ALGEBRA OF CURRENTS AND SOME APPLICATIONS TO ELEMENTARY PARTICLE PHYSICS

by

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PREFACE

The work presented in this thesis was carried out at the Department of Mathematics, University of Durham in the period from October 1964 to May 1967 under the supervision of Dr. D.B.Fairlie. The author wishes to express his sincere thanks to Dr. Fairlie for continued guidance and encouragement. The author also wishes to thank Professor E.J.Squires for help and advice. The author's gratitude also goes to his colleagues and especially to Anis Alam and Peter Watson for stimulating discussions.

Except where stated in the text, the work presented here is original and has not been submitted for any other degree in this or any other University. It is based essentially on three papers by the author.

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CONTENTS

ABSTRACT

We study the implications of the algebra of currents to elementary particle processes. In Chapter 1 we introduce the concepts of current algebra and discuss how the information contained in the current commutators can be used to set up sum rules and in particular to evaluate strong interaction renormalization effects.

In Chapter II we apply some of the techniques developed in Chapter 1 and obtain consistency conditions for the K-meson scattering amplitude.

In the third chapter we illustrate the methods further by calculating the F/D ratio and find good agreement with experiment.

Finally, in Chapter IV we are concerned with calculating the coupling constant of the so far hypothetical σ-meson to nucleon states. We find that the techniques of current algebras enable us to do so and we find a value for its coupling constant which should ultimately be testable experimentally should the σ-meson exist as a physical particle.

CHAPTER 1

The theory of current algebras has received a great deal of attention in the last two years and is very successful in explaining several features of elementary particle physics. It is somewhat surprising that although the current algebraic approach was invented and used by Gell-Mann[1] in his discussion of SU(3) symmetry in 1961, yet the full power of the current commutation relations was not recognized until much later.

The point of view advocated by Gell-Mann is that a broken symmetry is thought to be a manifestation of an exact algebraic structure of operators which do not all commute with the Hamiltonian. We begin with a brief discussion of exact symmetries:

Consider a set of Hermitian operators $\{F\}$ in the Hilbert space of physical states such that

a) The set $\{F\}$ is a linear space i.e. if $F_1, F_2 \in \{F\}$ then $(a_1F_1 + a_2F_2) \in \{F\}$ with a_1, a_2 real

b) $\{F\}$ is closed under commutation i.e. if $F_1, F_2 \in \{F\}$ then $i[F_1, F_2] \equiv i(F_1F_2 - F_2F_1)$ is Hermitian and belongs to $\{F\}$.

We shall concern ourselves with finite-dimensional spaces so that $\{F\}$ is then a Lie algebra. We choose a basis F^i $(i = 1,\ldots,N;$ N is the dimension of the algebra) in $\{F\}$ and form the following basic commutators

$$[F^i, F^j] = i\, c^{ijk} F^k \tag{1.1}$$

Repeated indices are summed over as usual. The numbers c^{ijk} thus generated are real because the F^i 's are Hermitian and are called the structure constants of the algebra.

The operators F^i generate a group of transformations G which also acts on the space of physical states. The infinitesimal elements of G are given by

$$U(\epsilon) = 1 + i \sum_{j=1}^{N} \epsilon^j F^j \tag{1.2}$$

where $\epsilon = (\epsilon', \ldots, \epsilon^N)$ is a set of N real infinitesimal parameters. The elements of the group further from the identity are generated by exponentiation. Since the F^i are Hermitian, $U(\epsilon)$ and hence all elements of G are unitary

$$U^+U = 1 , \quad U \in G \tag{1.3}$$

Let H be the Hamiltonian of a given physical system. If $\forall\, F^i \in \{F\}$

$$[F^i, H] = 0 \tag{1.4}$$

Which implies

$$[U, H] = 0 , \quad U \in G \tag{1.5}$$

then G is said to be an exact internal symmetry group for the considered physical system. It then follows that if $|A>$ is an arbitrary eigenstate of H then $U|A>$ is also an eigenstate of H, $\forall\ U \epsilon\ G$.

