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# Conserved charges of black holes in Weyl and Einstein–Gauss–Bonnet gravities

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**Abstract** An off-shell generalization of the Abbott–Deser–Tekin (ADT) conserved charge was recently proposed by Kim et al. They achieved this by introducing off-shell Noether currents and potentials. In this paper, we construct the crucial off-shell Noether current by the variation of the Bianchi identity for the expression of EOM, with the help of the property of Killing vector. Our Noether current, which contains an additional term that is just one half of the Lie derivative of a surface term with respect to the Killing vector, takes a different form in comparison with the one in their work. Then we employ the generalized formulation to calculate the quasi-local conserved charges for the most general charged spherically symmetric and the dyonic rotating black holes with AdS asymptotics in four-dimensional conformal Weyl gravity, as well as the charged spherically symmetric black holes in arbitrary dimensional Einstein–Gauss–Bonnet gravity coupled to Maxwell or nonlinear electrodynamics in AdS spacetime. Our results confirm those obtained through other methods in the literature.

## 1 Introduction

Modified gravity theories that involves higher curvature terms in the Lagrangian have been extensively investigated, generally motivated by the intriguing feature that these higher curvature terms render the gravity theories perturbatively renormalizable in the quantization process [1]. A very natural higher-order derivative modification of general relativity is the fourth-order theories of gravitation, which includes the well-known theories of Weyl gravity and Einstein–Gauss–Bonnet gravity. To the former, its Lagrangian contains the square of the Weyl tensor, so it is invariant under the local conformal transformation of the metric. The Lagrangian for Einstein–Gauss–Bonnet gravity includes up to the term with quadratic Riemann tensor, which can be thought of as the higher curvature correction to general relativity in the low energy limit of heterotic string theory. Due to the salient properties of the two gravity theories, together with the AdS/CFT correspondence, a lot of efforts have been made in seeking asymptotically AdS black hole solutions in Weyl gravity [2,3] and Einstein–Gauss–Bonnet gravity [4–8], to provide various interesting backgrounds of spacetime. Generally speaking, after obtaining a black hole solution, an important task is to identify its conserved charges, such as the energy and the angular momentum.

Till now several approaches have been proposed to compute the conserved charges of asymptotically AdS solutions, such as the so-called counterterm subtraction approach [9,10] generalized from the Brown–York method [11], the Ashtekar–Magnon–Das formalism [12,13], the covariant phase space approach [14,15], the method [16–19] developed by Barnich et al. and the Abbott–Deser–Tekin (ADT) formalism [20–23]. Particularly, the ADT formalism, which is defined by the Noether potential got through the linearized perturbation for the expression of EOM in a fixed background of AdS spacetime, has made some progress on computation scheme for conserved charges of asymptotically AdS black holes in fourth-order gravity theories. Since the background metric is a vacuum solution of the equation of motion (EOM), the Noether potential in ADT formalism is on-shell. Recently, in Ref. [24], Kim, Kulkarni and Yi proposed a quasi-local formulation of conserved charges by generalizing the on-shell Noether potential in the ADT formalism to off-shell level, as well as Refs. [16–19] to incorporate a single parameter path in the space of solutions into their definition. These modifications make it more operable to evaluate the Noether potential in terms of the corresponding current. The generalized formalism for the quasi-local conserved charges provides a more efficient way to compute the ADT conserved charges

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for covariant theories of gravity, and it has been extended to the theory of gravity with a gravitational Chern–Simons term [25] and the gravity theory in the presence of matter fields [26]. In [27], it was utilized to obtain the mass of the three- and five-dimensional Lifshitz black holes. To compare with the original ADT formalism, it is meaningful to employ this generalized quasi-local formulation to study the conserved charges in higher-order derivative gravity theories.

In this paper, to provide a deep understanding on the generalized ADT formalism proposed in [24], we derive the off-shell Noether current that educes the Noether potential finally entering into the formulation of conserved charges from different perspective. Our derivation endows the off-shell Noether current with a natural connection with its corresponding potential. Then we extend this formalism to investigate the quasi-local conserved charges of charged (rotating) black holes with AdS asymptotics in the two typical fourth-order derivative gravity theories: conformal Weyl gravity and Einstein–Gauss–Bonnet gravity. The remainder of this paper goes as follows. In Sect. 2, we give a brief review of the method in [24]. However, unlike there, we derive the off-shell Noether current and its corresponding potential through the variation of the Bianchi identity for the expression of EOM. Our results are formally different from those in [24]. In Sect. 3, we first present the explicit expressions of the off-shell Noether potentials in Weyl gravity. Then these quantities are applied to compute the mass of the most general static black hole and both the mass and the angular momentum of the dyonic rotating black hole in four-dimensional Weyl gravity. In Sect. 4, we calculate the energy of the general charged spherically symmetric black hole in arbitrary dimensional Einstein–Gauss–Bonnet gravity, coupled to Maxwell or non-linear electrodynamics in AdS spacetime. The general formalizations of the Noether potentials for Einstein–Gauss–Bonnet gravity are also given. The last section is for our conclusions.

## 2 The generalized ADT formalism

In this section, we shall review the formulation of conserved charges in [24], which can be thought of as the off-shell extension of the ADT formalism since both the Noether current and the potential there were constructed without the requirement that the gravitational fields must satisfy the equation of motion. What is more, another obvious difference from the ADT formalism is that one parameter path in the solution space was introduced to present the final definition of the quasi-local conserved charge in [24]. However, unlike in [24], where the starting point to derive the off-shell Noether current and potential is the variation and diffeomorphism transformation of the action, we shall give a different derivation of the two quantities by directly varying the Bianchi identity and using the property of a Killing vector. We proceed by considering the Lagrangian of a $D$-dimensional generally diffeomorphism covariant gravity theory [14,15],

$$\mathcal{L} = \sqrt{-g}L[g_{\mu\nu}, R, R_{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla R, \nabla R_{\mu\nu}, \nabla R_{\mu\nu\rho\sigma}, \cdots], \tag{1}$$

which includes no other matter fields. The variation of Eq. (1) generally yields

$$\delta\mathcal{L} = \sqrt{-g}\mathcal{E}_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}\nabla_\mu\Theta^\mu(g;\delta g), \tag{2}$$

where $\mathcal{E}_{\mu\nu}$ is the expression of the equation of motion (EOM), and $\Theta^\mu(g;\delta g)$[1] is a surface term. To preserve the diffeomorphism, $\mathcal{E}_{\mu\nu}$ satisfies the Bianchi identity

$$\nabla_\mu\mathcal{E}^{\mu\nu} = 0. \tag{3}$$

Varying the Bianchi identity (3), we get

$$\nabla_\mu\delta\mathcal{E}^{\mu\nu} + \delta\Gamma^\mu_{\mu\lambda}\mathcal{E}^{\lambda\nu} + \delta\Gamma^\nu_{\mu\lambda}\mathcal{E}^{\mu\lambda} = 0, \tag{4}$$

multiplying the above equation by a Killing vector $\xi^\mu$, we further obtain

$$\nabla_\mu\left(\delta\mathcal{E}^{\mu\nu}\xi_\nu + \mathcal{E}^{\mu\nu}\xi^\lambda\delta g_{\nu\lambda} + \frac{1}{2}\mathcal{E}^{\mu\nu}\xi_\nu g^{\alpha\beta}\delta g_{\alpha\beta} - \frac{1}{2}\xi^\mu\mathcal{E}^{\alpha\beta}\delta g_{\alpha\beta}\right) + \frac{1}{2}\delta g_{\alpha\beta}\mathcal{L}_\xi\mathcal{E}^{\alpha\beta} = 0, \tag{5}$$

