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FINITE ELEMENT ANALYSIS

FOR THE NAVIER - STOKES EQUATIONS

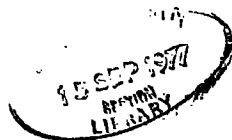
A thesis submitted for the degree of
Master of Science
in the
University of Durham

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by

Ken Y-K Cheng

June 1977



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ABSTRACT

The finite element method was employed to solve two-dimensional, unsteady, incompressible, viscous fluid flow problems. A practical computation procedure is presented. A complete finite element computer program has been developed. The numerical technique is based upon a general formulation for the Navier-Stokes equations making use of a combined variational principle finite element approach. Solution to the system of algebraic equations is approached by the Gaussian elimination scheme. The time-dependent Navier-Stokes equations are expressed in terms of a stream function equation and a transport equation. A variational functional of the stream function and a pseudo-variational functional of the vorticity of the respective boundary value problem is presented. The pressure distribution and velocity profile are determined from stream function. Two numerical examples are presented and compared with present papers. Some new ideas about the numerical method, obtained through numerical experiments, are presented and discussed.

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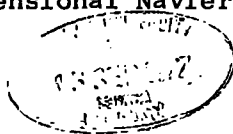
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Chapter 1 INTRODUCTION

The Navier-Stokes equations governing the fluid flow problems, are known to have applications to a large class of engineering problems. Exact solutions of such a viscous fluid flow problem are not currently available. The necessity of providing reasonable estimates for complicated flow phenomena leaves research engineers very little choice. The numerical approach seems to be one of the very few acceptable tools.

Even the numerical methods face difficulties arising from the non-linearity and complexity of the boundary involved. The present high-speed, large storage digital computers have now made it possible to solve the Navier-Stokes equations. Numerical solution of the Navier-Stokes equations utilising modern high-speed computers have been developed by a number of investigators.

A finite-difference approach presented by Fromm and Harlow (57) has had considerable success in solving problems. Lee and Fung (79) used a method which combines the conformal mapping and finite-difference technique to analysis viscous flow problems. Mills (91) employed the finite-difference scheme to solve viscous flow through a pipe orifice at low Reynolds numbers. Rimon (105) also used this scheme to get solutions of the incompressible time-dependent viscous fluid flow past a thin oblate spheroid. Dennis and Chang (35) employed it to study problems of steady flow past a circular cylinder. Using this approach, Greenspan (51) was doing numerical studies of steady, viscous, incompressible flow in a channel with a step. He also presented in a second paper (52) some useful equations to determine wall vorticity at some special solid wall surface. Lin, Pepper and Lee (83) employed the finite-difference techniques to analyze separated flows around a circular cylinder. Macagno and Hung (87) made a study of a captive annular eddy using the finite-difference method. Roscoe (108) has been using a new finite-difference approach to study the three-dimensional Navier-Stokes equations. Carlson



and Hornbeck (24) analysed the laminar entrance flow of an incompressible viscous fluid in a square duct using the finite-difference procedures.

From these numerous studies, it seems to show that the applications of the finite-difference method have been limited due to complexities in the developed computational procedures. It seems to require large amounts of computer time and storage. Another important disadvantage of the finite-difference methods is the fact that these methods rely mostly on meshes of very regular and symmetric patterns. Great computational difficulties are encountered if the geometric configuration of the fluid flow is complicated and cannot be readily transformed into a mesh of rectangular pattern (25,62,63,77).

These difficulties can be overcome with the finite-element method. The finite-element has, in general, certain advantages over the finite-difference approach. These are the ease with which irregular geometries, non-uniform meshes and imposition of appropriate boundary conditions can be applied (4,16,21,37,62,96,97,100,117).

The finite-element method, developed initially for structural and solid mechanics, has been applied to some fluid flow problems. Structural and non-structural elements may often be identical in shape and, further, be represented by similar mathematical expressions. The major difference between the elasticity and fluid flow problems lies in the boundary conditions to be satisfied (14,21,22,100).

Oden (95) has presented a theoretical finite element analogue for the Navier-Stokes equations, but without a practical numerical method. Olson (100) presented a numerical procedure to investigate steady incompressible flow problems using stream function formulation. Some useful practical techniques can be learned from his paper. Yamada, Yokouchi and Ohtsubo (129) used the pressure-velocity formulation to analyse steady flow problems. Tong (124) presented results for steady flow using this method with pressure and velocities as dependent variables. Skiba employed

a variational approach and rectangular elements to obtain results for steady convection flow in a rectangular cavity. R,T-S Cheng (25) suggested a versatile and widely applicable quasi-variational formulation to solve the time-dependent Navier-Stokes equations. Bratanow, Ecer and Kobiske (16,17) studied unsteady incompressible flow problems using a perturbation technique for treatment of the nonlinearities in the variational formulation of the vorticity transport equation, and employing higher-order finite elements for a consistent solution of the governing equations and in describing the boundary conditions. Baker (4,5,6) used a Galerkin method and triangular elements for unsteady flow.

Atkinson, Brocklebank, Card and Smith (2) studied creeping flow around a sphere, flow through a converging conical section, and developing flow in a circular pipe using the stream function formulation. They employed three-node triangular elements with stream function and its first two derivatives specified at each node. This kind of formulation requires less computer storage than velocity-pressure formulations, since there is only a single equation to be solved. However, the necessity for first order continuous ($C^{(1)}$) elements would tend to make extension to three dimensional work difficult. Tong and Fung (124) used the stream function formulation as well to investigate slow-viscous flow in a capillary in the presence of moving particles suspended in the flow. Their work has direct application to the biomedical problem of determining the influence of red blood cells on the flow in capillary blood vessels. Taylor and Hood employed the pressure-velocity formulation to study the problem of shear induced fluid flow past a cavity. Because the same interpolation functions were used for both pressure and velocity for this problem, the accuracy of the solution is open to question. They have recently presented a formulation using higher order shape functions for velocities than pressure (117). Tay and Davis (116) used variational principle to

study the problem of convection heat transfer between parallel planes. Bratanow and Ecer (20) employed a variational approach to analyse the three-dimensional unsteady flow around oscillating wings, and to study unsteady aerodynamics (17).

Using quadratic polynomials shape functions for velocity and linear polynomials shape functions for pressure, Kawahara and Yoshimura (71) solved steady flow problems by the Newton-Raphson method and perturbation method, and analysed unsteady flow problems by the perturbation method. Laskaris (77) developed a numerical procedure to study two-dimensional compressible and incompressible, steady state, viscous fluid flow and heat transfer problems. The numerical scheme he presented is based on a general formulation for the system of hydrodynamic equations, taking into full account nonlinear convective terms, viscous terms, and heat conduction terms, and using the method of weighted residuals applied over discrete, distorted rectangular elements of the fluid flow regions. Leonard (80) employed the Galerkin's method to solve perturbed compressible flow problems. Issacs (69) used a transformation similar to that used for quadrilateral isoparametric elements to derive a curved cubic triangular element which has as nodal parameters the value of the function and its two derivatives, and employed this kind of element to study potential flow problems. He compared the triangular element with a standard isoparametric element, and concluded that this kind of triangular element will give similar accuracy at a significantly lower cost. Brebbia and Smith (110) employed linear interpolation functions, a lumped mass system and a simple Euler time integration scheme to analyse the two-dimensional, unsteady, incompressible, viscous Navier-Stokes equations. The results are not only extremely accurate to describe the natural physical phenomena of the problem of vortex street development behind a rectangular obstruction but also highly economic in computer time.

There are still many other good papers concerning numerical treatments of the Navier-Stokes equations. Some of them are chosen and will be presented in the references.

In the present work, the finite element method was employed to solve two-dimensional, unsteady, incompressible, viscous fluid flow problems. A practical computation procedure is presented. A complete finite element computer program has been developed. The formulation can be modified to cover a number of different situations. The same computer program can be used with only minor modification to solve other similar problems. The numerical technique is based upon a general formulation for the Navier-Stokes equations making use of a combined variational principle-finite element approach, applied over discrete finite elements of the fluid flow domain where the unknown fluid variables are expressed continuously in terms of interpolation functions and unknown parameters. Solution to the system of algebraic equations is approached by the Gaussian elimination scheme. It is believed that this numerical procedure is also suitable for a general three-dimensional problem. The time-dependent Navier-Stokes equations are expressed in terms of a stream function equation and a vorticity transport equation. A variational functional of the stream function and a pseudovariational functional of the vorticity of the respective boundary value problem will be presented. The pressure distribution and velocity profiles are determined from stream function.

As in the conventional procedure of time-dependent fluid flow, analysis was often carried out by the incremental method, assuming that the values calculated in the preceding step keep constant during the subsequent small time increments. This commonly used idea is also followed here. To circumvent the nonlinearity in the Navier-Stokes equations, the unsteady flow problem is assumed to be linear in the stream

function and vorticity at each time step. Steady-state solutions are achieved by allowing the time-dependent solutions to converge.

To demonstrate the effectiveness of this numerical scheme, two numerical examples are presented and compared with present papers. The numerical procedure used seems to be fairly stable, and flow trends seem to be well represented. Some new ideas about the numerical method, obtained through numerous numerical experiments, are presented and discussed. Although their validity for all kinds of numerical schemes has not been ascertained yet, it is hoped to bring these observations to people's attention.

Chapter 2 VARIATIONAL FORMULATION OF NAVIER-STOKES EQUATIONS

2.1 Principles of Variational Calculus

In this section, some fundamental principles of variational calculus, which will be used in the subsequent analysis are presented.

Variational calculus is concerned primarily with theory of maxima and minima, but the functions to be minimised or maximised are functionals. The variational calculus, in general, has always been closely associated with realistic problems of continuum mechanics. Usually the functionals whose extreme values are sought are expressions of some form of system energy. For example, in fluid mechanics, for an incompressible, inviscid flow, the kinetic energy is a minimum. And another example is the principle of minimum total potential energy for elastic continua.(45,63,127)

Let us consider a simple functional expressed as

$$\phi = \int_{x_1}^{x_2} F(x, \psi, \psi_x, \psi_{xx}) dx \quad (2-1)$$

where

F is an arbitrary function of one independent variable, X.

$$\begin{aligned} \psi &= \psi(x) \\ \psi_x &= \frac{\partial \psi}{\partial x} \\ \psi_{xx} &= \frac{\partial^2 \psi}{\partial x^2} \end{aligned}$$

Now the variation of the functional is defined in a manner similar to the calculus definition of a total differential

$$\delta F = \frac{\partial F}{\partial \psi} \delta \psi + \frac{\partial F}{\partial \psi_x} \delta \psi_x + \frac{\partial F}{\partial \psi_{xx}} \delta \psi_{xx} \quad (2-2)$$

It is obvious that there is an analogy between finding the minimum or maximum of a function via ordinary calculus and finding the minimum or maximum of a functional via variational calculus.(37,63,127)

So extending the concept of ordinary calculus, the following equation is obtained:

$$\delta\phi = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \psi} \delta\psi + \frac{\partial F}{\partial \psi_x} \delta\psi_x + \frac{\partial F}{\partial \psi_{xx}} \delta\psi_{xx} \right) dx = 0 \quad (2-3)$$

Extending the principles of ordinary calculus again, it can be learned that the first variation is also a commutative operator with both differentiation and integration if the integration limits are not to be varied. (37, 63, 127)

So the following equations may be written.

$$\delta \left(\int F dx \right) = \int (\delta F) dx \quad (2-4)$$

$$\delta \left(\frac{d\psi}{dx} \right) = \frac{d}{dx} (\delta\psi) \quad (2-5)$$

And equation (2-3) becomes

$$\delta\phi = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \psi} \delta\psi + \frac{\partial F}{\partial \psi_x} \frac{\partial}{\partial x} (\delta\psi) + \dots \right) dx = 0 \quad (2-6)$$

Integrating each item by parts, the following equation is obtained.

$$\begin{aligned} \delta\phi &= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \psi} - \frac{d}{dx} \left(\frac{\partial F}{\partial \psi_x} \right) + \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial \psi_{xx}} \right) \right) \delta\psi dx \\ &\quad + \left(\frac{\partial F}{\partial \psi_x} - \frac{d}{dx} \left(\frac{\partial F}{\partial \psi_{xx}} \right) \right) \delta\psi \Big|_{x_1}^{x_2} \\ &\quad + \left[\left(\frac{\partial F}{\partial \psi_{xx}} \right) \delta\psi_x \right]_{x_1}^{x_2} \\ &= 0 \end{aligned} \quad (2-7)$$

Because $\delta\psi$ and $\delta\psi_x$ are arbitrary admissible variations, the integrand and remaining terms of equation (2-7) must vanish. Thus the necessary conditions for $\psi(x)$ to minimise $\phi(x)$ are as follows:

$$\frac{\partial F}{\partial \psi} - \frac{d}{dx} \left(\frac{\partial F}{\partial \psi_x} \right) + \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial \psi_{xx}} \right) = 0 \quad (2-8)$$

$$\left[\frac{\partial F}{\partial \psi_x} - \frac{d}{dx} \left(\frac{\partial F}{\partial \psi_{xx}} \right) \right] \delta \psi \Big|_{x_1}^{x_2} = 0 \quad (2-9)$$

$$\left[\left(\frac{\partial F}{\partial \psi_{xx}} \right) \delta \psi_x \right]_{x_1}^{x_2} = 0 \quad (2-10)$$

Equation (2-8) is the governing differential equation for the problem and is called the Euler-Lagrange equation, or just the Euler equation. The other two conditions give the necessary boundary conditions. From equation (2 - 9) the following equation may be written. (37,63,127)

$$\int \left[\frac{\partial F}{\partial \psi_x} - \frac{d}{dx} \left(\frac{\partial F}{\partial \psi_{xx}} \right) \right]_{x_1}^{x_2} = 0 \quad (2-11)$$

or

$$\psi(x_1) = 0$$

$$\psi(x_2) = 0 \quad (2-12)$$

and from equation (2 - 10) either

$$\frac{\partial F}{\partial \psi_{xx}} \Big|_{x_1}^{x_2} = 0 \quad (2-13)$$

or

$$\frac{\partial \psi}{\partial x} \Big|_{x_1}^{x_2} = 0 \quad (2-14)$$

Equation (2 - 11) and (2 - 13) are called natural boundary conditions. If they are satisfied, they are called free boundary conditions. Equations (2- 12) and (2 - 14) are called geometric boundary conditions or forced boundary conditions. It may be mentioned here that the Euler-Lagrange equation expresses only a necessary and not a sufficient condition for a minimum. So the solution of an Euler-Lagrange equation may not yield a function that minimises a given functional. (See later Section 2.3 Variational Formulation).

One of the principal advantages of the finite element method employing a suitable, valid variational principle is that only the geometric boundary conditions need to be specified. The natural boundary conditions are automatically incorporated in the formulation. That is why all the boundary conditions have only been enforced on rigid boundaries in this work and why the 'natural' boundary conditions are always left for the program to approximate when a suitable variational principle-finite element method is employed to deal with test problems. Through numerical experiments it has been found that when a combined variational principle-finite element method is employed, the 'natural' boundary conditions had better not be specified again, otherwise the results may be in error. (21,37,42,63,100,127,131) (see Chapter 4).

A functional of two independent variables has the form

$$\phi(\psi) = \iint_A F(x, y, \psi, \psi_x, \psi_y, \psi_{xx}, \psi_{xy}, \psi_{yy}) dx dy \quad (2-15)$$

Proceeding in a similar way, it is not difficult to derive the Euler-Lagrange equations and boundary conditions for the above functional.

The Euler equation for equation (2-15) is

$$\begin{aligned} \frac{\partial^2}{\partial x^2} \left(\frac{\partial F}{\partial \psi_{xx}} \right) + \frac{\partial^2}{\partial x \partial y} \left(\frac{\partial F}{\partial \psi_{xy}} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{\partial F}{\partial \psi_{yy}} \right) \\ - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial \psi_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial \psi_y} \right) + \frac{\partial F}{\partial \psi} = 0 \end{aligned} \quad (2-16)$$

Similarly, Euler-Lagrange equations and boundary conditions for other functionals may be derived. Some more detailed discussions and applications will be presented in Section 2.3.

2.2. Navier-Stokes Equations

The full Navier-Stokes equations representing a balance of viscous forces, inertia forces, and pressure forces are capable of describing

some of the most interesting phenomena in fluid mechanics. For unsteady, incompressible, two-dimensional, viscous fluid flow with inertia, the Navier-Stokes equations for analysing the motion of the fluids can be written as

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot (\nabla \vec{u}) = \frac{1}{\rho} \vec{F} - \frac{1}{\rho} \nabla P + \nu \nabla^2 \vec{u} \quad (2-17)$$

where

- \vec{u} = velocity vector = $[u, v]$
- t = time
- ∇ = differential operator = $\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$
- ρ = density of the fluid
- \vec{F} = body force vector
- ν = kinematic viscosity of the fluid
- P = pressure
- ∇^2 = Laplacian operator = $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
- u = velocity in X direction
- v = velocity in Y direction

in consistent units.

The equation of continuity for incompressible fluid is

$$\nabla \cdot \vec{u} = 0 \quad (2-18)$$

The velocity components u and v may be expressed in terms of a stream function ψ as

$$\begin{aligned} u &= \frac{\partial \psi}{\partial y} \\ v &= -\frac{\partial \psi}{\partial x} \end{aligned} \quad (2-19)$$

The vector

$$\vec{u} = [u, v]$$

may be written as

$$\vec{u} = \left[\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right] \quad (2-20)$$

Using equation (2-19) , fluid rotation or vorticity is defined as the average angular velocity of any two mutually perpendicular line elements of a fluid particle.

In vector notation, the following equation can be written:

$$\vec{\omega} = \nabla \times \vec{u} \quad (2-21)$$

A well-known vector identity shows that for any function P having continuous first and second derivatives,

$$\nabla \times \nabla P = 0 \quad (2-22)$$

At the same time, the following equation may be written

$$\nabla \times \vec{F} = 0 \quad (2-23)$$

Taking the curl ($\nabla \times$) of both sides of equation (2-17) gives

$$\frac{\partial \omega}{\partial t} + (\vec{u} \cdot \nabla) \omega = \nu \nabla^2 \omega \quad (2-24)$$

From equations (2-20) and (2-21), the stream functions are related to vorticity as follows:

$$\nabla^2 \psi = -\omega \quad (2-25)$$

Now the pressure distribution is to be calculated. The pressure field can be obtained by integrating the momentum equation. But, in general, it seems that a Poisson type equation yields more accurate numerical computation results and uses less computer time than the direct methods based on the momentum equation (16,21,77). So the Poisson type equation will be derived first. Equation (2-17) can be written as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho} \vec{F}_x - \frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] \quad (2-26)$$

Differentiating this equation with respect to x gives

$$\begin{aligned} & \left(\frac{\partial}{\partial x} \frac{\partial u}{\partial t} \right) + \left(\frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + u \frac{\partial^2 u}{\partial x^2} \right) + \left(\frac{\partial v}{\partial x} \frac{\partial u}{\partial y} + v \frac{\partial^2 u}{\partial x \partial y} \right) \\ &= \frac{1}{\rho} \frac{\partial F_x}{\partial x} - \frac{1}{\rho} \frac{\partial^2 P}{\partial x^2} + \nu \left(\frac{\partial^2 \partial u}{\partial x^2 \partial x} + \frac{\partial^2 \partial u}{\partial y^2 \partial x} \right) \end{aligned} \quad (2-27)$$

Similarly, from equation (2-17)

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \frac{1}{\rho} F_y - \frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right] \quad (2-28)$$

Differentiating it with respect to y gives

$$\begin{aligned} & \frac{\partial}{\partial y} \frac{\partial v}{\partial t} + \left(\frac{\partial u}{\partial y} \frac{\partial v}{\partial x} + u \frac{\partial^2 v}{\partial x \partial y} \right) + \left(\frac{\partial v}{\partial y} \frac{\partial v}{\partial y} + v \frac{\partial^2 v}{\partial y^2} \right) \\ &= \frac{1}{\rho} \frac{\partial F_y}{\partial y} - \frac{1}{\rho} \frac{\partial^2 P}{\partial y^2} + \nu \left(\frac{\partial^2}{\partial x^2} \frac{\partial v}{\partial y} + \frac{\partial^2}{\partial y^2} \frac{\partial v}{\partial y} \right) \end{aligned} \quad (2-29)$$

Combining equations (2-29) and (2-27) gives

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \left(\frac{\partial u}{\partial x} \right)^2 + u \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} + v \frac{\partial^2 u}{\partial x \partial y} \\ & + u \frac{\partial^2 v}{\partial x \partial y} + \left(\frac{\partial v}{\partial y} \right)^2 + v \frac{\partial^2 v}{\partial y^2} \\ &= \frac{1}{\rho} \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} \right) - \frac{1}{\rho} \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right) + \nu \left[\frac{\partial^2}{\partial x^2} \frac{\partial u}{\partial x} \right. \\ & \left. + \frac{\partial^2 \partial u}{\partial y^2 \partial x} + \frac{\partial^2}{\partial x^2} \frac{\partial v}{\partial y} + \frac{\partial^2}{\partial y^2} \frac{\partial v}{\partial y} \right] \end{aligned} \quad (2-30)$$

the following can be written

(2-18)

$$\nabla \cdot \vec{u} = 0$$

From equation (2-18) it is easy to get following relations:

$$\frac{\partial^2}{\partial x^2} \frac{\partial u}{\partial x} + \frac{\partial^2}{\partial y^2} \frac{\partial u}{\partial x} + \frac{\partial^2}{\partial x^2} \frac{\partial v}{\partial y} + \frac{\partial^2}{\partial y^2} \frac{\partial v}{\partial y} = 0 \quad (2-31)$$

$$\frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0 \quad (2-32)$$

and

$$u \frac{\partial^2 u}{\partial x^2} + v \frac{\partial^2 u}{\partial x \partial y} + u \frac{\partial^2 v}{\partial x \partial y} + v \frac{\partial^2 v}{\partial y^2} = 0 \quad (2-33)$$

Substituting equations (2-31), (2-32) and (2-33) into equation (2-30)

gives

$$\left(\frac{\partial u}{\partial x} \right)^2 + 2 \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} + \left(\frac{\partial v}{\partial y} \right)^2 = \frac{1}{\rho} \nabla \cdot \vec{F} - \frac{1}{\rho} \nabla^2 p \quad (2-34)$$

In the absence of body forces, the following equation can be obtained

$$\nabla^2 p = -\rho \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + 2 \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} \right] \quad (2-35)$$

or

$$\nabla^2 p = -2\rho \left[\frac{\partial v}{\partial x} \frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \right] \quad (2-36)$$

$$= 2\rho \left[\frac{\partial^2 \psi}{\partial y^2} \frac{\partial^2 \psi}{\partial x^2} - \left(\frac{\partial^2 \psi}{\partial x \partial y} \right)^2 \right] \quad (2-37)$$

$$= -\rho Q \quad (2-38)$$

where

$$Q = 2 \left[\frac{\partial v}{\partial x} \frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \right] \quad (2-39)$$

$$= -2 \left[\frac{\partial^2 \psi}{\partial y^2} \frac{\partial^2 \psi}{\partial x^2} - \left(\frac{\partial^2 \psi}{\partial x \partial y} \right)^2 \right]$$

The analysis of unsteady, incompressible, two-dimensional, viscous fluid flow involves the simultaneous solution of equations (2-24) and (2-25). Once the ψ and ω field are known, the pressure distribution can be calculated from equation (2-37).

2.3 Variational Formulation

Basically, the finite element method represents an approximate procedure for satisfying the problem in terms of its variational formulation. If the governing differential equations were all linear the variational formulation would be straightforward. The non-linear terms in the Navier-Stokes equations seem to have precluded the existence of an associated variational principle of the classical kind. It is found that the Navier-Stokes equations cannot be derived from a classical variational principle unless one of the terms $\vec{u} \times (\nabla \times \vec{u})$ or $\vec{u} \cdot (\nabla \vec{u})$ is zero (41). However, it has been shown that 'pseudo' principles can be obtained provided some terms are not allowed to vary when the first variation is performed. The pseudo-variational functional finite-element method has the advantage of simplicity and reduced computation. It is not a true variational method since from another point of view it can be regarded as a Galerkin method used with a particular approximation scheme. (21, 25, 63, 100, 127)

Equation (2-25) and (2-37) are in the form of Poisson's equation, for which a direct variational formulation exists. From theorems of variational methods, by inspection, the variational functionals for equations (2-25) and (2-37) can be written as follows

$$\phi_2 = \frac{1}{2} \int_A \left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right] dA - \int_A \omega \psi dA \quad (2-40)$$

$$\phi_3 = \frac{1}{2} \int_A \left[\left(\frac{\partial P}{\partial x} \right)^2 + \left(\frac{\partial P}{\partial y} \right)^2 \right] dA - \int_A P P Q dA \quad (2-41)$$

where

$$Q = -2 \left[\frac{\partial^2 \psi}{\partial y^2} \frac{\partial^2 \psi}{\partial x^2} - \left(\frac{\partial^2 \psi}{\partial x \partial y} \right)^2 \right]$$

and A is the region of interest. The variational functionals ϕ_2 and ϕ_3 exist such that the Euler-Lagrangian equations of equations (2-40) and (2-41) are simply equations (2-25) and (2-37). The functions ψ and P which satisfy the required boundary conditions and which give the functional integrals ϕ_2 and ϕ_3 are solutions of equations (2-25) and (2-37).

Similarly, by taking $\frac{\partial \omega}{\partial t}$ as an invariance, a variational functional ϕ_1

$$\begin{aligned} \phi_1 = \int_A \omega \frac{\partial \omega}{\partial t} dA + \int_A \left[u \omega \frac{\partial \omega}{\partial t} + v \omega \frac{\partial \omega}{\partial y} \right] dA \\ + \frac{\nu}{2} \int_A \left[\left(\frac{\partial \omega}{\partial x} \right)^2 + \left(\frac{\partial \omega}{\partial y} \right)^2 \right] dA \end{aligned} \quad (2-42)$$

exists (16,19,25,41) such that upon taking the first variation of ϕ_1 , the vorticity transport equation, equation (2-24) will be recovered. The function ω satisfying equation (2-24) and its boundary conditions minimise ϕ_1 . Segregating stream function and vorticity solutions according to different instants of time reduces the problem to one of consecutively minimising ϕ_1 and ϕ_2 . This can be conveniently accomplished by the finite element method. (21,25,63,100,127)

A disadvantage of the procedure is that it is not known whether or not a particular pseudo-functional will yield convergence or not until it has been tried, since a mathematical criterion for convergence is not yet available. (21,25,93,100)

Norrie and Vries (93) suggested that if a certain functional does not converge, one has no choice but to modify it; such a change alters the

stiffness matrix and may result in convergence.

Some experience with the method assists in choosing an appropriate functional on intuitive grounds. Norrie and Vries postulate that: "The process will converge if the terms which dominate the physical behaviour of the system are included are those terms in the functional which are not iterated upon but are used only in the minimisation procedure". (93)

One of the disadvantages of the use of stream function-vorticity formulation is that unless the velocities are entirely prescribed on all boundaries it is often impossible to establish the values of stream functions on some positions of the boundary. This is particularly serious in multiple connected boundaries, such as are presented by flow around obstacles etc. To overcome these difficulties it is necessary to introduce additional constraints on the rate of boundary work. (see Section 6.1, example one)

Chapter 3 FINITE ELEMENT MODEL

3.1 Introduction

The finite element method is based on the use of series expansions within subdomains or elements, into which the domain of interest is divided. It is a general numerical technique which provides an approximate piecewise continuous representation of the unknown field variables in terms of polynomials, sometimes called interpolation functions or shape functions and model parameters (22,33,37,77,127).

The continuous region is subdivided into a finite number of elements where the nodal values and/or the partial derivatives of the dependent variables at prescribed points, nodes of elements, become the unknown parameters of the problem. The finite-element representation of the dependent field variable must be able to provide an improved approximation to the true solution as successive subdivisions of the domain using smaller and smaller elements is attempted.

The basic steps of the solution procedure are as follows (22,33,37,127).

1. Discretisation of the continuum.
2. Selection of interpolation functions.
3. Evaluation of the matrices of the elements.
4. Assembly of the element equations.
5. Application of the boundary conditions.
6. Solution of the system equations.
7. Calculation of any other unknown field variables.

3.2 Matrix Formulation

Different finite-element models were chosen for representing the variations of streamfunctions, vorticities and pressure. Fig.3-1 shows a typical finite element. The stream functions and vorticities were assumed to vary linearly over each finite element as

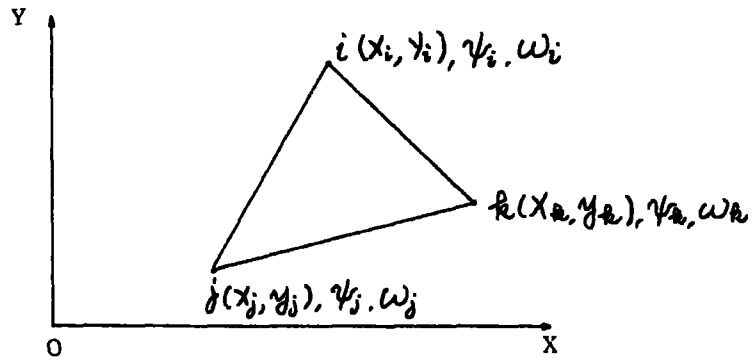


Figure 3-1 Triangular finite element

$$\psi^{(e)}(x, y, t) = [H_1^{(e)}(t), H_2^{(e)}(t), H_3^{(e)}(t)] \begin{Bmatrix} T_1^{(e)}(x, y) \\ T_2^{(e)}(x, y) \\ T_3^{(e)}(x, y) \end{Bmatrix} \quad (3-1)$$

$$\omega^{(e)}(x, y, t) = [H_1^{(e)}(t), H_2^{(e)}(t), H_3^{(e)}(t)] \begin{Bmatrix} Q_1^{(e)}(x, y) \\ Q_2^{(e)}(x, y) \\ Q_3^{(e)}(x, y) \end{Bmatrix} \quad (3-2)$$

where the T's and Q's are the trial functions and the superscript (e) indicates the eth element. At the nodal points of this element, points i, j and k in Fig.3-1, $\psi_i, \omega_i, \psi_j, \omega_j$, and ψ_k, ω_k are

$$\begin{aligned} \psi_i(t) &= H_1^{(e)}(t) T_1^{(e)}(x_i, y_i) + H_2^{(e)}(t) T_2^{(e)}(x_i, y_i) \\ &\quad + H_3^{(e)}(t) T_3^{(e)}(x_i, y_i) \\ \omega_i(t) &= H_1^{(e)}(t) Q_1^{(e)}(x_i, y_i) + H_2^{(e)}(t) Q_2^{(e)}(x_i, y_i) \\ &\quad + H_3^{(e)}(t) Q_3^{(e)}(x_i, y_i) \\ \psi_j(t) &= H_1^{(e)}(t) T_1^{(e)}(x_j, y_j) + H_2^{(e)}(t) T_2^{(e)}(x_j, y_j) \\ &\quad + H_3^{(e)}(t) T_3^{(e)}(x_j, y_j) \\ \omega_j(t) &= H_1^{(e)}(t) Q_1^{(e)}(x_j, y_j) + H_2^{(e)}(t) Q_2^{(e)}(x_j, y_j) \\ &\quad + H_3^{(e)}(t) Q_3^{(e)}(x_j, y_j) \\ \psi_k(t) &= H_1^{(e)}(t) T_1^{(e)}(x_k, y_k) + H_2^{(e)}(t) T_2^{(e)}(x_k, y_k) \\ &\quad + H_3^{(e)}(t) T_3^{(e)}(x_k, y_k) \\ \omega_k(t) &= H_1^{(e)}(t) Q_1^{(e)}(x_k, y_k) + H_2^{(e)}(t) Q_2^{(e)}(x_k, y_k) \\ &\quad + H_3^{(e)}(t) Q_3^{(e)}(x_k, y_k) \end{aligned} \quad (3-3)$$

In matrix form, equations (3-1) may be written as

$$\begin{Bmatrix} \psi_i(t) \\ \psi_j(t) \\ \psi_k(t) \end{Bmatrix} = [C^{(e)}] \begin{Bmatrix} H_1^{(e)}(t) \\ H_2^{(e)}(t) \\ H_3^{(e)}(t) \end{Bmatrix} \quad (3.4)$$

and

$$\begin{Bmatrix} \omega_i(t) \\ \omega_j(t) \\ \omega_k(t) \end{Bmatrix} = [D^{(e)}] \begin{Bmatrix} H_1^{(e)}(t) \\ H_2^{(e)}(t) \\ H_3^{(e)}(t) \end{Bmatrix}$$

where

$$[C^{(e)}] = \begin{bmatrix} T_1^{(e)}(x_i, y_i) & T_2^{(e)}(x_i, y_i) & T_3^{(e)}(x_i, y_i) \\ T_1^{(e)}(x_j, y_j) & T_2^{(e)}(x_j, y_j) & T_3^{(e)}(x_j, y_j) \\ T_1^{(e)}(x_k, y_k) & T_2^{(e)}(x_k, y_k) & T_3^{(e)}(x_k, y_k) \end{bmatrix} \quad (3-5)$$

and

$$[D^{(e)}] = \begin{bmatrix} Q_1^{(e)}(x_i, y_i) & Q_2^{(e)}(x_i, y_i) & Q_3^{(e)}(x_i, y_i) \\ Q_1^{(e)}(x_j, y_j) & Q_2^{(e)}(x_j, y_j) & Q_3^{(e)}(x_j, y_j) \\ Q_1^{(e)}(x_k, y_k) & Q_2^{(e)}(x_k, y_k) & Q_3^{(e)}(x_k, y_k) \end{bmatrix}$$

In order to express H_n 's in terms of ψ_n 's or ω_n 's uniquely, equations (3-4) can be written as

$$\begin{Bmatrix} H_1^{(e)}(t) \\ H_2^{(e)}(t) \\ H_3^{(e)}(t) \end{Bmatrix} = [C^{(e)}]^{-1} \begin{Bmatrix} \psi_i(t) \\ \psi_j(t) \\ \psi_k(t) \end{Bmatrix} \quad (3-6)$$

and

$$\begin{Bmatrix} H_1^{(e)}(t) \\ H_2^{(e)}(t) \\ H_3^{(e)}(t) \end{Bmatrix} = [D^{(e)}]^{-1} \begin{Bmatrix} \omega_i(t) \\ \omega_j(t) \\ \omega_k(t) \end{Bmatrix}$$

So equations (3-1) and (3-2) give

$$\begin{aligned}
 \psi^{(e)}(x, y, t) &= [T_1^{(e)}(x, y), T_2^{(e)}(x, y), T_3^{(e)}(x, y)] \begin{Bmatrix} H_1^{(e)}(t) \\ H_2^{(e)}(t) \\ H_3^{(e)}(t) \end{Bmatrix} \\
 &= [T_1^{(e)}(x, y), T_2^{(e)}(x, y), T_3^{(e)}(x, y)] [C^{(e)}]^{-1} \begin{Bmatrix} \psi_i(t) \\ \psi_j(t) \\ \psi_k(t) \end{Bmatrix} \\
 &= [N_1^{(e)}(x, y), N_2^{(e)}(x, y), N_3^{(e)}(x, y)] \begin{Bmatrix} \psi_i(t) \\ \psi_j(t) \\ \psi_k(t) \end{Bmatrix}
 \end{aligned} \tag{3-7}$$

where

$$[N_1^{(e)}(x, y), N_2^{(e)}(x, y), N_3^{(e)}(x, y)] = [T_1^{(e)}(x, y), T_2^{(e)}(x, y), T_3^{(e)}(x, y)] [C^{(e)}]^{-1}$$

and

$$\begin{aligned}
 \omega^{(e)}(x, y, t) &= [Q_1^{(e)}(x, y), Q_2^{(e)}(x, y), Q_3^{(e)}(x, y)] \begin{Bmatrix} H_1^{(e)}(t) \\ H_2^{(e)}(t) \\ H_3^{(e)}(t) \end{Bmatrix} \\
 &= [Q_1^{(e)}(x, y), Q_2^{(e)}(x, y), Q_3^{(e)}(x, y)] [D^{(e)}]^{-1} \begin{Bmatrix} \omega_i(t) \\ \omega_j(t) \\ \omega_k(t) \end{Bmatrix} \\
 &= [L_1^{(e)}(x, y), L_2^{(e)}(x, y), L_3^{(e)}(x, y)] \begin{Bmatrix} \omega_i(t) \\ \omega_j(t) \\ \omega_k(t) \end{Bmatrix}
 \end{aligned} \tag{3-8}$$

where

$$[L_1^{(e)}(x, y), L_2^{(e)}(x, y), L_3^{(e)}(x, y)] = [Q_1^{(e)}(x, y), Q_2^{(e)}(x, y), Q_3^{(e)}(x, y)] [D^{(e)}]^{-1}$$

For a triangular element, if Tn's and Qn's are taken to be 1, x and y

then

$$N_i^{(e)}(x, y) = L_i^{(e)}(x, y) = (a_i + b_i x + c_i y) / 2\Delta \tag{3-9}$$

in which

$$\begin{aligned}
 a_i &= x_j y_k - x_k y_j \\
 b_i &= y_j - y_k \\
 c_i &= x_k - x_j
 \end{aligned}$$

with the other coefficients obtained by a cyclic permutation of subscripts in order i, j, k and where

$$2\Delta = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix}$$

where Δ is the area of the element.