Thus the basic ingredients of a symmetry group are a set of operators $\{F\}$ which form a Lie algebra and commute with the Hamiltonian. A broken symmetry is then described by a set of operators which also close under commutation i.e. still form a Lie algebra but do not all commute with the Hamiltonian. It is important to realize that irrespective of how badly the symmetry is broken the algebra is exact.

In order to demonstrate the fact that a set of operators could exist and have a simple and exact algebraic structure even if they do not all commute with the Hamiltonian (i.e. are not all conserved) we consider the simplest and oldest algebra known to elementary particle physicists, namely that of the isotopic spin.

We consider an elementary field theoretic model with two basic particles having the same quantum numbers as the proton and neutron and in interaction with an electromagnetic field A_μ. The field describing such a doublet is written $\psi_\mu^a(x)$ where $a = 1, 2$ is an index describing the internal degree of freedom and μ is the usual spinor index $\mu = 0, \ldots, 3$. We write a Lagrangian for this system as

$$\mathcal{L} = \mathcal{L}_0 + \frac{ie}{2}\bar{\psi}(1 + \tau_3)\,\gamma_\mu \psi\, A_\mu \qquad (1.6)$$

Here τ_3 is the usual Pauli matrix. $\mathcal{L}_0$ is given by the sum of free Lagrangians for the spinor and electromagnetic fields

$$\mathcal{L}_0 = \bar{\psi}(\gamma + m)\psi + \tfrac{1}{4} F_{\mu\upsilon} F^{\mu\upsilon} \qquad (1.7)$$

The fields ψ_μ^a are Fermi fields and obey the following anti-commutation rules at equal times

$$\left.\begin{aligned} \{ \psi_\mu^{a+}(x), \psi_\upsilon^b(y) \}_{x_0=y_0} &= \delta_{ab}\delta_{\mu\upsilon}\delta^3(\underline{x} - \underline{y}) \\ \{ \psi_\mu^a(x), \psi_\upsilon^b(y) \}_{x_0=y_0} &= \{ \psi_\mu^{+a}(x), \psi_\upsilon^{+b}(y) \} = 0 \end{aligned}\right\} \qquad (1.8)$$

We now define a set of currents by

$$j_\mu^i = \bar{\psi}(x)\tau^i\gamma_\mu\psi(x) , \quad i = 1,2,3 \qquad (1.9)$$

and three operators I^i as the spatial integrals of the time components of the above currents i.e.

$$I^i = \int j_0^i(x) d^3x \qquad (1.10)$$

We then evaluate the commutator $[I^i, I^j]$ using the anticommutation properties of the Fermi fields (1.8) and the commutation properties of the Pauli matrices. If we evaluate the commutator formally, disregarding any complications arising from the fact that the product of several field operators at a single space-time point is an ill-defined quantity, we arrive at the result that

$$[I^i, I^j] = i \epsilon^{ijk} I^k \qquad (1.11)$$

This is immediately recognized as the SU(2) algebra. This result enables us to identify the set I^i with the generators of the isospin group. It is easy to see from the Lagrangian that I_3 is conserved while I_1 and I_2 are not. We have thus demonstrated our initial assertion. As will be discussed further later on, the basic philosophy of this approach to the problem of broken symmetries is to set up a model which is at least consistent with the basic principles of field theory, relativity and causality. This model is usually taken to be a "Lagrangian quark model". We then construct the currents given by such a model and work out the commutation relations. We finally abstract these properties and assume their validity in general and discard the model.