where $\mathcal{L}_\xi$ denotes the Lie derivative with respect to the Killing vector $\xi^\mu$. In order to get Eq. (5), the equation of the Killing vector, $\nabla_{(\mu}\xi_{\nu)} = 0$, is used and the variation of the gravitational field $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ is assumed to preserve the Killing vector, namely, $\delta\xi^\mu = 0$. Obviously, the component in the bracket of Eq. (5) is just the off-shell Noether current presented in [24,28]. This makes it possible to construct the Noether current from Eq. (5). Actually, combined with the equation

$$\mathcal{L}_\xi\delta(\sqrt{-g}L) = -\sqrt{-g}\delta g_{\mu\nu}\mathcal{L}_\xi\mathcal{E}^{\mu\nu} + \sqrt{-g}\nabla_\mu[\mathcal{L}_\xi\Theta^\mu(g;\delta g)] = 0, \tag{6}$$

which is the Lie derivative of Eq. (2) in terms of the Killing vector $\xi^\mu$, one can rewrite Eq. (5) as

$$\nabla_\mu\left[\delta\mathcal{E}^{\mu\nu}\xi_\nu + \mathcal{E}^{\mu\nu}\xi^\lambda\delta g_{\nu\lambda} + \frac{1}{2}\mathcal{E}^{\mu\nu}\xi_\nu g^{\alpha\beta}\delta g_{\alpha\beta} - \frac{1}{2}\xi^\mu\mathcal{E}^{\alpha\beta}\delta g_{\alpha\beta} + \frac{1}{2}\mathcal{L}_\xi\Theta^\mu(g;\delta g)\right] = 0. \tag{7}$$

Thus, the off-shell Noether current can be defined by

$$\mathcal{J}^\mu = \delta\mathcal{E}^{\mu\nu}\xi_\nu + \mathcal{E}^{\mu\nu}\xi^\lambda\delta g_{\nu\lambda} + \frac{1}{2}\mathcal{E}^{\mu\nu}\xi_\nu g^{\alpha\beta}\delta g_{\alpha\beta} - \frac{1}{2}\xi^\mu\mathcal{E}^{\alpha\beta}\delta g_{\alpha\beta} + \frac{1}{2}\mathcal{L}_\xi\Theta^\mu(g;\delta g). \tag{8}$$

[1] It multiplied by $\sqrt{-g}$ is equal to the surface term $\Theta^\mu(g;\delta g)$ defined in [24].

Note that the last term in the above equation, which is one half of the Lie derivative of the surface term $\Theta^\mu(g;\delta g)$ with respect to the Killing vector $\xi^\mu$, is an additional term compared with the Noether current in [24,28]. Of course, it is feasible to define the off-shell Noether current $\mathcal{J}^\mu$ the same as that of [24,28] in terms of Eq. (7). One only needs to add a term $-\nabla_\mu\left[\delta(\sqrt{-g}\Theta^\mu(g;\mathcal{L}_\xi g))/\sqrt{-g}\right]/2$ to the left side of Eq. (7), together with the result $\mathcal{L}_\xi(\sqrt{-g}\Theta^\mu(g;\delta g)) - \delta(\sqrt{-g}\Theta^\mu(g;\mathcal{L}_\xi g)) = 0$ given in [14,15]. To see this clearly, under the diffeomorphism $\xi^\mu$, the metric $g_{\mu\nu}$ transforms as $\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = 0$ and $\delta(\sqrt{-g}L) = \mathcal{L}_\xi(\sqrt{-g}L) = 0$, so Eq. (2) becomes $\nabla_\mu\Theta^\mu(g;\mathcal{L}_\xi g) = 0$, whose variation leads to

$$\delta(\nabla_\mu\Theta^\mu(g;\mathcal{L}_\xi g)) = \nabla_\mu\left[\frac{1}{\sqrt{-g}}\delta(\sqrt{-g}\Theta^\mu(g;\mathcal{L}_\xi g))\right] = 0.$$

Next, we shall derive the off-shell Noether potential $Q^{\mu\nu}_{\mathrm{ADT}}$, which is associated with the off-shell Noether current $\mathcal{J}^\mu$ through the relation

$$\mathcal{J}^\mu = \nabla_\nu Q^{\mu\nu}_{\mathrm{ADT}}. \tag{9}$$

To do this, substituting the Lie derivative of the surface term $\Theta^\mu(g;\delta g)$

$$\mathcal{L}_\xi\Theta^\mu(g;\delta g) = -2\nabla_\nu\xi^{[\mu}\Theta^{\nu]}(g;\delta g) + \xi^\mu\nabla_\nu\Theta^\nu(g;\delta g) \tag{10}$$

and Eq. (2) into the off-shell current (8), we present the current $\mathcal{J}^\mu$ in the form

$$\sqrt{-g}\mathcal{J}^\mu = \delta\left(\sqrt{-g}\mathcal{E}^{\mu\nu}\xi_\nu + \frac{1}{2}\sqrt{-g}\xi^\mu L\right) - \sqrt{-g}\nabla_\nu\xi^{[\mu}\Theta^{\nu]}(g;\delta g). \tag{11}$$

Following [29] we define another off-shell current $J^\mu$ as

$$J^\mu = 2\mathcal{E}^{\mu\nu}\xi_\nu + \xi^\mu L = \nabla_\nu K^{\mu\nu}, \tag{12}$$

where $K^{\mu\nu}$ is the off-shell Noether potential corresponding to the current $J^\mu$, and it is easy to verify that $\nabla_\mu J^\mu = 0$. We further cast the current $\mathcal{J}^\mu$ into the form

$$\sqrt{-g}\mathcal{J}^\mu = \partial_\nu\left[\frac{1}{2}\delta(\sqrt{-g}K^{\mu\nu}) - \sqrt{-g}\xi^{[\mu}\Theta^{\nu]}(g;\delta g)\right] = \partial_\nu\left(\sqrt{-g}Q^{\mu\nu}_{\mathrm{ADT}}\right). \tag{13}$$

The above equation yields the important relation between ADT potential and the off-shell Noether potential $K^{\mu\nu}$, namely,

$$\sqrt{-g}Q^{\mu\nu}_{\mathrm{ADT}} = \frac{1}{2}\delta(\sqrt{-g}K^{\mu\nu}) - \sqrt{-g}\xi^{[\mu}\Theta^{\nu]}(g;\delta g), \tag{14}$$

or equally,

$$Q^{\mu\nu}_{\mathrm{ADT}} = \frac{1}{2}\delta K^{\mu\nu} + \frac{1}{4}K^{\mu\nu}g^{\alpha\beta}\delta g_{\alpha\beta} - \xi^{[\mu}\Theta^{\nu]}(g;\delta g). \tag{15}$$

In comparison with the results in [24], although the off-shell Noether currents $\mathcal{J}^\mu$ and $J^\mu$ obtained by the variation of the Bianchi identity (3) are formally different from their corresponding quantities given there, each of them is correspondingly equal since it can easily be verified that $\Theta^\mu(g;\mathcal{L}_\xi g) = \mathcal{L}_\xi\Theta^\mu(g;\delta g) = \mathcal{L}_\xi\mathcal{E}^{\mu\nu} = 0$ for the generally diffeomorphism covariant Lagrangian (1). The merit of our formulation for the Noether current $\mathcal{J}^\mu$ is that it becomes more natural to get the off-shell Noether potential $Q^{\mu\nu}_{\mathrm{ADT}}$. In fact, to finally get the relation between $Q^{\mu\nu}_{\mathrm{ADT}}$ and $K^{\mu\nu}$, the condition $\mathcal{L}_\xi(\sqrt{-g}\Theta^\mu(g;\delta g)) - \delta(\sqrt{-g}\Theta^\mu(g;\mathcal{L}_\xi g)) = 0$ related to the symplectic current [14,15] has been used in [24], but it is not needed in our case. Particularly, when the background metric satisfies the vacuum equation of motion, $\mathcal{E}_{\mu\nu} = 0$, all the Noether currents and potentials become the conventional ones in the ADT formalism [20–23].