Equations(3-7) and (3-8) may be expressed by the following matrix equations.

$$\psi^{(e)}(x, y, t) = \{N^{(e)}(x, y)\}^T \{\psi(t)\} \quad (3-10)$$

$$\omega^{(e)}(x, y, t) = \{N^{(e)}(x, y)\}^T \{\omega(t)\} \quad (3-11)$$

where T denotes the transpose to the column matrix.

The gradients of $\psi^{(e)}$ and $\omega^{(e)}$ are

$$\frac{\partial \psi^{(e)}}{\partial x} = \left\{ \frac{\partial N^{(e)}}{\partial x} \right\}^T \{\psi(t)\} \quad (3-12)$$

$$\frac{\partial \psi^{(e)}}{\partial y} = \left\{ \frac{\partial N^{(e)}}{\partial y} \right\}^T \{\psi(t)\} \quad (3-13)$$

$$\frac{\partial \omega^{(e)}}{\partial x} = \left\{ \frac{\partial N^{(e)}}{\partial x} \right\}^T \{\omega(t)\} \quad (3-14)$$

$$\frac{\partial \omega^{(e)}}{\partial y} = \left\{ \frac{\partial N^{(e)}}{\partial y} \right\}^T \{\omega(t)\} \quad (3-15)$$

The finite element models were employed in discretising the variational functionals in equations (2-40), (2-41) and (2-42). Substitution of equations (3-10), (3-11) and the last equations into equations (2-40) and (2-42) gives

$$\begin{aligned}
 \phi_1 &= \int_{A^{(e)}} \left[\{N^{(e)}\}^T \{\omega\} \right] \frac{\partial [\{N^{(e)}\}^T \{\omega\}]}{\partial t} dA^{(e)} \\
 &+ \int_{A^{(e)}} \left(\left[\left\{ \frac{\partial N^{(e)}}{\partial y} \right\}^T \{\psi\} \right] \left[\left\{ \frac{\partial N^{(e)}}{\partial x} \right\}^T \{\omega\} \right] \right. \\
 &- \left. \left[\left\{ \frac{\partial N^{(e)}}{\partial x} \right\}^T \{\psi\} \right] \left[\left\{ \frac{\partial N^{(e)}}{\partial y} \right\}^T \{\omega\} \right] \right) \left[\{N^{(e)}\}^T \{\omega\} \right] dA^{(e)} \\
 &+ \frac{\nu}{2} \int_{A^{(e)}} \left(\left[\left\{ \frac{\partial N^{(e)}}{\partial x} \right\}^T \{\omega\} \right]^2 + \left[\left\{ \frac{\partial N^{(e)}}{\partial y} \right\}^T \{\omega\} \right]^2 \right) dA^{(e)} \\
 &= \int_{A^{(e)}} \left[\{N^{(e)}\}^T \{\omega\} \right] \frac{\partial [\{N^{(e)}\}^T \{\omega\}]}{\partial t} dA^{(e)} \tag{3-16} \\
 &+ \int_{A^{(e)}} \left\{ \left(\frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \omega}{\partial y} \right) \left[\{N^{(e)}\}^T \{\omega\} \right] \right\} dA^{(e)} \\
 &+ \frac{\nu}{2} \int_{A^{(e)}} \left(\left[\left\{ \frac{\partial N^{(e)}}{\partial x} \right\}^T \{\omega\} \right]^2 + \left[\left\{ \frac{\partial N^{(e)}}{\partial y} \right\}^T \{\omega\} \right]^2 \right) dA^{(e)}
 \end{aligned}$$

and

$$\begin{aligned}
 \phi_2 &= \frac{1}{2} \int_{A^{(e)}} \left(\left[\left\{ \frac{\partial N^{(e)}}{\partial x} \right\}^T \{\psi\} \right]^2 + \left[\left\{ \frac{\partial N^{(e)}}{\partial y} \right\}^T \{\psi\} \right]^2 \right) dA^{(e)} \\
 &- \int_{A^{(e)}} \left(\left[\{N^{(e)}\}^T \{\omega\} \right] \left[\{N^{(e)}\}^T \{\psi\} \right] \right) dA^{(e)} \tag{3-17}
 \end{aligned}$$

The finite element solution to the problem involves picking the nodal values of ω and ψ so as to make stationary the functionals ϕ_1 and ϕ_2 . To make ϕ_1 and ϕ_2 stationary with respect to the nodal values of vorticity and stream function respectively, the following conditions are required.

$$\delta \phi_1(\omega) = \sum_{i=1}^{NN} \frac{\partial \phi_1(\omega)}{\partial \omega_i} \delta \omega_i = 0 \tag{3-18}$$

$$\delta \phi_2(\psi) = \sum_{i=1}^{NN} \frac{\partial \phi_2(\psi)}{\partial \psi_i} \delta \psi_i = 0 \tag{3-19}$$

where NN is the total number of nodes. Since the $\delta\omega_i$'s and $\delta\psi_i$'s are independent, equations (3-18) and (3-19) can hold only if

$$\frac{\partial \phi_1(\omega)}{\partial \omega_i} = 0 \quad (3-20)$$

$$\frac{\partial \phi_2(\psi)}{\partial \psi_i} = 0 \quad (3-21)$$

Hence from equations (3-21) and (3-17), for a typical node i the following equation is required

$$\begin{aligned} \frac{\partial \phi_2(\psi)}{\partial \psi_i} &= \int_{A^{(e)}} \left[\left\{ \frac{\partial N^{(e)}}{\partial x} \right\}^T \{\psi\} \frac{\partial N_i}{\partial x} + \left\{ \frac{\partial N^{(e)}}{\partial y} \right\}^T \{\psi\} \frac{\partial N_i}{\partial y} \right] dA^{(e)} \\ &\quad - \int_{A^{(e)}} \left[\left\{ N^{(e)} \right\}^T \{\omega\} N_i \right] dA^{(e)} \\ &= 0 \end{aligned} \quad (3-22)$$

Utilising equation (3-11), equation (3-22) may be written as:

$$\begin{aligned} \int_{A^{(e)}} \left[\left\{ \frac{\partial N^{(e)}}{\partial x} \right\}^T \frac{\partial N_i}{\partial x} + \left\{ \frac{\partial N^{(e)}}{\partial y} \right\}^T \frac{\partial N_i}{\partial y} \right] \{\psi\} dA^{(e)} \\ - \int_{A^{(e)}} \omega^{(e)} N_i dA^{(e)} = 0 \end{aligned} \quad (3-23)$$

In matrix form, for the entire element the following equation is obtained

$$[K_\psi]^{(e)} \{\psi\}^{(e)} + \{S_\psi\}^{(e)} = \{0\} \quad (3-24)$$

where

$$K_{\psi_{ij}}^{(e)} = \int_{A^{(e)}} \left[\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right] dA^{(e)}$$

$$S_{\psi_i}^{(e)} = - \int_{A^{(e)}} \omega^{(e)} N_i dA^{(e)}$$

Similarly, from equations (3-20) and (3-16), for a typical node i the following equation is required

$$\begin{aligned} \frac{\partial \Phi(\omega)}{\partial \omega_i} &= \int_{A^{(e)}} N_i N_j \frac{\partial \omega}{\partial t} dA \\ &+ \int_{A^{(e)}} \left(\frac{\partial \psi^{(e)}}{\partial y} \frac{\partial \omega^{(e)}}{\partial x} - \frac{\partial \psi^{(e)}}{\partial x} \frac{\partial \omega^{(e)}}{\partial y} \right) N_i dA^{(e)} \\ &+ \frac{\nu}{2} \int_{A^{(e)}} \left(2 \left[\left\{ \frac{\partial N}{\partial x} \right\}^T \{ \omega \} \right] \frac{\partial N_i}{\partial x} + 2 \left[\left\{ \frac{\partial N}{\partial y} \right\}^T \{ \omega \} \right] \frac{\partial N_i}{\partial y} \right) dA^{(e)} \\ &= 0 \end{aligned}$$

In matrix form, for the entire element the following equation is obtained.

$$[K\omega]^{(e)} \{ \omega \}^{(e)} + [K\dot{\omega}]^{(e)} \{ \dot{\omega} \}^{(e)} + \{ S\omega \}^{(e)} = \{ 0 \}$$

where

(3-25)

$$K_{\omega_{ij}}^{(e)} = \nu K_{\psi_{ij}}^{(e)}$$

$$K_{\dot{\omega}_{ij}}^{(e)} = \int_{A^{(e)}} N_i N_j dA^{(e)}$$

$$S_{\omega_{ij}}^{(e)} = \int_{A^{(e)}} N_i \left(\frac{\partial \psi^{(e)}}{\partial y} \frac{\partial \omega^{(e)}}{\partial x} - \frac{\partial \psi^{(e)}}{\partial x} \frac{\partial \omega^{(e)}}{\partial y} \right) dA^{(e)}$$

In solving a fluid flow problem with the foregoing elements, the usual assemblage process for finite elements is followed as well. For the assembled system the foregoing equations become

$$[K_\psi] \{ \psi \} + \{ S_\psi \} = \{ 0 \} \tag{3-26}$$

$$[K_\omega] \{ \omega \} + [K_{\dot{\omega}}] \{ \dot{\omega} \} + \{ S_\omega \} = \{ 0 \} \tag{3-27}$$

where

$$K_{\psi_{ij}} = \sum_{e=1}^M \int_{A^{(e)}} \left[\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right] dA^{(e)} \tag{3-28}$$

$$S_{\psi_i} = - \sum_{e=1}^M \int_{A^{(e)}} \omega^{(e)} N_i dA^{(e)} \quad (3-29)$$

$$K_{\omega_{ij}} = \nu K_{\psi_{ij}} \quad (3-30)$$

$$K_{\dot{\omega}_{ij}} = \sum_{e=1}^M \int_{A^{(e)}} N_i N_j dA^{(e)} \quad (3-31)$$

$$S_{\omega_i} = \sum_{e=1}^M \int_{A^{(e)}} N_i \left(\frac{\partial \psi^{(e)}}{\partial y} \frac{\partial \omega^{(e)}}{\partial x} - \frac{\partial \psi^{(e)}}{\partial x} \frac{\partial \omega^{(e)}}{\partial y} \right) dA^{(e)} \quad (3-32)$$

where M is the total number of elements.

3.3 Integration of the Matrix Equation

To integrate equations (2-24) and (2-25) with respect to time, the method suggested by R.T-S Cheng (25) is used. Cheng considered two solutions ψ_n and ω_n at the Nth time step and ψ_{n+1} and ω_{n+1} at a time increment Δt later. Then the governing equations may be expressed as follows:

$$\frac{\partial \omega_n}{\partial t} + \frac{\partial \psi_{n+1}}{\partial y} \frac{\partial \omega_n}{\partial x} - \frac{\partial \psi_{n+1}}{\partial x} \frac{\partial \omega_n}{\partial y} = \nu \nabla^2 \omega_{n+1} \quad (3-33)$$

and

$$\nabla^2 \psi_{n+1} = -\omega_n \quad (3-34)$$

And the assembled system equations may be written as follows:

$$[K_{\psi}] \{\psi\}_{n+1} + \{S_{\psi}\}_n = \{0\} \quad (3-35)$$

$$[K_{\omega}] \{\omega\}_{n+1} + [K_{\dot{\omega}}] \{\dot{\omega}\}_{n+1} + \{S_{\omega}\}_n = \{0\} \quad (3-36)$$

where

$$K_{\psi_{ij}} = \sum_{e=1}^M \int_{A^{(e)}} \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) dA^{(e)} \quad (3-37)$$

$$S_{\psi_i} = - \sum_{e=1}^M \int_{A^{(e)}} \omega_n^{(e)} N_i dA^{(e)} \quad (3-38)$$

$$K_{\omega_{ij}} = \nu K_{\psi_{ij}} \quad (3-39)$$

$$K_{\dot{\omega}_{ij}} = \sum_{e=1}^M \int_{A^{(e)}} N_i N_j dA^{(e)} \quad (3-40)$$

$$S_{\dot{\omega}_i} = \sum_{e=1}^M \int_{A^{(e)}} N_i \left(\frac{\partial \psi_{n+1}^{(e)}}{\partial y} \frac{\partial \omega_n^{(e)}}{\partial x} - \frac{\partial \psi_{n+1}^{(e)}}{\partial x} \frac{\partial \omega_n^{(e)}}{\partial y} \right) dA^{(e)} \quad (3-41)$$

The iterative solution procedure starts by assuming an initial values for ω_n (i.e., ω_0). Then the ψ_{n+1} (i.e., ψ_1) is found from equation (3-35) and used as the source function to determine the ω_{n+1} (i.e., ω_1) from equation (3-36). This process is repeated until steady state is reached. Using a two-point finite difference formula, the term $\{\dot{\omega}\}_{n+1}$ can be written as follows (25)

$$\{\dot{\omega}\}_{n+1} = \left\{ \frac{\partial \omega}{\partial t} \right\}_{n+1} = \left\{ \frac{\omega_{n+1} - \omega_n}{\Delta t} \right\} \quad (3-42)$$

so that equation (3-36) becomes

$$\left[[K_{\omega}] + \frac{1}{\Delta t} [K_{\dot{\omega}}] \right] \{\omega\}_{n+1} = \frac{1}{\Delta t} [K_{\dot{\omega}}] \{\omega\}_n - \{S_{\omega}\}_n \quad (3-43)$$

which can be solved at successive time steps for the column vector of nodal values of vorticity. The coefficient $\left[[K_{\omega}] + \frac{1}{\Delta t} [K_{\dot{\omega}}] \right]$ is symmetric, banded, and positive definite. Numerical solutions of equations (3-35) and (3-36) were obtained by the Gaussian elimination method. Cheng reports that the iteration procedure was always stable for sufficiently small Δt . As a guideline, Δt should be chosen so that

$$\Delta t < 0.1 (\Delta \ell)^2 Re \quad (3-44)$$

where Δ is the characteristic length of an element.(25,63,127)

It has been observed that such a procedure for time dependent problems does not fulfil the requirements of the calculus of variation. This is because during variations of vorticity, the term $\frac{\partial \omega}{\partial t}$ is treated as an invariance. This principle is, therefore, referred to as pseudo-variational principle.

3.4 Evaluation of the Matrices of the Elements

Equations (3-35) and (3-41) may now be explicitly evaluated using the definition of the interpolation functions. For linear triangular elements the following equations can be obtained.

$$\begin{aligned} N_1^{(e)}(x,y) &= \frac{a_1 + b_1x + c_1y}{2\Delta} \\ N_2^{(e)}(x,y) &= \frac{a_2 + b_2x + c_2y}{2\Delta} \\ N_3^{(e)}(x,y) &= \frac{a_3 + b_3x + c_3y}{2\Delta} \end{aligned} \tag{3-45}$$

where $2\Delta = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 2x$ (area of triangle)

$$\begin{aligned} a_1 &= x_2y_3 - x_3y_2 \\ b_1 &= y_2 - y_3 \\ c_1 &= x_3 - x_2 \end{aligned} \tag{3-46}$$

The other coefficients are obtained by cyclically permuting the subscripts. From equations (3-45) the following equations can be obtained

$$\begin{aligned} \frac{\partial N_i}{\partial x} &= \frac{b_i}{2\Delta} \\ \frac{\partial N_i}{\partial y} &= \frac{c_i}{2\Delta} \quad i = 1, 2, 3. \end{aligned} \tag{3-47}$$

Substitution of equations (3-48) into equation (3-37) gives:

$$K_{nkij} = \sum_{e=1}^M \int_{A^{(e)}} \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) dA^{(e)}$$

$$\begin{aligned}
 &= \sum_{e=1}^M \int_{A^{(e)}} \left(\frac{b_i}{2\Delta} \frac{b_j}{2\Delta} + \frac{c_i}{2\Delta} \frac{c_j}{2\Delta} \right) dA^{(e)} \\
 &= \sum_{e=1}^M \frac{1}{4\Delta^2} \left(b_i b_j \int_{A^{(e)}} dA^{(e)} + c_i c_j \int_{A^{(e)}} dA^{(e)} \right) \\
 &= \sum_{e=1}^M \frac{1}{4\Delta} (b_i b_j + c_i c_j) \quad (3-48)
 \end{aligned}$$

To evaluate

$$S_{\psi_i} = - \sum_{e=1}^M \int_{A^{(e)}} \omega_n^{(e)} N_i dA^{(e)} \quad (3-38)$$

Some special considerations are required. $\omega_n^{(e)}$ can be treated as a constant within the element, and from the integration formula (22,33,61,63) the following equation may be obtained.

$$S_{\psi_i} = - \sum_{e=1}^M \frac{\Delta}{3} \omega_n^{(e)} \quad (3-49)$$

or the term $\omega_n^{(e)}$ may be linearly interpolated in terms of its nodal values as (25,63,127)

$$\omega_n^{(e)} = \omega_{n1}^{(e)} N_1 + \omega_{n2}^{(e)} N_2 + \omega_{n3}^{(e)} N_3 \quad (3-50)$$

in which case equation (3-38) may be written as follows:

$$S_{\psi_i} = - \sum_{e=1}^M \int_{A^{(e)}} (\omega_{n1}^{(e)} N_1 N_i + \omega_{n2}^{(e)} N_2 N_i + \omega_{n3}^{(e)} N_3 N_i) dA^{(e)} \quad (3-51)$$

Again, employing the integration formula, the following equation may be obtained

(3-52)

$$\{S_{\psi}\}_n = - \sum_{e=1}^M \frac{\Delta}{12} \left\{ \begin{array}{ccc} 2\omega_{n1}^{(e)} & + & \omega_{n2}^{(e)} & + & \omega_{n3}^{(e)} \\ \omega_{n1}^{(e)} & + & 2\omega_{n2}^{(e)} & + & \omega_{n3}^{(e)} \\ \omega_{n1}^{(e)} & + & \omega_{n2}^{(e)} & + & 2\omega_{n3}^{(e)} \end{array} \right\}$$

Similarly, from the integration formula, equation (3-40) may be written as

$$K_{\bar{w}_{ij}} = \sum_{e=1}^M \int_{A^{(e)}} N_i N_j dA^{(e)} \tag{3-40}$$

$$= \sum_{e=1}^M \frac{\Delta}{12} \tag{3-53}$$

Now the equation (3-41) is to be evaluated.

$$S_{w_i} = \sum_{e=1}^M \int_{A^{(e)}} N_i \left(\frac{\partial \psi_{n+1}^{(e)}}{\partial y} \frac{\partial \omega_n^{(e)}}{\partial x} - \frac{\partial \psi_{n+1}^{(e)}}{\partial x} \frac{\partial \omega_n^{(e)}}{\partial y} \right) dA^{(e)} \tag{3-41}$$

Substitution of equations (3-47) into equations (3-12) - (3-15) gives

$$\frac{\partial \psi_{n+1}^{(e)}}{\partial x} = \left\{ \frac{\partial N^{(e)}}{\partial x} \right\}^T \{ \psi_{n+1} \} = \frac{b_1}{2\Delta} \psi_{1(n+1)} + \frac{b_2}{2\Delta} \psi_{2(n+1)} + \frac{b_3}{2\Delta} \psi_{3(n+1)}$$

$$\frac{\partial \psi_{n+1}^{(e)}}{\partial y} = \left\{ \frac{\partial N^{(e)}}{\partial y} \right\}^T \{ \psi_{n+1} \} = \frac{c_1}{2\Delta} \psi_{1(n+1)} + \frac{c_2}{2\Delta} \psi_{2(n+1)} + \frac{c_3}{2\Delta} \psi_{3(n+1)}$$

(3-54)

$$\frac{\partial \omega_n^{(e)}}{\partial x} = \left\{ \frac{\partial N^{(e)}}{\partial x} \right\}^T \{ \omega_n \} = \frac{b_1}{2\Delta} \omega_{1n} + \frac{b_2}{2\Delta} \omega_{2n} + \frac{b_3}{2\Delta} \omega_{3n}$$

$$\frac{\partial \omega_n^{(e)}}{\partial y} = \left\{ \frac{\partial N^{(e)}}{\partial y} \right\}^T \{ \omega_n \} = \frac{c_1}{2\Delta} \omega_{1n} + \frac{c_2}{2\Delta} \omega_{2n} + \frac{c_3}{2\Delta} \omega_{3n}$$

Substitution of the last equations into equation (3-41) gives

$$\begin{aligned}
 S w_i &= \sum_{e=1}^M \int_{A^{(e)}} N_i \left[\left(\frac{C_1}{2\Delta} \psi_{1(n+1)} + \frac{C_2}{2\Delta} \psi_{2(n+1)} + \frac{C_3}{2\Delta} \psi_{3(n+1)} \right) \right. \\
 &\quad \left(\frac{b_1}{2\Delta} w_{1n} + \frac{b_2}{2\Delta} w_{2n} + \frac{b_3}{2\Delta} w_{3n} \right) \\
 &\quad - \left(\frac{b_1}{2\Delta} \psi_{1(n+1)} + \frac{b_2}{2\Delta} \psi_{2(n+1)} + \frac{b_3}{2\Delta} \psi_{3(n+1)} \right) \\
 &\quad \left. \left(\frac{C_1}{2\Delta} w_{1n} + \frac{C_2}{2\Delta} w_{2n} + \frac{C_3}{2\Delta} w_{3n} \right) \right] dA^{(e)} \\
 &= \sum_{e=1}^M \frac{1}{4\Delta^2} \left[\left(C_1 \psi_{1(n+1)} + C_2 \psi_{2(n+1)} + C_3 \psi_{3(n+1)} \right) \right. \\
 &\quad \left(b_1 w_{1n} + b_2 w_{2n} + b_3 w_{3n} \right) \\
 &\quad - \left(b_1 \psi_{1(n+1)} + b_2 \psi_{2(n+1)} + b_3 \psi_{3(n+1)} \right) \\
 &\quad \left. \left(C_1 w_{1n} + C_2 w_{2n} + C_3 w_{3n} \right) \right] \int_{A^{(e)}} N_i dA^{(e)} \\
 &= \sum_{e=1}^M \frac{1}{12\Delta} \left[\left(C_1 \psi_{1(n+1)} + C_2 \psi_{2(n+1)} + C_3 \psi_{3(n+1)} \right) \right. \\
 &\quad \left(b_1 w_{1n} + b_2 w_{2n} + b_3 w_{3n} \right) \\
 &\quad - \left(b_1 \psi_{1(n+1)} + b_2 \psi_{2(n+1)} + b_3 \psi_{3(n+1)} \right) \\
 &\quad \left. \left(C_1 w_{1n} + C_2 w_{2n} + C_3 w_{3n} \right) \right]
 \end{aligned} \tag{3-55}$$

At present, the governing equations and elements can easily be incorporated into the computer programs.

3.5 Pressure and Velocity Distributions

Now the pressure and velocity distributions are to be calculated.

It may be mentioned here that serious attention must be given to the choice of interpolation functions for pressure and stream functions. To achieve the same order of approximation for stream functions and pressure, the interpolation functions for stream functions should be higher by one order than the interpolation functions for pressure. So quadratic triangular elements were used for the stream functions, and linear triangular elements for the pressure (see Fig.3.2).

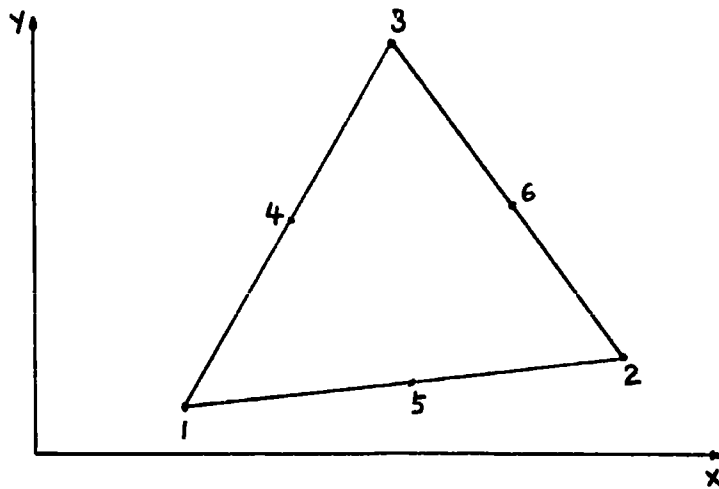


Figure 3-2: Triangular element with corner nodes 1,2,3 and mid-edge nodes 4,5,6.

$$P^{(e)}(x,y,t) = \sum_{i=1}^3 N_i^P(x,y) P_i(t)$$

$$\psi^{(e)}(x,y,t) = \sum_{i=1}^6 N_i(x,y) \psi_i(t)$$

$$N_1 = L_1^2 - L_1(L_2 + L_3)$$

$$N_2 = L_2^2 - L_2(L_3 + L_1)$$

$$N_3 = L_3^2 - L_3(L_1 + L_2)$$

$$N_4 = 4 L_1 L_3$$

$$N_5 = 4 L_1 L_2$$

$$N_6 = 4 L_2 L_3$$

$$N_i^P = L_i = \text{natural coordinates}$$

$$\begin{aligned}
 P^{(e)}(x, y, t) &= \sum_{i=1}^3 N_i^P(x, y) P_i(t) \\
 &= \{N^P(x, y)\}^T \{P(t)\}
 \end{aligned}
 \tag{3-56}$$

and

$$\begin{aligned}
 \psi^{(e)}(x, y, t) &= \sum_{i=1}^6 N_i(x, y) \psi_i(t) \\
 &= \{N(x, y)\}^T \{\psi(t)\}
 \end{aligned}
 \tag{3-57}$$

where

$$\begin{aligned}
 N_1 &= L_1^2 - L_1(L_2 + L_3) \\
 N_2 &= L_2^2 - L_2(L_3 + L_1) \\
 N_3 &= L_3^2 - L_3(L_1 + L_2) \\
 N_4 &= 4L_1L_3 \\
 N_5 &= 4L_1L_2 \\
 N_6 &= 4L_2L_3 \\
 N_i^P &= L_i = \text{natural coordinates}
 \end{aligned}
 \tag{3-58}$$

The gradients of $P^{(e)}$ are

$$\begin{aligned}
 \frac{\partial P^{(e)}}{\partial x} &= \left\{ \frac{\partial N^P}{\partial x} \right\}^T \{P(t)\} \\
 \frac{\partial P^{(e)}}{\partial y} &= \left\{ \frac{\partial N^P}{\partial y} \right\}^T \{P(t)\}
 \end{aligned}
 \tag{3-59}$$

Substitution of equations (3-59) and (3-56) into equation (2-41) gives:

$$\begin{aligned}
 \phi_3 &= \frac{1}{2} \int_{A^{(e)}} \left(\left[\left\{ \frac{\partial N^P}{\partial x} \right\}^T \{P(t)\} \right]^2 + \left[\left\{ \frac{\partial N^P}{\partial y} \right\}^T \{P(t)\} \right]^2 \right) dA^{(e)} \\
 &\quad - \int_{A^{(e)}} \rho \left(\{N^P\}^T \{P(t)\} \right) Q dA^{(e)}
 \end{aligned}
 \tag{3-60}$$

Minimisation of the functional gives

$$\begin{aligned}
 \frac{\partial \phi_3(P)}{\partial P_i} &= \int_{A^{(e)}} \left(\left[\left\{ \frac{\partial N^P}{\partial x} \right\}^T \{P(t)\} \right] \frac{\partial N_i^P}{\partial x} \right. \\
 &\quad \left. + \left[\left\{ \frac{\partial N^P}{\partial y} \right\}^T \{P(t)\} \right] \frac{\partial N_i^P}{\partial y} \right) dA^{(e)} - \int_{A^{(e)}} \rho Q N_i^P dA^{(e)} \\
 &= 0
 \end{aligned}
 \tag{3-61}$$

The last equation may be written concisely as

$$[K_p]^{(e)} \{P(t)\}^{(e)} + \{S_p\}^{(e)} = \{0\} \quad (3-62)$$

where

$$K_{P_{ij}}^{(e)} = \int_{A^{(e)}} \left(\frac{\partial N_i^P}{\partial x} \frac{\partial N_j^P}{\partial x} + \frac{\partial N_i^P}{\partial y} \frac{\partial N_j^P}{\partial y} \right) dA^{(e)} \quad (3-63)$$

$$S_{P_i}^{(e)} = - \int_{A^{(e)}} P N_i^P Q dA^{(e)} \quad (3-64)$$

$$Q = -2 \left[\frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial y^2} - \left(\frac{\partial^2 \psi}{\partial x \partial y} \right)^2 \right] \quad (3-65)$$

Referring to equations (3-57) and (3-58), the derivatives in equation (3-65) may be written as

$$\begin{aligned} \frac{\partial^2 \psi}{\partial x^2} &= \left\{ \frac{\partial^2 N}{\partial x^2} \right\}^T \{ \psi \} \\ &= \frac{b_1}{2\Delta^2} (b_1 - b_2 - b_3) \psi_1 \\ &\quad + \frac{b_2}{2\Delta^2} (b_2 - b_3 - b_1) \psi_2 \\ &\quad + \frac{b_3}{2\Delta^2} (b_3 - b_1 - b_2) \psi_3 \\ &\quad + \frac{2b_1 b_3}{\Delta^2} \psi_4 + \frac{2b_1 b_2}{\Delta^2} \psi_5 + \frac{2b_2 b_3}{\Delta^2} \psi_6 \end{aligned} \quad (3-66)$$

$$\begin{aligned} \frac{\partial^2 \psi}{\partial y^2} &= \left\{ \frac{\partial^2 N}{\partial y^2} \right\}^T \{ \psi \} \\ &= \frac{c_1}{2\Delta^2} (c_1 - c_2 - c_3) \psi_1 \end{aligned}$$

$$\begin{aligned}
 & + \frac{C_2}{2\Delta^2} (C_2 - C_3 - C_1) \psi_2 \\
 & + \frac{C_3}{2\Delta^2} (C_3 - C_1 - C_2) \psi_3 \\
 & + \frac{2C_1C_3}{\Delta^2} \psi_4 + \frac{2C_1C_2}{\Delta^2} \psi_5 + \frac{2C_2C_3}{\Delta^2} \psi_6
 \end{aligned} \tag{3-67}$$

and

$$\begin{aligned}
 \frac{\partial^2 \psi}{\partial x \partial y} &= \left\{ \frac{\partial^2 N}{\partial x \partial y} \right\}^T \{ \psi \} \\
 &= \frac{1}{4\Delta^2} (2C_1b_1 - C_1b_3 - C_3b_1 - C_1b_2 - C_2b_1) \psi_1 \\
 &+ \frac{1}{4\Delta^2} (2C_2b_2 - C_2b_3 - C_3b_2 - C_1b_2 - C_2b_1) \psi_2 \\
 &+ \frac{1}{4\Delta^2} (2C_3b_3 - C_3b_1 - C_1b_3 - C_3b_2 - C_2b_3) \psi_3 \\
 &+ \frac{1}{\Delta^2} (C_1b_3 + C_3b_1) \psi_4 \\
 &+ \frac{1}{\Delta^2} (C_1b_2 + C_2b_1) \psi_5 \\
 &+ \frac{1}{\Delta^2} (C_2b_3 + C_3b_2) \psi_6
 \end{aligned} \tag{3-68}$$

To simplify these calculations as much as possible, the following equations are assumed.

$$\begin{aligned}
 \psi_4 &= \frac{\psi_1 + \psi_3}{2} \\
 \psi_5 &= \frac{\psi_2 + \psi_1}{2} \\
 \psi_6 &= \frac{\psi_2 + \psi_3}{2}
 \end{aligned} \tag{3-69}$$

For a solution domain of M elements, the system equations are of the form

$$[K_P] \{P(t)\} + \{S_P\} = \{0\} \tag{3-70}$$

which can be solved for the column vector of nodal values of pressure, $\{ P \}$.

Now the velocity distribution is to be calculated. The most usual procedure of describing incompressible velocities is by using the following equations (3-71) in a direct manner. It is straightforward. At present, an alternative way is presented here. For a two-dimensional fluid flow, the velocity components u and v may be expressed in terms of a stream function $\psi(x,y)$ as

$$\begin{aligned} u &= \frac{\partial \psi}{\partial y} \\ v &= - \frac{\partial \psi}{\partial x} \end{aligned} \quad (3-71)$$

The variational functionals for equations (3-71) can be written as follows:

$$\begin{aligned} \phi_u(u) &= \int_{A^{(e)}} \left(\frac{1}{2} u^2 - \frac{\partial \psi}{\partial y} u \right) dA^{(e)} \\ \phi_v(v) &= \int_{A^{(e)}} \left(\frac{1}{2} v^2 + \frac{\partial \psi}{\partial x} v \right) dA^{(e)} \end{aligned} \quad (3-72)$$

The interpolation functions for stream functions should also be higher by one order than N_i^u and N_i^v , the interpolation functions for u and v . So a quadratic interpolation function for stream function and a linear one for fluid velocity components are adopted.

$$\begin{aligned} u^{(e)} &= \sum_{i=1}^3 N_i^u u_i \\ v^{(e)} &= \sum_{i=1}^3 N_i^v v_i \\ \psi^{(e)} &= \sum_{i=1}^6 N_i \psi_i \end{aligned} \quad (3-73)$$

where

$$N_i^u = N_i^v = L_i = \text{natural coordinates}$$

$$N_1 = L_1^2 - L_1(L_2 - L_3)$$

$$\begin{aligned}
 N_2 &= L_2^2 - L_2(L_3 + L_1) \\
 N_3 &= L_3^2 - L_3(L_1 + L_2) \\
 N_4 &= 4L_1L_3 \\
 N_5 &= 4L_1L_2 \\
 N_6 &= 4L_2L_3
 \end{aligned}
 \tag{3-74}$$

Substitution of equations (3-73) and (3-74) into equations (3-72) gives:

$$\begin{aligned}
 \phi_u(u) &= \frac{1}{2} \int_{A^{(e)}} \left([\{N^u\}^T \{u\}]^2 - 2 \left[\left\{ \frac{\partial N}{\partial y} \right\}^T \{\psi\} \right] \right. \\
 &\quad \left. [\{N^u\}^T \{u\}] \right) dA^{(e)}
 \end{aligned}
 \tag{3-75}$$

$$\begin{aligned}
 \phi_v(v) &= \frac{1}{2} \int_{A^{(e)}} \left([\{N^v\}^T \{v\}]^2 + 2 \left[\left\{ \frac{\partial N}{\partial x} \right\}^T \{\psi\} \right] \right. \\
 &\quad \left. [\{N^v\}^T \{v\}] \right) dA^{(e)}
 \end{aligned}$$

Minimization of equations (3-75) gives

$$\begin{aligned}
 \frac{\partial \phi_u(u)}{\partial u_i} &= \int_{A^{(e)}} \{N^u\}^T \{u\} N_i^u dA^{(e)} - \int_{A^{(e)}} \left\{ \frac{\partial N}{\partial y} \right\}^T \{\psi\} N_i^u dA^{(e)} \\
 &= 0
 \end{aligned}
 \tag{3-76}$$

and

$$\begin{aligned}
 \frac{\partial \phi_v(v)}{\partial v_i} &= \int_{A^{(e)}} \{N^v\}^T \{v\} N_i^v dA^{(e)} + \int_{A^{(e)}} \left\{ \frac{\partial N}{\partial x} \right\}^T \{\psi\} N_i^v dA^{(e)} \\
 &= 0
 \end{aligned}$$

From equations (3-76), the element equations may be written as follows:

$$\begin{aligned}
 [R_u]^{(e)} \{u\}^{(e)} - [Q_u]^{(e)} \{\psi\}^{(e)} &= \{0\} \\
 [R_v]^{(e)} \{v\}^{(e)} + [Q_v]^{(e)} \{\psi\}^{(e)} &= \{0\}
 \end{aligned}
 \tag{3-77}$$

where

$$R_{u_{ij}}^{(e)} = \int_{A^{(e)}} N_i^u N_j^u dA^{(e)} \quad (3-78)$$

$$R_{v_{ij}}^{(e)} = \int_{A^{(e)}} N_i^v N_j^v dA^{(e)}$$

$$Q_{u_{ij}}^{(e)} = \int_{A^{(e)}} \frac{\partial N_j}{\partial y} N_i^u dA^{(e)}$$

$$Q_{v_{ij}}^{(e)} = \int_{A^{(e)}} \frac{\partial N_j}{\partial x} N_i^v dA^{(e)} \quad (3-79)$$

The assembled system equations become

$$[R_u]\{u\} - [Q_u]\{\psi\} = \{0\} \quad (3-80)$$

$$[R_v]\{v\} + [Q_v]\{\psi\} = \{0\}$$

which can be solved for the column vector of nodal values of velocity components u and v . Matrices $[Q_u]$ and $[Q_v]$ have been evaluated using equation (3-73), (3-74) and (3-79). Matrices $[Q_u]$ and $[Q_v]$ are given in Appendix A.