Generalization to SU(3) and Chiral SU(3) ⊗ SU(3) :

The original motivation for current algebra came from the success of the universal V - A theory of weak interactions by Feynmann and Gell-Mann[2] and others[3]. According to this theory the basic Hamiltonian density for weak interactions is of the current-current type

$$H = \frac{G_0}{\sqrt{2}} \sum_{n,m} \left(j_\mu^{(n)} j_\mu^{+(m)} + \text{h.c} \right) \qquad (1.12)$$

where n,m run over basic lepton and hadron fields. This is analogous to the electromagnetic interaction between charged particles after the electromagnetic field has been eliminated. The current j involves only left-handed components of the fields. By basic fields here we mean (e, ν_e) and (μ, ν_μ) for the leptons and some fundamental hadronic field for the baryons and mesons. The simplest such field we can assume is the quark triplet $q = (q^1, q^2, q^3)$. We shall take the quarks to be spin $\frac{1}{2}$ particles obeying Fermi statistics. In terms of quark fields we construct the vector and axial vector currents of the weak interactions by

$$\left.\begin{aligned} j^i_\mu(x) &= \bar{q}(x)\gamma_\mu\lambda^i q(x) \\ j^i_{5\mu}(x) &= \bar{q}(x)\gamma_\mu\gamma_5\lambda^i q(x) \end{aligned}\right\} \quad (1.13)$$

where the λ^i are the Gell-Mann matrices obeying the SU(3) algebra

$$\left.\begin{aligned} [\lambda^i, \lambda^j] &= i\, f_{ijk}\, \lambda^k \\ \text{and} \qquad \{\lambda^i, \lambda^j\} &= d_{ijk}\, \lambda^k \end{aligned}\right\} \quad (1.14)$$

where i runs from one to eight. Using the basic commutation properties (1-8) of Dirac fields together with the relations (1.14) we find with the help of the identity

$$[A \otimes B, A' \otimes B'] = \frac{1}{2}\left\{ [A,A'] \otimes \{B,B'\} + \{A,A'\} [B,B'] \right\} \quad (1.15)$$

that the following commutation relations hold

$$\delta(x_o - y_o)[\, j_o^{\,i}(x),\ j_\mu^j(y)\,] = i\, f_{ijk}\, j_\mu^k(x)\delta^4(x - y) \qquad (1.16)$$

$$\delta(x_o - y_o)[\, j_o^{\,i}(x),\ j_{5\mu}^j(y)] = i\, f_{ijk} j_{5\mu}^k(x)\delta^4(x - y) \qquad (1.17)$$

$$\delta(x_o - y_o)[\, j_{50}^i(x),\ j_{5\mu}^j(y)] = i\, f_{ijk}\, j_\mu^k(x)\delta^4(x - y) \qquad (1.18)$$

These relations follow by formal application of the canonical commutation properties (1.18). Schwinger[4] pointed out that extra terms proportional to the spatial derivatives of delta-functions could also be present on the R.H.S. of equations (1.16) - (1.18) . We shall return to a discussion of such terms later. Generalizing the definition of the isospin operators (1.10) we define vector and axial vector charges in the following way:

$$Q^i(t) = \int_{x_o=t} d^3x\ j_o^i(x) \qquad (1.19)$$

$$Q_5^i(t) = \int_{x_o=t} d^3x\ j_{50}^i(x) \qquad (1.20)$$

With these definitions the following relations are derived

$$[\, Q^i(t),\ j_\mu^j(x)\,]_{x_o=t} = i\, f_{ijk} j_\mu^k(x) \qquad (1.21)$$

$$[\, Q^i(t),\, j^j_{5\mu}(x)\,]_{x_o = t} = i\, f_{ijk}\, j^k_{5\mu}(x) \tag{1.22}$$

$$[\, Q^i_5(t),\, j^j_{\mu}(x)\,]_{x_o = t} = i\, f_{ijk}\, j^k_{5\mu}(x) \tag{1.23}$$

$$[\, Q^i_5(t),\, j^j_{5\mu}(x)\,]_{x_o = t} = i\, f_{ijk}\, j^k_{\mu}(x) \tag{1.24}$$

plus

$$[\, Q^i(t),\, Q^j(t)\,] = i\, f_{ijk}\, Q^k(t) \tag{1.25}$$

$$[\, Q^i(t),\, Q^j_5(t)\,] = i\, f_{ijk}\, Q^k_5(t) \tag{1.26}$$

$$[\, Q^i_5(t),\, Q^j_5(t)\,] = i\, f_{ijk}\, Q^k(t) \tag{1.27}$$

From (1.25) we see that the charges $Q^i(t)$ at equal times satisfy the algebra of SU(3). The Lie algebra formed by the 16 charges $Q^i(t)$ and $Q^i_5(t)$ can be decomposed into two disjoint algebras. By defining the new operators $Q^{\pm}_i(t)$ by :