Finally, one can propose a formulation of the conserved charge in terms of the integral of the Noether potential (15) over the boundary of a spatial hypersurface under the conditions that a background metric that is a vacuum solution of EOM is fixed and the perturbation of the given metric is taken as the divergence between it and the fixed background metric, like the ADT method [20–23]. However, Ref. [24] gave a different definition from Refs. [16–19] to incorporate a single parameter path characterized by a parameter $s$ $(0 \le s \le 1)$ in the space of solutions. This path interpolates between the given solution and the background solution through parameterizing a set of free parameters $\mathcal{C}$ in the space for the solutions of EOM as $s\mathcal{C}$. On the basis of the Noether potential $Q^{\mu\nu}_{\mathrm{ADT}}$ in Eq. (14), by integrating the variable $s$, one can define the quasi-local conserved charge by

$$\begin{aligned}\mathcal{Q} &= \frac{1}{8\pi}\int_0^1 \mathrm{d}s\int \mathrm{d}\Sigma_{\mu\nu}Q^{\mu\nu}_{\mathrm{ADT}}(g;s)\\ &= \frac{1}{16\pi}\int d^{(D-2)}x_{\mu\nu}\triangle\hat{K}^{\mu\nu}\\ &\quad -\frac{1}{8\pi}\int_0^1 \mathrm{d}s\int \mathrm{d}\Sigma_{\mu\nu}\xi^{[\mu}\Theta^{\nu]}(g;s),\end{aligned} \tag{16}$$

where $\mathrm{d}\Sigma_{\mu\nu} = \frac{1}{2}\frac{1}{(D-2)!}\epsilon_{\mu\nu\mu_1\mu_2\cdots\mu_{(D-2)}}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_{(D-2)}}$ with $\epsilon_{012\cdots(D-1)} = \sqrt{-g}$ and

$$\triangle\hat{K}^{\mu\nu} = \sqrt{-g}K^{\mu\nu}\big|_{s=1} - \sqrt{-g}K^{\mu\nu}\big|_{s=0}$$

is the finite difference between the given solution and the background solution, i.e. the two end points of the single parameter path. Equation (16) might be proposed for the conserved charge, defined in the interior region or at the asymptotical infinity, for any covariant gravity theory with the Lagrangian (1) whenever its integration is well defined.[2]

[2] It is not always guaranteed that the conserved charge defined by Eq. (16) is integrable. It has to satisfy some integrability conditions. For details see [19].

In the following sections, we shall make use of Eq. (16) to calculate the mass and angular momenta of charged static and rotating black holes in Weyl and Einstein–Gauss–Bonnet gravities although Eq. (16) is defined in terms of the Lagrangian for pure gravity, without any matter fields. We can do this since the terms associated with the gauge fields fall off fast enough at asymptotic infinity to guarantee that the integration is finite.

## 3 Conserved charges of black holes in four-dimensional Weyl gravity

In this section, we make use of the generalized ADT formalism in the previous section to calculate the quasi-local conserved charges of the most general charged spherically symmetric black hole and the charged rotating black hole in four-dimensional Weyl gravity. The Lagrangian for Weyl gravity takes the form

$$\mathcal{L}_W=\frac{1}{2}\alpha\sqrt{-g}C^2=\frac{1}{2}\alpha\left(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}-2R^{\mu\nu}R_{\mu\nu}+\frac{1}{3}R^2\right), \tag{17}$$

where $\alpha$ is a coupling constant, and the Weyl tensor $C_{\mu\nu\rho\sigma}$ is given by

$$C_{\mu\nu\rho\sigma}=-(g_{\mu[\rho}R_{\sigma]\nu}-g_{\nu[\rho}R_{\sigma]\mu})+\frac{1}{3}Rg_{\mu[\rho}g_{\sigma]\nu}+R_{\mu\nu\rho\sigma} \tag{18}$$

in four dimensions. The Weyl tensor has the same symmetry properties as the Riemann curvature but it is traceless, i.e. $C^{\rho}_{\ \mu\rho\nu}=0$. The expression of EOM from the Lagrangian (17) is

$$\mathcal{E}^W_{\mu\nu}=2\alpha B_{\mu\nu}=-\alpha(2\nabla^\rho\nabla^\sigma+R^{\rho\sigma})C_{\mu\rho\sigma\nu}, \tag{19}$$

where $B_{\mu\nu}$ is just the Bach tensor. By using the properties for Lie derivative along a Killing vector $\nabla_\mu\mathcal{L}_\xi=\mathcal{L}_\xi\nabla_\mu$ and $\mathcal{L}_\xi R_{\mu\nu\rho\sigma}=0$, one can check that $\mathcal{L}_\xi\mathcal{E}^W_{\mu\nu}=0$.

Now we present some quantities tightly related to our calculation, such as the surface term $\Theta^\mu_W(g;\delta g)$ and the off-shell Noether potentials $K^{\mu\nu}_W$, $Q^{\mu\nu}_W$ for four-dimensional Weyl gravity. More general results for higher derivative gravity theories can be found in [24,46]. The surface term $\Theta^\mu_W(g;\delta g)$ and the potential $K^{\mu\nu}_W$ are read off as

$$\Theta^\mu_W(g;h)=2\alpha C^{\mu\nu\rho\sigma}\nabla_\sigma h_{\nu\rho}+2\alpha h_{\nu\sigma}\left(\nabla^{[\mu}R^{\nu]\sigma}+\frac{1}{6}g^{\sigma[\mu}\nabla^{\nu]}R\right), \tag{20}$$

$$K^{\mu\nu}_W=2\alpha C^{\mu\nu\rho\sigma}\nabla_{[\rho}\xi_{\sigma]} \tag{21}$$

$$-4\alpha\xi_\sigma\left(\nabla^{[\mu}R^{\nu]\sigma}+\frac{1}{6}g^{\sigma[\mu}\nabla^{\nu]}R\right), \tag{22}$$

where $h_{\mu\nu}\equiv\delta g_{\mu\nu}$ denotes the variation of the metric and its indices are lowered or raised by the background metric $g_{\mu\nu}$ or $g^{\mu\nu}$. It is easy to verify that $\Theta^\mu_W(g;\xi)=0$ when $h_{\mu\nu}=\mathcal{L}_\xi g_{\mu\nu}=0$ and $\mathcal{L}_\xi\Theta^\mu_W(g;h)=0$. The off-shell Noether potentials $Q^{\mu\nu}_W$ is

$$Q^{\mu\nu}_W=\frac{1}{2}\delta K^{\mu\nu}_W+\frac{1}{4}hK^{\mu\nu}_W-\xi^{[\mu}\Theta^{\nu]}_W(g;h), \tag{23}$$

where $h$ in the second term denotes $h=g^{\alpha\beta}\delta g_{\alpha\beta}$. For the explicit expression of $\delta K^{\mu\nu}_W$ see Eq. (45) in the appendix.

