\frac{1}{2}} + \frac{(J+3/2)}{J} P_{J-3/2} \right]$$

The other elements of $d^J(\theta)$ are obtained from (A.31) using the following symmetry relations:

$$d^J_{\lambda\mu}(\theta) = d^J_{-\mu,-\lambda}(\theta) = (-1)^{\lambda-\mu} d^J_{\mu\lambda}(\theta) = (-1)^{\lambda-\mu} d^J_{-\lambda,-\mu}(\theta)$$

$$d^J_{\lambda\mu}(\theta) = (-1)^{J+\lambda} d^J_{\lambda,-\mu}(\pi-\theta) \quad \text{(C.32)}$$

The above representations of the matrices $d^J(\theta)$ will be very often used in the calculations of exchanged Born terms.

APPENDIX D

Some Useful Relations for the Calculations of Exchange Born Terms

We have,

$$E' = E - w; \quad k' = 2k \cos \theta/2; \quad E'^2 = k'^2 + m^2 \qquad (D.1)$$

$$a_1 = (E' + m)k - (E + m)k'; \quad a_2 = (E' + m)k + (E + m)k' \qquad (D.2)$$

$$a_3 = (E + m)(E' + m) - kk'; \quad a_4 = (E + m)(E' + m) + kk' \qquad (D.3)$$

$$E = \sqrt{k^2 + m^2}; \quad w = \sqrt{k^2 + \mu^2} \qquad (D.4)$$

Let,

$$c = (E + m)(E' + m) \qquad (D.5)$$

Then using the above relations we obtain the following:

$$\left.\begin{array}{l} a_1^2 + a_2^2 = 4c(EE' - m^2); \quad a_1^2 - a_2^2 = -4c\, kk' \\ a_1\, a_2 = 2m\, c\,(E - E') \end{array}\right\} \qquad (D.6)$$

$$\left.\begin{array}{l} a_3^2 + a_4^2 = 4c(EE' + m^2); \quad a_3^2 - a_4^2 = -4kk'c \\ a_3\, a_4 = 2m\, c(E + E') \end{array}\right\} \qquad (D.7)$$

$$\left.\begin{aligned} a_1 a_3 &= 2c(E'k - Ek'); \quad a_1 a_4 = 2mc(k - k') \\ a_2 a_3 &= 2mc(k + k'); \quad a_2 a_4 = 2c(E'k + Ek') \end{aligned}\right\} \quad (D.8)$$

$$\left.\begin{aligned} (a_1 k - a_3 E)^2 + (a_2 k - a_4 E)^2 &= 4m^2 c(EE' + m^2) \\ (a_1 k - a_3 E)^2 - (a_2 k - a_4 E)^2 &= 4m^2 c\, kk' \end{aligned}\right\} \quad (D.9)$$

$$\left.\begin{aligned} a_3(a_2 k - a_4 E) + a_4(a_1 k - a_3 E) &= -4mc(EE' + m^2) \\ a_3(a_2 k - a_4 E) - a_4(a_1 k - a_3 E) &= 4mckk' \end{aligned}\right\} \quad (D.10)$$

$$\left.\begin{aligned} a_4(a_2 k - a_4 E) + a_3(a_1 k - a_3 E) &= -4m^2 c(E + E') \\ a_4(a_2 k - a_4 E) - a_3(a_1 k - a_3 E) &= 0 \\ (a_1 k - a_3 E)(a_2 k - a_4 E) &= 2m^3 c(E + E') \end{aligned}\right\} \quad (D.11)$$

$$\left.\begin{aligned} a_1(a_2 k - a_4 E) + a_2(a_1 k - a_3 E) &= -4mckE' \\ a_1(a_2 k - a_4 E) - a_2(a_1 k - a_3 E) &= 4mck'E \end{aligned}\right\} \quad (D.12)$$

$$\left.\begin{aligned} a_1(a_1k - a_3E) + a_2(a_2k - a_4E) &= -4m^2 c k \\ a_1(a_1k - a_3E) - a_2(a_2k - a_4E) &= 4m^2 c k' \end{aligned}\right\} \qquad (D.13)$$

$$\left.\begin{aligned} a_1 a_3 + a_2 a_4 &= 4c E' k \\ a_2 a_4 - a_1 a_3 &= 4 c E k' \end{aligned}\right\} \qquad (D.14)$$

$$\left.\begin{aligned} a_2 a_3 + a_1 a_4 &= 4m c k \\ a_2 a_3 - a_1 a_4 &= 4m c k' \end{aligned}\right\} \qquad (D.15)$$

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