CHAPTER 4 BOUNDARY CONDITIONS AND NUMERICAL PROCEDURES

4.1 Boundary Conditions

In this section, some general ideas about boundary conditions are discussed. A fluid flow problem, governed by a system of partial differential equation, is defined only when a proper set of initial and/or boundary conditions is given. The boundary conditions are such an important part of the definition of a problem that the patterns of two flow fields can be completely different from one another simply due to some differences in the flow boundaries, in spite of the fact that both flow obey the same system of indefinite differential equations. One cannot exaggerate the importance of the effects that boundary conditions have on the fluid flow analysis. In mixed initial- and boundary - value problems major troubles must arise if the boundary conditions are not properly handled (33,92,128).

Through theoretical considerations and numerical experiments, it is found that even though the patterns of two fluid flow fields are the same, boundary conditions may still be different. It seems that boundary conditions depend not only on fluid patterns but also on the kind of finite element formulation used or on the kind of field variables employed, and on which kind of interpolation function adopted. For example, to solve the same fluid flow problem, if velocity and pressure are used as field variables, then the boundary conditions for this formulation are different from those when stream function and vorticity are used as field variables. For another example, when a fluid flow problem is solved by using a velocity-pressure formulation, unless a high order element is used the value of second derivative $\frac{\partial^2 U}{\partial x^2}$ has to be assumed zero which normally would be valid for creeping flow only. This seems to be equivalent to a high order element being used and $\frac{\partial^2 U}{\partial x^2} = 0$ boundary condition being assumed in this interpolation function at the same time (62,132).

The boundary conditions for a stream function-vorticity formulation problem are of three general types; 1) the specifications of the values which stream function or vorticity must assume along the boundary; 2) the specifications of the values of the component of the gradient of ψ or ω at the normal to the boundary; 3) the provision of some algebraic relation which connects the value of ψ or ω to the values of their normal components along the boundary (16,55). In fact, it is usual for values to be specified at some parts of a boundary and for gradients to be specified at other parts (see example one, Chapter 6). When several differential equations are to be solved simultaneously, there is no need for the boundary conditions for each equation to be of the same type (55, 63).

In example one (see Section 6.1), since the flow is considered to be fully developed at the downstream end, the gradients of stream function and vorticity with respect to the flow direction should vanish at the boundary. These provide the normal boundary conditions at the downstream end. There is no need to specify the values of stream function and vorticity at this boundary. (21,25,55,63,100,110,127)

The mixed type boundary value problem, such as that appearing in example one, causes no difficulty in itself, provided a scheme can be found for specifying boundary conditions associated with first derivatives of stream function and vorticity. For example, to impose the boundary conditions for the example one, the relations $\frac{\partial \psi}{\partial x} = \frac{\partial \omega}{\partial x} = 0$ can just be incorporated into the element stiffness matrix. This will give an alternate form for the element stiffness matrix which can then be used for elements having node on downstream edge. The disadvantage is that such elements must then be incorporated into the computer program and used as need arises. This scheme has been used in this computer program. An alternate way is to consider the boundary conditions as constraints.

The set of equations is expressed as the gross assemblage stiffness equation. The detailed procedure can be found from the papers of Bratanow (16,18) or Martin (88).

The 'natural' boundary conditions are somewhat arbitrary, since there is little agreement in the field of fluid mechanics as to what the proper ones are. The choice is therefore made on the basis of practicality (33,100). Usually, the 'rigid' boundary conditions are used. And the programme is left to seek its own approximation of the 'natural' boundary conditions (see Chapter 6). (100)

A rigid wall may be either of two types, no-slip or free-slip. The latter may be considered to represent a plane of symmetry, rather than a true wall (58,128). In the examples to be presented, no-slip boundary condition would be considered.

To satisfy the condition of no-slip at solid walls, the normal and tangential gradients of stream function must vanish at these boundaries. The tangential conditions are satisfied by setting stream function constant along these boundaries. However special attention is required to determine the boundary formulae for the normal conditions. To determine the boundary values for the wall vorticity, application of the no-slip boundary condition alone is not enough. At a point (X_0, Y_0) on the wall, the vorticity may be calculated from the relation. (25,63,121)

$$\omega(X_0, Y_0) = - \frac{\partial^2 \psi}{\partial n^2}(X_0, Y_0) \quad (4-1)$$

where n is the coordinate normal to the wall. Using a Taylor series expansion, at a point (X_1, Y_1) along the normal direction, a small distance from the wall, the following equation may be obtained.

$$\psi(X_1, Y_1) = \psi(X_0, Y_0) + n \frac{\partial \psi}{\partial n}(X_0, Y_0) + \frac{n^2}{2} \frac{\partial^2 \psi}{\partial n^2}(X_0, Y_0) \quad (4-2)$$

Since the no-slip condition dictates that

$$\frac{\partial \psi}{\partial n}(x_0, y_0) = 0 \quad (4-3)$$

then the vorticity on the wall may be calculated by

$$\omega(x_0, y_0) = \frac{2}{\Delta^2} \left[\psi(x_0, y_0) - \psi(x_1, y_1) \right] \quad (4-4)$$

Wall vorticity is then given in terms of the stream function evaluated at the wall and a small distance away from the wall. If $(X1, Y1)$ is not a nodal point, the stream function $\psi(X1, Y1)$ may be obtained by interpolation between stream functions of the neighbouring nodal points of $(X1, Y1)$ (25,63).

There are several other ways to compute the surface vorticity, which can easily be found (4,33,110,128). Equation (4-4) is called the first-order one-sided difference formula. This formula gave numerical results in excellent agreement with the exact solution (128). It is found that the second-order one-sided difference formula sometimes led to unstable results. The detailed discussions may be found from Wu's latest paper (128). To play safe, this program employs the first-order one-sided difference formula. When using this formula, use of a finer mesh round the corners is to be encouraged. This is not only because there are big variations of values of stream function round the corners but also because it is hoped to force the effects of the corners to spread into the fluid in every direction, by using a finer mesh in regions adjacent corners (see Sec.7.6.1.).

For each system of equations there are a number of sufficient and necessary boundary conditions. For example, for a viscous flow the condition of no-slip is sufficient to determine the flow field. No other condition may be imposed on the rigid wall (92,100).

When a problem of viscous flow is treated by a numerical technique. A certain mesh is used. At the interior mesh-points the governing equations are substituted by a matrix equation. At each computational step, information is transmitted from each point to its neighbouring points via the numerical computation. In this way, the boundary mesh-points influence their neighbours and transmit the effects of the boundary conditions into the flow field. Moretti (92) maintained that if boundary conditions are not proceeded properly, the risk of over-specifying the boundary conditions themselves is faced and, in all probability these overspecified boundary conditions will not be consistent with the natural of the boundary and the limiting forms of the equations of motion (62,92).

It is not possible to isolate any portion of a fluid field and obtain the solution in only that portion. The difficulty arises from the boundary conditions. It is easier to deal with a fluid flow problem on a larger or entire flow field than just on a part of the whole flow field.

Some more detailed discussions about boundary conditions will be presented in example problems later.

4.2 Numerical Procedures

The present scheme for solution of the assembled system equations (3-35) and (3-43) uses an iterative method to obtain self-consistent stream function and vorticity fields.

The solution to equation (3-35) requires specification of stream function or its normal derivatives on all boundaries. The initial vorticity is not known anywhere. Usually an initial guess of zero vorticity is often appropriate. Equation (3-35) is then able to be solved for the stream function. Using the results of stream function, the wall vorticity can be obtained from equation (4-4). And then we can use values of wall vorticities and stream functions to solve equation

(3-43) for the vorticity. The velocity boundary conditions provide derivative boundary conditions on stream function. On the boundary, stream function and the normal and tangential derivatives may all be specified. After the vorticity field has been determined, the values of vorticity can be treated as a source function, and solve the stream function for the next time instant from equation (3-35).

Once the new stream function field has been determined the wall vorticity for the new time instant can be obtained by solving equation (4-4) again. Then equation (3-43) is resolved by adjusting the boundary vorticity. This procedure is repeated until a self-consistent stream function and vorticity field is obtained. This procedure not only circumvents the nonlinearity of the governing equation but also leads to a linear algebraic system (25,63).

Employing the foregoing procedure, solutions of stream function and vorticity can be obtained for creeping flow. Once a convergent Stokes solution was determined, this solution can be used as the initial conditions of vorticity in the calculation for a solution of the governing equations at a small Reynolds number. These solutions of stream function and vorticity for a small Reynolds number are considered as the initial conditions when the numerical solution of vorticity and stream function for a bit higher new Reynolds number are solved. This procedure is repeated such that the solution at a lower Reynolds number is used as the initial conditions for the higher Reynolds number until the solutions of stream function and vorticity for a desired Reynolds number are reached (25,63).

To get an accurate result, the smaller the increment of Reynolds number is, the better.

The procedures of the numerical solution are summarised as follows:

} These boundary conditions must be kept through this calculation.

- (1) Define suitable boundary conditions for both stream function and vorticity.
- (2) Assume an initial vorticity distribution.
- (3) Compute stream function from equation (3-35).
- (4) Find the boundary vorticity values from equation (4-4)
- (5) Solve the vorticity of Stokes flow for the next time instant. from equation (3-43).

Convergence ?

- (6) Read a new small Reynolds number.
- (7) Treat the vorticity field of Stokes flow as a new initial vorticity distribution.
- (8) Compute stream function from equation (3-35).
- (9) Find the boundary vorticity values from equation (4-4).
- (10) Solve the vorticity for the next time instant at the small Reynolds number from equation (3-43).

Convergence ?

- (11) Read a bit higher Reynolds number
-
- Repeat the foregoing procedure until desired Reynolds number is reached.
-
- (12) Compute pressure distribution.
- (13) Compute velocity distribution.
- (14) Solution is complete.

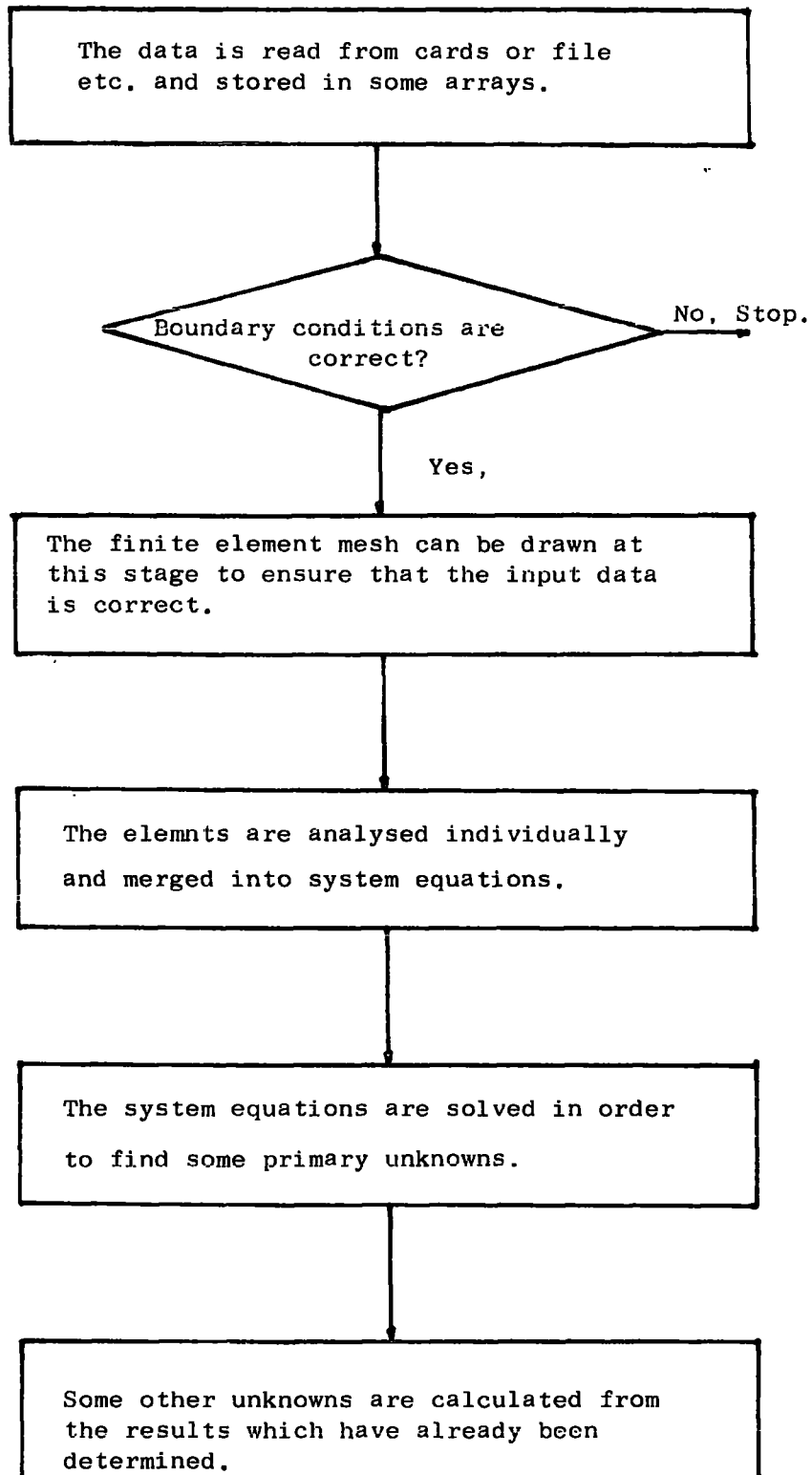


Figure 5.1. Simplified Flow Chart

the computer unit of the University has been used.

This method makes use of the fact that FORTRAN allows the definition of arrays with unknown dimensions within sub-programmes. The old main program is converted into a subroutine having as arguments the name and size of each array requiring dynamic allocation. This subroutine is called MAINPR. A small main program is required to write. This small main program will read in the array dimensions and calculate the space required for each array and pass this information to the subroutine DYNMIC, the arguments of which are identical to those for MAINPR. Subroutine DYNMIC acquires space for the arrays and calls MAINPR passing the arguments given to it, having inserted the correct addresses for the arrays.

The subroutine SOLMIX is used for solving matrix equations. Within SOLMIX a subroutine SOLVE is called. The subroutine SOLVE solves this kind of equation.

$$A(I,J) \times X(J) = C(I)$$

for $X(J)$ by Gaussian elimination scheme. There are a lot of this kind of present sub-program which can be used. The subroutines SOLMIX and SOLVE used here were written by Professor J.F.Booker of Cornell University (63).

The main advantages of the finite element program are as follows:

- (1) Complicated boundary conditions can be involved without difficulty.
- (2) Changing the type of boundary condition requires only the change of input data, and there is no need to change the computer programmes.
- (3) The convergence of the calculation can be observed by printing the values at selected points after each iteration.

The main disadvantage is that the achievement of a successful solution depends on the choice of the correct convergence parameters. For some problems, it might take a long time by trial and error before what kinds of values of parameters are the most appropriate to use are known. So it

is important to write down and keep the information about these control factors. It may be useful when this program is used later.

5.3 Simplified Flow Diagram for the Finite-Element Programmes

A flow chart outlining the finite element procedure is presented as follows:

Finite Element Analysis of Incompressible Unsteady Viscous Flow.

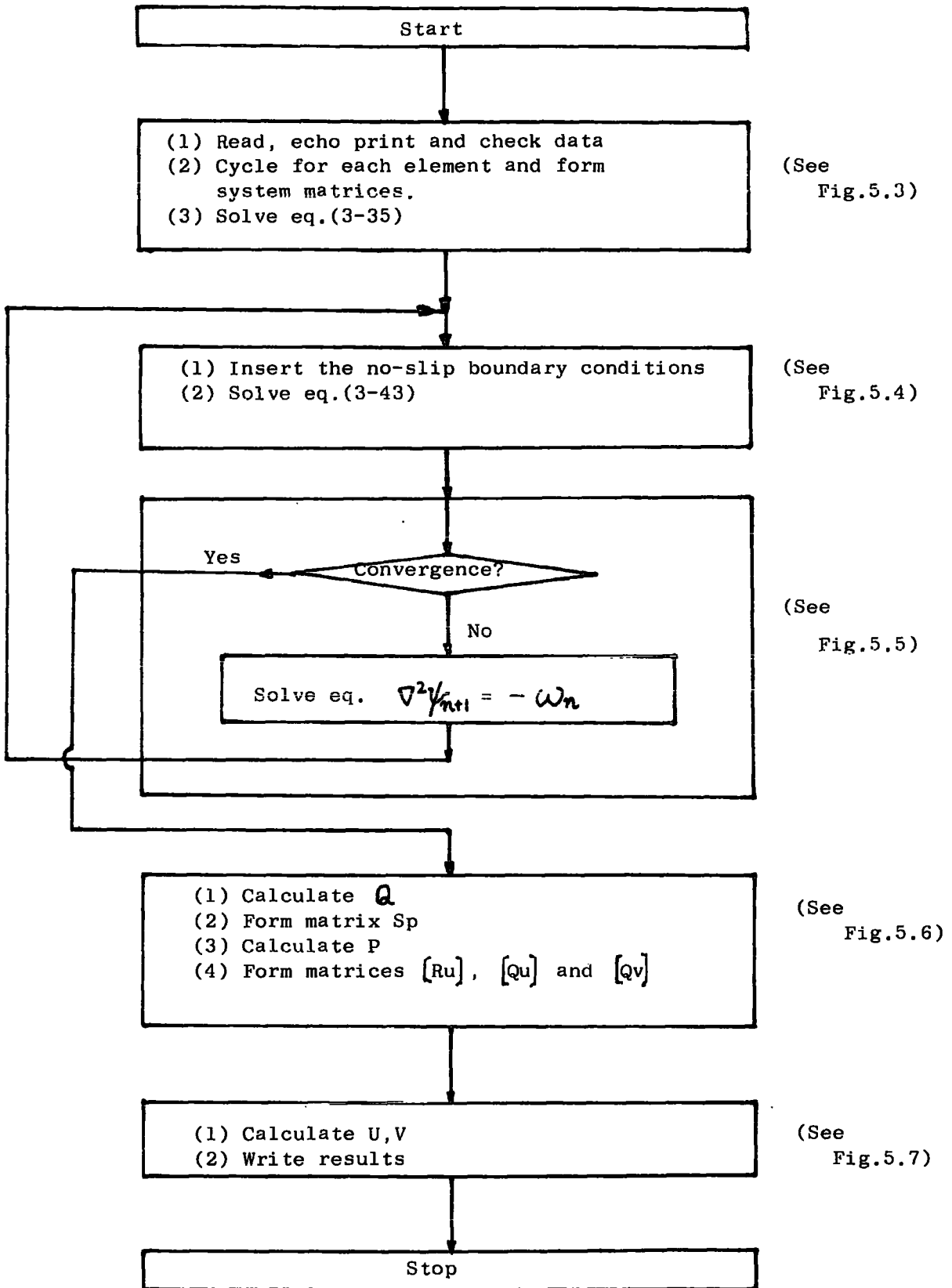


Figure 5.2

Figure 5.3

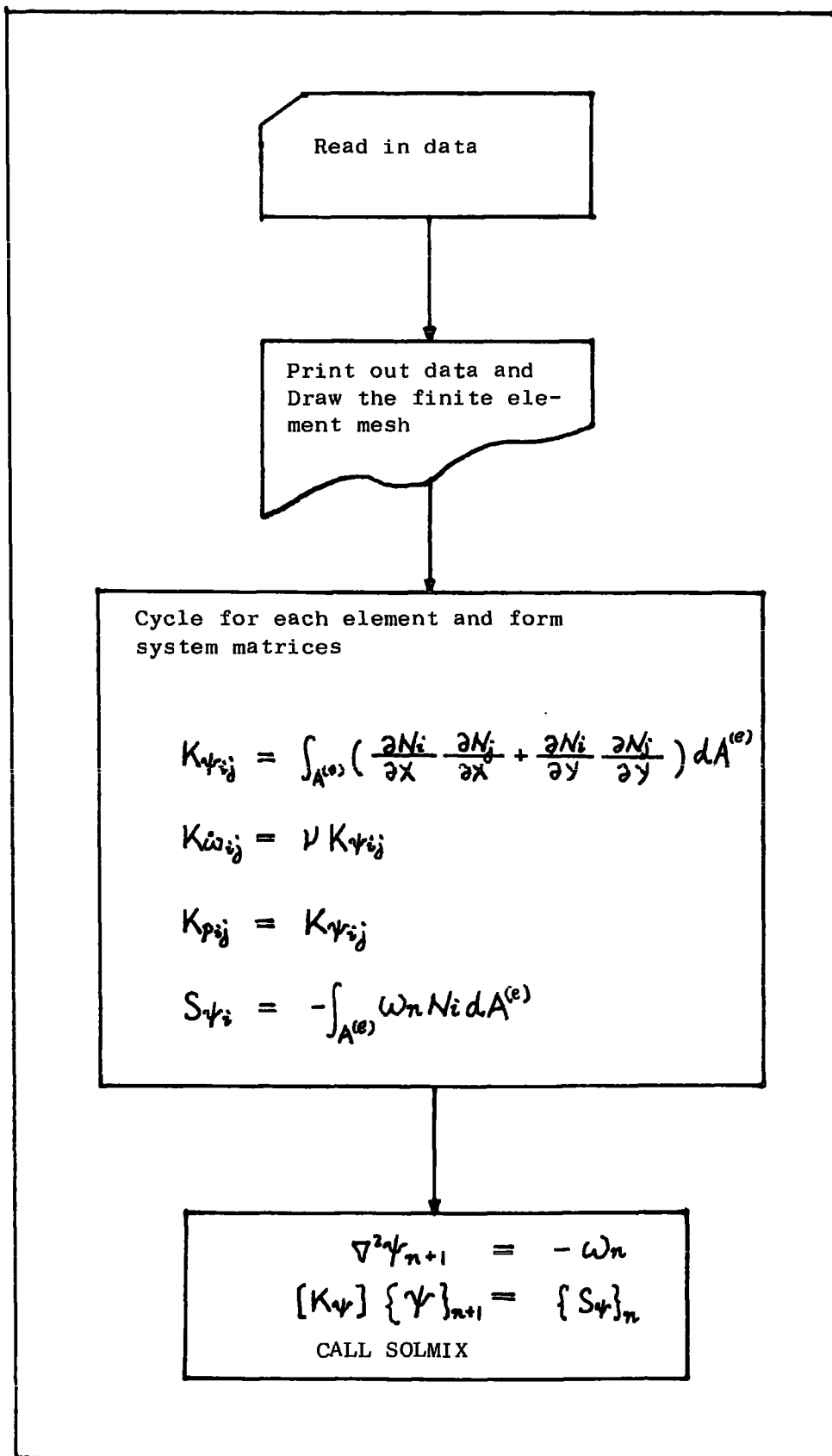


Figure 5.4

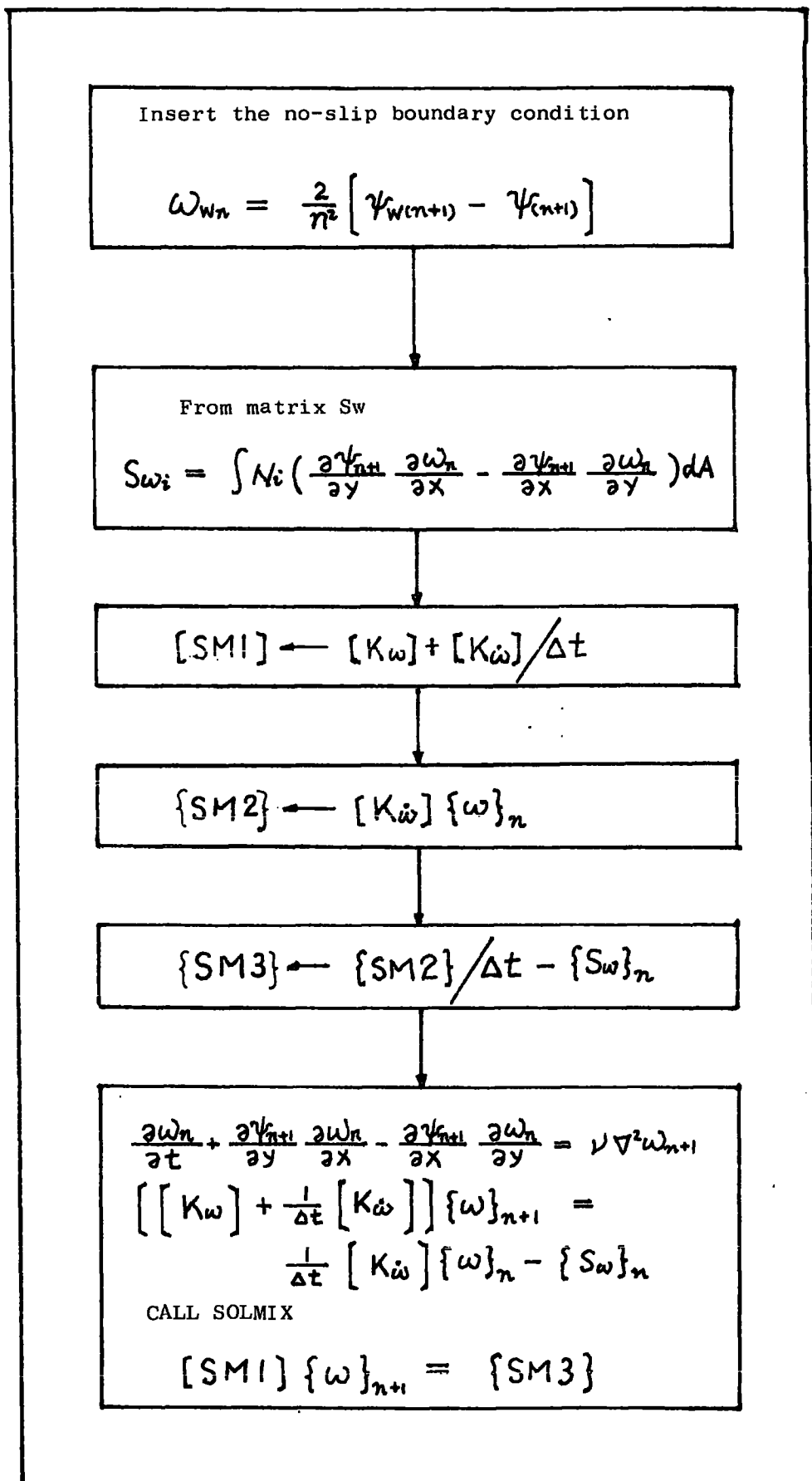


Figure 5.5

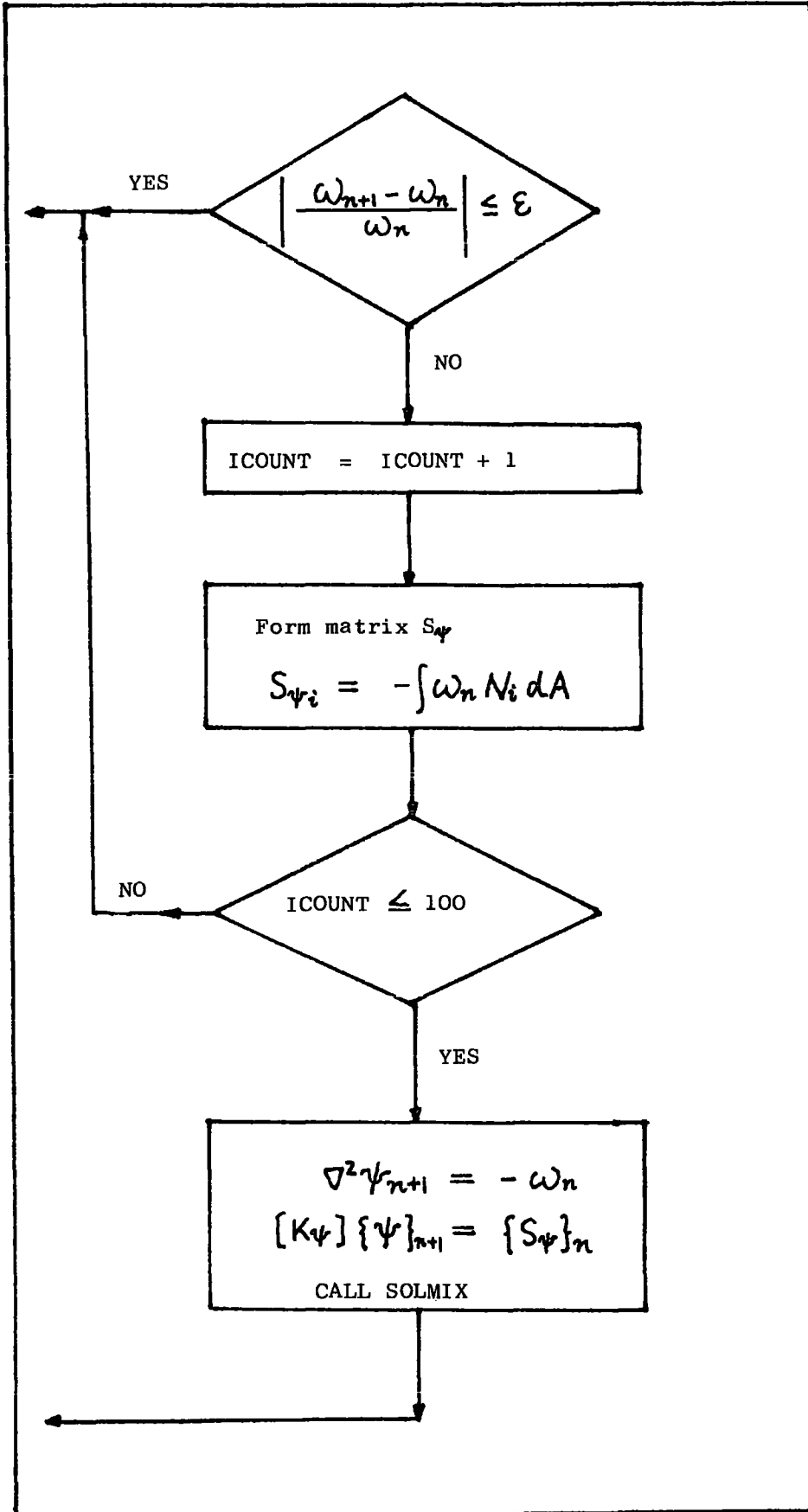


Figure 5.6

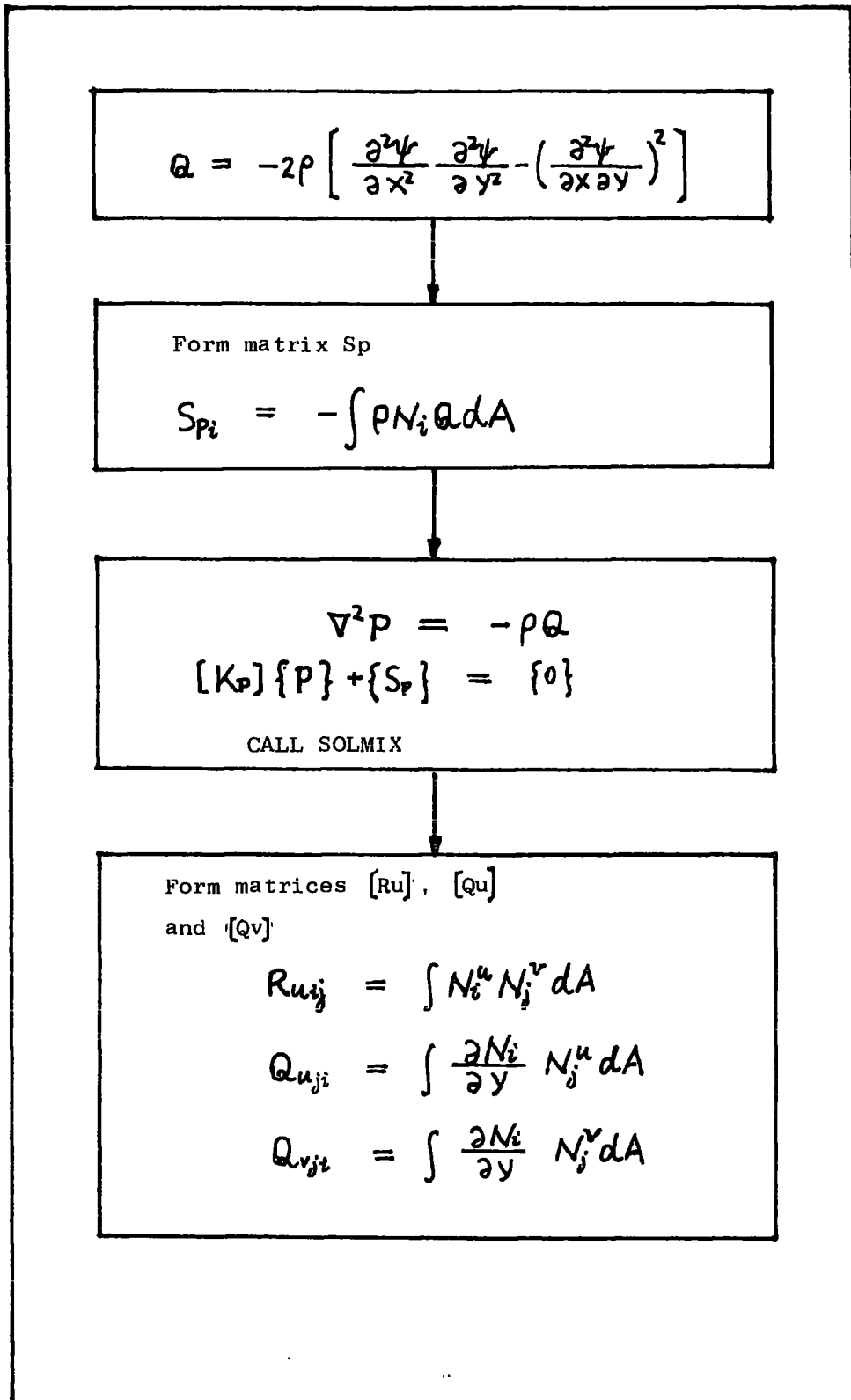
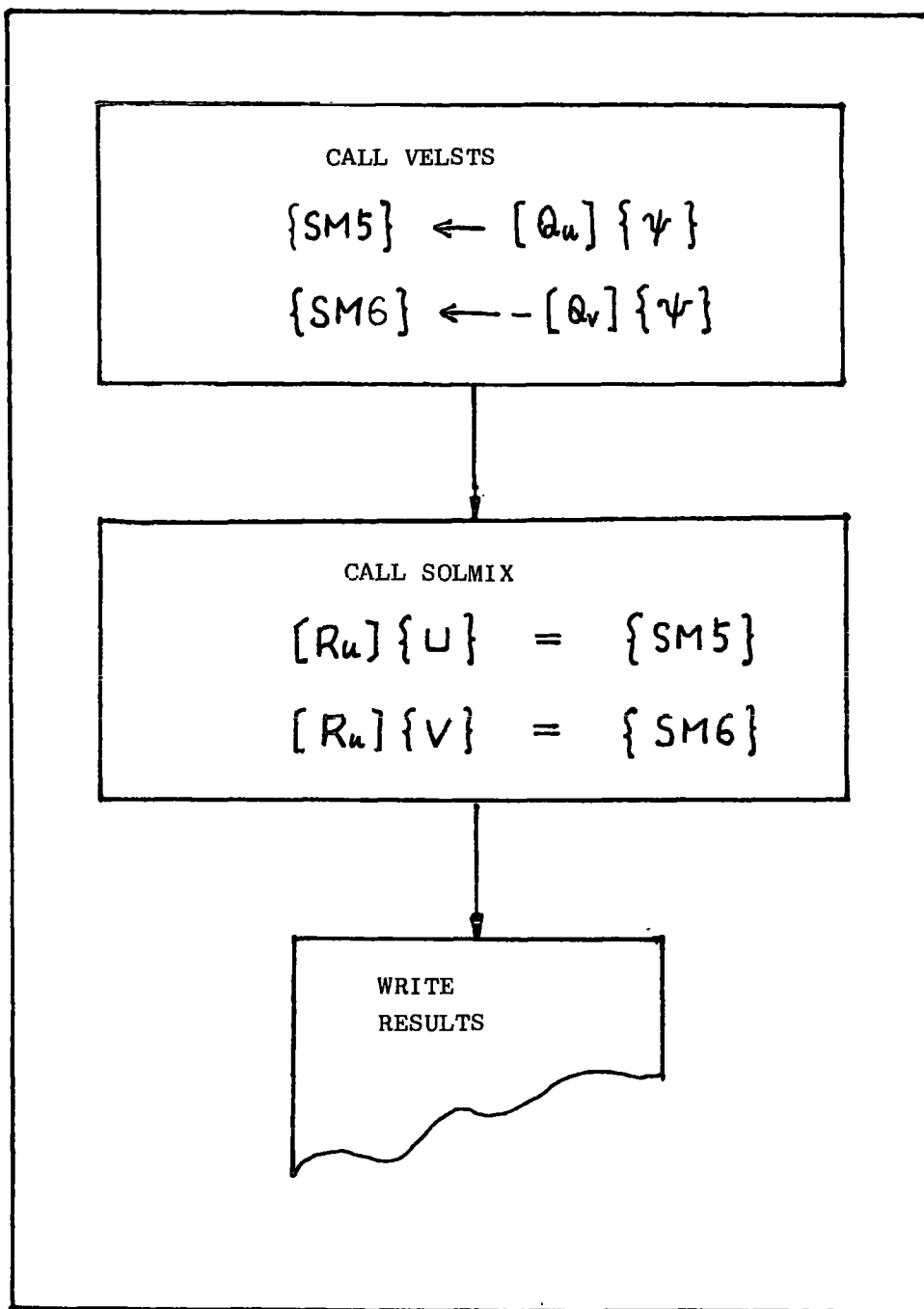


Figure 5.7



Chapter 6 TEST EXAMPLES

6.1 Example One

6.1.1. Introduction

To test the effectiveness of this formulation, and program, and therefore to assess solution accuracy, convergence, and stability, a fluid flow between plane, parallel plates was chosen. The main reason why it was selected as the first example is that this problem retains the total non-linear character of the Navier-Stokes equations (4,6,25). For the reason of symmetry, only one-half the problem region was considered. The coarsest finite element mesh which was plotted by the computer is illustrated in Figure 6-1.