### 3.1 The conserved charge of the charged spherically symmetric black hole

In this subsection, we take into account of the quasi-local charge of the charged black hole in static case. The total Lagrangian is $\mathcal{L}_{\rm total}=\mathcal{L}_W+L_{EM}$, where $L_{EM}=\frac{1}{3}\alpha\sqrt{-g}F_{\mu\nu}F^{\mu\nu}$ is the Lagrangian for the gauge field $A_\mu$ and the field strength $F_{\mu\nu}=2\partial_{[\mu}A_{\nu]}$. The most general charged spherically symmetric black hole in this four-dimensional Weyl gravity theory has the form [2]

$$\begin{aligned}
&\mathrm{d}s^2=-f(r)\mathrm{d}t^2+\frac{\mathrm{d}r^2}{f(r)}+r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2),\\
&A=-\frac{q}{r}\mathrm{d}t,\\
&f(r)=m+\frac{b}{r}+ar-\frac{1}{3}\Lambda r^2,
\end{aligned} \tag{24}$$

where the four constants $(a,b,m,q)$ are constrained by $3ab+1+q^2=m^2$. When $a\neq0$, which is the case we first consider, we get $b=(m^2-1-q^2)/(3a)$ from the constraint.

To make use of the formulation (16) to calculate the conserved charge of the static black hole (24), we choose an infinitesimal parametrization of a single parameter path in the solution space by letting the constants $(m,q)$ change as

$$m\to m+\mathrm{d}m,\quad q\to q+\mathrm{d}q. \tag{25}$$

Under such a parametrization and choice of the Killing vector $\xi^\mu=(-1,0,0,0)$, the $(t,r)$ components of both Noether potentials, $K^{\mu\nu}_W$ and $Q^{\mu\nu}_W$, are given by

$$\begin{aligned}
\sqrt{-g}K^{\rm tr}_W&=\frac{2\alpha\sin\theta[2\Lambda(m^2-q^2-1)+3a^2(m-1)]}{9a}+\mathcal{O}\left(\frac{1}{r}\right),\\
\sqrt{-g}Q^{\rm tr}_W&=\frac{\alpha(3a^2+4m\Lambda)\sin\theta\mathrm{d}m}{9a}-\frac{4\alpha q\Lambda\sin\theta\mathrm{d}q}{9a}+\frac{2\alpha q\sin\theta\mathrm{d}q}{3r}.
\end{aligned} \tag{26}$$

Then the mass of the charged spherically symmetric black hole (24) can be computed as

$$M = \frac{1}{4}\int\int_0^{\pi}\left(\lim_{r=\infty}\sqrt{-g}Q^{\text{tr}}_W\right)\text{d}\theta = \frac{\alpha(3ma^2+2\Lambda m^2-2q^2\Lambda)}{18a}. \quad (27)$$

The temperature $T$ and the entropy $S$ [30] of the black hole are

$$T = \frac{1}{12\pi}\frac{1+q^2-m^2+3a^2r_+^2-2a\Lambda r_+^3}{ar_+^2},$$
$$S = \frac{2\pi\alpha[1+q^2-m^2-(m+2)ar_+]}{3ar_+},$$

respectively, where $r_+$ is the event horizon, given by the equation $f(r_+)=0$. Both the electric charge $Q_e$ and the electric potential $\Phi_e$ are

$$Q_e = -\frac{1}{3}\alpha q, \quad \Phi_e = \frac{q}{r_+}.$$

One can verify that the mass (27) satisfies the first law:

$$\text{d}M = T\text{d}S + \Phi_e\text{d}Q_e. \quad (28)$$

The above first law shows that the generalized ADT formulation (16) is applicable to the static black hole in four-dimensional Weyl gravity. However, in [32], it was claimed that ADT formalism fails to give a finite result if one trivially chooses the static AdS metric as the background to calculate the mass of the black hole (24) when $a \neq 0$, so the standard Noether method to the Lagrangian of Weyl gravity was adapted, namely, they directly used the Noether potential $K^{\mu\nu}_W$ to define the conserved charge as

$$\hat{\mathcal{Q}} = \frac{1}{16\pi}\int_{r=\infty}\text{d}\Sigma_{\mu\nu}K^{\mu\nu}_W. \quad (29)$$

In fact, due to the disappearance of the $(t,r)$ component of the surface term $\xi^{[\mu}\Theta^{\nu]}_W(g;h)$ at $r=\infty$, one can only utilize the Noether potential $K^{\mu\nu}_W$, like Eq. (29), to calculate the mass of the static black hole, but there exists a divergence $M_0 = \hat{M} - M = (3a^2+2\Lambda)/(18a)$ between the mass $M$ and the one $\hat{M}$ got through the definition (29), since the quantity $\sqrt{-g}K^{\text{tr}}_W|_{m,q=0}$ does not vanish at $r=\infty$. In the uncharged case, the mass $\hat{M}$ agrees with the one computed from the conserved current that consists of the holographic response functions in [33].

At the end of this subsection, we take into account the $a=0$ case. For convenience, we recast the function $f(r)$ as

$$f(r) = \sqrt{1+q^2} - \frac{2m}{r} - \frac{1}{3}\Lambda r^2.$$

Performing the parallel analysis as the case where $a \neq 0$, we obtain the mass $M = -2\alpha m\Lambda/3$, which can also be got through the ADT formalism and the Ashtekar–Magnon–Das method [12,13,40].

### 3.2 Conserved charges of the dyonic rotating black holes

In four-dimensional conformal Weyl gravity with the Lagrangian $\mathcal{L}_{\text{total}} = \mathcal{L}_W + L_{EM}$, the charged rotating black hole solution, found in [3], takes the form

$$\text{d}s^2 = -\frac{\Delta}{\rho^2(r,\theta)}\left(\text{d}t - \frac{a\sin^2\theta}{\Xi}\text{d}\phi\right)^2 + \frac{\rho^2(r,\theta)}{\Delta}\text{d}r^2 + \frac{\rho^2(r,\theta)}{F(\theta)}\text{d}\theta^2 + \frac{F(\theta)\sin^2\theta}{\rho^2(r,\theta)}\left(a\text{d}t - \frac{r^2+a^2}{\Xi}\text{d}\phi\right)^2$$
$$\Delta = (r^2+a^2)(1+r^2\ell^2) - 2mr + \frac{1}{6}m(p^2+q^2)r^3,$$
$$\rho^2(r,\theta) = r^2 + a^2\cos^2\theta, \quad F(\theta) = 1 - a^2\ell^2\cos^2\theta,$$
$$\Xi = 1 - a^2\ell^2, \quad (30)$$

where the constants $a, m, p, q$ denote the mass, angular momentum, and magnetic and electric charges, respectively. When $m = p = q = 0$, the metric (30) becomes the conventional $AdS_4$ spacetime, with the negative cosmological constant $\Lambda = -3\ell^2$. Comparing this black hole solution with the conventional four-dimensional Kerr–Newman-AdS solution in Einstein–Maxwell gravity, one finds that the last term related to the magnetic and electric charge parameters $(p,q)$ in the function $\Delta$ is not the usual combination $p^2+q^2$ for the Kerr–Newman-AdS black hole. Such a difference makes the solution (30) has some new interesting properties [30,31].

We first calculate the energy and angular momentum of the black hole (30) in neutral case, namely, $p=q=0$. In such a case, we take an infinitesimal parametrization of a single parameter path by letting the constants $(m,a)$ fluctuate as

$$m \to m + \text{d}m, \quad a \to a + \text{d}a.$$

In addition to the Killing vector $\xi^{\mu}_{WM} = (-1,0,0,0)$, the $(t,r)$ components of the Noether potentials related to the energy are given by

$$\sqrt{-g}K^{\text{tr}}_{WM} = \frac{4\alpha m\ell^2(3\sin^2\theta + 3\Xi\cos^2\theta - \Xi)\sin\theta}{\Xi^2} + \mathcal{O}\left(\frac{1}{r^2}\right),$$
$$\sqrt{-g}Q^{\text{tr}}_{WM} = \frac{6\alpha am\ell^4[4-\Xi-(4-3\Xi)\cos^2\theta]\sin\theta\text{d}a}{\Xi^3} + \frac{2\alpha\ell^2[3-\Xi-3(1-\Xi)\cos^2\theta]\sin\theta\text{d}m}{\Xi^2} + \mathcal{O}\left(\frac{1}{r^2}\right). \quad (31)$$

Utilizing the definition of the quasi-local conserved charge (16), we obtain the energy of the neutral rotating black hole

$$M_{NR} = \frac{2m\alpha\ell^2}{\Xi^2}. \tag{32}$$

To calculate the angular momentum, the spacelike Killing vector is chosen as $\xi^{\mu}_{WJ} = (0, 0, 0, 1)$. Then the $(t, r)$ components of the Noether potentials related to the angular momentum $J_{NR}$ are presented by

$$\begin{aligned}\sqrt{-g}K^{\text{tr}}_{WJ} &= \frac{12\alpha m a\ell^2 \sin^3\theta}{\Xi^2} + \mathcal{O}\left(\frac{1}{r^2}\right),\\ \sqrt{-g}Q^{\text{tr}}_{WJ} &= \frac{6\alpha m\ell^2(4-3\Xi)\sin^3\theta \text{d}a}{\Xi^3}\\ &\quad + \frac{6\alpha a\ell^2\sin^3\theta \text{d}m}{\Xi^2} + \mathcal{O}\left(\frac{1}{r^2}\right),\end{aligned} \tag{33}$$

and the angular momentum is

$$J_{NR} = \frac{2\alpha m a\ell^2}{\Xi^2} = M_{NR}a. \tag{34}$$

The mass $M_{NR}$ and the angular momentum $J_{NR}$ in the neutral case coincide with the ones presented in [33].

Next, we consider the conserved charges of the general dyonic rotating black hole (30). The perturbation of the metric is determined by the change of the free parameters $(a, m, p, q)$ through

$$m \to m+\text{d}m, \quad a \to a+\text{d}a, \quad p \to p+\text{d}p, \quad q \to q+\text{d}q,$$

and both the timelike and the spacelike Killing vectors are identical with those in the neutral case. Under these conditions, after a bit computation, we get the energy and angular momentum

$$\begin{aligned}M_{DR} &= \frac{\alpha m\ell^2(12+a^2p^2+a^2q^2)}{6\Xi^2},\\ J_{DR} &= \frac{\alpha m a(12\ell^2+p^2+q^2)}{6\Xi^2}.\end{aligned} \tag{35}$$

If the parameters $(\ell, p, q)$ are rescaled as

$$\ell \to \frac{1}{\ell}, \quad p \to \frac{p}{m}, \quad q \to \frac{q}{m},$$

the energy $M_{DR}$ and the angular momentum $J_{DR}$ agree with those obtained via the definition (29) in [3], where it is demonstrated that both the energy and the angular momentum satisfy the first law of thermodynamics. Such a match arises from that the integral of the quantity $\xi^{[t}\Theta^{r]}_{W}(g;h)$ and $\sqrt{-g}K^{\text{tr}}_{W}\big|_{m,a,p,q=0}$ vanish at asymptotical infinity for the dyonic rotating black hole.

## 4 Conserved charges of black holes in Einstein–Gauss–Bonnet gravity

In this section, we discuss calculations on the conserved charges of charged spherically symmetric black holes in $d$-dimensional ($d > 4$) Einstein–Gauss–Bonnet gravity. The Lagrangian has the form

$$\begin{aligned}\mathcal{L}_{(\text{GB})} &= \sqrt{-g}(R - 2\Lambda + \alpha L_{(\text{GB})}),\\ L_{(\text{GB})} &= R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2,\end{aligned} \tag{36}$$

where the Gauss–Bonnet term $L_{(\text{GB})}$ can be interpreted as a quadratic curvature correction to general relativity and the negative cosmological constant $\Lambda$ is expressed as $\Lambda = -(d-1)(d-2)\ell^2/2$ in terms of the radius $1/\ell$ of the AdS spacetime. The variation of the Lagrangian (36) yields the expression of EOM

$$\begin{aligned}\mathcal{E}^{(\text{GB})}_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} + 2\alpha\Big[R_{\mu}^{\ \lambda\rho\sigma}R_{\nu\lambda\rho\sigma}\\ &\quad + RR_{\mu\nu} - 2(R^{\rho\sigma}R_{\mu\rho\nu\sigma} + R_{\mu}^{\ \lambda}R_{\nu\lambda}) - \frac{1}{4}g_{\mu\nu}L_{(\text{GB})}\Big].\end{aligned} \tag{37}$$

Like before, we now derive the surface term and the off-shell Noether potentials. For convenience, we introduce a tensor $P^{\mu\nu\rho\sigma}_{(\text{GB})}$, defined by

$$\begin{aligned}P^{\mu\nu\rho\sigma} &= \frac{\partial\mathcal{L}_{(\text{GB})}}{\partial R_{\mu\nu\rho\sigma}}\\ &= g^{\mu[\rho}g^{\sigma]\nu} + 2\alpha\Big[R^{\mu\nu\rho\sigma} - 2\Big(g^{\mu[\rho}R^{\sigma]\nu}\\ &\quad - g^{\nu[\rho}R^{\sigma]\mu}\Big) + Rg^{\mu[\rho}g^{\sigma]\nu}\Big].\end{aligned} \tag{38}$$

It is easy to prove that the tensor $P^{\mu\nu\rho\sigma}$ has the following properties:

$$\begin{aligned}&P_{\mu\nu\rho\sigma} = -P_{\nu\mu\rho\sigma} = -P_{\mu\nu\sigma\rho} = P_{\rho\sigma\mu\nu}\\ &P_{\mu[\nu\rho\sigma]} = 0, \quad \nabla_{\sigma}P^{\mu\nu\rho\sigma} = 0.\end{aligned} \tag{39}$$