6.1.2. Entry Length

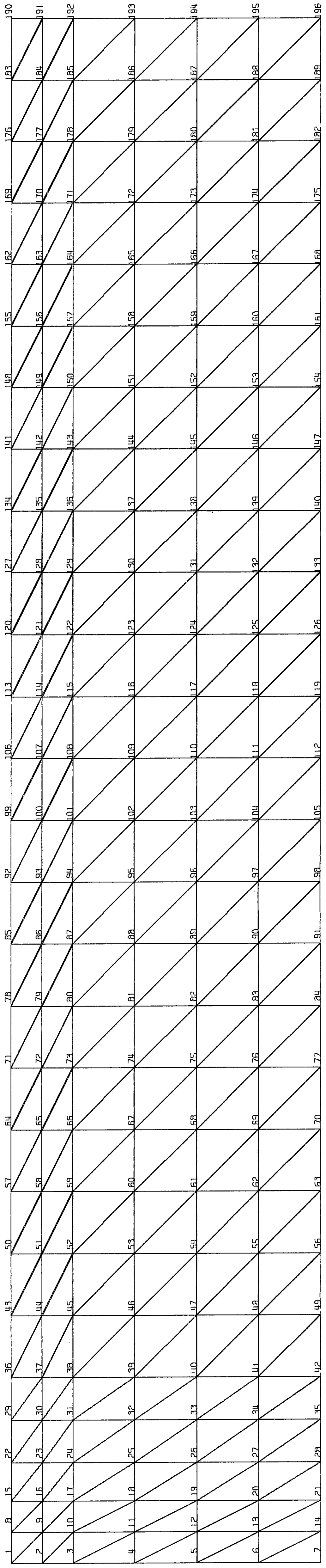
The entry length plays an important role in this kind of problem. The fact that the duct length is insufficient can lead to unstable results. So the entry length will be discussed first. The entry length is defined as the axial position at which the centre-line velocity reaches 99% of its fully developed value (11,24,55,56). This length can be determined by experimentation in which every parameter but the entrance length is held fixed (14,39,55,56). Schlichting (136) has shown that the entrance length is only a linear function of the Reynolds number for parallel plate channels and pipes. This is true only if the shape of the inlet velocity profile is kept the same (55). Hai (55) concluded that the entrance length necessary for flow development is a function of channel height, Reynolds number and shape of the entrance velocity profile for the flow regions considered here. Laughaar (39) suggested that the entry length for an incompressible isothermal laminar of a Newtonian fluid flow in a circular pipe can be obtained from the following simple equation.

$$X_c = Re \times D \times K \qquad (6-1)$$

where

FIG. 6 - 1
Finite Element Mesh for fluid flow between parallel plates

FINITE ELEMENT MESH



X_c = hydrodynamic entry length for circular pipes.

Re = Reynolds number = $\frac{V_{av} \cdot D \cdot \rho}{\mu}$

ρ = Density

μ = dynamic viscosity

D = pipe diameter

K = constant (Laughaar suggested that $K = 0.057$)

After X_c is derived, the entrance length for a fluid flow between plane, parallel plates can be obtained.

6.1.3. Initial and Boundary Conditions

Now initial and boundary conditions will be discussed.

Initial Condition:

The duct is initially filled with water of density of 1.0, which is instantaneously accelerated by application of a uniform velocity of unity upstream of the duct. The equivalent vorticity initial condition is vanishing everywhere. Of course, this is by no means the only initial condition which can be used. But in this problem, it is employed to test the program.

Boundary Conditions:

Referring to what has been discussed in chapter 4, the following conditions are employed (see Fig.6-2).

- (i) To make sure that there is no mass injection cross the upper wall, the following equation should be given:

$$\psi = \text{constant} \quad \text{along } \overline{AB} \quad (6-2)$$

where the constant is determined from the mass flux entering the duct.

- (ii) One of the difficulties of this problem is that it is not suitable and not even possible to define values of velocity on the downstream edge. This does mean it is difficult to establish the values of stream functions along downstream edge. To overcome this difficulty, some additional constraints along this edge were introduced.

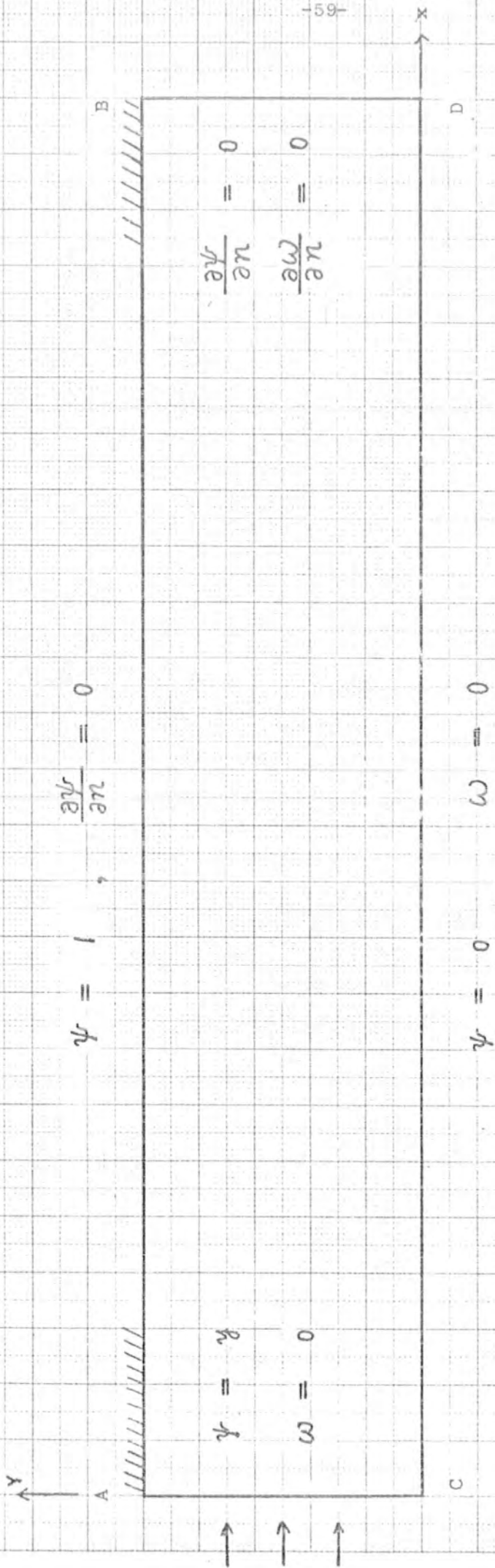


FIGURE 6-2. Boundary conditions for flow between parallel plates.

Because the flow becomes parallel along the downstream edge, the normal derivatives of both vorticity and stream function must be vanishing to enforce the flow parallel at the exit. So,

$$\begin{aligned} \frac{\partial \psi}{\partial n} &= 0 & \text{along } \overline{BD} & \quad (6-3) \\ \frac{\partial \omega}{\partial n} &= 0 \end{aligned}$$

(iii) From symmetry, in the centre-line of the duct, the vorticity and streamfunction may be defined as follows:

$$\begin{aligned} \psi &= 0 \\ \omega &= 0 & \text{along } \overline{CD} & \quad (6-4) \end{aligned}$$

(iv) For uniform flow at the duct entrance, the following equations can be given

$$\begin{aligned} \psi &= KY \\ \omega &= 0 & \text{along } \overline{AC} & \quad (6-5) \end{aligned}$$

where K is constant.

(v) The no-slip condition is most important (see Chapter 4). The formula used to calculate wall vorticity is as follows:

$$\omega_w(x_0, y_0) = \frac{2}{n^2} \left[\psi(x_0, y_0) - \psi(x_1, y_1) \right] \quad (4-4)$$

6.1.4. Iteration Technique

To obtain a starting solution, Stokes' flow was assumed. And stream function and vorticity fields were calculated by giving $Re=0$. After the Stokes' flow solution was obtained, the iteration process was used to calculate the flow at successively larger Reynolds numbers. The solutions of stream function and vorticity at a lower Reynolds number are used as the initial conditions for the solutions at the next higher Reynolds number .

It was found that the numerical scheme was stable if a sufficient small Δt was chosen.

6.1.5. Discussion and Conclusion

At the entry section where a velocity discontinuity is occurred at

point A (see Figure 6-3(a)), the singularity introduces a considerable disturbance to the solution. Serious numerical errors may be encountered in the calculation unless sufficiently small elements are used.

In fact, because the boundary conditions are contradictory at the point A, the approximate solution will not be able to satisfy such boundary conditions exactly.

In actual computation, two types of entrance conditions at the discontinuous point have been tested.

Case 1. U decreases to zero from point E to point A as a parabola function. The velocity profile for this kind of entrance condition was shown in Figure 6-3(b).

Case 2. U decreases to zero from point E to point A as a linear function. The velocity profile for the entrance condition was presented in Figure 6-3(c).

The phenomena shown in Figures 6-3(b) and 6-3(c) seem to agree well with Tong and Fung's (124) results (see Figure 6-4).

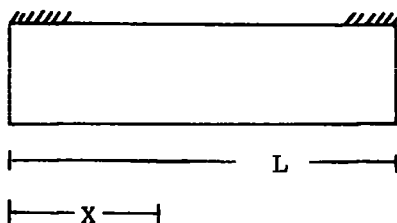
The input data used is presented in Appendix I. Its main results for streamfunction and vorticity from this computer program are given in Table 6-1. From this table, it is found that the results seem to be along with those of Baker and Gasman (4). The streamline contours are presented in Fig.6-5. Its main contour program is given in Appendix C. Results for velocity and pressure are shown in Figures 6-6, 6-7 and 6-8. From Figure 6-6 and 6-8, it is found that the velocity and pressure results from this program seem to be along with those of Goldstein (49) and Yamada (129) as well. Some streamline contours from this program for slightly higher Reynolds numbers are presented in Appendix D. The streamline contours of Baker (4) for Reynolds number of 200 are shown in Appendix E. It is seen that the contours compare reasonably well thus indicating that the present program seems to be accurately representing the phenomenon.

TABLE 6-1

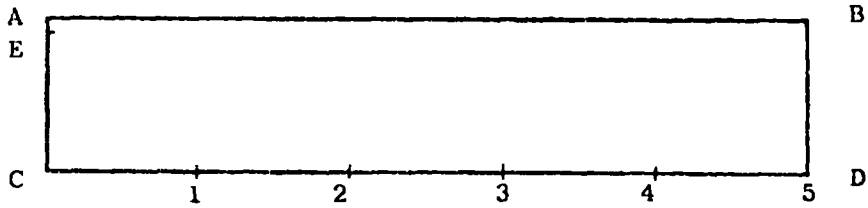
Streamfunction and wall vorticity distribution

X/L	Streamfunction ψ			Wall vorticity ω_w		
	F.E.M.* (Baker)	F.D.M.* (Gosman)	This ** prog.	F.E.M.*	F.D.M.*	This** prog.
0.047	0.980	0.976	0.967	4.04	5.05	6.67
0.095	0.982	0.980	0.981	3.55	4.44	3.54
0.156	0.983	0.982	0.982	3.57	4.14	3.43
0.228	0.983	0.983	0.984	3.58	3.71	3.10
0.379	0.983	0.984	0.984	3.60	3.28	3.08
0.521	0.984	0.985	0.984	3.30	3.12	3.08
1.000	0.983	0.985	0.984	3.51	3.06	3.15

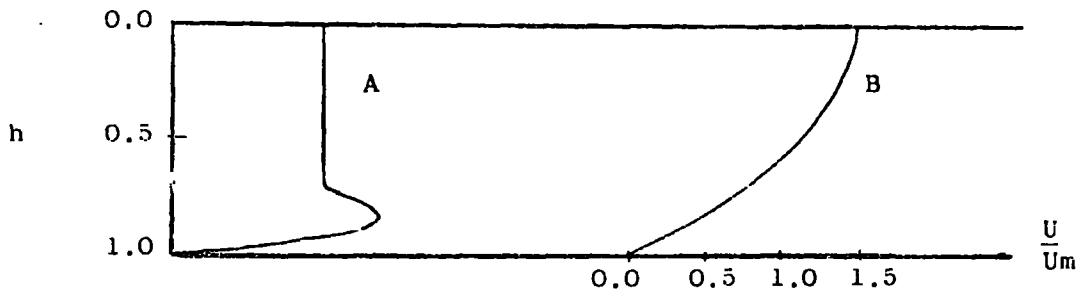
(i)



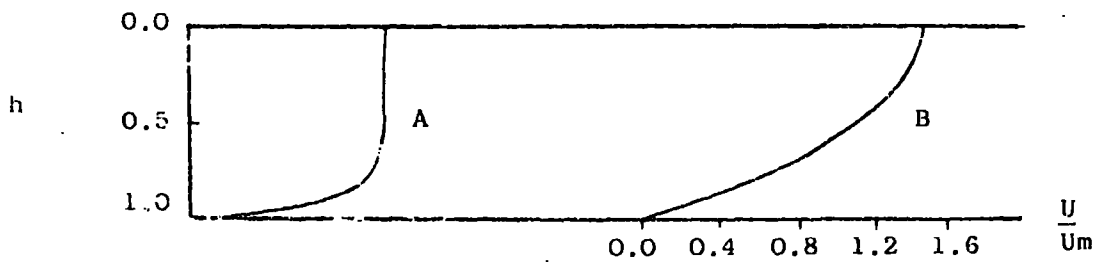
- (2) * : Re = 2 00
 ** : Re = 0.002



(a)



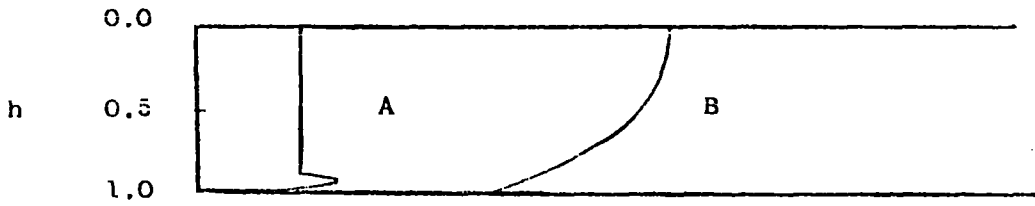
(b)



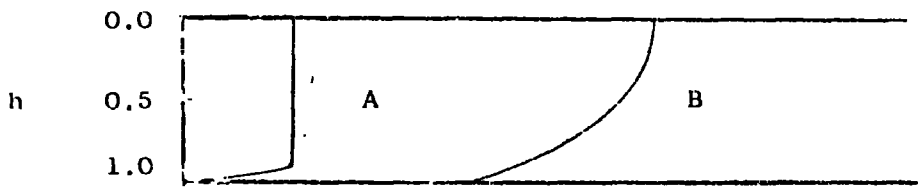
(c)

Figure G-3 Velocity distribution for flow between parallel plates.

- A: Velocity distribution at the entrance.
- E: Velocity distribution for fully developed flow.



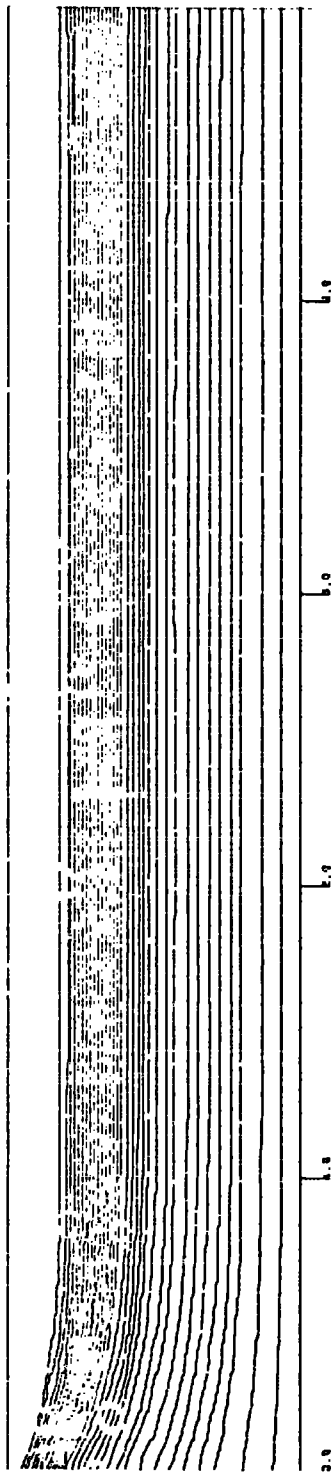
(a) Case 1



(b) Case 2

Figure 6-4 Tong & Fung's Results (124)

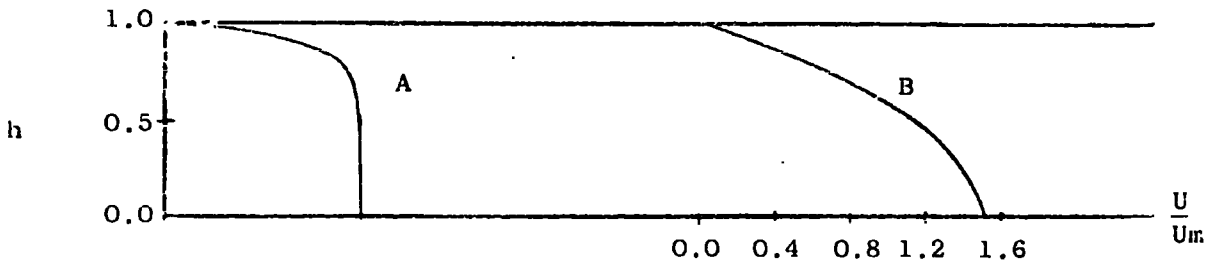
- A: Velocity distribution at the entrance.
- B: Velocity distribution for fully developed flow.



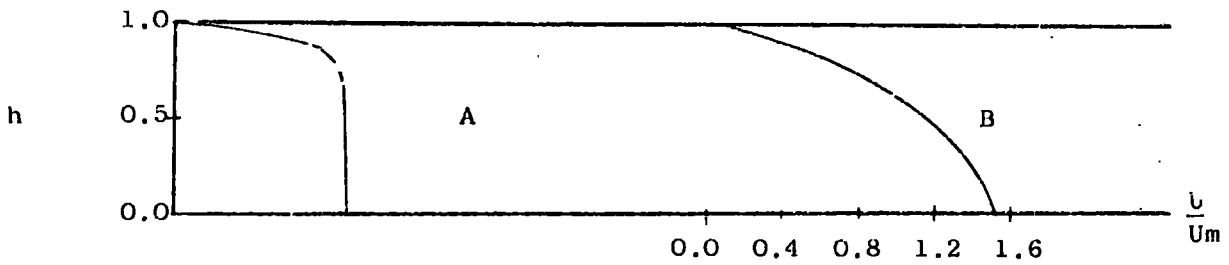
STREAM LINES

Figure 6-5

Streamline contours in flow between parallel plates.



(a) Results from this program



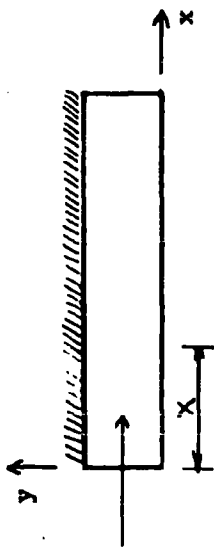
(b) Results of Goldstein

A: Velocity distribution at the entrance.
B: Velocity distribution for fully developed flow.

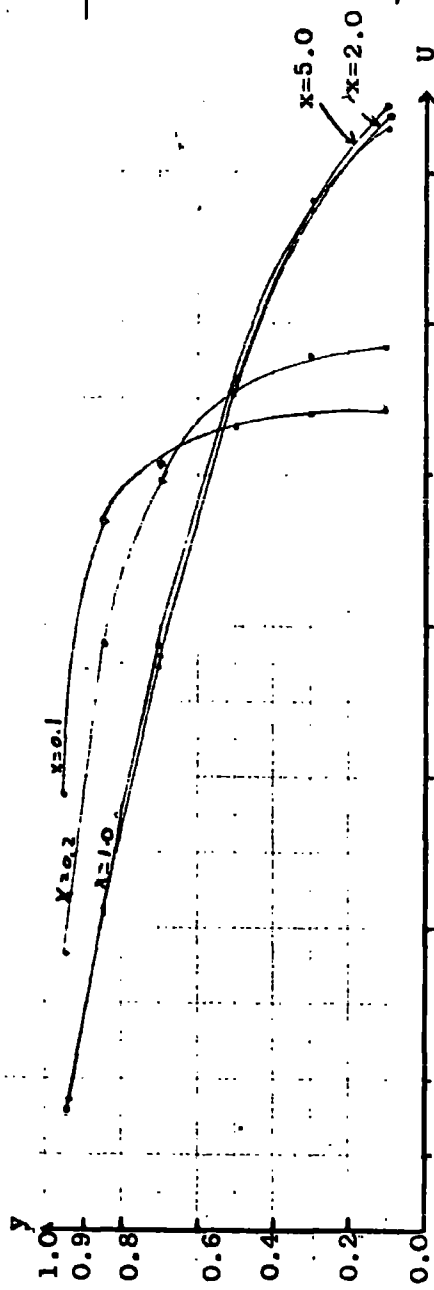
Figure 6-b

Velocity distribution for flow between parallel plates

FIGURE: 6-7 Velocity distribution for flow between parallel plates

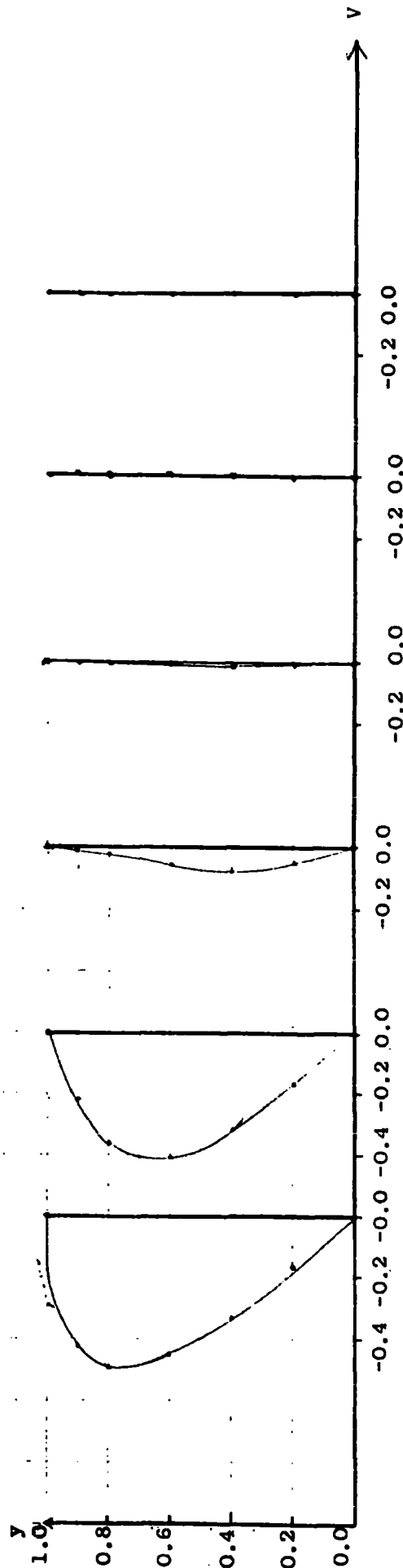


U: the axial component of velocity
 V: the radial component of velocity



0.0 0.1 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.5

(a)

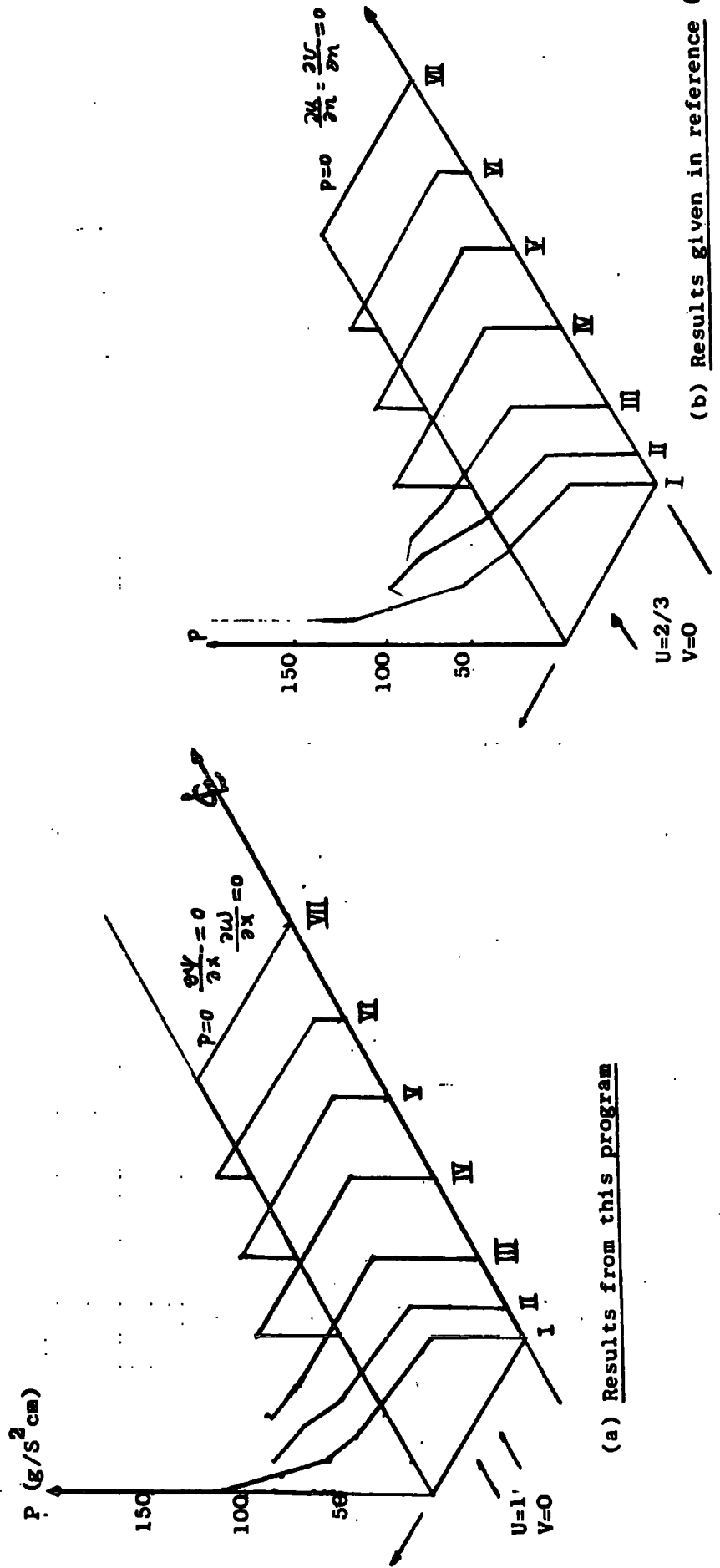


(x=0.05) (x=0.15) (x=0.7) (x=1.5) (x=2.0) (x=4.9)

(b)

Figure 6.8

Pressure distribution for flow between parallel plates



6.2 Example Two

6.2.1. Introduction

The next application was to the internal fluid flow in a channel of arbitrary cross section. The geometry of an example flow passage was shown in Fig.6-9. This type of geometry could provide a good test and demonstration for this solution procedure because the constriction causes rapid changes in stream function and vorticity near the constriction region. The coarsest finite element discretization evaluated were illustrated in Figure 6-10.

6.2.2. Boundary Conditions and Iteration Technique

It has been shown that the entrance length is reduced as the shape of the entrance velocity profile approaches that of the fully developed profile. And this entrance length is reduced to zero when the entrance velocity profile is identical with the fully developed profile at which point the flow is fully developed at this entrance to the channel (55). So Poiseuille type flow was used at the upstream edge in view of the fact that larger values of X_1 (see Figure 6-9) would imply a rapid growth of computer time required. That is

$$\psi = \frac{3}{2} \left(y - \frac{1}{3} y^3 \right) \quad \text{on } \overline{AF} \quad (6-6)$$

At the downstream edge (\overline{EG}) Poiseuille type flow was also assumed. It is worth emphasizing here that the values of X_3 need be provided large enough. Extending the concept of entrance length discussed in example one to this example, it is known that at higher Reynolds number, after passing the constriction, the fluid flow would have to travel a much longer distance X_3 before it was returned to a Poiseuille type flow. If the provision of X_3 was not adequately specified in the finite element mesh, the numerical procedure may become unstable. Even when the calculations are convergent, wrong results or results which are not expected, for example, a result from a different boundary condition, may still be obtained.

From the governing equation $\nabla^2 \psi = -\omega$ and equation (6-6), the

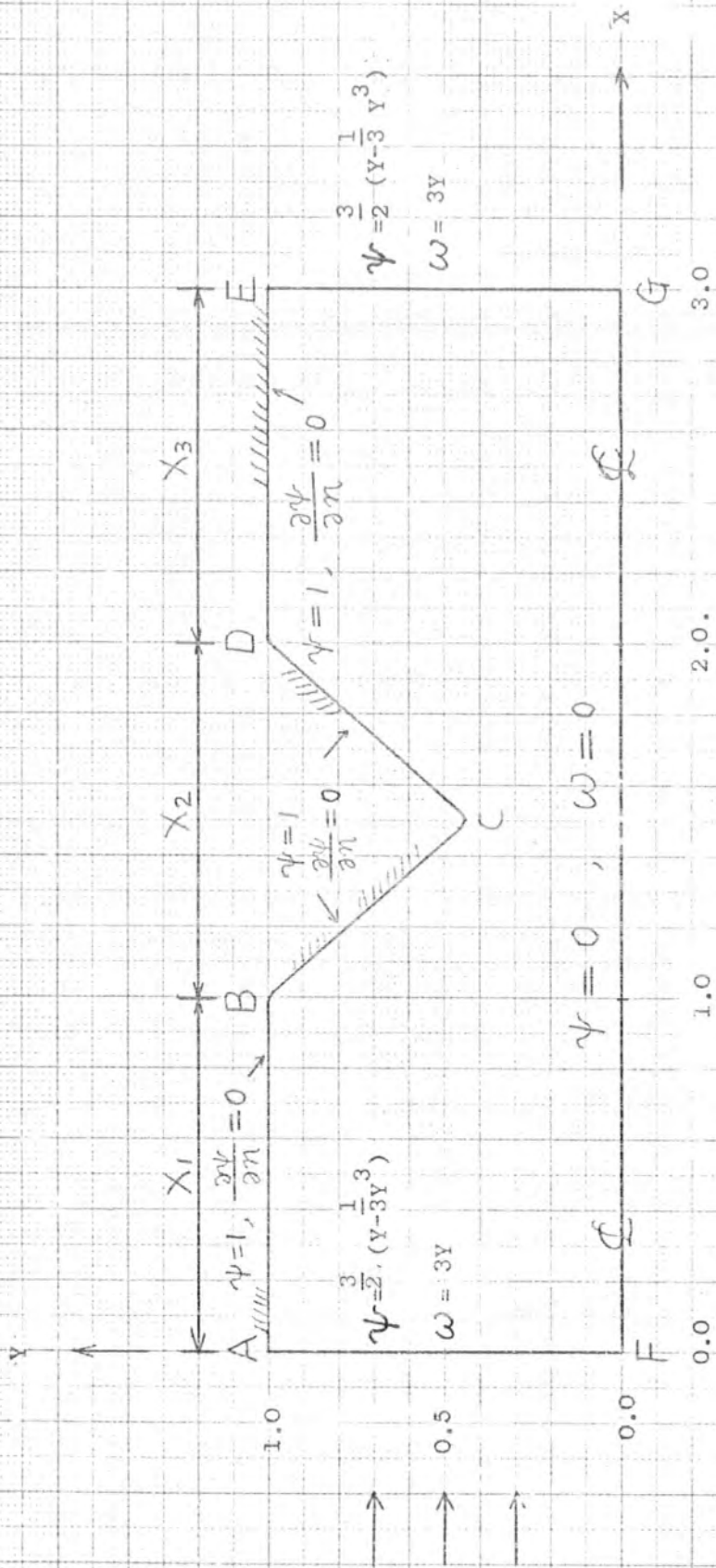


FIGURE 6 - 9

The Geometry and boundary conditions for internal fluid flow in a constricted channel.