With the help of the tensor $P^{\mu\nu\rho\sigma}$, the surface term $\Theta^{\mu}_{(\text{GB})}$ and the off-shell Noether potential $K^{\mu\nu}_{(\text{GB})}$ for Einstein–Gauss–Bonnet gravity are presented as

$$\Theta^{\mu}_{(\text{GB})}(g;h) = 2P^{\mu\nu\rho\sigma}\nabla_{\sigma}h_{\nu\rho}, \quad K^{\mu\nu}_{(\text{GB})} = 2P^{\mu\nu\rho\sigma}\nabla_{[\rho}\xi_{\sigma]}, \tag{40}$$

respectively, while the ADT Noether $Q^{\mu\nu}_{(\text{GB})}$ is given by

$$\begin{aligned}Q^{\mu\nu}_{(\text{GB})} &= \frac{1}{2}\delta K^{\mu\nu}_{(\text{GB})} + \frac{1}{2}hP^{\mu\nu\rho\sigma}\nabla_{[\rho}\xi_{\sigma]}\\ &\quad - 2\xi^{[\mu}P^{\nu]\lambda\rho\sigma}\nabla_{\sigma}h_{\lambda\rho},\end{aligned} \tag{41}$$

where $\delta K^{\mu\nu}_{(\text{GB})}$, the variation of the off-shell Noether potential $K^{\mu\nu}_{(\text{GB})}$, is given by Eq. (50) in the appendix.

In the remainder of this section, we utilize the above Noether potentials and the formulation of the quasi-local conserved charge (16) to calculate the energy of the general charged spherically symmetric black hole in $d$-dimensional ($d > 4$) Einstein–Gauss–Bonnet gravity, coupled to Maxwell or nonlinear electrodynamics in AdS spacetime. We first consider the case for the coupling of Maxwell electrodynamics. The metric of the asymptotically AdS black hole takes the form [4–8]

$$
\begin{aligned}
\mathrm{d}s^2 &= -H(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{H(r)} + r^2\mathrm{d}\Omega^2_{d-2},\\
A &= \frac{1}{d-3}\frac{q}{r^{d-3}}\mathrm{d}t,\\
H(r) &= 1 + \frac{r^2[1-U(r)]}{2\tilde{\alpha}},\\
U(r) &= \sqrt{1 - 4\tilde{\alpha}\ell^2 + \frac{4\tilde{\alpha}m}{r^{d-1}} - 2\alpha\frac{d-4}{d-2}\frac{q^2}{r^{2d-4}}},\\
\mathrm{d}\Omega^2_{d-2} &= \mathrm{d}\theta^2_{d-2} + \sum_{i=1}^{d-3}\sin^2\theta_{i+1}\cdots\sin^2\theta_{d-2}\mathrm{d}\theta_i^2,\\
&\quad 0 \le \theta_1 < 2\pi, \quad 0 \le \theta_i < \pi,
\end{aligned}
\tag{42}
$$

where $\tilde{\alpha} = (d-3)(d-4)\alpha$. The total Lagrangian for this black hole solution is $\mathcal{L}_{(\mathrm{GB})} - \frac{1}{4}\sqrt{-g}F_{\mu\nu}F^{\mu\nu}$.

To get the energy of the black hole (42), we take the same infinitesimal parametrization of a single parameter path like in Eq. (25). The timelike Killing vector is $\xi^{\mu}_{GB} = (-1, 0, \ldots, 0)$. Then a bit calculation gives the $(t, r)$ components of the Noether potentials $K^{\mu\nu}_{(\mathrm{GB})}$ and $Q^{\mu\nu}_{(\mathrm{GB})}$ as

$$
\begin{aligned}
\sqrt{-g}K^{\mathrm{tr}}_{(\mathrm{GB})} &= -\frac{\omega_{d-2}}{\tilde{\alpha}(d-4)}r^{d-1}[-2+(d-2)U(r)]\\
&\quad\times\left(-1+U(r)+\frac{r}{2}\frac{\mathrm{d}U}{\mathrm{d}r}\right),\\
\sqrt{-g}Q^{\mathrm{tr}}_{(\mathrm{GB})} &= \frac{(d-2)\omega_{d-2}}{8\tilde{\alpha}}r^{d-1}\left(\frac{\partial U^2(r)}{\partial m}\mathrm{d}m\right.\\
&\quad\left.+\frac{\partial U^2(r)}{\partial q}\mathrm{d}q\right),
\end{aligned}
\tag{43}
$$

where $\omega_{d-2} = \sin\theta_2\sin^2\theta_3\cdots\sin^{d-4}\theta_{d-3}\sin^{d-3}\theta_{d-2}$ is the square root of the determinant of the line element $\mathrm{d}\Omega^2_{d-2}$. Note that $\sqrt{-g}K^{\mathrm{tr}}_{(\mathrm{GB})}$ and $\sqrt{-g}Q^{\mathrm{tr}}_{(\mathrm{GB})}$ in Eq. (43) still hold even if $U(r)$ is a more general function related to $r$, $m$ and $q$. By utilizing Eq. (16), it is very easy to obtain the energy

$$
M_{(\mathrm{GB})} = \frac{(d-2)V_{d-2}}{16\pi}m,
\tag{44}
$$

where $V_{d-2} = 2\pi^{\frac{d-1}{2}}/\Gamma(\frac{d-1}{2})$ is the volume of the $(d-2)$-sphere. The mass (44) coincides with the one via ADT method [22,23] or other methods [34–42]. Such coincidences should be expected. For the conserved charges obtained from the formulas defined on basis of Noether theorem and exact Killing vectors, it was demonstrated that the conserved charges should take the same forms in [16,43] if the Noether currents are equivalent,[3] namely, they differ by a trivial current. In fact, it can be proved that the generalized ADT potential (14) matches that got from the covariant phase space approach [14,15]. This means that the Noether potentials for Weyl gravity and Einstein–Gauss–Bonnet gravity can be rederived along the line of [44], which gives an explicit form of conserved charges for general higher derivative gravity theory. What is more, the generalized ADT potential (14) is also equivalent to the one by the method developed by Barnich et al. [16–19] for the theories of general relativity and $(2n+1)$-dimensional supergravity [26,50].

From Eq. (44), one sees that the gauge field makes no contribution to the energy of the static black hole (42). This is attributed to that the term with the electric charge parameter $q$ in function $U(r)$ falls off much faster than the one with the mass parameter $m$ when $r \to \infty$ so that its contribution to the energy can be neglected. The same situation takes place for the charged spherically symmetric black hole in $d$-dimensional ($d > 4$) Einstein–Gauss–Bonnet gravity coupled to nonlinear electrodynamics in [45]. The metric of this black hole can be reexpressed as the same form as Eq. (42) except for the term associated with the electric field in $U(r)$, whose contribution to $U(r)$ is smaller than the one from the mass term. Therefore, the term including the mass parameter still plays a dominant role in determining the energy of the black hole. By utilizing the formulation of the conserved charge (16), combined with Eq. (43), we obtain the energy of the static black hole in the case for the nonlinear coupling of electrodynamics, the same as that in Eq. (44). It also matches the energy in [45].

At the end of this section, it is worth mentioning that the Gauss–Bonnet term $L_{(\mathrm{GB})}$ in the Lagrangian (36) is a surface term in $d = 4$ dimensions, namely, it makes no contribution to the equation of motion. This implies that all the black hole solutions in four-dimensional Einstein gravity are also the ones in Einstein–Gauss–Bonnet gravity. We have applied the generalized ADT formalism (16) to compute the masses and angular momenta of the four-dimensional Kerr(–AdS) and Kerr–Newman black holes corrected by the Gauss–Bonnet term. However, our results show that this term makes no corrections to all the conserved charges, compared with their corresponding ones in Einstein gravity.

## 5 Conclusions and discussions

In this paper, we have extended the off-shell generalization of the conventional ADT formalism proposed in [24] to calculate the quasi-local conserved charges for the most general charged static and the dyonic rotating black holes with AdS asymptotics in four-dimensional conformal Weyl gravity, as

[3] We thank the anonymous referee for pointing this.

well as the charged spherically symmetric black holes in higher dimensional Einstein–Gauss–Bonnet gravity coupled to Maxwell or nonlinear electrodynamics in AdS spacetime. Our results confirm those through other methods in the literature. To do this, we first directly vary the Bianchi identity (3), together with the help of the property of a Killing vector, to get the off-shell Noether currents (8) and (12), which are formally different from the ones in [24]. But they are actually equal to each other since both the surface term $\Theta^\mu(g;\mathcal{L}_\xi g)$ and the Lie derivative of the surface term $\Theta^\mu(g;\delta g)$ with respect to the Killing vector $\xi$ vanish for the generally diffeomorphism gravity theory. The Noether current (8) makes it natural to derive the corresponding off-shell Noether potential (15), without a further requirement of the property for the symplectic current. Next, we present the explicit expressions of the surface term and Noether potential for Weyl gravity as the ones in Eqs. (20) and (22). Utilizing these quantities, we obtain the mass (27) of the most general static black hole in four-dimensional Weyl gravity, as well as the mass and angular momentum in Eq. (35) for the dyonic rotating black hole, although a naive application of the original ADT method fails to give a finite result in the static case. Finally, as the case of Weyl gravity, we start by deriving the surface term and the off-shell Noether potential in Eq. (40) for the general Lagrangian (36) and then utilize them to gain the energy (44) of the general charged static asymptotically AdS black hole in higher dimensional Einstein–Gauss–Bonnet gravity coupled to Maxwell or nonlinear electrodynamics. The energy (44) is independent on the electric parameter due to a fast fall-off of the term related to electric field.

Weyl gravity and Einstein–Gauss–Bonnet gravity are two typical fourth-order derivative gravity theories. It has been proposed that the general fourth-order gravity admits a critical theory in [47,48]. The application of the ADT formalism to the critical theory demonstrates that the mass and angular momenta of all asymptotically Kerr–AdS and Schwarzschild–AdS black holes vanish at the critical point [40,48]. We expect to learn whether the generalized formulation of the ADT charge supports this or not.

Although the formulation (16) for the quasi-local conserved charge is defined by only taking into account of the contribution from the pure gravity part, our analysis on charged black holes implies that it may be applicable to the black holes with matter fields, if the terms including matter fields in the given metric fall off fast enough at asymptotic infinity to ensure that the formulation (16) is convergent. Otherwise the effect of matter fields must be considered [26]. For instance, the formulation (16) fails to give a finite mass when it is utilized to the charged rotating Gödel-type black hole [49] in five-dimensional minimal supergravity since the $(t,r)$ component of the Noether potential (14) is divergent at infinity if the contribution from gauge field is omitted. To get finite conserved charges for the Gödel-type black hole, the effect of the gauge field has to be incorporated into the definition like in [50]. What is more, even if the conserved charge through the expression (16) is well defined in the presence of matter fields, it is possible for one to omit a finite value[4] from the actions of the matter fields. In order to overcome these, the contribution from the matter fields has to be taken into account in future work.

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## 6 Appendix: Variations of the potentials $K_W^{\mu\nu}$ and $K_{(GB)}^{\mu\nu}$

In this appendix, we give the explicit expressions for the variation of the Noether potential $K^{\mu\nu}$ in both the Weyl and the Einstein–Gauss–Bonnet gravity theories. Note that the Riemann curvature in our conventions is defined through $(\nabla_\mu\nabla_\nu-\nabla_\nu\nabla_\mu)\omega_\rho = R_{\mu\nu\rho}{}^{\lambda}\omega_\lambda$, where $\omega_\rho$ is an arbitrary vector.

For four-dimensional Weyl gravity, $\delta K_W^{\mu\nu}$ reads

$$\begin{aligned}\delta K_W^{\mu\nu} =\ & 2\alpha\delta C^{\mu\nu\rho\sigma}\nabla_{[\rho}\xi_{\sigma]}\\ & +2\alpha C^{\mu\nu\rho\sigma}(\xi^\lambda\nabla_{[\rho}h_{\sigma]\lambda}-h_{\lambda[\rho}\nabla_{\sigma]}\xi^\lambda)\\ & -4\alpha h_{\sigma\rho}\xi^\rho\left(\nabla^{[\mu}R^{\nu]\sigma}+\frac{1}{6}g^{\sigma[\mu}\nabla^{\nu]}R\right)\\ & -4\alpha\xi_\sigma\bigg(-h^{\rho[\mu}\nabla_\rho R^{\nu]\sigma}+\nabla^{[\mu}\delta R^{\nu]\sigma}\\ & +2R^{\rho\sigma}\nabla^{[\mu}h^{\nu]}{}_\rho-R^{\rho[\mu}\nabla^{\nu]}h^\sigma{}_\rho-R^{\rho[\mu}\nabla_\rho h^{\nu]\sigma}\\ & +R^{\rho[\mu}\nabla^{|\sigma|}h^{\nu]}{}_\rho-\frac{1}{6}h^{\sigma[\mu}\nabla^{\nu]}R-\frac{1}{6}g^{\sigma[\mu}h^{\nu]\rho}\nabla_\rho R\\ & +\frac{1}{6}g^{\sigma[\mu}\nabla^{\nu]}\delta R\bigg),\end{aligned} \tag{45}$$

where the variation of the Weyl tensor $C^{\mu\nu\rho\sigma}$ is given by

$$\begin{aligned}\delta C^{\mu\nu\rho\sigma} =\ & \frac{1}{3}\left(\delta R g^{\mu[\rho}g^{\sigma]\nu}-Rh^{\mu[\rho}g^{\sigma]\nu}-Rg^{\mu[\rho}h^{\sigma]\nu}\right)\\ & +2\left(h^{[\mu|[\rho}R^{\sigma]|\nu]}-g^{[\mu|[\rho}\delta R^{\sigma]|\nu]}\right)\\ & +\delta R^{\mu\nu\rho\sigma}.\end{aligned} \tag{46}$$

[4] We have considered the effect from the action $L_{EM}=\alpha\sqrt{-g}F_{\mu\nu}F^{\mu\nu}$ along the line of [14,15,26]. It makes no contribution to the total conserved charges of the black holes in this work.