FIG. 6 - 10
Finite Element Mesh for Fluid Flow in a constricted channel

FINITE ELEMENT MESH

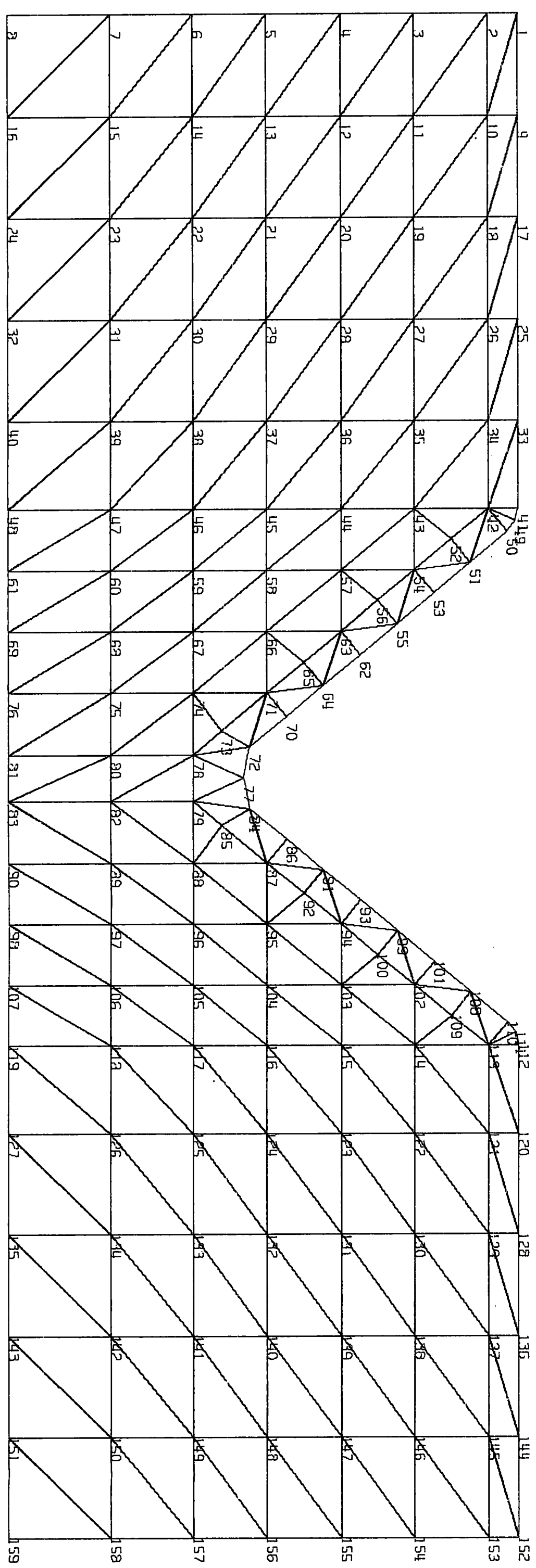
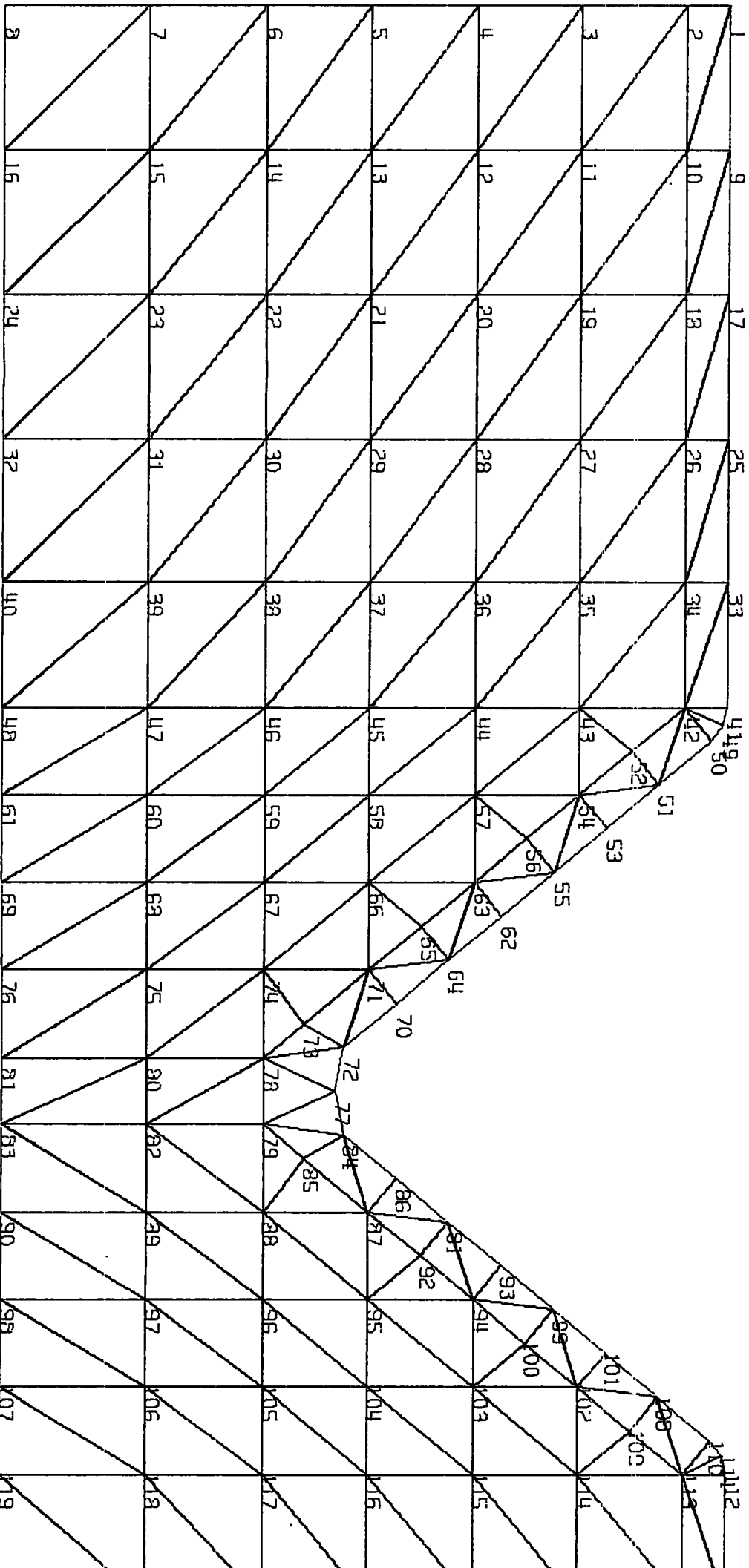


FIG. 6 - 10
Finite Element Mesh for fluid flow in a constricted channel

FINITE ELEMENT MESH



following boundary condition should be specified

$$\omega = 3Y \quad \text{on } \overline{AF} \text{ and } \overline{EG} \quad (6-7)$$

On the solid wall, the no-slip condition was applied as well. And the first-order one-sided difference formula (equation (4-4)) was employed to calculate wall vorticities. It is worth mentioning here that this wall vorticity formula is mainly not suitable for fields containing corners with high curvatures. Thom and Apelt in their useful book "Field Computations in Engineering and Physics" (121) suggested some helpful schemes to deal with this difficulty. Greenspan (51,52) also suggested some other formulae to solve this problem. However, no matter what kind of scheme is being used, it should be stressed that it is most important to use a finer mesh in the constriction regions. If the elements used are small enough, then the first-order one-sided difference formula can still be employed. The key point is that the effects of the corners must be forced to spread into the fluid field in every direction. It is worth emphasizing here that the finite element method is simply a man-made method. It is important to help the method by letting "him" work as close to the phenomenon which occurs physically as possible.

Referring to chapter 4 again, in which some ideas about boundary conditions have been presented, the following boundary conditions may be specified (see Figure 6-9).

$$\psi = 0 \quad \text{on } \overline{FG} \quad (6-8)$$

$$\psi = 1 \quad \text{on } \overline{ABCDE} \quad (6-9)$$

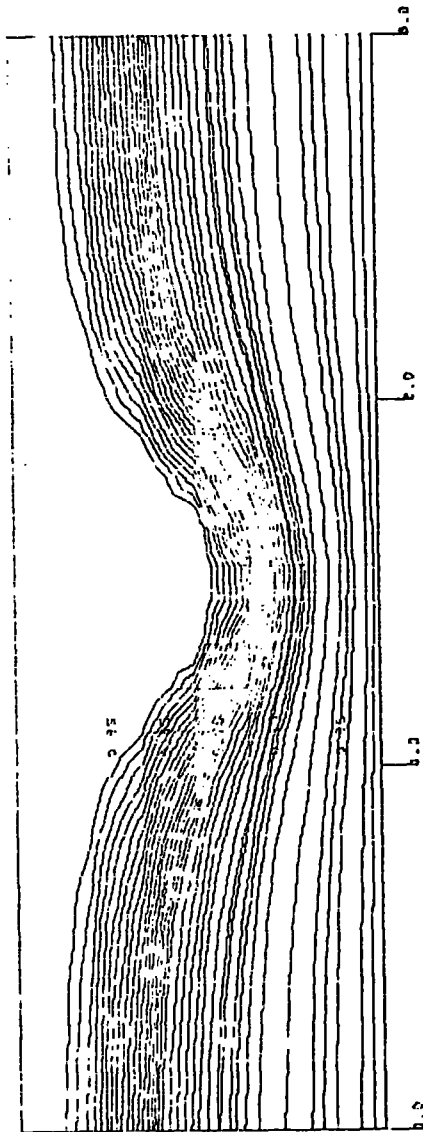
$$\omega = 0 \quad \text{on } \overline{FG} \quad (6-10)$$

At time $t=0$, values of vorticities are assumed to be zero everywhere. And these values are considered as the initial conditions when the numerical solutions of stream function and vorticity for creeping flow are solved. After the convergent creeping flow solution was obtained, this solution was used for the initial conditions of stream function and vorticity in the iteration process to calculate the flow at a small Reynolds number.

This process was carried out such that the solution at a low Reynolds number was used as the initial conditions for the solution at a higher Reynolds number (see Chapter 4). It has been found that the iteration procedure was always stable for sufficiently small Δt . Through numerical experiments, it is found that a high curvature at point C (see Figure 6-9) has profound adverse effects on both numerical stability and accuracy. Even with the same mesh size, the high-curvature body requires a time step that is an order of magnitude smaller than that for a low-curvature body in order to achieve stability (105). This is due mainly to the very steep gradients of the vorticity in the field created by the high curvature. The same gradients also cause severe problems with respect to loss of accuracy (105). In order to avoid large errors, and accelerate the speed of convergence, it seems that two kinds of time steps, Δt , and a smaller one Δt_2 , may be used in low-curvature regions and high-curvature ones respectively. (In this example, this kind of scheme has not been employed yet.)

6.2.3. Conclusion

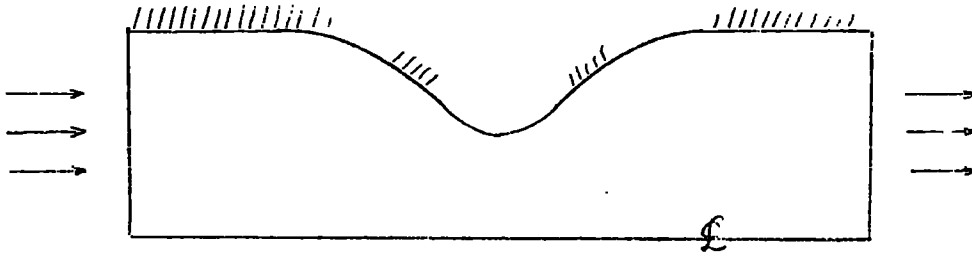
The input data used is presented in Appendix F. Its main results for stream function and vorticity from this program are given in Appendix G. The stream line contours are presented in Figure 6-11. The contours seem to be along with those of Lee and Fung (79) (see Figure 6-12). Results for wall vorticity are shown in Figure 6-13. From this figure, it is found that in the Stokes limit the flow patterns are symmetric before and after the constriction, no separation occurred at this limit. This result seems to agree well with both the solutions of Cheng (25) and those by Lee and Fung (79). The slight discrepancies among the results for the vorticity on the wall of Stokes flow from this program and those of Cheng (25) and Lee and Fung (79) are probably due to the differences in the geometries of constrictions and the coarseness of these meshes being used. The velocity results are presented in Figure 6-14. They seem to be in



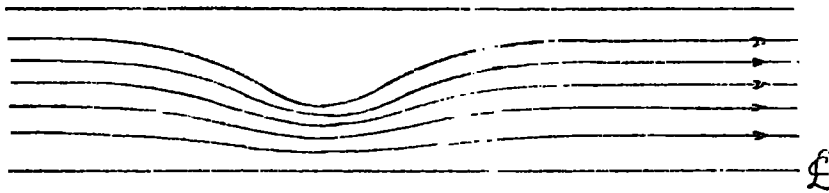
STREAM LINES

FIG. 6-11

Streamline Contours

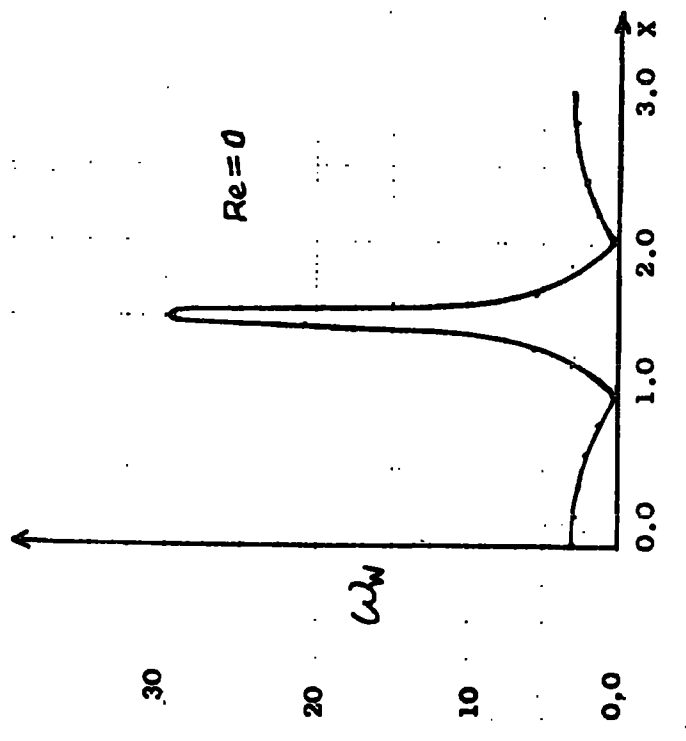


(a) The geometry used by
Lee & Fung

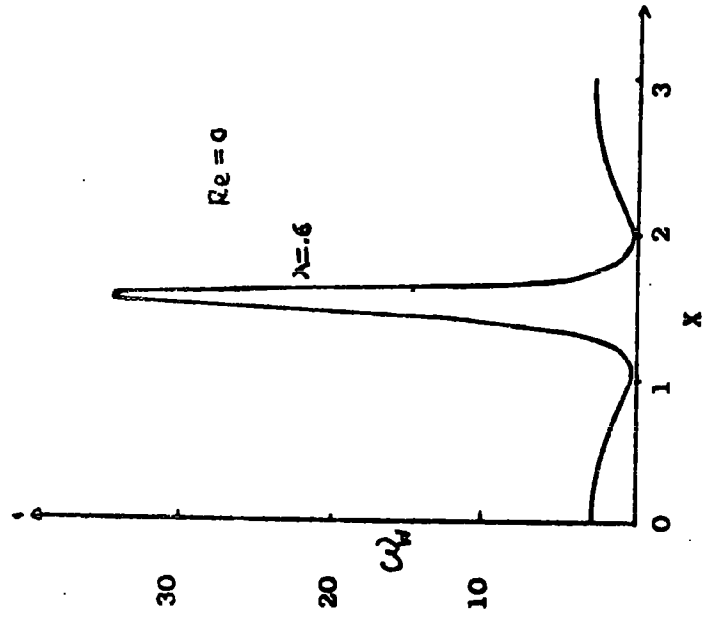


(b) Streamlines Contours

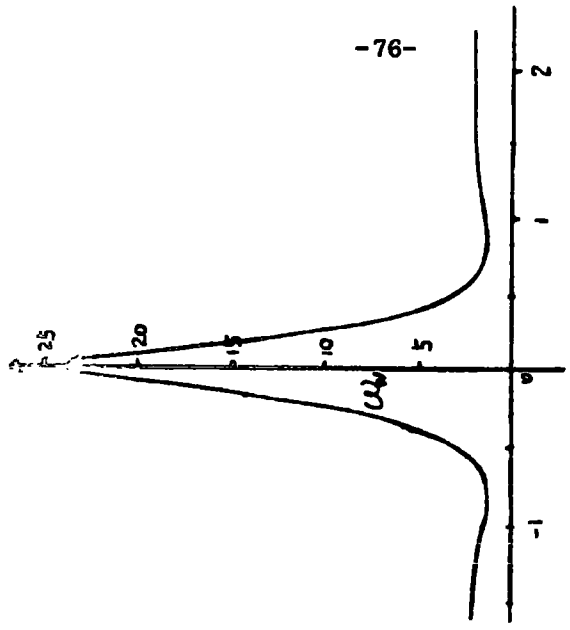
FIG.6-12 Results of Lee & Fung



(a) Results from this program



(b) By R T-S Cheng

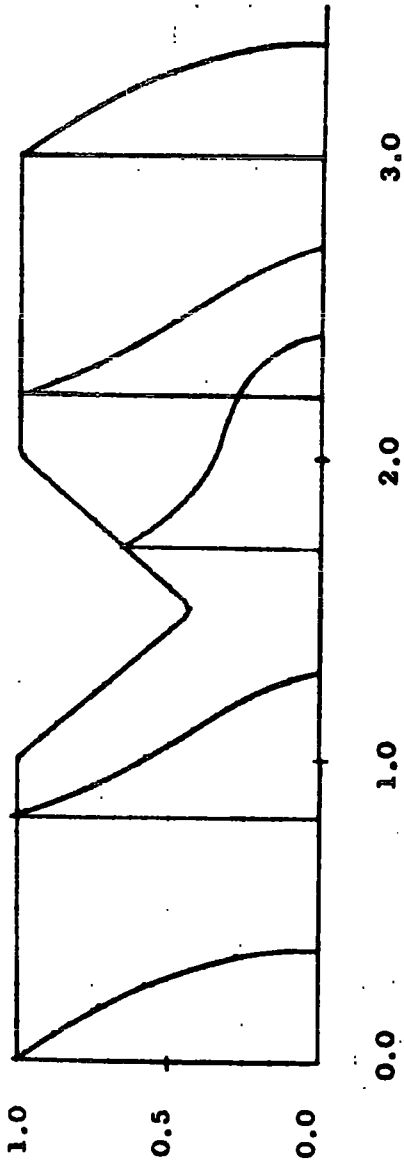


(c) By Lee and Fung

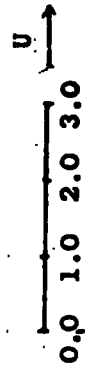
FIG. 6-13

Distribution of the vorticity on the wall

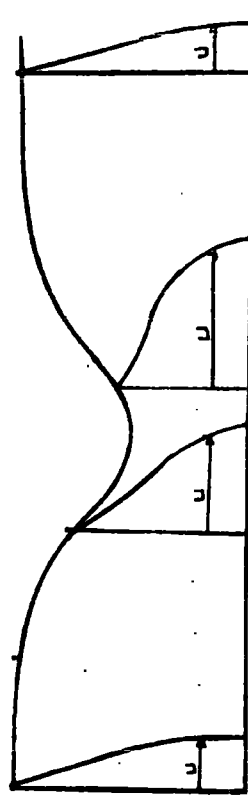
VEL PROFILE



Results from this program



(a)

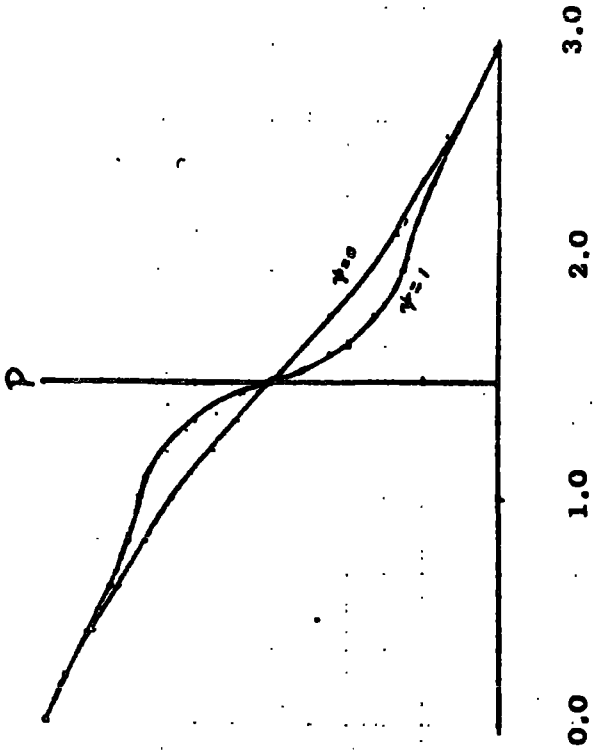


Results of Lee & Fung

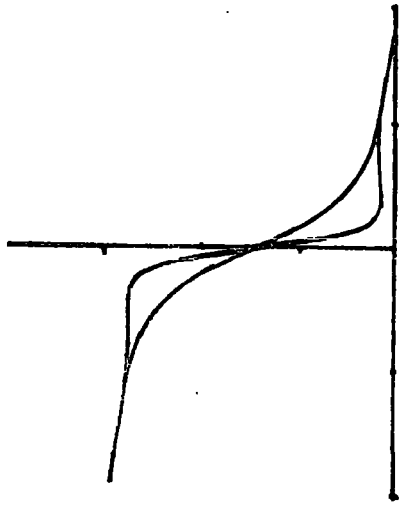
(b)

FIG. 6-14

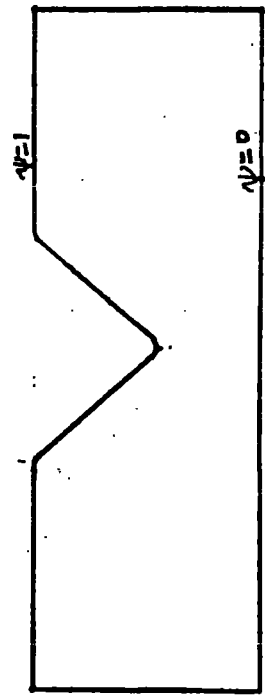
Velocity distribution in a
constricted channel



(a)

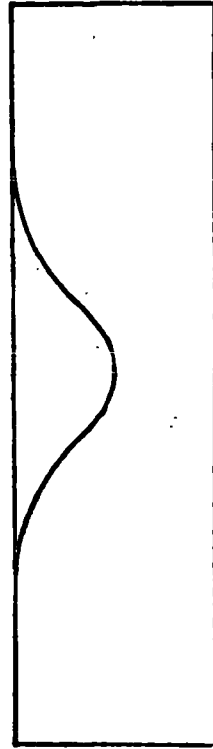


(b)



(c) Results from this program

(FEM)



(d) Results of Lee & Fung

(FDM)

FIG. 6-15 Pressure distribution.

good agreement with the solution of Cheng (25). The pressure results are given in Figure 6-15. The slight discrepancies between the solutions for pressure from this program and those of Lee and Fung (79) are also probably due to the differences in the geometries of constrictions and the coarseness of the finite element grid.

The results seem to compare reasonably well thus indicating that this program seems to be accurately representing the physical phenomenon.

Chapter 7 DISCUSSION

Although the numerical schemes to solve the Navier-Stokes equations have been discussed, some realistic difficulties can still occur when the schemes are carried out. In this chapter, some problems will be discussed. And some observations about the numerical procedure, obtained through numerous numerical experiments, will be presented and discussed. These observations may be useful in the finite element analysis.

7.1 Convergence problems

The problem of convergence is one of most important problems in a numerical analysis. When the foregoing process is applied to the Navier-Stokes equations, divergence can take place in parts of the field. Emphasis has already been put on the need for care with the method of determining the boundary values of vorticity, but even if these are known and fixed the field may still diverge if the mesh size is too large (33,42,100,121). To prevent divergence, Thom (121) suggested that at each point the value of ω may be adjusted from the old value ω_n to the newly calculated value ω_{n+1} . In other words, ω its full movement may be given. Thom claims that the movement should be restricted by combining ω_n and ω_{n+1} in the proportion $r:1$ where r is positive. However, if r is too large, it is obvious that the rate of advance would be very slow.

In the calculations for wall vorticity, sometimes, it happens that to repeat this operation many times will result in an unstable oscillation of the field. Thom and Apelt (121) suggested that new boundary values

$$\omega_w = (\omega_n)_w + K [(\omega_{n+1})_w - (\omega_n)_w]$$

may be used instead of original ones, where K is less than unity. The best value of K can only be estimated by trial and experience. As a start they suggested $K = 0.5$. In other words the boundary values are only moved about one-half of the amount indicated by equation (4-4).

Lee and Fung (79) also employed a similar manipulation. They combined the conformal mapping and finite-difference techniques to investigate

problems of viscous flow in a locally constricted tube. They found that the calculation was straightforward for $Re \leq 15$. But when $Re \geq 20$, the numerical procedure failed to converge. To improve the matter, they used an under-relaxation factor. They claim that the values of the vorticity ω were given one half their theoretical changes in each iteration. In this way they got the results for $Re=25$.

Although incorporation of the under-relaxation technique seems to be able to accelerate the speed of convergence, and improve the possibility of convergence, there is still no, to the best knowledge of the author, apparent theoretical justification for such a manipulation of the relaxation factor.

Some comparisons between the results of using a relaxation factor and not using this kind of factor have been made, and presented in appendices. Note that when $TURF1=1$ (see Appendix D) the relaxation factors were never used, and when $TURF1 = 0.1$, the relaxation factors used were 0.1.

7.2 Storage problems

In many finite element problems the amount of core store required is too great for the computer being used and it is necessary to use backing store. Sometimes a peripheral such as a magnetic disc on a magnetic tape deck can be used automatically within a program. The non-linear matrix, because of its size, may be stored out of core, the high-speed disc being the next best place.

There are lots of ways to improve storage problems: such as: employing the techniques of the front solution, substructuring, overlaying, equivalence, or dynamic allocation, etc. Some important methods of them will be briefly discussed here.

7.2.1 Front Solution (33,37,67,134)

Theoretically the front solution is quite simple. There is a large linear set of simultaneous equations with the n stream functions $\{\psi\}$ of the fluid flow field as unknowns. When all the information relating to

a particular variable is complete then that variable may be eliminated since an expression for it in terms of the other variables in the problem can be obtained. Some unknowns can be eliminated before the complete "stiffness matrix" is formed and therefore the whole of the "stiffness matrix" is never needed in core at one time. Irons (67) has developed a good front solution program for finite element analysis to solve symmetric positive-definite equations. Hood (134) has also presented another front solution program which may be used for the solution of unsymmetric matrix equations. Using these schemes, core requirements and computer time may be considerably reduced.

7.2.2. Banded Solution (33,37,116)

Using the fact that the stiffness matrix is square and symmetric and all non-zero terms are concentrated in a narrow band either side of the leading diagonal, great economies of storage are possible by storing only the band.

To solve equations (3-35) and (3-43) directly by using, say, the Gauss-Jordan elimination procedure would be very inefficient in terms of computer time and storage, since that does not take advantage of the banded nature of $[K_{\psi}]$ and $\left[[K_w] + [K\dot{w}] / \Delta t \right]$.

It appears that the most efficient procedure is to store those elements of $[K_{\psi}]$ and $\left[[K_w] + [K\dot{w}] / \Delta t \right]$ those are within the band rowwise in two vectors respectively, say $[A]$ and $[B]$, and employ a modified Gaussian elimination scheme with back substitution which takes advantage of the banded nature of $[K_{\psi}]$ and $\left[[K_w] + [K\dot{w}] / \Delta t \right]$. In this procedure, Gaussian elimination and back substitution need only be carried out up to the lower and upper edges respectively of the bands. Thus the zeros of $[K_{\psi}]$ and $\left[[K_w] + [K\dot{w}] / \Delta t \right]$ outside the band are not operated upon and are actually not stored in $[K_{\psi}]$ and $\left[[K_w] + [K\dot{w}] / \Delta t \right]$. With this method of solution, it will be necessary to know the width of the bands and the location of the diagonal elements within the bands at every

row of $[K_{\psi}]$ and $[[K_w] + [K\dot{w}] / \Delta t]$.

7.3 Computer time problems (6,14,25)

There are also a lot of methods to save computer time. But if the computer program has been prepared, then the following suggestion may be a helpful way to reduce the computation time used.

It is found that the initial flow configuration does not change the steady-state solution. Regardless of the initial conditions employed, the procedure does indeed converge to the same solution. Therefore, provided that there is no interest in the transient solution to the problem, the analysis can be started at a good initial guess with attendant saving in computer time. The solution obtained with a coarse mesh can be utilized as the guess for a finer mesh.

7.4 Boundary conditions

It has been shown that all boundary conditions require detailed attention. But sometimes, even if the significance of boundary conditions has been noticed, it is still not impossible to face the problems of how to specify suitable boundary conditions for a given problem. A practical way, but not a good way from points of view of computer storage, to overcome this kind of problem is presented here.

It is pointed out that the difficulty in boundary conditions, sometimes can be by-passed if the region to be considered is changed. Usually, a bigger region or the whole region of the problem can be used instead of a smaller or half the region. For example, to study the problem of vortex street development behind some obstructions in channel of finite width, the boundary conditions in the centre line are not quite obvious. So in this case, the best way to deal with this kind of problem is just to use the total region of the flow field instead of the half symmetric one. (see Figure 7-1). (110)

7.5 Finite-element mesh

A likely distribution of contours of field variables of interest is best be predicted in advance. Then the mesh is arranged to be as similar to

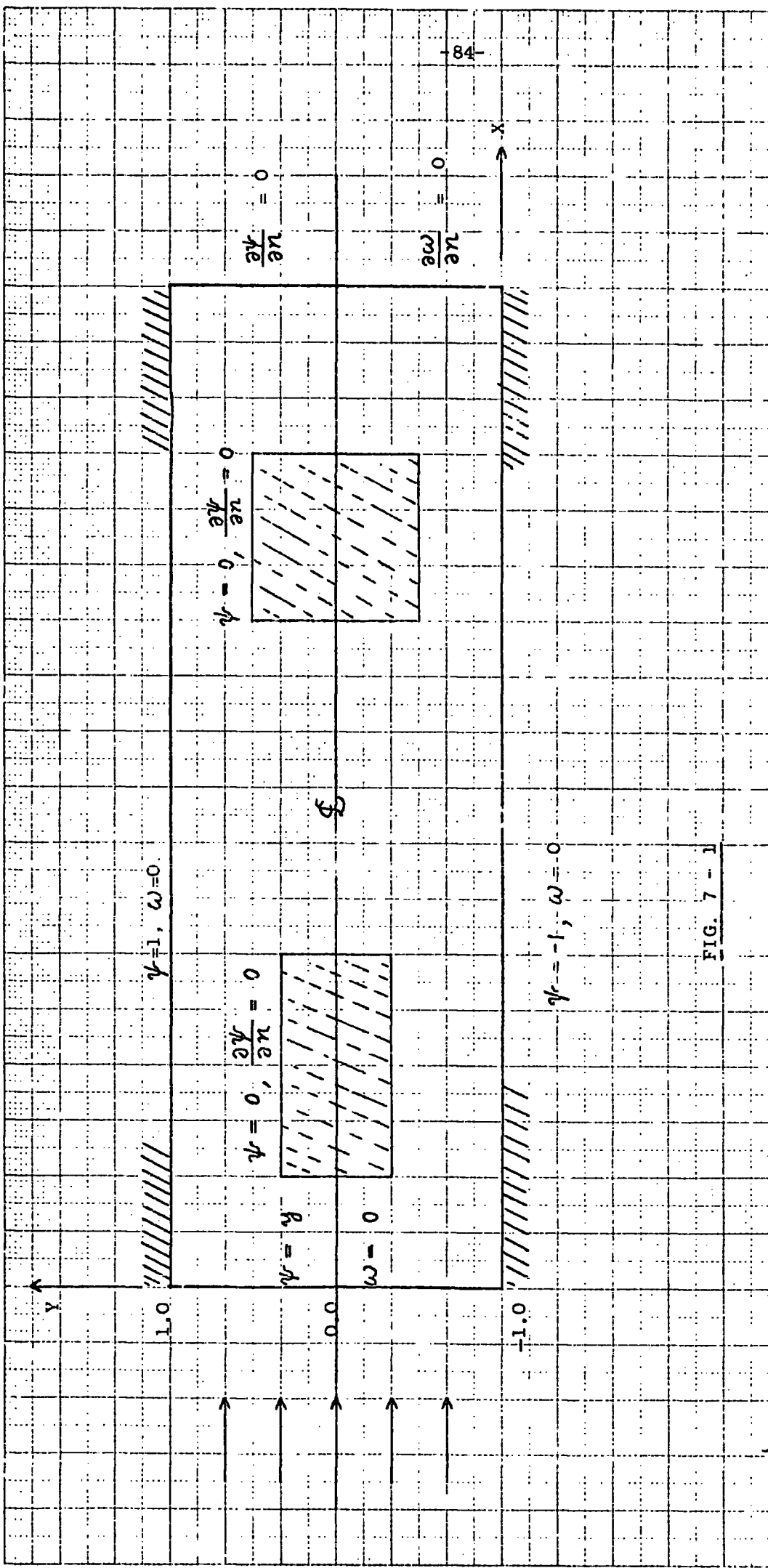


FIG. 7 - 1

Boundary conditions for flow in a channel of finite width with some obstructions.

the predicted distribution as possible. For example, a finer mesh should be arranged for the regions where the variations of streamfunctions or vorticities are more pronounced.

If there is no idea at all about the distribution of contours of field variables, an alternative way suggested here is that a coarse mesh can first be made to analyse the problem. It is possible to get some ideas of values of results from this analysis based on the coarse mesh. And then, these values may be used as a guide to arrange a better mesh. Bearing in mind that an appropriate mesh used can save computer storage, and computation time, but a bad mesh can even make the calculation unstable.

7.6 Some Observations

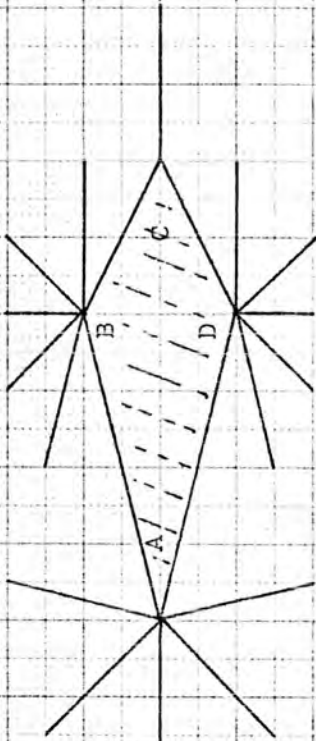
Through numerous numerical experiments, some observations were made. They may be useful in the future applications of the finite element method. Although their validity for all kinds numerical schemes has not been ascertained yet, it is hoped to bring these observations to people's attention.

7.6.1. The transmission phenomena of a mesh line

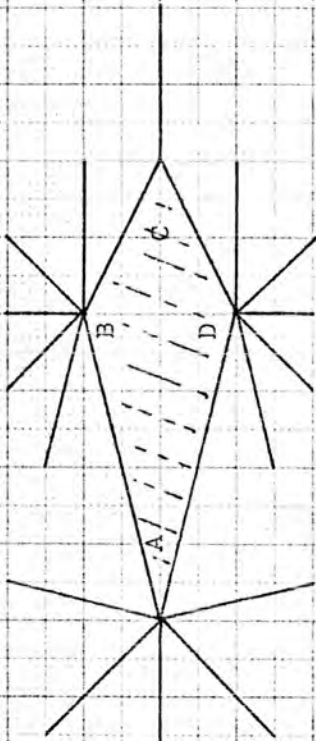
To save computer time and storage, it is best to reduce the number of elements to as few as possible. However to improve and guarantee the accuracy and stability of a calculation, it is hoped to increase the number of elements used to as many as possible. This situation is like a famous Chinese proverb which says "It is difficult to make a horse fat without giving it enough food". Usually, if the number of elements is reduced too much, then inevitably some of the accuracy will be lost, and convergent results can not even be reached. However, even if this is the case, a coarse mesh is often forced to be employed in a complex region, even though a finer grid should have been used, in view of the limitation imposed by the computer. In this case, the best way to do this is to improve the "quality" of the mesh being used. It is found that mesh lines seem to have an ability of giving effect to the calculation by

transmitting or spreading newly calculated values into nodes of neighbouring elements. It is known that any numerical method is simply an instrument to help people do what they want to do, and analyse what they wish to analyse. If the instrument is hoped to work properly, then it should be used in a proper manner. If the finite element scheme, the instrument, is helped by using a good mesh employing the nature of a mesh line, it is not impossible, while adopting a coarse gridwork in a calculation, to get a satisfactory and accurate and stable result. It is found through numerical experiments that in a region on which a high curvature is found, more mesh lines should be used there. The higher a curvature is, the more mesh lines should be used. And it is better to do the mesh symmetrically unless the effect of the mesh on some directions is hoped to intensify through the calculation. To illustrate the idea, an example is presented.