In the above two equations, $\delta R^{\mu\nu\rho\sigma}$, $\delta R^{\mu\nu}$ and $\delta R$ are presented by

$$\delta R^{\mu\nu\rho\sigma} = -h^{\lambda[\mu} R_{\lambda}{}^{\nu]\rho\sigma} - 2h^{\lambda[\rho} R_{\lambda}{}^{\sigma]\mu\nu} + \nabla^{[\rho}\nabla^{|\nu|} h^{\sigma]\mu} - \nabla^{[\rho}\nabla^{|\mu|} h^{\sigma]\nu}, \quad (47)$$

$$\delta R^{\mu\nu} = -2h^{\rho(\mu} R^{\nu)}{}_{\rho} + \nabla_{\rho}\nabla^{(\mu} h^{\nu)\rho} - \frac{1}{2}\left(\nabla^{\rho}\nabla_{\rho} h^{\mu\nu} + \nabla^{\mu}\nabla^{\nu} h\right), \quad (48)$$

$$\delta R = \nabla_{\mu}\nabla_{\nu} h^{\mu\nu} - \nabla^{\rho}\nabla_{\rho} h - h^{\mu\nu} R_{\mu\nu}. \quad (49)$$

For Einstein–Gauss–Bonnet gravity, the variation of the off-shell Noether potential $K^{\mu\nu}_{(\mathrm{GB})}$ takes the form

$$\begin{aligned}\delta K^{\mu\nu}_{(\mathrm{GB})} &= 2\delta P^{\mu\nu\rho\sigma}\nabla_{[\rho}\xi_{\sigma]} + 2P^{\mu\nu\rho\sigma}(\xi^{\lambda}\nabla_{[\rho} h_{\sigma]\lambda} - h_{\lambda[\rho}\nabla_{\sigma]}\xi^{\lambda}) \\ \delta P^{\mu\nu\rho\sigma} &= 2\alpha\Big[\delta R^{\mu\nu\rho\sigma} + 4\left(h^{[\mu|[\rho} R^{\sigma]|\nu]} - g^{[\mu|[\rho}\delta R^{\sigma]|\nu]}\right) + \delta R g^{\mu[\rho} g^{\sigma]\nu} - R\left(h^{\mu[\rho} g^{\sigma]\nu} + g^{\mu[\rho} h^{\sigma]\nu}\right)\Big] \\ &\quad - \left(h^{\mu[\rho} g^{\sigma]\nu} + g^{\mu[\rho} h^{\sigma]\nu}\right). \end{aligned} \quad (50)$$

## References

1. K.S. Stelle, Phys. Rev. D **16**, 953 (1977)
2. R.J. Riegert, Phys. Rev. Lett. **53**, 315 (1984)
3. H.S. Liu, H. Lu, JHEP **1302**, 139 (2013)
4. D.G. Boulware, S. Deser, Phys. Rev. Lett. **55**, 2656 (1985)
5. J.T. Wheeler, Nucl. Phys. B **268**, 737 (1986)
6. D.L. Wiltshire, Phys. Lett. B **169**, 36 (1986)
7. R.G. Cai, Phys. Rev. D **65**, 084014 (2002)
8. M. Cvetic, S. Nojiri, S.D. Odintsov, Nucl. Phys. B **628**, 295 (2002)
9. M. Henningson, K. Skenderis, JHEP **9807**, 023 (1998)
10. V. Balasubramanian, P. Kraus, Commun. Math. Phys. **208**, 413 (1999)
11. J.D. Brown, J.W. York, Phys. Rev. D **47**, 1407 (1993)
12. A. Ashtekar, A. Magnon, Class. Quantum Grav. **1**, L39 (1984)
13. A. Ashtekar, S. Das, Class. Quantum Grav. **17**, L17 (2000)
14. V. Iyer, Robert M. Wald, Phys. Rev. D **50**, 846 (1994)
15. R.M. Wald, A. Zoupas, Phys. Rev. D **61**, 084027 (2000)
16. G. Barnich, F. Brandt, Nucl. Phys. B **633**, 3 (2002)
17. G. Barnich, Class. Quantum Grav. **20**, 3685 (2003)
18. G. Barnich, G. Compere, Phys. Rev. D **71**, 044016 (2005). Erratum ibid **73**, 029904(2006)
19. G. Barnich, G. Compere, J. Math. Phys. **49**, 042901 (2008)
20. L.F. Abbott, S. Deser, Nucl. Phys. B **195**, 76 (1982)
21. L.F. Abbott, S. Deser, Phys. Lett. B **116**, 259 (1982)
22. S. Deser, B. Tekin, Phys. Rev. Lett. **89**, 101101 (2002)
23. S. Deser, B. Tekin, Phys. Rev. D **67**, 084009 (2003)
24. W. Kim, S. Kulkarni, S.-H. Yi, Phys. Rev. Lett. **111**, 081101 (2013)
25. W. Kim, S. Kulkarni, S.-H. Yi, Phys. Rev. D **88**, 124004 (2013)
26. S. Hyun, J. Jeong, S.A Park, S.-H. Yi, arXiv:1406.7101 [hep-th]
27. Y. Gim, W. Kim, S.-H. Yi, JHEP **1407**, 002 (2014)
28. A. Bouchareb, G. Clement, Class. Quantum Grav. **24**, 5581 (2007)
29. T. Padmanabhan, Rept. Prog. Phys. **73**, 046901 (2010)
30. M. Cvetic, H. Lu, C.N. Pope, Phys. Rev. D **88**, 044046 (2013)
31. G.M. Deng, Gen. Rel. Grav. **46**, 1757 (2014)
32. H. Lu, Y. Pang, C.N. Pope, J. Vazquez-Poritz, Phys. Rev. D **86**, 044011 (2012)
33. D. Grumiller, M. Irakleidou, I. Lovrekovic, R. McNees, Phys. Rev. Lett. **112**, 111102 (2014)
34. N. Deruelle, J. Katz, S. Ogushi, Class. Quantum Grav. **21**, 1971 (2004)
35. A.N. Petrov, Class. Quantum Grav. **22**, L83 (2005)
36. N. Okuyama, J. Koga, Phys. Rev. D **71**, 084009 (2005)
37. G. Kofinas, R. Olea, Phys. Rev. D **74**, 084035 (2006)
38. H. Maeda, M. Nozawa, Phys. Rev. D **77**, 064031 (2008)
39. R.G. Cai, L.M. Cao, Y.P. Hu, N. Ohta, Phys. Rev. D **80**, 104016 (2009)
40. Y. Pang, Phys. Rev. D **83**, 087501 (2011)
41. M.H. Dehghani, G.H. Bordbar, M. Shamirzaie, Phys. Rev. D **74**, 064023 (2006)
42. Y. Brihaye, E. Radu, JHEP **0809**, 006 (2008)
43. G. Compere, arXiv:gr-qc/0611129
44. T. Azeyanagi, G. Compere, N. Ogawa, Y. Tachikawa, S. Terashima, Prog. Theor. Phys. **122**, 355 (2009)
45. O. Miskovic, R. Olea, Phys. Rev. D **83**, 024011 (2011)
46. G. Lopes Cardoso, B. de Wit, T. Mohaupt, Fortsch. Phys. **48**, 49 (2000)
47. H. Lu, C.N. Pope, Phys. Rev. Lett. **106**, 181302 (2011)
48. S. Deser, H.S. Liu, H. Lu, C.N. Pope, T.C. Sisman, B. Tekin, Phys. Rev. D **83**, 061502 (2011)
49. S.Q. Wu, Phys. Rev. Lett. **100**, 121301 (2008)
50. G. Barnich, G. Compere, Phys. Rev. Lett. **95**, 031302 (2005)