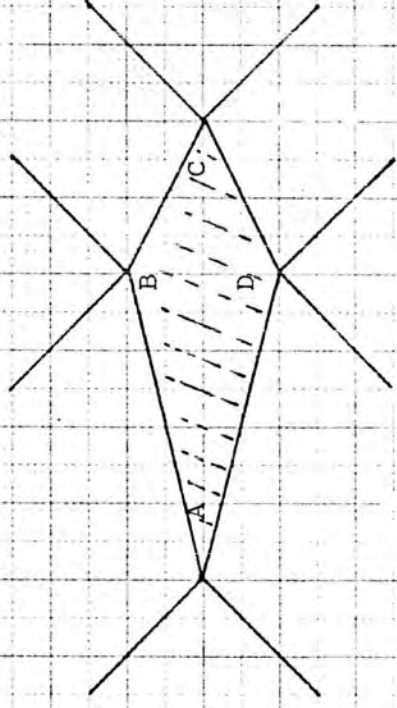
Figure 7-2 shows an obstacle in a fluid flow or a hole on a plate. The curvatures of points B and D are the same, and so the same number of mesh lines is suggested. The curvature at point A is the highest, then the number of mesh lines used should be more than those at points B, C and D. Note that all the mesh lines are symmetrical. Furthermore, not only to save storage but also to reduce computer time, in a region where bigger unsymmetric variations of results are expected, the number of mesh lines there should be increased unsymmetrically. It is significant for a user of a finite element program to learn the effect of a shape function on the choice of a mesh for a calculation and to have an idea about the actual physical phenomena described by the problem. For example, from the foregoing discussions, it is obvious to conclude that the optimal mesh for a viscous fluid flow depends also on Reynolds number. See Figure 7-3. A mesh for case A should be different from that for case B. If the same mesh is used to analyse the two cases, then there will be a lot of computer time and storage wasted in analysing for case B. Bearing in mind that to help the procedure predict the actual physical phenomena, a suitable mesh



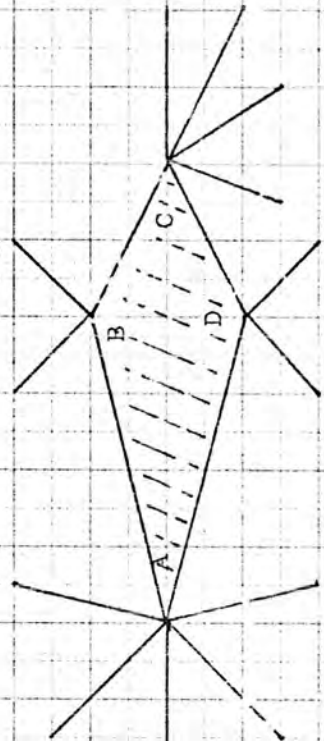
(a) Proper



(2) Improper



(3) Improper

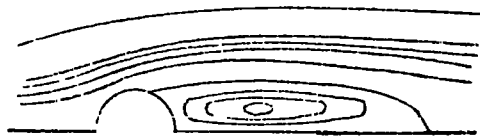


(4) Improper

FIG. 7 - 2 Mesh lines for a hole on a plate or an obstacle in a fluid flow.



(1) Case A, Re = 10



(2) Case B, Re = 60.0

FIG. 7 - 3

Flow round a cylinder

density for each node must be used to enable the procedure to force and spread the effect of governing equations in the regions of interest.

7.6.2. Maximum stable time step Δt_{max} .

Through numerical experiments it is found that the maximum stable time step Δt_{max} also depends on Reynolds number, mesh size, surface curvatures of bodies, etc.

For the example one, if the input data shown in Appendix I is used, the values for Δt_{max} are

Re	=	0.002	,	Δt_{max}	=	0.00001
Re	=	1.	,	Δt_{max}	=	0.0001
Re	=	2.	,	Δt_{max}	=	0.0001
Re	=	3.	,	Δt_{max}	=	0.0002
Re	=	4.	,	Δt_{max}	=	0.0002
Re	=	5.	,	Δt_{max}	=	0.0004
Re	=	6.	,	Δt_{max}	=	0.0004

It may be mentioned here that it is difficult to determine exactly when Δt_{max} is reached. Thus, all values must be considered as approximate. The parameters which affect the zones of convergence would also include maximum stable time step, Reynolds number, mesh size, etc. To accelerate the speed of convergence and to make the results stable, these parameters should be tried.

7.7. General discussions

It has been found that the finite element solution algorithm is capable of predicting some natural physical phenomena without resort to special devices. The finite element method is able to define the nodal points and elements arbitrarily to permit flexibility and easy accommodation of the complex boundary. Employing the knowledge of fluid dynamics concerning the anticipated solution distribution, a smaller element in regions containing larger spatial derivatives of the dependent variables can be used. Some of the other major advantages of the finite-element method

over finite difference ones are that different shapes may be represented automatically, various boundary conditions may be satisfied in a straight forward manner, different element sizes may be used to get maximum efficiency. A finer mesh can be used to gain details of the fluid flow field exactly where desired. Generally significantly fewer equations are required to provide a given accuracy. (16,62,77,100).

As far as computational efforts are concerned, a computer program for the finite element computation seems to be more complicated than its counterpart for the finite difference method. However, the complication stems from the intrinsic generality of the finite element. The generality of the finite element method usually leads to the computer program to be applicable to a class of similar problems (19,25,33).

However, just as a Chinese saying has it "There is nothing in the world which is perfect!" The finite element method also suffers from some disadvantages. It is known that error analysis is very important in numerical methods. However, up to now, there does not seem to be a method which can be used to calculate the truncation error incurred by using a particular kind of element shape. With the finite difference method, on the other hand, the truncation error involved in any finite difference formula can be analysed using the calculus of finite difference. However, it is expected that the truncation error incurred will be comparable with that of a finite difference mesh of the same size, so it is possible to get some ideas about the order of approximation of a particular element shape. When a higher order of approximation to the unknown function is sought, the situation may become more complicated with the finite element method. With the finite difference method, increasing the order of approximation presents no real difficulty (25,116).

It can be shown that the finite element approach converges to the exact solution as the number of elements is increased. Solution convergence with finer finite element mesh is very significant for numerical solution of non-linear equations like the Navier-Stokes equations. An insufficient

number of points on the solid boundary, at which no-slip boundary conditions are specified may cause unsatisfactory results. The no-slip condition determines the amount of vorticity created at the solid surface. The highest values of the wall vorticity and the vorticity gradient, which govern the spreading of vorticity in the fluid flow field are found on the solid wall (20,25,79,121).

The coefficient matrix $[[Kw] + [K\dot{w}] / \Delta t$ is symmetric, banded, and positive definite. To keep the bandwidth of the coefficient matrix to a minimum, the nodal points should be ordered in such a way that the difference in nodal point numbers for any element be a minimum (25,33).

The three-node triangular element seemed to give quite accurate results. If the sizes of the elements are small enough, the approximation of the unknown function with the element is adequate. Owing to the simpler formulation and the ability to cater for arbitrary boundary shape, the three-node triangular element seems to be adequate for most purposes (63,116).

Although convergence is expected for higher Reynolds number, such a study was discontinued, in view of the fact that for higher Reynolds number fluid flow field, the channel between parallel plates or a constricting internal passage must be elongated and finer mesh must be used to get stable results, thus necessitating that the number of mesh points be increased so much as to be impracticable for this computer.

Chapter 8 CONCLUSIONS

A general numerical procedure for the analysis of two-dimensional, time-dependent, incompressible, viscous fluid flow is presented. A finite-element computer program is developed.

Using a combined variational principle-finite element method, difficulties arising from the nonlinearity of time-dependent Navier-Stokes equations have been remedied. The numerical results obtained by the method have revealed very similar properties to known solutions of similar problems. In the Stokes limit, the flow patterns are symmetric before and after the constriction, no separation occurred at the limit. The high-curvature body requires a time step that is an order of magnitude smaller than that for a low-curvature body in order to achieve stability. In order to avoid large errors, an extremely fine mesh must be used in the regions of large gradients of the vorticity.

The accuracy of the finite element scheme depends basically on the number of nodal points in the finite element mesh, the time step, and order of the numerical integration procedure. Stable results can be obtained for a sufficiently small time step. The simple time integration scheme was found to be sufficiently accurate for present tests. To maintain the accuracy of the calculation, the number of iterations required increases slightly with Reynolds number.

Finally, some important points would be stressed as follows:

- (1) The maximum stable time step Δt_{max} also depends on Reynolds number, grid sizes and the shape of a body.
- (2) Even if the finite element formulation used is the same, the zones of convergence for different problems may not be identical. The zones of convergence also depend on the nature of the flow problem, Reynolds number, mesh sizes, time step, the way of constructing a mesh, and the shapes of obstacles, etc.
- (3) The boundary conditions at the body surface play a decisive part in the solution procedure.

(4) The topology and number of elements of a mesh also depend on the natural boundary conditions used. Many elements would be required to reasonably approximate such natural boundary conditions as occurred in test problem one.

(5) The fact that the numerical procedure will be convergent if Δt used is small enough means that the inertia terms play a stabilizing role in the scheme.

Using the finite element method, the region of low pressure of a body in flow, which accounts for most of the drag force may numerically be calculated. When the velocity of the fluid increases, a symmetric eddies can be produced behind the body which are alternatively shed. For low Reynolds number cases, this phenomenon can be predicted and the shedding frequencies can be found by employing the finite element method (135). These problems are interesting in the design of offshore structures.

It appears that the finite element method may be powerful to predict the natural physical phenomena of a fluid flow field. The next major fields for study would include stability and convergence problems as well as further research into applications of the method to the design of offshore structures.

REFERENCES

1. Adey, R. and Brebbia, C.A.,
"Finite Element Solution for Effluent Dispersion"
Numerical Methods in Fluid Dynamics, C. Brebbia and J.J. Connor
(Eds.), Pentech Press, 1974.
2. Atkinson, B., Brocklebank, M.P., Card, C.C.H. and Smith, J.M.
"Low Reynolds Number Developing Flows",
Am. Inst. Chem. Eng. J., Vol. 15, July 1969.
3. Atkinson, B., Card, C.C.H. and Irons, B.M.,
"Application of the Finite Element Method to Creeping Flow Problems",
Trans. Inst. Chem. Eng., Vol. 48, 1970.
4. Baker, A.J.,
"Finite Element Solution Algorithm for Viscous Incompressible
Fluid Dynamics",
Int. J. Numer. Methods Eng., Vol. 6, No. 1, 1973.
5. Baker, A.J.,
"A Highly Stable Explicit Integration Technique for Computational
Continuum Mechanics",
Numerical Methods in Fluid Dynamics C. Brebbia and J.J. Connor (Eds.)
Pentech Press, 1974.
6. Baker, A.J.,
"Finite Element Solution Algorithm for Incompressible Fluid Dynamics"
Proc. Int. Symp. on Finite Element Methods in Flow Problems, Swansea,
1974.
7. Baker, A.J.,
"Predictions in Environmental Hydrodynamics. Using the Finite
Element Method, I. Theoretical Development,"
J. A. I. A. A. Vol. 13, No. 1, 1975.
8. Bazeley, G.P., Cheung, Y.K., Irons, B.M., and Zienkiewicz, O.C.,
"Triangular Elements in Plate Bending-Conforming and Non-
Conforming Solutions",
Proceedings of Conference on Matrix Methods and Structural
Mechanics, AFFDL-TR-66-80, Oct. 26-28, 1965, Wright-Patterson Air Force
Base, Ohio, p. 547-576.
9. Batchelor, G.K.,
An Introduction to Fluid Dynamics,
Cambridge University Press, 1967.
10. Bateman, H.,
"Notes on a Differential Equation which Occurs in Two-Dimensional
Motion of a Compressible Fluid and Its Associated Variational
Principles",
Proceedings of the Royal Society, London, Series A, Vol. 125, 1929,
p. 598-618.
11. Beavers, G.S., Sparrow, E.M., and Magnuson, R.A.,
"Experiments on Hydrodynamically Developing Flow in Rectangular
Ducts of Arbitrary Aspect Ratio",
International Journal of Heat and Mass Transfer, Vol. 13, 1970, p. 689-702

12. Bird, R.B.,
"New Variational Principle for Incompressible Non-Newtonian Flow"
Phys.Fluids, 3, p.539-541, 1960.
13. Bogner, F.K., Fox, R.L., and Schmit, L.A.,
"The Generation of Interelement Compatible Stiffness and Mass
Matrices by the Use of Interpolation Formulae",
Proceedings of Conference on Matrix Methods and Structural
Mechanics, Oct.26-28.1965, Wright-Patterson Air Force Base, Ohio.
14. Bowley, W.W., and Prince, J.F.,
"Finite Element Analysis of General Fluid"
A.I.A.A. Paper No.71-602, 1971.
15. Bramlette, T.T.,
"Plane Poiseuille Flow of a Rarefied Gas Based on the Finite Element
Method",
Phys.Fluids, Vol.14, No.2, 1971.
16. Bratanow, T., Ecer, A., and Kobiski, M.,
"Finite Element Analysis of Unsteady Incompressible Flow around
an Oscillating Obstacle of Arbitrary Shape",
J.A.I.A.A., Vol.11, No.11, Nov.1973, p.1471-1477.
17. Bratanow, T., and Ecer, A.,
"On the Application of the Finite Element Method in Unsteady
Aerodynamics",
J.A.I.A.A., Vol.12, No.4, April 1974, p.503-510.
18. Bratanow, T., and Ecer, A.,
"Analysis of Moving Body Problems in Aerodynamics",
Proc.Int.Symp.on Finite Element Methods in Flow Problems, Swansea, 1974.
19. Bratanow, T., and Ecer, A.,
"Suitability of the Finite Element Method for Analysis of Unsteady
Flow around Oscillating Airfoils",
International Conference on Numerical Methods in Fluid Dynamics,
University of Southampton, Southampton, England, Sept.26-28, 1973.
20. Bratanow, T., and Ecer, A.,
"Analysis of Three-Dimensional Unsteady Viscous Flow around
Oscillating Wings",
J.A.I.A.A., Vol.12, No.11, Nov.1974.
21. Brebbia, C.A.,
"Some Applications of Finite Elements for Flow Problems",
Int.Conf.Variational Meth.Engng., Southampton, Univ.C.A.Brebbia
and H.Tottenham (Eds.) Vol. I p.5/1-5/26, 1973.
22. Brebbia, C.A., and Connor, J.J.,
Fundamentals of Finite Element Techniques for Structural Engineers,
Butterworths, 1973.
23. Burggraf, O.R.,
"Analytic and Numerical Studies of the Structure of Steady Separated
Flows",
J.Fluid Mech., 24, p.113-151, 1966.

24. Carlson,G.A., and Hornbeck,R.W.,
"A Numerical Solution for Laminar Entrance Flow in a Square Duct",
J.of Applied Mechanics, March 1973 , p.25-30, Transation of the ASME.
25. Cheng,R.T.,
"Numerical Solution of the Navier-Stokes Equations by the Finite
Element Method",
Phys.Fluids, Vol.15, No.12, 1972.
26. Cheng,R.T.,
"Numerical Investigation of Lake Circulation Around Islands by the
Finite Element Method",
Int.J.Numer.Methods Engng., Vol.5, No.1, 1972.
27. Cheng,S., and Hoff,N.J.,
"Bending of Thin Circular Rings"
Int.J.Solids Structures, Vol.11, p.139-152, 1975.
28. Cheng,S.I.,
"Accuracy of Difference Formulation of Navier-Stokes Equations"
The Physics of Fluids, Suppl.II, Vol.12, No.12, Dec.1969, p.34.
29. Cheng,S.I., and Rimon,Y.,
"Numerical Solution of a Uniform Flow over a Sphere at Intermediate
Reynolds Numbers",
The Physics of Fluid, Vol.12, No.5, May 1969, p.949.
30. Chorin,A.J.,
"Numerical Study of Slightly Viscous Flow",
J.of Fluid Mechanics, Vol.57, Part 4, March 1973, p.785-796.
31. Chorlton,F.,
Textbook of Fluid Dynamics
D.Van Nostrand Company Ltd., 1967.
32. Collins,M. and Schowalter,W.R.,
Physics of Fluids 5, p.1122-1124, 1962.
33. Connor,J.J. and Brebbia,C.A.
Finite Element Techniques for Fluid Flow,
Butterworth & Co.Ltd., 1976.
34. Davis,R.T.
"Numerical Solution of the Navier-Stokes Equations for Symmetric
Laminar Incompressible Flow Past a Parabola",
J.of Fluid Mechanics, Vol.51, Part 3, p.417-433, 1972.
35. Dennis,S.C.R. and Chang,G.Z.
"Numerical Solutions for Steady Flow Past a Circular Cylinder at
Reynolds Numbers Up to 100",
J.of Fluid Mechanics, Vol.42, Part 3, 1970.
36. Dennis,S.C.R. and Walsh,J.D.
"Numerical Solutions for Steady Symmetric Viscous Flow Past a
Parabolic Cylinder in a Uniform Stream",
J.Fluid Mechanics, Vol.50, Part 4, p.801-814, 1971.

37. Desai, C.S., and Abel, J.F.
Introduction to the Finite Element Method: A Numerical Method for Engineering Analysis,
Van Nostrand-Reinhold, New York, 1972.
38. Doctors, L.J.
"An Application of the Finite Element Technique to Boundary Value Problems of Potential Flow,"
Int. J. Numer. Methods Engng., Vol. 2, No. 2, 1970.
39. Fargie, D. and Martin, B.W.
"Developing Laminar Flow in a Pipe of Circular Cross-Section",
Proc. Roy. Soc. Lond. A, 321, p. 461-476, 1971.
40. Fenner, R.T.
"Finite Element Methods for Engineers"
The Macmillan Press Limited, 1975.
41. Finlayson, B.A.
"Existence of Variational Principles for the Navier-Stokes Equations",
The Physics of Fluids, Vol. 15, No. 6, June 1972.
42. Finlayson, B.A.
"Weighted Residual Methods and their Relation to Finite Element Methods in Flow Problems",
Proc. Int. Symp. on Finite Element Methods in Flow Problems, Swansea, 1974.
43. Finlayson, B.A. and Scriven, L.E.
"The Method of Weighted Residuals - a Review",
Applied Mech. Rev. Vol. 19, No. 9, p. 735-748, 1966.
44. Finlayson, B.A. and Scriven, L.E.
"On the Search for Variational Principles",
Int. J. Heat Mass Transfer, 10, p. 799-821, 1967.
45. Farray, M.J.
Variational Calculus in Science and Engineering,
New York, McGraw-Hill Book Co., 1968.
46. Fromm, J.E.
"Practical Investigation of Convective Difference Approximations of Reduced Dispersion",
The Physics of Fluids, Suppl. II, Vol. 12, No. 12, Dec. 1969, p. 3.
47. Gelder, D.
"Solution of the Compressible Flow Equations",
Int. J. Numer. Methods Engng., Vol. 3, No. 1, 1971.
48. Ghia, U. and Davis, R.T.
"Navier-Stokes Solutions for Flow Past a Class of Two-Dimensional Semi-Infinite Bodies",
J. A. I. A. A., Vol. 12, No. 12, Dec. 1974, p. 1658-1665.
49. Goldstein, S.
"Modern Developments in Fluid Dynamics",
Oxford.

50. Goldstein, R.J. and Kreid, D.K.
"Measurement of Laminar Flow Development in a Square Duct
Using a Laser-Doppler Flow-meter",
Journal of Applied Mechanics, Vol.34, Trans.ASME, Vol.89, 1967, p.813-818
51. Greenspan, D.
"Numerical Studies of Steady, Viscous, Incompressible Flow in a
Channel with a Step",
Journal of Engineering Mathematics, Vol.3 No.1 Jan. 1969, p.21-28.
52. Greenspan, D.
"Numerical Studies of Viscous Incompressible Flow Through an Orifice
for Arbitrary Reynolds Number",
Int.J.Numer.Methods.Eng.Vol.6, p.486-496, 1973.
53. Guran, R.J., Ujihara, B.H. and Welch, P.W.
"Hydroelastic Analysis of Axisymmetric Systems by a Finite Element
Method",
Proceedings of 2nd Conference on Matrix Methods in Structural
Mechanics, Wright-Patterson Air Force Base, Dayton, Ohio, 1968.
54. Gurtin, M.E.
"Variational Principles for Linear Initial Value Problems",
Quart.Appl.Maths, 22, p.252-256, 1964.
55. Hai, S.M.
"Pressure Development in the Entrance Region and Fully Developed
Region of Generalised Channel Turbulent Flows",
J.of Applied Mechanics, March 1976, p.13-19. Trans of the ASME.
56. Han, L.S.
"Hydrodynamics Entrance Lengths for Incompressible Flow in
Rectangular Ducts",
J.of Applied Mechanics, Vol.27. Trans.ASME, Vol.82, Series, E.1960.
p.403-409.
57. Harlow, F.H. and Fromm, J.E.
"Numerical Solution of the Problem of Vortex Street Development",
The Physics of Fluids, Vol.6, No.7, July 1963.
58. Harlow, F.H. and Welch, J.E.
"Numerical Calculations of Time Dependent Viscous Incompressible
Flow of Fluid with Free Surface",
The Physics of Fluids, Vol.8, No.12, 1965, p.2182-2189.
59. Hildebrand, F.B.
Introduction to Numerical Analysis,
McGraw-Hill, New York, 1956.
60. Hirsch, C.H. and Warzee, G.
"A Finite Element Method for through Flow-Calculations in Turbo-
machines",
J.of Fluids Engineering, Sept.1976, p.403-421, Trans.of the ASME.
61. Holand, I. and Bell, K. eds.
Finite Element Methods in Stress Analysis,
Tapir, Trondheim, Norway.

62. Hood, P.
PhD. Thesis, University of Wales, Swansea, U.K. 1974.
63. Huebner, K.H.
The Finite Element Method for Engineers,
John Wiley and Sons, Inc., 1975.
64. Hussey, M.J.L., Thatcher, R.W. and Bernal, M.J.M.
"On the Construction and Use of Finite Elements",
J. Inst. Maths. Applics. 6, p. 263-282, 1970.
65. Ikegawa, M. and Washizer, K.
"Finite Element Method Applied to Analysis Flow over a Spillway Crest",
Int., J. Numer. Methods. Eng., Vol. 6, No. 2, 1973.
66. Irons, B.M.,
"Economical Computer Techniques for Numerically Integrated Finite
Elements"
Int. J. Numer. Methods Eng., Vol. 1, p. 201-203, 1969.
67. Irons, B.M.,
"A Frontal Solution Program for Finite Elements Analysis",
Int. J. Numer. Methods Eng., Vol. 2, No. 1, 1970.
68. Irons, B.M., de Oliveria, E.A. and Zienkiewicz, O.C.
Comments on the paper: Theoretical Foundations of the Finite Element
Methods,
Int. J. Solids Struct. 6, p. 695-697.
69. Isaacs, L.T.
"A Curved Cubic Triangular Finite Element for Potential Flow Problems"
Int. J. Numer. Methods Eng. Vol. 7, p. 337-344, 1973.
70. Jain, P.C. and Rao, K.S.
"Numerical Solution of Unsteady Viscous Incompressible Flow Past
a Circular Cylinder",
Physics of Fluids Supplement II, p. 57-64, 1969.
71. Kawahara, M. and Yoshimura, N.
"Steady and Unsteady Finite Element Analysis of Incompressible
Viscous Fluid",
Int. J. Numer. Methods Eng. Vol. 10, p. 437-456, 1976.
72. Kinney, R.B. and Paolino, M.A.
"Flow Transient Near the Leading Edge of a Semi-Infinite Flat
Plate Moving Through a Viscous Fluid",
Journal of Applied Mechanics, Vol. 42, No. 4, Dec. 1974, p. 919-924.
73. Kitchens, C.W.
"Calculation of Low Reynolds Number Flow Past a Square Protuberance",
J. AIAA, July 1974, p. 1005-1007, Technical Notes.
74. Kuwahara, K. and Imai, I.
"Steady, Viscous Flow within a Circular Boundary",
The Physics of Fluids Supplement II, p. 94-104, 1969.
75. Lamb, H.
Hydrodynamics.
Dover, N.Y., 1945.

76. Langhaar, H.L.
"Steady Flow in the Transition Length of a Straight Tube",
Journal of Applied Mechanics, Vol. 9, Trans. ASME, Vol. 64, 1942,
p. 55-58.
77. Laskaris, T.E.
"Finite Element Analysis of Compressible and Incompressible
Viscous Flow and Heat Transfer Problems"
The Physics of Fluids, Vol. 18, No. 12, Dec. 1975. p. 1639-1648.
78. Lee, C.H.
"Finite Element Method for Transient Linear Viscous Flow Problems"
Proc. Int. Conf. Numerical Methods in Fluid Dynamics, Southampton, 1973.
79. Lee, J.S. and Fung, Y.C.
"Flow in Locally Constricted Tubes at Low Reynolds Number",
Journal of Applied Mechanics, Trans. of the ASME, March 1970, p. 9-16.
80. Leonard, J.W.
"Finite Element Analysis of Perturbed Compressible Flow",
Int. J. Numer. Methods Eng., Vol. 4, No. 1, 1972.
81. Lew, H.S. and Fung, Y.C.
"On the Low Reynolds Number Entry Flow Into a Circular Cylindrical
Tube",
Journal of Biomechanics, Vol. 2, 1969, p. 105-119.
82. Lew, H.S. and Fung, Y.C.
"Arbitrary Reynolds Numbers",
Journal of Biomechanics, Vol. 3, 1970, p. 23-38.
83. Lin, C.L., Pepper, D.W. and Lee, S.C.
"Numerical Methods for Separated Flow Solutions around a Circular
Cylinder",
J. AIAA, Vol. 14, No. 7, 1976, p. 900-907.
84. Loitsyanskii, L.G.
Mechanics of Liquids and Gases,
Pergamon Press, 1966.
85. Lundgren, T.S., Sparrow, E.M. and Starr, J.B.
"Pressure Drop due to the Entrance Regions in Ducts of Arbitrary
Cross Sections",
Journal of Basic Engineering, Trans. ASME, Series D, Vol. 86, 1964, p. 620-
626.
86. Lyness, J.F., Owen, D.R.J. and Zienkiewicz, O.C.
"Finite Element Analysis of the Steady Flow of Non-Newtonian Fluids
through Parallel Sided Conducts",
Int. Symp. on Finite Element Methods in Flow Problems, Swansea, 1974.
87. Macagno, E.O. and Hung, T.K.
"Computational and Experimental Study of a Captive Annular Eddy",
Journal of Fluid Mechanics, Vol. 28, No. 1, Jan. 1967, p. 43-64.
88. Martin, H.C.
"Finite Element Analysis of Fluid Flows",
Proceedings of Second Conference on Matrix Methods in Structural
Mechanics, AFFDL-TR68-150, Wright-Patterson Air Force Base, Dayton,
Ohio, October 1968.

89. Martin, H.C. and Carey, G.F.
Introduction to Finite Element Analysis,
McGraw-Hill, 1973.
90. Meissner, V.
"A Mixed Finite Element Model for Use in Potential Flow Problems",
Int. J. Numer. Methods Eng., Vol. 6, No. 4, 1973.
91. Mills, R.D.
"Numerical Solutions of Viscous Flow through a Pipe Orifice at
Low Reynolds Numbers",
Journal of Mechanical Engineering Sciences, Vol. 10, 1968, p. 133-140.
92. Moretti, G.
"Importance of Boundary Conditions in the Numerical Treatment of
Hyperbolic Equations,"
Physics of Fluids, Vol. 12, No. 12, Dec. 1969, p. 11.13-20.
93. Norrie, D.H. and de Vries, G.
"Application of the Pseudo-Functional Finite Element Method to
Non-Linear Problems,"
Proc. Int. Symp. on Finite Element Methods in Flow Problems, Swansea,
1974.
94. Norrie, D.H. and de Vries, G.
The Finite Element Method, Fundamentals and Applications,
Academic Press, 1973.
95. Oden, J.T.
"A General Theory of Finite Elements",
Int. J. Numer. Methods Eng. Vol. 1, p. 247-259, 1969.
96. Oden, J.T.
"A Finite Element Analog of Navier-Stokes Equations",
Proc. ASCE, 96, EM4, p. 529-534, 1970.
97. Oden, J.T.
Finite Elements of Nonlinear Continua,
McGraw-Hill, N.Y., 1972.
98. Oden, J.T. and Somogyi, D.
"Finite Element Applications in Fluid Dynamics",
J. Eng. Mech. Div. Proc. ASCE, 95, EM4, p. 821-826, 1969.
99. Oliveira, E.R.A.
"Convergence of Finite Element Solutions in Viscous Flow Problems"
Int. J. Numer. Methods Eng., Vol. 9, p. 739-763, 1975.
100. Olson, M.D.
"Variational-Finite Element Methods for Two-Dimensional and
Axisymmetric Navier-Stokes Equations",
Proc. Int. Symp. on Finite Element Methods in Flow Problems,
Swansea, 1974.
101. Palit, K. and Fenner, R.T.
"Finite Element Analysis of Slow Non-Newtonian Channel Flow",
A. I. Ch. E. J., 18, p. 628-633, 1972.



102. Payne, R.B.
"Calculations of Unsteady Viscous Flow Past a Circular Cylinder",
Journal of Fluid Mechanics, Vol.4, Part 1, May 1958, p.81-86.
103. Pearson, C.E.
"A Computational Method for Viscous Flow Problems",
Journal of Fluid Mechanics, p.611, Vol.21, 1965.
104. Pian, T.H.H. and Tong, P.
"Basis of Finite-Element Methods for Solid Continua",
Int.J.Numer.Methods Eng., Vol.1 1969, p.3-28.
105. Rimon, Y.
"Numerical Solution of the Incompressible Time-Dependent Viscous
Flow past a Thin Oblate Spheroid",
The Physics of Fluids Supplement II, 1969, p.65-75.
106. Roache, P.J. and Mueller, T.J.
"Numerical Solutions of Laminar Separated Flows",
J.AIAA, Vol.8, No.3, March 1970, p.530-538.
107. Rouse, H.
Advanced Mechanics of Fluids,
John Wiley & Sons, Inc., 1959.
108. Roscoe, D.F.
"The Solution of the Three-Dimensional Navier-Stokes Equations
Using a New Finite Difference Approach",
Int.J.Numer.Methods Eng., Vol.10, p.1299-1308, 1976.
109. Schmall, R.A. and Kinney, R.B.
"Numerical Study of Unsteady Viscous Flow Past a Lifting Plate",
J.AIAA, Vol.12, Nov.1974, p.1573-1576.
110. Smith, S.L. and Brebbia, C.A.
"Finite-Element Solution of Navier-Stokes Equations for Transient
Two-Dimensional Incompressible Flow",
Journal of Computational Physics, Vol.17, p.235-245, 1975.
111. Sparrow, E.M., Hixon, C.W. and Shavit, G.
"Experiments on Laminar Flow Development in Rectangular Ducts",
Journal of Basic Engineering, Trans.ASME, Series D, Vol.89, No.1,
March 1967, p.116-124.
112. Sparrow, E.M., Lin, S.H. and Lundgren, T.S.
"Flow Development in the Hydrodynamic Entrance Region of Tubes
and Ducts",
Physics of Fluids, vol.7.1964, p.338-347.
113. Strang, G. and Fix, G.J.
"An Analysis of the Finite Element Method",
Prentice-Hall, Inc., Englewood Cliffs, N.J., 1973.
114. Takaisi, Y.
"Numerical Studies of a Viscous Liquid past a Circular Cylinder",
The Physics of Fluids, Supplement V, Vol.12, No.12, Dec.1969, p.86.
115. Takami, H. and Keller, H.B.
"Steady Two-Dimensional Viscous Flow of an Incompressible Fluid
past a Circular Cylinder",
The Physics of Fluids, Supplement II, Vol.12, No.12, Dec.1969, p.51-56

116. Tay,A.O. and de Vahl Davis,G.
"Application of Finite Element Method to Convection Heat Transfer Between Parallel Plates",
Int.J.Heat Mass Transfer,Vol.14,No.8,1971.
117. Taylor,C. and Hood,P.
"A Numerical Solution of the Navier-Stokes Equations Using the Finite Element Technique",
Computers and Fluids,Vol.1,p.73-100,1973.
118. Thompson,E.C. and Hague,M.I.
"A High-Order Finite Element for Completely Incompressible Creeping Flow",
Int.J.Numer.Methods Eng.,Vol.6,No.3,1973.
119. Thompson,J.F.,Shanks,S.P., and Wu,J.C.
"Numerical Solution of Three-Dimensional Navier-Stokes Equations Showing Trailing Tip Vortices",
J.AIAA,Vol.12,June 1974,p.787-794.
120. Thom,A.
"The Flow Past Circular Cylinders at Low Speeds",
Proc.Royal Society,London,Series A,Vol.126,1933,p.651-669.
121. Thom,A and Apelt,C.F.
Field Computations in Engineering and Physics
Van Nostrand,Princeton,N.J.,1961.
122. Thoman,D.C. and Szewczyk,A.A.
"Time Dependent Viscous Flow over a Circular Cylinder",
The Physics of Fluids,Supplement V,Vol.12,No.12,p.76-85,1969.
123. Tong,P.
"The Finite Element Method for Fluid Flow"
Recent Advances in Matrix Method of Structural Analysis and Design,R.Gallagher et al (Eds.) 1971.
124. Tong,P. and Fung,Y.C.
"Slow Particulate Viscous Flow in Channels and Tubes - Applications to Biomechanics"
Trans.ASME,Series E,Journal of Applied Mechanics,Vol.38,Dec.1971.
125. Vallentine,H.R.
Applied Hydrodynamics,
Butterworths,London,1959.
126. Wiginton,C.L. and Dalton,C.
"Incompressible Laminar Flow in the Entrance Region of a Rectangular Duct",
Journal of Applied Mechanics,Vol.37,Trans.ASME,Vol.92,1970,p.854-856.
127. Wilson,J.M.
Lectures on Finite Element Methods,Dept.of Engineering Science,
University of Durham,England,1976.
128. Wu,J.C.
"Numerical Boundary Conditions for Viscous Flow Problems",
J.AIAA,Vol.14,No.8,p.1042-1049,1976.

129. Yamada, Y. Yokouchi, K. Ito, Y. and Ohtsubo, T.
"Finite Element Analysis of Steady Fluid and Metal Flow",
Proc. Int. Symp. on Finite Element Methods in Flow Problems,
Swansea, 1974.
130. Zlamal, M.
"The Finite Element Method in Domains with Curved Boundaries",
Int. J. Numer. Methods Eng. Vol. 5, p. 367-373, 1973.
131. Ziekiewicz, O. C.
"The Finite Element Method in Engineering Science",
McGraw-Hill, London, 1971.
132. Ziekiewicz, O. C. and Cheung, Y. K.
"Finite Elements in the Solution of Field Problems",
The Engineer, 220, p. 507-510, 1965.
133. Ziekiewicz, O. C. and Godbole, P. N.
"Flow of Plastic and Viscoplastic Solids with Special Reference
to Extrusion and Forming Processes",
Int. J. Numer. Methods Eng., Vol. 8, p. 3-16, 1974.
134. Hood, P.
"Frontal Solution Program for Unsymmetric Matrices",
Int. J. Numer. Methods Eng. Vol. 10, p. 379-400, 1976.
135. Brebbia, C. A.
"Fluid Structure Interaction Problems", in Vibrations of
Engineering Structures, Computational Mechanics Ltd.,
Southampton, U.K., 1975.
136. Schlichting, H.
Boundary Layer Theory, 6th ed.,
McGraw-Hill New York, 1968.

$$\begin{bmatrix}
 2(c_1 - c_2 - c_3) & -(c_1 + c_2 + c_3) & -(c_1 + c_2 + c_3) & 4(2c_3 + c_1) & 4(c_3 + c_2) \\
 -(c_1 + c_2 + c_3) & 2(c_2 - c_3 - c_1) & -(c_1 + c_2 + c_3) & 4(c_2 + 2c_1) & 4(2c_3 + c_2) \\
 -(c_1 + c_2 + c_3) & -(c_1 + c_2 + c_3) & 2(c_3 - c_1 - c_2) & 4(c_2 + c_1) & 4(c_3 + 2c_2)
 \end{bmatrix}$$

$$[Q_u]^{(e)} = \frac{1}{24}$$

$$\begin{bmatrix}
 2(b_1 - b_2 - b_3) & -(b_1 + b_2 + b_3) & -(b_1 + b_2 + b_3) & 4(2b_3 + b_1) & 4(b_3 + b_2) \\
 -(b_1 + b_2 + b_3) & 2(b_2 - b_3 - b_1) & -(b_1 + b_2 + b_3) & 4(b_2 + 2b_1) & 4(2b_3 + b_2) \\
 -(b_1 + b_2 + b_3) & -(b_1 + b_2 + b_3) & 2(b_3 - b_1 - b_2) & 4(b_2 + b_1) & 4(b_3 + 2b_2)
 \end{bmatrix}$$

$$[Q_v]^{(e)} = \frac{1}{24}$$

APPENDIX A

FINITE-ELEMENT ANALYSIS OF INCOMPRESSIBLE, UNSTEADY,

VISCIOUS FLOW

CC

SUBROUTINE MAINPFI

- 1 XC,M1,YC,M2,NCUR,IAL,M15,M17,V15,M4,M25,M5,M75,M8,M25,M7,V75,M75
- 2 N45,M5,V45,M10,M55,M11,V71,M11,M55,M17,M75,M17,M75,M17,M75,M17,M75
- 3 YECU,IAB,IAE,IAE,IAE,IAE,IAE,IAE,IAE,IAE,IAE,IAE,IAE,IAE,IAE,IAE,IAE
- 4 M25,M25,M25,M25,M25,M25,M25,M25,M25,M25,M25,M25,M25,M25,M25,M25
- 5 V71,M17,V71,M17,M17,M17,M17,M17,M17,M17,M17,M17,M17,M17,M17,M17
- 6 ARU,IDS,ANU,IDS,ANU,IDS,ANU,IDS,ANU,IDS,ANU,IDS,ANU,IDS,ANU,IDS
- 7 XECL,IA2,ASSTA,IB3,KK6,M17,V75,M14,ASSTA,IB3,KK6,M17,V75,M14
- 8 NEI,M11,NE2,M12,NE3,M13,NE4,M14,NE5,M15,NE6,M16,NE7,M17,NE8
- 9 ASSTN1,M25,V71,IA10,DUUJ,M30,VVV,M31,ASSTN1,M25,V71,IA10,DUUJ
- 1 RP,M35,NPM,M40,MU,M41,NUM,M42,NV,M43,M44,M45,M46,M47,M48,M49
- 2 NUA,M46,ASUA,M47,NMUA,M48,MUR,M49,NCUR,M50,NMUR,M51,M52,M53
- 3 AEKPT,IC10,AEKUT,IC11,VPS,M54,PI5,M55,PS,M50

COMMON M,NL,L,IL

INTERF PTC,FK,RTCL,TFST,TESI

DATA STOP/'STOP'/
 DATA SLF1/'SLF1'/
 DATA SLF2/'SLF2'/
 DATA VTF1/'VTF1'/
 DATA BK/' BK'/
 DATA RTCL/'RTCL'/
 DATA TESI/'TESI'/

REAL*4 NCNC

DIMENSION XC(M),YC(M),
 DIMENSION NCUR(NL,IL),
 DIMENSION NLS(M),V15(M),N25(M),V25(M),
 DIMENSION NE5(M),V45(M),N45(M),V45(M),
 DIMENSION N55(M),V71(M),N75(M),V75(M),
 DIMENSION NE7(M),NE8(M),
 DIMENSION XCL(3),YCL(3),
 DIMENSION XECL(4),YECU(NL,IL),
 DIMENSION AA(3),PB(3),CC(3),
 DIMENSION AA(NL,IL),PB(NL,IL),CC(NL,IL),
 DIMENSION DE1E(NL)

COMPCN M,NL,L,IL
 REAL*4 NCNC
 EQUIVALENCE (P1,M2,M3,M4,M5,M6,M7,M8,M9,M10,M11,
 M12,M13,M14,M15,M16,M17,M18,M19,M20,M21,M22,M23,
 M24,M25,M26,M27,M28,M29,M30,M31,M32,M33,M34,M35,M36,M37,M38,
 M39,M40,M41,M42,M43,M44,M45,M46,M47,M48,M49,M50,M51,M52,M53,
 M54,M55,M56,M)

EQUIVALENCE (IA1,IA2,IA3,IA4,IA5,IA6,IA7,IA8,IA9,IA10,IA)
 EQUIVALENCE (IB1,IB2,IB3,IB)
 EQUIVALENCE (IC1,IC2,IC3,IC)
 EQUIVALENCE (ID1,IC2,IC2,IC4,IC5,IC6,IC7,IC8,IC)
 EQUIVALENCE (IL,NL1,NL2,NL3,NL4)

RFZ(1:5002) M,NL,L,IL
 WRITE(6,5002) M,NL,L,IL
 5002 FORPAT(415)

IA=NL+IL
 IB=L*4
 IC=M*N
 IC=N1+IL+IL
 IF=NL*6
 M1=M
 IA1=IA
 IB1=IB
 IC1=IC
 ID1=D
 IL1=IL

CALL DYNFIC

- 1 XC,M1,YC,M2,NCUR,IAL,M15,M17,V15,M4,M25,M5,M75,M8,M25,M7,V75,M75
- 2 N45,M5,V45,M10,M55,M11,V71,M11,M55,M17,M75,M17,M75,M17,M75
- 3 YECU,IAB,IAE,IAE,IAE,IAE,IAE,IAE,IAE,IAE,IAE,IAE,IAE,IAE,IAE,IAE
- 4 M25,M25,M25,M25,M25,M25,M25,M25,M25,M25,M25,M25,M25,M25,M25
- 5 V71,M17,V71,M17,M17,M17,M17,M17,M17,M17,M17,M17,M17,M17,M17
- 6 ARU,IDS,ANU,IDS,ANU,IDS,ANU,IDS,ANU,IDS,ANU,IDS,ANU,IDS,ANU,IDS
- 7 XECL,IA2,ASSTA,IB3,KK6,M17,V75,M14,ASSTA,IB3,KK6,M17,V75,M14
- 8 NEI,M11,NE2,M12,NE3,M13,NE4,M14,NE5,M15,NE6,M16,NE7,M17,NE8
- 9 ASSTN1,M25,V71,IA10,DUUJ,M30,VVV,M31,ASSTN1,M25,V71,IA10,DUUJ
- 1 RP,M35,NPM,M40,MU,M41,NUM,M42,NV,M43,M44,M45,M46,M47,M48,M49
- 2 NUA,M46,ASUA,M47,NMUA,M48,MUR,M49,NCUR,M50,NMUR,M51,M52,M53
- 3 AEKPT,IC10,AEKUT,IC11,VPS,M54,PI5,M55,PS,M50

STOP
END

APPENDIX B

```

DIMENSION ASST(1,1),ASSTAT(1,1),ASST(1,1)
DIMENSION NP(M),PFI(M),APM(M),NI(M)
DIMENSION NUP(M),NV(M),NVA(M)
DIMENSION EKVT(1,3),
1
2
3
4
5
6
7
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11
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N5S(I)=N51
V5(N5S(I))=V1
IF (WORD.EQ.SLFP1) GO TO 1222
V1=5*(YC(N5S(I))-YC(N5S(I)+2)/3.0)
V5(N5S(I))=V1
1222 IF (WORD.EQ.SLFP2) GO TO 12
V1=YC(N5S(I))
V5(N5S(I))=V1
12 CONTINUE
I=NN+1
15 N5T=I-1
C
C READ BOUNDARY CONDITION 2 (N22=ACCE NUMBER,
C V2=THE DERIVATIVE OF THE STREAM FUNCTION WITH RESPECT TO X)
C
DO 17 I=1,NN
RTAF(5,18) WCRD,N22,V2
IF (WORD.EQ.STOP) GO TO 513
N2S(I)=N22
V2S(N2S(I))=V2
17 CONTINUE
I=NN+1
513 N2T=I-1
C
C READ BOUNDARY CONDITION 3 (N23=ACCE NUMBER,
C V3=THE DERIVATIVE OF THE STREAM FUNCTION WITH RESPECT TO Y)
C
DO 20 I=1,NN
RTAF(5,21) WCRD,N23,V3
IF (WORD.EQ.STOP) GO TO 521
N3S(I)=N23
V3S(N3S(I))=V3
20 CONTINUE
I=NN+1
521 N3T=I-1
C
C READ BOUNDARY CONDITION 4
C
DO 23 I=1,NN
RTAF(5,24) WCRD,N44,V4
IF (WORD.EQ.STOP) GO TO 524
N4S(I)=N44
V4S(N4S(I))=V4
IF (WORD.EQ.SLFP1) GO TO 25
V4=1.5*(1.0-YC(N4S(I))*2)
V4S(N4S(I))=V4
23 CONTINUE
I=NN+1
524 N4T=I-1
C
C READ BOUNDARY CONDITION 5
C
DO 22 I=1,NN
RTAF(5,29) WCRD,N55,V5
IF (WORD.EQ.STOP) GO TO 30
N5S(I)=N55
V5S(I)=N5S(I)

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I=NN+1
30 N5T=I-1
C
C READ INITIAL CONDITIONS
C
DO 32 I=1,NN
RTAF(5,33) WCRD,N66,V6
IF (WORD.EQ.STOP) GO TO 34
N6S(I)=N66
V6=3.0*YC(N6S(I))
VT(I,CCUR,N6S(I))=V6
32 CONTINUE
I=NN+1
34 N6T=I-1
C
C READ BOUNDARY CONDITION 7
C
DO 30 I=1,NN
RTAF(5,40) WCRD,N77,V7
IF (WORD.EQ.STOP) GO TO 41
N7S(I)=N77
V7S(N7S(I))=V7
30 CONTINUE
I=NN+1
41 N7T=I-1
40 FORMAT(6X,A4,I10,F15.7)
C
DO 45 I=1,NN
RTAF(5,46) I,NF8
N8E=1
N8C(I)=NF8
45 CONTINUE
C
DO 48 I=1,NN
PFAT(5,53) WCRD,N59,V9
IF (WORD.EQ.STOP) GO TO 47
DO 48 J=1,L
ICCLAT=J
N9S(I)=N95
I1=N9T+I
I1=1
N9S(I1)=N99
VT(I,CCUR,N9S(I))=V9
VKE[P(N9S(I))]=VT(I,CCUR,N9S(I))
IF (WORD.EQ.V1F1) GO TO 48
VT(I,CCUR,N9S(I))=3.0*YC(N9S(I))
VKE[P(N9S(I))]=VT(I,CCUR,N9S(I))
46 CONTINUE
I=NN+1
47 N9T=I-1
N9S(N9T+N95)
N9S(N9T)
JJ=C
DO 7234 IJ=1,NN
IJ=IJ
I=1
DO 7235 I=1,N9ST
IF (I1.EQ.N9S(I)) GO TO 7234

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7234 CCNTINUE
      PSMT=JJ
      WRITE(8,7237) NN,KSST,ASMT
7237 FORWRT( , ,316)
C
      ICCUNT=1
      DO 7240 I=1,NA
      READ(5,33) WCPC,MPP,VPR
      IF (WCPC.EC.STCP) GO TO 7241
      NP(I)=MPP
      PRINT(1)=VPR
7240 CCNTINUE
7241 NPRT=I-1
      JJ=C
      DO 7246 IJ=1,NA
      II=IJ
      I=1
      DO 7247 I=1,NPPT
      IF (II.EC.NP(I)) GO TO 7246
7247 CCNTINUE
      JJ=JJ+1
      NFN(JJ)=II
7246 CCNTINUE
      NPMT=JJ
      WRITE(8,7237) NN,APPT,NPMT
C
      DO 7242 I=1,NA
      READ(5,33) WCPC,NUU,VU
      IF (WCPC.EC.STCP) GO TO 7243
      NU(I)=NUU
      NSUF(I)=NUU
      VU(I,NU(I))=VU
      KSLA(I)=NUU
7242 CCNTINUE
7243 NUUT=I-1
      JJ=C
      DO 7547 IJ=1,NA
      II=IJ
      I=1
      DO 7248 I=1,NUUT
      IF (II.EC.NU(I)) GO TO 7547
7248 CCNTINUE
      JJ=JJ+1
      NFN(JJ)=II
      NMT=JJ
7547 CCNTINUE
      NMT=JJ
      WRITE(8,7237) NN,NUUT,NMT
C
      DO 7244 I=1,NA
      READ(5,33) WCPC,NVV,VV
      IF (WCPC.EC.STCP) GO TO 7245
      NV(I)=NVV
      VV(NV(I))=VV
7244 CCNTINUE
7245 NVVT=I-1
      JJ=C
      DO 7249 IJ=1,NA

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      DO 7250 I=1,NVVT
      IF (II.EC.NV(I)) GO TO 7249
7250 CCNTINUE
      JJ=JJ+1
      NVN(JJ)=II
7249 CCNTINUE
      NVMT=JJ
      WRITE(8,7237) NN,NVMT,NVMT
      DO PC00 I=1,NN
      PFAI(5,33) WCPC,IUA,UAC
      IF (WCPC.EC.STCP) GO TO 8001
      NUA(I)=IUA
8000 CCNTINUE
8001 NSLCT=I-1
      NSUAR=MSUAT+NUIT
      JJ=C
      DO PC02 IJ=1,NA
      II=IJ
      I=1
      DO PC03 I=1,PSUAR
      IF (II.EC.KSLA(I)) GO TO PC02
8003 CCNTINUE
      JJ=JJ+1
      NPUA(JJ)=II
8002 CCNTINUE
      NPUP=JJ
      WRITE(8,7237) NN,PSUAR,NPCLAB
C
      DO SC04 I=1,NA
      READ(5,33) WCPC,IUR,UBC
      IF (WCPC.EC.STEP) GO TO 8005
      SUP(I)=IUR
      I=I+1
      MSUR(II)=IUR
      UUU(NUR(II))=UBC
8004 CCNTINUE
8005 NSLCT=I-1
      NSUER=NSURT+RUOT
      JJ=C
      DO PC06 IJ=1,NA
      II=IJ
      I=1
      DO PC07 I=1,NSURP
      IF (II.EC.NSLP(I)) GO TO 8006
8007 CCNTINUE
      JJ=JJ+1
      NPLF(JJ)=II
      NMURP=JJ
8006 CCNTINUE
      WRITE(8,7237) NN,NSURP,NMURP
C
      READ(5,1051) TIME
      READ(5,1051) THRE1
      READ(5,1055) TUSFP,TEST
      RFF(5,1055) RH,ATC
      FORWRT( , ,F15,9,A4)
      READ(5,1051) EPSL
      READ(5,1051) PCST

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C FORM MATRIX S (EQ. 3-52)
C
  ASST(ICOUNT,1)=(DEL/12.0)*N(2)*VT(ICOUNT,NODE(K,1))
  1 +VT(ICOUNT,NODE(K,2))+VT(ICOUNT,NODE(K,3))
  ASST(ICOUNT,2)=(DEL/12.0)*(VT(ICOUNT,NODE(K,1))
  1 +2.0*VT(ICOUNT,NODE(K,2))+VT(ICOUNT,NODE(K,3)))
  ASST(ICOUNT,3)=(DEL/12.0)*(VT(ICOUNT,NODE(K,1))
  1 +VT(ICOUNT,NODE(K,2))+2.0*VT(ICOUNT,NODE(K,3)))
C
C FORM MATRIX K (EQ. 3-53)
C
  DO 125 I=1,3
  DO 125 J=1,3
  EKVT(I,J)=DEL/12.0
  EKVTE(K,I,J)=EKVT(I,J)
125 CONTINUE
C
C ASSEMBLY SYSTEM EQUATIONS
C
  N(1)=N1
  N(2)=N2
  N(3)=N3
  NEF(K,1)=N1
  NEC(K,2)=N2
  NEE(K,3)=N3
  DO 130 IL=1,2
  I=N(IL)
  ASSTAT(I)=ASSTAT(I)+ASST(ICOUNT,I)
  DO 130 JL=1,3
  J=N(JL)
  EKVT(I,J)=EKVT(I,J)+EKVT(IL,JL)
  EKST(I,J)=EKST(I,J)+EKST(IL,JL)
  AEKPT(I,J)=AEKPT(I,J)+EKPT(IL,JL)
  AKUT(I,J)=AKUT(I,J)+EKUT(IL,JL)
  AKVT(I,J)=AKVT(I,J)+EKVT(IL,JL)
130 CONTINUE
C
1000 CONTINUE
  CALL SCLMIX(PA,AEKUT,PIS,PS,J,YKA,NA,EKA,P,Z,C)
C
C EQ. 3-35 1C BE SOLVED
C
  CALL SCLMIX(MA,AFKST,VIS,ASSTAT,MIT,MIS,IMP,MAS,N,Z,C)
C
2000 CONTINUE
  DO 3471 I=1,NA
  DO 3481 J=1,NK
  3491 AWP(I,J)=J.0
  DO 3477 K=1,NE
  N(1)=NEC(K,1)
  N(2)=NEE(K,2)
  N(3)=NEE(K,3)
  DO 3478 IL=1,3

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  7AE(K,1)=AA(1)
  7AE(K,2)=AA(2)
  7AE(K,3)=AA(3)
C
  EP(1)=YCL(2)-YCL(3)
  EP(2)=YCL(3)-YCL(1)
  EP(3)=YCL(1)-YCL(2)
C
  RBE(K,1)=RE(1)
  RBE(K,2)=RE(2)
  RBE(K,3)=RE(3)
C
  CC(1)=XCL(3)-XCL(2)
  CC(2)=XCL(1)-XCL(3)
  CC(3)=XCL(2)-XCL(1)
C
  CCE(K,1)=CC(1)
  CCE(K,2)=CC(2)
  CCE(K,3)=CC(3)
C
  DCL=ABS(C.5*(XCL(1)*(YCL(2)-YCL(3))+
  1XCL(2)*(YCL(3)-YCL(1))+
  2XCL(3)*(YCL(1)-YCL(2))))
  DELET(K)=DEL
C
C FORM MATRIX K (EQ. 3-68)
C FORM MATRIX K (EQ. 3-62)
C FORM MATRIX K (EQ. 3-78)
C
  DO 101 I=1,3
  DO 102 J=1,3
  EKST(I,J)=C(1)*(RBR(J)+CC(1)*CC(1))/(4.0*DEL)
  EKUT(I,J)=R(1)*RE(J)/(4.0*DEL)
  EKPT(I,J)=EKST(I,J)
  DO 102 II=1,3
  EA=NODE(K,II)
  DO 1024 IKF=1,NM
  KK(I)=ASST(I,K)
  IF (EKST,NE,PK) GO TO 8024
  EKPT(I,J)=RE(1)*RE(J)/(4.0*DEL)
  EKST(I,J)=C(1)*CC(1)/(4.0*DEL)
8024 CONTINUE
8025 CONTINUE
  DO 8027 JJ=1,2
  JKK=NODE(K,JJ)
  DO 8026 KKK=1,NA
  KKPT=JJA(KKK)
  IF (KKPT,AE,JKK) GO TO 8026
  EKPT(I,J)=C(1)*CC(1)/(4.0*DEL)
  EKST(I,J)=R(1)*RE(J)/(4.0*DEL)
8026 CONTINUE
8027 CONTINUE
  EKVT(I,J)=CNEIJ*EKST(I,J)
  EKST(K,I,J)=EKST(I,J)
  EKVTE(K,I,J)=EKVT(I,J)

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215 CONTINUE
C
CALL SCLMIX(MN,SM1,VTN,SM3,ASST,NS5,NSMT,NSV,5,7,C)
C
C ARE THE VALUES OF VOPTICITIES SATISFACTORY ?
C
DO 5220 I=1,NN
DIFF(I)=4*PS(VTN(I))-VT(I*COUNT,I)
DIFF(I)=ABS(VTN(I))-VT(I*COUNT,I)
IF (DIFF(I).GT.EPSL) GO TO 222
5220 CONTINUE
DO 5225 I=1,NN
5225 VT(I*COUNT,I)=VTM(I)
GC TO 270
222 CONTINUE
C
ICOUNT=ICOUNT+1
8025 FORMAT(' N COUNT =',I3)
C
DO 226 I=1,NN
226 ASSTAT(I)=0.0
C
DO 225 K=1,NE
ASST(I*COUNT,I)=(DELE(K)/12.0)*(2.0-VTM(INODE(K,I))
+VTM(INODE(K,2)))+VTM(INODE(K,3))
ASST(I*COUNT,2)=DELE(K)/12.0*(VTM(INODE(K,1))
+2.0*VTM(INODE(K,2))+VTM(INODE(K,3)))
ASST(I*COUNT,3)=(DELE(K)/12.0)*(VTM(INODE(K,1))
+VTM(INODE(K,2))+2.0*VTM(INODE(K,3)))
DC 226 IL=1,3
I=NEE(K,IL)
ASSTAT(I)=ASSTAT(I)+ASST(I*COUNT,I)
236 CONTINUE
235 CONTINUE
IF (ICOUNT.GT.100) GC TO 265
C
CALL SCLMIX(MN,AFKST,VIS,ASSTAT,NIT,MIS,HOT,NS5,B,Z,C)
C
DC 262 I=1,NN
262 VT(I*COUNT,I)=VTM(I)
C
DC 267 I=1,NE
267 VT(I*COUNT,MIS(I))=VKFEP(NS5(I))
WRITE(8,266) ICOUNT
NIC=ICOUNT+2
GC TO 270
265 WRITE(8,266) ICOUNT
ICOUNT=ICOUNT-1
270 CONTINUE
DC FOR1 I=1,NN
8031 VPS(I)=VIS(I)
C
C CALCULATE THE PRESSURE DISTRIBUTION
C
DC 300 K=1,NE
DELESO=DELE(K)*2
PREKI=PRE(K,1)

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187 CONTINUE
196 CONTINUE
195 CONTINUE
C
IF (M2T.EQ.0) GO TO 9111
IF (M4T.EQ.0) GC TO 9111
DC 9112 I=1,M2T
DC 9114 J=1,M4T
IF (M2S(I).NE.M4S(J)) GO TO 9114
DO 9112 J1=1,NE
DC 9115 I1=1,NE
I=IODE(J1,I1)
I=NEE(K,I1)
IF (M2S(I).NE.KK) GO TO 9115
ESVT(J1,I1)=(1.0/6.0)*V4S(M4S(J1))*
1 (PRE(J1,1)+VT(I*COUNT,MODE(J1,1)))+
2 (PRE(J1,2)+VT(I*COUNT,MODE(J1,2)))+
3 (PRE(J1,3)+VT(I*COUNT,MODE(J1,3)))
9115 CONTINUE
9113 CONTINUE
9114 CONTINUE
9112 CONTINUE
9111 CONTINUE
C
DO 188 I=1,NN
188 AFS(I)=0.0
DC 190 I=1,NE
DC 191 IL=1,3
I=NEE(K,IL)
AESVT(I)=AESVT(I)+ESVT(K,IL)
191 CONTINUE
190 CONTINUE
C
CURTI=TIME*ICOUNT
WRITE(8,1053) CURTI
1053 FORMAT(' ***** AT TIME =',F10.2,EX,' SFCORD',
2////, ' *****')
C
DC 195 I=1,NN
DO 195 J=1,NN
SM2(I,J)=SERVT(I,J)+SERVT(I,J)/TIME
195 CONTINUE
C
DC 200 I=1,NN
KKK(I)=I
VTM(I)=VT(I*COUNT,I)
200 CONTINUE
C
CALL SOLMIX(MN,AFNVTT,VTM1,SM2,MN,KK3,C,KVN,R,Z,C)
C
DC 205 I=1,NN
SM2(I)=SM2(I)/TIME-AESVT(I)
205 CONTINUE
C
C EO. 3-43 IS READY TO BE SOLVED

```

```

C CCEK1=CCE(K,1)
C CCEK2=CCE(K,2)
C CCEK3=CCE(K,3)

C
C CCEK1=VIS(MODE(K,1))
C CCEK2=VIS(MODE(K,2))
C CCEK3=VIS(MODE(K,3))

C
C VISXX(1)=(PBEK1/12.0*(DELESC1))*
1 (PBEK1-PBEK2-PBEK3)*V1SK1
2 +(PBEK2/12.0*(DELESC1))*
3 (PBEK2-PBEK1-PBEK3)*V1SK2
4 +(PBEK3/12.0*(DELESC1))*
5 (PBEK3-PBEK1-PBEK2)*V1SK3
6 +(2.0*(PBEK1-PBEK2)/DELESC1)*(V1SK1+
7 V1SK3)/2.0+(2.0*(PBEK1-PBEK3)/DELESC1)*
8 (V1SK1+V1SK2)/2.0
9 +(2.0*(PBEK2-PBEK3)/DELESC1)*
10 (V1SK2+V1SK3)/2.0

C
C VISYY(1)=(CCEK1/12.0*(DELESC1))*
1 (CCEK1-CCEK2-CCEK3)*V1SK1
2 +(CCEK2/12.0*(DELESC1))*
3 (CCEK2-CCEK3-CCEK1)*V1SK2
4 +(CCEK3/12.0*(DELESC1))*
5 (CCEK3-CCEK1-CCEK2)*V1SK3
6 +(2.0*(CCEK1-CCEK2)/DELESC1)*
7 (V1SK1+V1SK3)/2.0
8 +(2.0*(CCEK1-CCEK3)/DELESC1)*
9 (V1SK1+V1SK2)/2.0
10 +(2.0*(CCEK2-CCEK3)/DELESC1)*
11 (V1SK2+V1SK3)/2.0

C
C VISXY(1)=(2.0*(CCEK1-PBEK1-CCEK1+PBEK3-
1 CCEK3+PBEK1-CCEK1+PBEK2-
2 CCEK2+PBEK1)/4.0*(DELESC1))*V1SK1
3 +(2.0*(CCEK2-PBEK2-CCEK2+PBEK3-
4 CCEK3+PBEK2-CCEK1+PBEK2-
5 CCEK2+PBEK1)/4.0*(DELESC1))*V1SK2
6 +(2.0*(CCEK3-PBEK3-CCEK3+PBEK1-
7 CCEK1+PBEK3-CCEK3+PBEK2-
8 CCEK2+PBEK3)/4.0*(DELESC1))*V1SK3
9 +(CCEK1-PBEK2+CCEK2+PBEK1)/
10 (DELESC1)*(V1SK1+V1SK3)/2.0
11 +(CCEK1+PBEK2+CCEK2+PBEK1)/
12 (DELESC1)*(V1SK1
13 +V1SK2)/2.0+(CCEK2+PBEK3+CCEK3
14 +PBEK2)/(DELESC1)*(V1SK3+V1SK2)/2.0

C
C CCG(1)=2.0*(VISXX(1)+VISYY(1)-VISXY(1))*2)
C ESP=DELE(CCE(K,1)/3.0)

C
C DC 220 IL=1,2
C I=NEEK,IL)
C AESPP(1)=AESPP(1)+ESP
320 CCNTINUE
300 CCNTINUE

C
C CCEK1=CCE(K,1)
C CCEK2=CCE(K,2)
C CCEK3=CCE(K,3)

C
C CALCULATE FLOW VELOCITIES

C
DC 8064 I=1,NN
IF (-1*(YC(I+1)-YC(I))*LT.0.0) GC TC 8065
8064 CCNTINUE
8065 IK=I
IANN=NN-1
DC 8066 I=1,NN
IF (-1*(YC(I+1)-YC(I))*LT.0.0) GC TC 8066
IF (-1*(YC(I+1)-YC(I))*EQ.0.0) GC TC 8066
U(I)=(VIS(I+1)-VIS(I))/(YC(I+1)-YC(I))
WRITE(7,8066) I,U(I)
8066 FORMAT(' I =',I3,10X,'VFL(U) =',F10.3)
8066 CCNTINUE
IANN=NN-1
DC 8062 I=1,NN
IF (1*(XC(IK+1)-XC(I))*GT.0.0) GC TC 8062
V2(I)=(VIS(IK+1)-VIS(I))/(1-(XC(IK+1)-XC(I)))
WRITE(7,8067) I,V2(I)
8067 FORMAT(' I =',I3,10X,'VEL(V) =',F10.3)
8062 CCNTINUE
C
DC 235 I=1,NN
DO 235 J=1,NN
ACU(I,J)=0.0
AGV(I,J)=0.0
APUL(I,J)=0.0
335 CCNTINUE
C
DO 233 K=1,NI
DC 234 J=1,3
DO 234 I=1,3
RU(J,I)=DELE(K)/12.0
334 CCNTINUE
C
DC 238 IL=1,2
I=NEEK,IL)
DO 238 JL=1,2
J=NEEK,IL)
APUL(I,J)=APUL(I,J)+PU(IL,JL)
338 CCNTINUE
333 CCNTINUE
C
DC 8030 I=1,NST
VPS(N5S(I))=F15(N5S(I))
8030 CCNTINUE
CALL VELSTS(ERI,CCE,CU,CV,NICE,ACH,ACHV,NE,NCHC,V15,SMS,SM6
1 ,NA,VIS)
CALL SCLMIX(NB,APU,ULU,SM5,PULT,NU,PURT,NUK,P,7,C)
CALL SCLMIX(IN,AFI,VVV,SM6,AVVT,AV,AVNT,RV,F,7,C)
ILCOP=1
8011 CCNTINUE
DO 360 I=1,NN
PRI(I)=FCST-((UUU(I))*2+(VVV(I))*2)*DFI*0.5
360 CCNTINUE
DO 1061 I=1,NN

```



```

SUBROUTINE SCLMIX(N,A,X,Y,NX=LISTX,NY,LISTY,R,Z,C)
DIMENSION A(I,N),X(I),Y(N),LISTX(N),LISTY(N)
DIMENSION R(N,NY),Z(NY),C(NY)
C
IF (N.EQ.(NX*NY)) GO TO 100
WRITE(6,111) N,NX,NY
111 FORMAT(10X,'N = ',I2,'10X','NX = ',I2,'10X','NY = ',I2)
STOP
100 CONTINUE
C
IF (NX.NE.0) GO TO 120
CALL SOLVE(N,A,Y,X)
RETURN
120 CONTINUE
C
IF (NY.NE.0) GO TO 140
DC 130 I=1,N
Y(I)=0.0
DC 120 J=1,N
120 Y(I)=Y(I)+A(I,J)*X(J)
PLOTUPN
140 CONTINUE
C
DO 200 I=1,NY
DC 200 J=1,NY
200 R(I,J)=A(LISTX(I),LISTY(J))
C(I)=Y(LISTY(I))
DO 300 J=1,NX
300 C(I)=C(I)-A(LISTY(I),LISTX(J))*X(LISTX(J))
C
CALL SOLVE(NY,B,C,Z)
DC 400 I=1,NY
400 X(LISTY(I))=Z(I)
C
DC 800 I=1,NX
C(I)=0.0
DC 600 J=1,NY
600 C(I)=C(I)+A(LISTX(I),LISTY(J))*X(LISTY(J))
DO 700 J=1,NY
700 C(I)=C(I)+A(LISTX(I),LISTX(J))*X(LISTX(J))
800 Y(LISTX(I))=C(I)
RETURN
END

```

```

DO 8041 I=1,NX
IF (YC(I).EQ.YC(1)) GO TO 8042
IF (YC(I).EQ.YC(2)) GO TO 8043
IF (YC(I).EQ.YC(3)) GO TO 8044
IF (YC(I).EQ.YC(4)) GO TO 8045
IF (YC(I).EQ.YC(5)) GO TO 8046
IF (YC(I).EQ.YC(6)) GO TO 8047
IF (YC(I).EQ.YC(7)) GO TO 8048
GO TO 8041
8042 SLP(I,J1)=V1S(I)
J1=J1+1
GO TO 8041
8043 SLP(I,J2)=V1S(I)
J2=J2+1
GO TO 8041
8044 SLP(I,J3)=V1S(I)
J3=J3+1
GO TO 8041
8045 SLP(I,J4)=V1S(I)
J4=J4+1
GO TO 8041
8046 SLP(I,J5)=V1S(I)
J5=J5+1
GO TO 8041
8047 SLP(I,J6)=V1S(I)
J6=J6+1
GO TO 8041
8048 SLP(I,J7)=V1S(I)
J7=J7+1
8041 CONTINUE
DC 8050 IJ=1,7
WRITE(11,8052) (SLP(IJ,II),II=1,28)
8052 FORMAT( 2(E15.8,2X) )
8050 CONTINUE
8040 CONTINUE
RETURN
END

```

```

C SUBROUTINE MATMLT(A,I,J,T,N,N,P,NL)
  INTEGER P
  DIMENSION A(3,6),U(6,1),T(3,1)
  DO 1 I=1,N
  DO 1 J=1,P
  DO 2 I=1,M
  DO 2 J=1,P
  DO 2 K=1,N
  T(I,J)=A(I,K)*U(K,J)+T(I,J)
  1 CONTINUE
  RETURN
  END

```

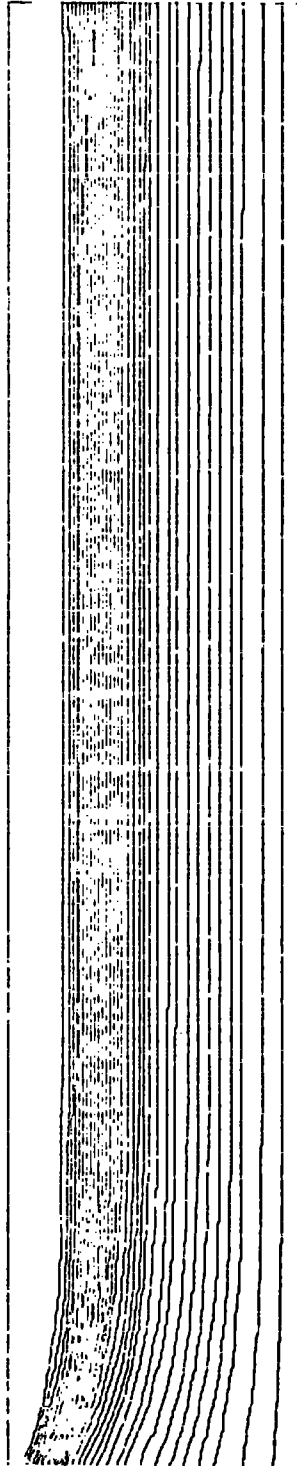
```

C SUBROUTINE SOLVE(N,A,C,X)
  DIMENSION A(I,N),X(N),C(N)
  DO 400 K=1,N
  BIG=ABS(A(K,X))
  IBIG=K
  DO 100 I=K,N
  SIZE=ABS(A(I,K))
  IF (SIZE.LT.BIG) GO TO 100
  IBIG=I
  100 CONTINUE
  IF (K.EQ.IBIG) GO TO 280
  DO 200 J=K,N
  ABIG=A(IBIG,J)
  A(IBIG,J)=A(K,J)
  200 A(K,J)=ABIG
  CBIG=C(IBIG)
  C(IBIG)=C(K)
  C(K)=CBIG
  280 CONTINUE
  IF (A(K,K).EQ.0.0) GO TO 600
  DO 400 I=1,K
  IF (I.EQ.K) GO TO 400
  RATIO=A(I,K)/A(K,K)
  DO 300 J=K,N
  A(I,J)=A(I,J)-RATIO*A(K,J)
  C(I)=C(I)-RATIO*C(K)
  300 CONTINUE
  DO 500 K=1,N
  X(K)=C(K)/A(K,K)
  RETURN
  600 WRITE (6,666)
  666 FORMAT(10X,'SINGULAR MATRIX')
  700 X(I)=0.0
  RETURN
  END

```


1	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
2	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
3	0.9947	0.9949	0.9953	0.9961	0.9977	1.0	1.0	1.0	1.0	1.0	1.0
4	1.0	1.0	1.0	1.0	1.0	0.9977	0.9961	0.9953	0.9949	0.9947	0.9947
5	0.9416	0.9442	0.9486	0.9566	0.9703	0.9867	0.9982	1.0	1.0	1.0	1.0
6	1.0	1.0	1.0	0.9992	0.9868	0.9704	0.9566	0.9486	0.9442	0.9416	0.9416
7	0.8285	0.845	0.855	0.8728	0.903	0.9406	0.9704	0.996	1.0	1.0	1.0
8	1.0	1.0	0.996	0.9706	0.9408	0.9031	0.8729	0.8551	0.845	0.8385	0.8385
9	0.6943	0.7041	0.7185	0.7437	0.7864	0.8421	0.8919	0.9448	0.9902	1.0	1.0
10	1.0	0.9902	0.9451	0.8923	0.8425	0.7867	0.7439	0.7186	0.7042	0.6946	0.6946
11	0.5188	0.5289	0.5441	0.5704	0.6151	0.6750	0.7314	0.7973	0.8667	0.913	0.913
12	0.9082	0.8663	0.7977	0.732	0.6755	0.6155	0.5706	0.5442	0.53	0.5188	0.5188
13	0.296	0.3033	0.3142	0.3327	0.3643	0.4067	0.4462	0.4915	0.5365	0.5674	0.5674
14	0.5696	0.5376	0.4923	0.4467	0.4071	0.3646	0.3329	0.3142	0.3034	0.296	0.296
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

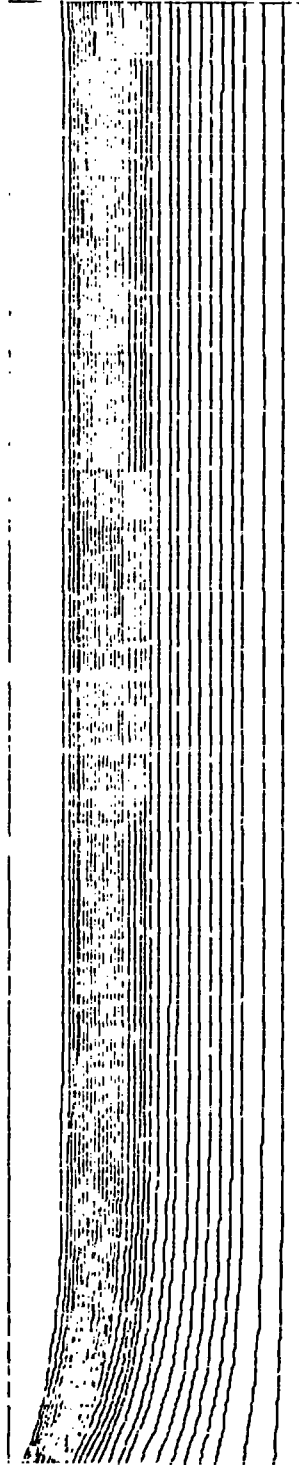
END OF FILE



TURFl = 1.0 Re = 1

FIG. D - 1

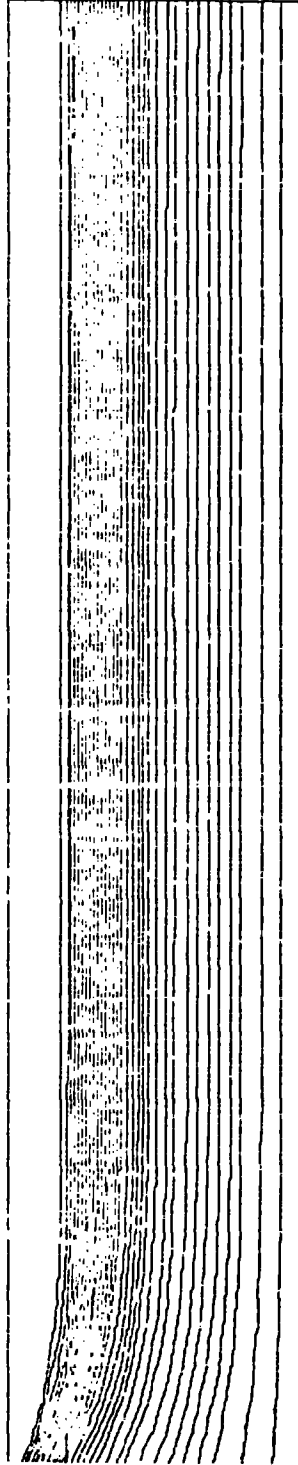
Streamline Contours in Flow between Parallel Plates.



TURFl = 0.1 Re = 1.0

FIG. D - 2

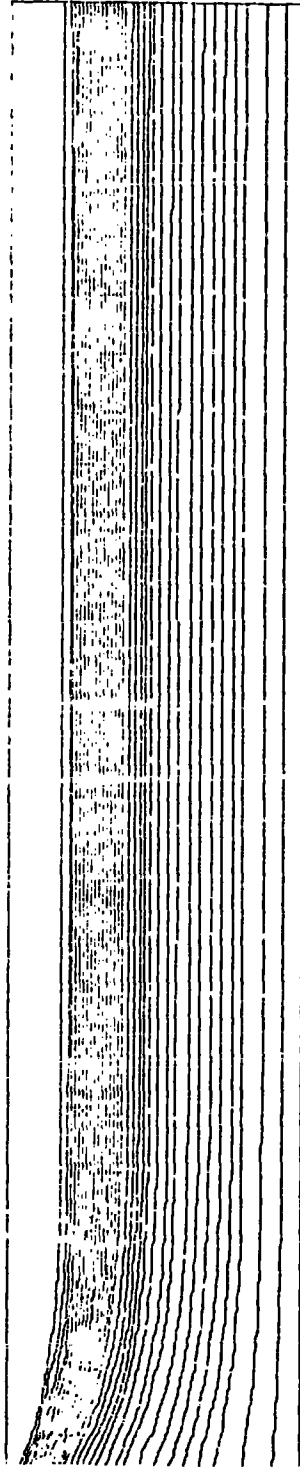
Streamline Contours in Flow between Parallel Plates



TURF1 = 1.0 Re = 2.0

FIG. D - 3

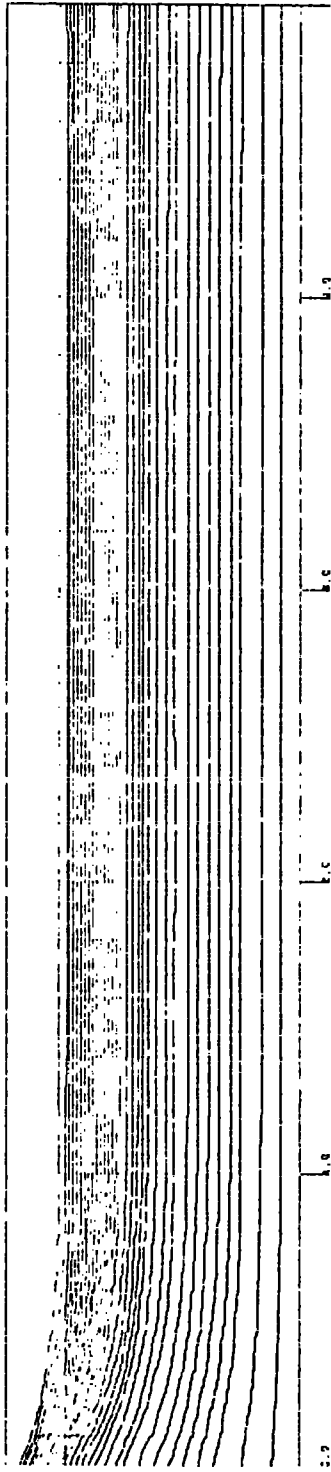
Streamline Contours in Flow between Parallel plates



TURFL = 0.1 Re = 2.0

FIG. D - 4

Streamline Contours in Flow between Parallel Plates

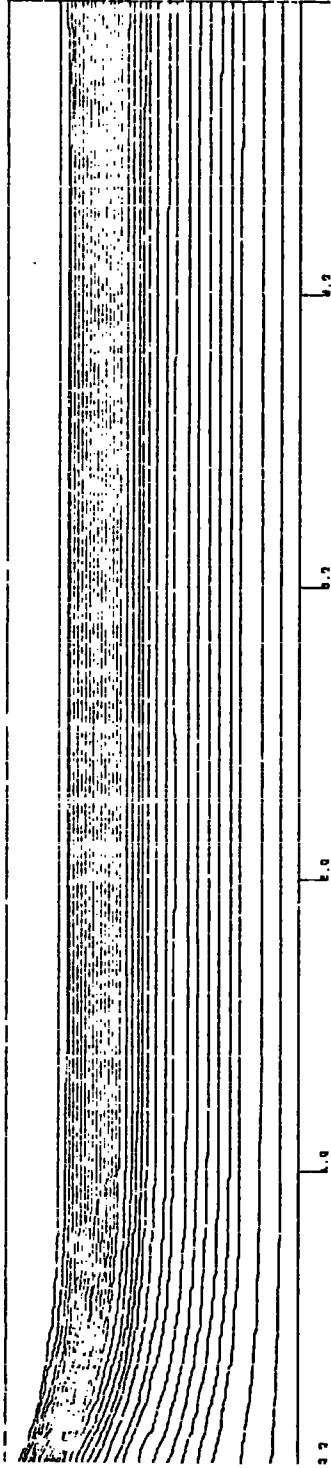


376,145 4,400.

TURFL = 1.0 Re = 3.0

FIG. D - 5

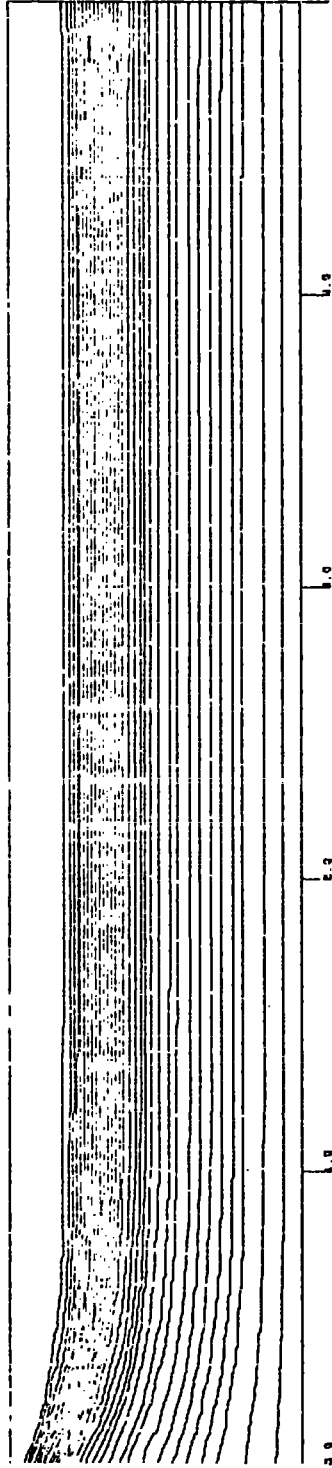
Streamline Contours in Flow between Parallel Plates



$$\frac{\text{TURFL} = 0.1}{\text{Re} = 3.0}$$

FIG. D - C

Streamline Contours in Flow between Parallel Plates

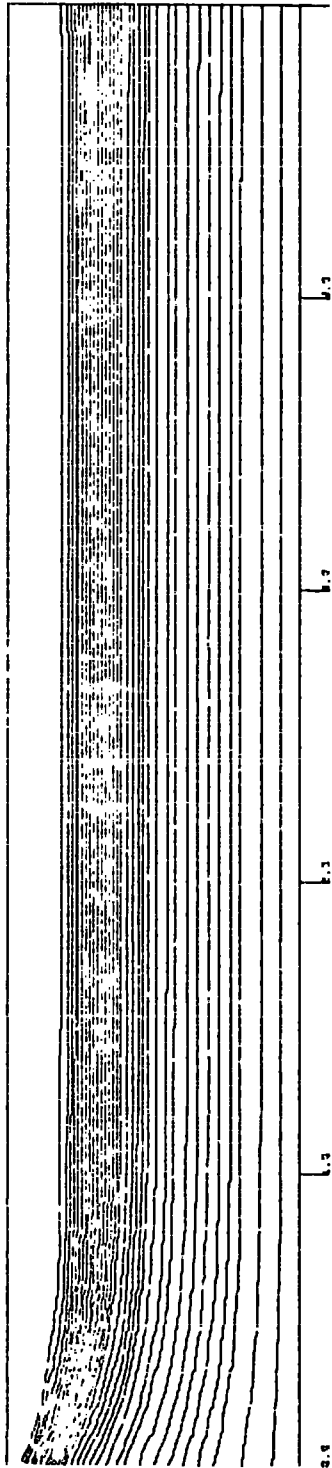


STREAM LINES

$TURFL = 1.0 \quad Re = 4.0$

FIG. D - 7

Streamline Contours in Flow between Parallel Plates

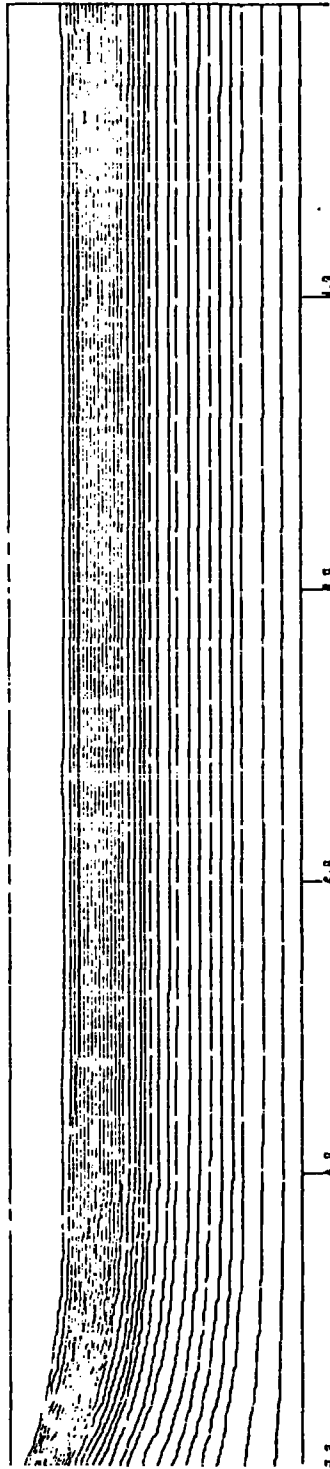


STREAM LINES

TURFL = 0.1 Re = 4.0

FIG. D - 8

Streamline Contours in Flow between Parallel Plates

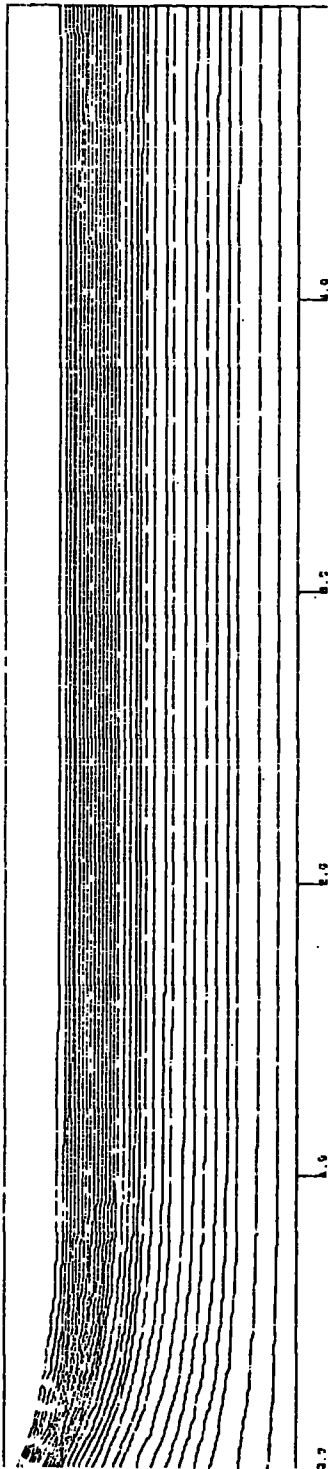


STREAK LINES

TURFL = 1.0 Re = 5

FIG. D - 9

Streamline Contours in Flow between Parallel Plates

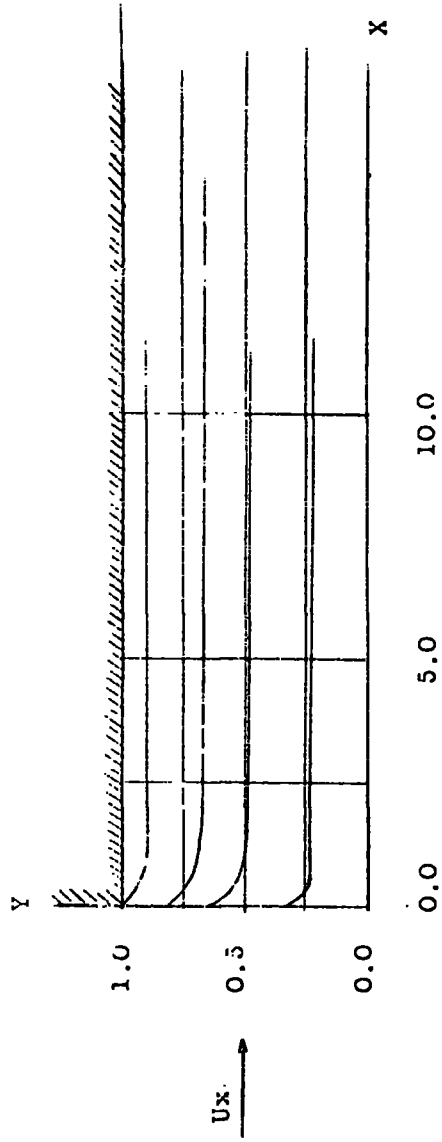


STREAM LINES

TURFL = 0.1 Re = 5

FIG. D - 10

Streamline Contours in Flow between Parallel Plates



Appendix E Re = 200

Streamline Contours

122	2.5	0.755	23	13	21
123	2.5	0.755	24	13	22
124	2.2	0.503	25	14	23
125	2.2	0.3616	26	14	24
126	2.2	0.2	27	15	25
127	2.2	0.0	28	16	26
128	2.4	1.0	29	17	27
129	2.4	0.94	30	17	28
130	2.4	0.755	31	18	29
131	2.4	0.659	32	18	30
132	2.4	0.503	33	19	31
133	2.4	0.3616	34	20	32
134	2.4	0.2	35	20	33
135	2.4	0.0	36	21	34
136	2.6	1.0	37	21	35
137	2.6	0.94	38	22	36
138	2.6	0.755	39	22	37
139	2.6	0.659	40	23	38
140	2.6	0.503	41	23	39
141	2.6	0.3616	42	24	40
142	2.6	0.2	43	25	41
143	2.6	0.0	44	25	42
144	2.8	1.0	45	26	43
145	2.8	0.94	46	26	44
146	2.8	0.755	47	27	45
147	2.8	0.659	48	27	46
148	2.8	0.503	49	28	47
149	2.8	0.3616	50	28	48
150	2.8	0.2	51	29	49
151	2.8	0.0	52	29	50
152	3.0	1.0	53	30	51
153	3.0	0.94	54	30	52
154	3.0	0.755	55	31	53
155	3.0	0.659	56	31	54
156	3.0	0.503	57	32	55
157	3.0	0.3616	58	33	56
158	3.0	0.2	59	33	57
159	3.0	0.0	60	34	58
1	1	0	61	34	59
2	1	10	62	35	60
3	2	10	63	35	61
4	3	11	64	36	62
5	3	11	65	37	63
6	4	12	66	37	64
7	4	12	67	38	65
8	4	13	68	38	66
9	5	13	69	39	67
10	5	14	70	39	68
11	6	14	71	40	69
12	6	15	72	41	70
13	7	15	73	42	71
14	7	16	74	42	72
15	8	16	75	43	73
16	8	17	76	43	74
17	9	17	77	44	75
18	9	18	78	44	76
19	10	18	79	45	77
20	11	19	80	45	78
		20	81	46	79
			82	46	80
			83	47	81
			84	47	82
			85	48	83
			86	48	84
			87	49	85
			88	49	86
			89	50	87
			90	50	88
			91	51	89
			92	51	90
			93	52	91
			94	52	92
			95	53	93
			96	53	94
			97	54	95
			98	54	96
			99	55	97
			100	55	98

87	44	58	87	143	61	92	94
84	44	58	84	144	62	95	94
85	45	59	85	145	67	96	97
86	45	59	86	146	67	97	98
87	46	60	87	147	68	98	99
88	46	60	88	148	68	99	99
89	47	61	89	149	69	99	99
90	47	61	90	150	69	99	99
91	47	61	91	151	69	99	99
92	47	61	92	152	69	99	99
93	47	61	93	153	69	99	99
94	47	61	94	154	69	99	99
95	47	61	95	155	69	99	99
96	47	61	96	156	69	99	99
97	47	61	97	157	69	99	99
98	47	61	98	158	69	99	99
99	47	61	99	159	69	99	99
100	47	61	100	160	69	99	99
101	47	61	101	161	69	99	99
102	47	61	102	162	69	99	99
103	47	61	103	163	69	99	99
104	47	61	104	164	69	99	99
105	47	61	105	165	69	99	99
106	47	61	106	166	69	99	99
107	47	61	107	167	69	99	99
108	47	61	108	168	69	99	99
109	47	61	109	169	69	99	99
110	47	61	110	170	69	99	99
111	47	61	111	171	69	99	99
112	47	61	112	172	69	99	99
113	47	61	113	173	69	99	99
114	47	61	114	174	69	99	99
115	47	61	115	175	69	99	99
116	47	61	116	176	69	99	99
117	47	61	117	177	69	99	99
118	47	61	118	178	69	99	99
119	47	61	119	179	69	99	99
120	47	61	120	180	69	99	99
121	47	61	121	181	69	99	99
122	47	61	122	182	69	99	99
123	47	61	123	183	69	99	99
124	47	61	124	184	69	99	99
125	47	61	125	185	69	99	99
126	47	61	126	186	69	99	99
127	47	61	127	187	69	99	99
128	47	61	128	188	69	99	99
129	47	61	129	189	69	99	99
130	47	61	130	190	69	99	99
131	47	61	131	191	69	99	99
132	47	61	132	192	69	99	99
133	47	61	133	193	69	99	99
134	47	61	134	194	69	99	99
135	47	61	135	195	69	99	99
136	47	61	136	196	69	99	99
137	47	61	137	197	69	99	99
138	47	61	138	198	69	99	99
139	47	61	139	199	69	99	99
140	47	61	140	200	69	99	99

39	1.14
40	0.59
41	1.574
42	2.25
43	2.72
44	2.65
45	1.75
46	1.222
47	1.24
48	2.781
49	3.664
50	3.75
51	2.702
52	2.442
53	3.236
54	5.178
55	5.77
56	3.41
57	5.276
58	7.572
59	5.47
60	5.09
61	14.61
62	14.61
63	6.422
64	14.665
65	6.433
66	14.8
67	7.431
68	5.592
69	5.114
70	5.268
71	2.272
72	5.155
73	5.032
74	3.611
75	2.712
76	1.817
77	5.915
78	1.571
79	2.287
80	2.722
81	2.74
82	1.76
83	1.864
84	1.671
85	1.921
86	1.906
87	1.767
88	1.142
89	2.068
90	1.923

1	1.14
9	0.59
17	1.574
25	2.25
33	2.72
41	2.65
49	1.75
57	1.222
65	1.24
73	2.781
81	3.664
89	3.75
97	2.702
105	2.442
113	3.236
121	5.178
129	5.77
137	3.41
145	5.276
153	7.572
161	5.47
169	5.09
177	14.61
185	14.61
193	6.422
201	14.665
209	6.433
217	14.8
225	7.431
233	5.592
241	5.114
249	5.268
257	2.272
265	5.155
273	5.032
281	3.611
289	2.712
297	1.817
305	5.915
313	1.571
321	2.287
329	2.722
337	2.74
345	1.76
353	1.864
361	1.671
369	1.921
377	1.906
385	1.767
393	1.142
401	2.068
409	1.923

2.68
2.277
1.192
1.634
1.035
0.42
2.437
2.1355
1.826
1.82
1.153
0.422
2.37
1.931
1.915
1.631
1.322
0.91
1.37
1.68

STOP

124	0.811	46
127	2.455	47
128	2.124	48
129	1.834	49
130	1.582	50
131	1.313	51
132	0.673	52
133	2.678	53
134	2.276	54
135	1.891	55
136	1.503	56
137	1.358	57
138	0.932	58
139		59
140		60
141		61
142		62
143		63
144		64
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APPENDIX G

Nodal points	Streamfunction	Vorticity
1	1.	2.934
2	0.995	2.820
3	0.942	2.387
4	0.839	1.953
5	0.695	1.519
6	0.519	1.085
7	0.296	0.560
8	0.0	0.0
9	1.0	2.857
10	0.994	2.679
17	1.0	2.633
18	0.995	2.467
25	1.0	2.177
26	0.996	2.071
33	1.0	1.271
34	0.998	1.368
42	0.999	0.588
49	1.0	0.007
72	1.0	20.785
77	1.0	29.091
78	0.913	14.613
79	0.908	14.665
84	1.0	21.323
111	1.0	0.005
113	0.999	0.585
120	1.0	1.267
121	0.998	1.364
128	1.0	2.173
129	0.996	2.068
136	1.0	2.631
137	0.995	2.465
144	1.0	2.855
145	0.994	2.678
152	1.0	2.934
153	0.995	2.820

Nodal points	Streamfunction	Vorticity
154	0.942	2.387
155	0.839	1.953
156	0.695	1.519
157	0.519	1.085
158	0.296	0.600
159	0.0	0.0

C LABELS OF CONTOUR LINE VALUES SOMEWHERE NEAR
C THE CONTOUR LINE ITSELF.

C CALL CONTOUR (X,IX,Y,IY,Z,IDX,CZ, NC, SWCHES, F9910, F9920, F9930)
C
C CALL CONTOUR TO DO THE PLOT.

C CALL IGTXT('STREAM LINES<>')
C CALL IGTAM('AMP', 'WINDOW', 0.0, 0.3, 0.0, 0.3, 0.0, 'MOVE', 0.0, 1.0, C,
C 'SCALE', 1.0)
C CALL IGRCN('CALCCMP')
C CALL CCMEND(NREGEN, NALLOC, MUSED)

C STOP
9810 PAUSE : CCONLEL RETURN 1'
9820 PAUSE : CCONLPL RETURN 2'
9810 PAUSE : CCONLPL RETURN 1'
9820 PAUSE : CCONLPL RETURN 2'
9830 PAUSE : CCONLPL RETURN 3'
C END

CC
C THIS PROGRAM CAN ONLY BE USED FOR PRODUCING
C CONTOUR LINES OF DATA SURFACE DEFINED ON
C RECTANGULAR GRID.

C CONTOUR ARE PLOTTED AS A SERIES OF STRAIGHT LINES
C OVER EACH ELEMENT CONSISTENT WITH THIS NODA DATA.

CC
EXTERNAL IGM2, IGR4
EXTERNAL CLABEL
LOGICAL SWCH(5) / 4* .TRUE., .TRUE. /

REAL X(20), Y(8), Z(20, 8), CZ(26)
REAL YLABS(11), XLABS(2)
REAL XC(15), YC(7)

C DATA X / C.0, 0.2, 0.4, 0.6, 0.8, 0.972,
F 1.0523, 1.2128, 1.3332, 1.4539, 1.5461, 1.6668,
F 1.7372, 1.9077, 2.0228, 2.2, 2.4, 2.6, 2.8, 3.0 /

C DATA Y / C.0, C.2, 0.2616, 0.5063, 0.6569, C.7955, 0.94, 1.0 /

C DATA CZ / 0.1, 0.2, C.5, C.7, 0.8, 0.85, 0.9, 0.95, 0.05,
C 1.2, 0.25, 0.57, C.56, 0.54, 0.53, C.92, C.91,
F C.99, 0.88, 0.87, C.86, 0.83, 0.84, 0.82, 0.81,
F C.72, 0.76, 0.74, C.72, C.67, C.64, 0.63, 0.6, 0.58,
C C.55, 0.47

C DATA XLABS/ 1.0, 2.9 /

NYLABS = 1
NYLABS=0
DIST=0.0
HC = 25
ICX = 20
IY = 8
IX = 10

DC SC JD=1, IY
J=IY+1-JD
PEAC(4,7) (Z(I,J), I = 1, IX)
70 FORMAT(10(F8.4,2X))
WRITE(8,70) (Z(I,J), I = 1, IX)
90 CONTINUE

C CALL IGM2
C CALL CCONSET(IGM2, IGR4, 6)

C CONSET IS CALLED TO SET UP THE NAMES OF THE FUNCTIONS
C TO DISPOSE OF PICTURE COORDINATES, AND TO PASS IN THE
C LOGICAL I/O UNIT UPON WHICH ERROR COMMENTS (IF ANY)
C WILL BE WRITTEN.

C CALL CONLBL(CLABEL, DIST, NXLABS, XLABS, NYLABS, YLABS, F9810, F9820

APPENDIX H

RETURN
END

62	1.2	0.2
63	1.2	0.0
64	1.4	1.0
65	1.4	0.9
66	1.4	0.2
67	1.4	0.4
68	1.4	0.4
69	1.4	0.2
70	1.4	0.0
71	1.6	1.0
72	1.5	0.9
73	1.4	0.8
74	1.6	0.4
75	1.6	0.4
76	1.6	0.2
77	1.7	0.0
78	1.8	1.0
79	1.8	0.0
80	1.8	0.8
81	1.8	0.4
82	1.8	0.4
83	1.8	0.3
84	1.8	0.0
85	2.0	1.0
86	2.0	0.8
87	2.0	0.8
88	2.0	0.0
89	2.0	0.4
90	2.0	0.2
91	2.0	0.0
92	2.2	1.0
93	2.2	0.0
94	2.2	0.8
95	2.2	0.4
96	2.2	0.4
97	2.2	0.2
98	2.2	0.0
99	2.4	1.0
100	2.4	0.9
101	2.4	0.4
102	2.4	0.4
103	2.4	0.4
104	2.4	0.2
105	2.4	0.0
106	2.6	1.0
107	2.6	0.8
108	2.6	0.8
109	2.6	0.4
110	2.6	0.4
111	2.6	0.2
112	2.6	0.0
113	2.8	1.0
114	2.8	0.8
115	2.8	0.8
116	2.8	0.0
117	2.8	0.4
118	2.8	0.2

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2	0.0	0.9
3	0.0	0.6
4	0.0	0.4
5	0.0	0.2
6	0.0	0.0
7	0.0	0.0
8	0.1	0.1
9	0.100	0.5
10	0.1	0.8
11	0.1	0.6
12	0.1	0.4
13	0.1	0.2
14	0.1	0.0
15	0.2	1.0
16	0.2	0.8
17	0.2	0.6
18	0.2	0.4
19	0.2	0.2
20	0.2	0.0
21	0.2	1.0
22	0.225	0.9
23	0.225	0.8
24	0.225	0.6
25	0.225	0.4
26	0.225	0.2
27	0.225	0.0
28	0.225	1.0
29	0.425	0.0
30	0.425	0.8
31	0.425	0.6
32	0.425	0.4
33	0.425	0.2
34	0.425	0.0
35	0.425	1.0
36	0.6	0.9
37	0.6	0.8
38	0.6	0.6
39	0.6	0.4
40	0.6	0.2
41	0.6	0.0
42	0.6	1.0
43	0.8	0.9
44	0.8	0.8
45	0.8	0.6
46	0.8	0.4
47	0.8	0.2
48	0.8	0.0
49	0.8	1.0
50	1.0	0.9
51	1.0	0.8
52	1.0	0.6
53	1.0	0.4
54	1.0	0.2
55	1.0	0.0
56	1.0	1.0
57	1.2	0.9
58	1.2	0.8

APPENDIX I

182	0.8	4.6	9	0.0
183	0.6	4.8	10	0.0
184	0.4	4.1	11	0.0
185	0.2	4.9	12	0.0
186	0.0	4.8	13	0.0
187	1.0	4.8	14	0.0
188	0.8	4.8	15	0.0
189	0.5	4.8	16	0.0
190	0.4	5.0	17	0.0
191	0.2	5.0	18	0.0
192	0.0	5.0	19	0.0
193	1.0	5.0	20	0.0
194	0.9	5.0	21	0.0
195	0.8	5.0	22	0.0
196	0.8	5.0	23	0.0
197	0.4		24	0.0
198	0.2		25	0.0
199	0.0		26	0.0
200	0.0		27	0.0
201	0.0		28	0.0
202	0.0		29	0.0
203	0.0		30	0.0
204	0.0		31	0.0
205	0.0		32	0.0
206	0.0		33	0.0
207	0.0		34	0.0
208	0.0		35	0.0
209	0.0		36	0.0
210	0.0		37	0.0
211	0.0		38	0.0
212	0.0		39	0.0
213	0.0		40	0.0
214	0.0		41	0.0
215	0.0		42	0.0

43	26	27	24	104	61	62	63
47	27	25	34	107	62	70	70
48	27	28	34	108	62	63	70
49	28	27	35	109	64	71	71
50	29	28	36	110	64	72	72
51	29	29	37	111	65	73	73
52	30	28	37	112	65	73	73
53	30	29	38	113	66	74	74
54	31	30	38	114	66	74	74
55	31	32	39	115	67	75	75
56	32	40	39	116	67	75	75
57	32	40	40	117	68	76	76
58	33	41	40	118	68	76	76
59	33	34	41	119	69	77	77
60	34	42	41	120	69	77	77
61	34	35	42	121	71	78	78
62	35	44	43	122	71	78	78
63	36	44	44	123	72	79	79
64	37	45	44	124	72	80	80
65	37	45	44	125	73	81	81
66	38	46	45	126	74	81	81
67	38	46	45	127	74	82	82
68	39	47	46	128	74	82	82
69	39	47	46	129	75	83	83
70	40	48	47	130	75	84	84
71	40	48	47	131	76	84	84
72	41	49	48	132	76	85	85
73	41	42	48	133	77	85	85
74	42	51	49	134	77	86	86
75	43	44	50	135	78	87	87
76	43	44	50	136	79	87	87
77	44	52	51	137	80	88	88
78	44	45	52	138	80	88	88
79	45	46	53	139	81	89	89
80	45	46	53	140	81	89	89
81	46	54	54	141	82	90	90
82	46	47	54	142	82	90	90
83	47	55	54	143	83	91	91
84	47	48	55	144	83	91	91
85	48	56	55	145	84	92	92
86	48	49	56	146	85	92	92
87	49	56	57	147	86	93	93
88	50	51	58	148	86	93	93
89	50	51	58	149	87	94	94
90	51	52	59	150	87	94	94
91	51	52	59	151	88	95	95
92	52	60	60	152	88	95	95
93	52	61	60	153	89	96	96
94	53	61	61	154	89	96	96
95	53	62	62	155	90	97	97
96	54	63	62	156	90	97	97
97	54	63	63	157	91	98	98
98	55	64	63	158	91	98	98
99	55	64	64	159	92	99	99
100	56	65	64	160	92	99	99
101	57	65	65	161	93	100	100
102	57	66	65	162	93	100	100
103	58	66	66	163	94	101	101
104	58	66	66	164	94	101	101
105	59	67	67	165	95	102	102
106	59	67	67	166	95	102	102

166	56	97	104	226	131	132	133
167	57	105	104	227	132	140	136
168	58	106	105	228	133	141	140
169	59	107	106	229	134	142	141
170	60	108	107	230	135	143	142
171	61	109	108	231	136	144	143
172	62	110	109	232	137	145	144
173	63	111	110	233	138	146	145
174	64	112	111	234	139	147	146
175	65	113	112	235	140	148	147
176	66	114	113	236	141	149	148
177	67	115	114	237	142	150	149
178	68	116	115	238	143	151	150
179	69	117	116	239	144	152	151
180	70	118	117	240	145	153	152
181	71	119	118	241	146	154	153
182	72	120	119	242	147	155	154
183	73	121	120	243	148	156	155
184	74	122	121	244	149	157	156
185	75	123	122	245	150	158	157
186	76	124	123	246	151	159	158
187	77	125	124	247	152	160	159
188	78	126	125	248	153	161	160
189	79	127	126	249	154	162	161
190	80	128	127	250	155	163	162
191	81	129	128	251	156	164	163
192	82	130	129	252	157	165	164
193	83	131	130	253	158	166	165
194	84	132	131	254	159	167	166
195	85	133	132	255	160	168	167
196	86	134	133	256	161	169	168
197	87	135	134	257	162	170	169
198	88	136	135	258	163	171	170
199	89	137	136	259	164	172	171
200	90	138	137	260	165	173	172
201	91	139	138	261	166	174	173
202	92	140	139	262	167	175	174
203	93	141	140	263	168	176	175
204	94	142	141	264	169	177	176
205	95	143	142	265	170	178	177
206	96	144	143	266	171	179	178
207	97	145	144	267	172	180	179
208	98	146	145	268	173	181	180
209	99	147	146	269	174	182	181
210	100	148	147	270	175	183	182
211	101	149	148	271	176	184	183
212	102	150	149	272	177	185	184
213	103	151	150	273	178	186	185
214	104	152	151	274	179	187	186
215	105	153	152	275	180	188	187
216	106	154	153	276	181	189	188
217	107	155	154	277	182	190	189
218	108	156	155	278	183	191	190
219	109	157	156	279	184	192	191
220	110	158	157	280	185	193	192
221	111	159	158	281	186	194	193
222	112	160	159	282	187	195	194

53	1.354
54	1.218
55	0.605
58	2.703
59	2.41
60	1.811
61	1.211
62	0.604
65	2.69
66	2.552
67	1.756
68	1.2
69	0.6
72	2.63
73	2.266
74	1.789
75	1.155
76	0.6
79	2.55
80	1.384
81	1.785
83	1.10
84	0.598
86	2.69
87	1.284
88	1.784
89	1.10
90	0.597
92	1.69
94	2.364
95	1.783
96	1.15
97	0.594
100	2.69
101	2.314
102	1.783
103	1.15
104	0.594
107	2.451
108	2.364
109	1.783
110	1.19
111	0.594
114	2.691
115	2.585
116	1.783
117	1.19
118	0.595
121	2.691
122	2.285
123	1.783
124	1.185
125	0.596
128	2.69
129	2.385
130	1.783
131	1.19

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STOP

5	4.556
10	2.5555
11	0.783
12	0.57
13	0.148
16	4.298
17	2.962
18	1.545
19	0.658
20	0.28
23	4.051
24	3.09
25	1.72
26	0.927
27	0.411
30	2.43
31	2.9
32	1.872
33	1.1
36	0.507
37	3.078
38	2.712
39	1.3
40	1.18
41	0.653
44	2.829
45	2.521
46	1.57
47	1.2167

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STOP

1.125
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STOP

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STOP
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