$$Q^{\pm}_i(t) = \frac{1}{\sqrt{2}}\, [\, Q^i(t) \pm Q^i_5(t)\,] \tag{1.28}$$

we find

$$[\, Q^+_i(t),\, Q^+_j(t)\,] = i\, f_{ijk}\, Q^+_k(t) \tag{1.29}$$

$$[\, Q^-_i(t),\, Q^-_j(t)\,] = i\, f_{ijk}\, Q^-_k(t) \tag{1.30}$$

$$[\, Q_i^+(t),\ Q_j^-(t) \,] = 0 \qquad (1.31)$$

The algebra generated by $Q_i^{\pm}(t)$ is that of chiral SU(3) $\otimes$ SU(3). We now turn to a discussion of the Schwinger terms.

As mentioned before the existence of such terms was first pointed out by Schwinger[4] who remarked that the commutator of the time component of the electromagnetic current with a space component must be non-vanishing otherwise an inconsistency arises in the theory. The presence of such terms is due to the necessity of defining the singular product of two field operators at the same point as the limit of the product of two operators at different points as the points approach each other. Johnson and Low[5] have looked for Schwinger terms in a simple theory of quarks interacting with a scalar neutral boson using perturbation theory to compute the commutators. While they find these extra terms in general they do not appear in the algebra of time components of vector and axial vector currents.

Here we give an elementary discussion which demonstrates the existence of such terms in the quark model. We want to evaluate the commutator

$$[\ \psi^+(\underline{x},0)M\ \psi(\underline{x},0),\quad \psi^+(\underline{x}',0)M'\ \psi(\underline{x}',0)\ ]$$

where M, M' are matrices acting on both Dirac and unitary spin indices. To define the commutator as a product of distributions we take the limit $\epsilon, \epsilon' \to 0$ of

$$[\, \psi^+_\alpha (\underline{x} - \underline{\epsilon}/2, 0) M_{\alpha\beta}\, \psi_\beta(\underline{x} + \underline{\epsilon}/2, 0),$$

$$\psi^+_{\alpha'}(\underline{x}' - \underline{\epsilon}'/2, 0) M_{\alpha'\beta'}\, \psi_{\beta'}(\underline{x}' + \underline{\epsilon}'/2, 0)\,] \,.$$

Using the canonical commutation relations (1.18) we simplify this to

$$\lim_{\substack{\epsilon \to 0 \\ \epsilon' \to 0}} \Big[\psi^+_\alpha (\underline{x} - \underline{\epsilon}/2, 0) M_{\alpha\beta} \delta_{\beta\alpha'} M'_{\alpha'\beta'}\, \psi_{\beta'}(\underline{x}' + \underline{\epsilon}'/2, 0)$$

$$\delta^3(\underline{x} + \underline{\epsilon}/2 - \underline{x}' + \underline{\epsilon}'/2) - \psi^+_\alpha (\underline{x} - \underline{\epsilon}/2, 0) M_{\alpha\beta}$$

$$\psi^+_{\alpha'}(\underline{x}' - \underline{\epsilon}'/2, 0)\, \psi_\beta(\underline{x} + \underline{\epsilon}/2, 0) M'_{\alpha'\beta'} \psi_{\beta'}(\underline{x}' + \underline{\epsilon}'/2, 0)$$

$$- \psi^+_{\alpha'} (\underline{x}' - \underline{\epsilon}'/2, 0) M'_{\alpha'\beta'} \delta_{\beta'\alpha} M_{\alpha\beta} \psi_\beta(\underline{x} + \underline{\epsilon}/2, 0)$$

$$\delta^3(\underline{x}' + \underline{\epsilon}'/2 - \underline{x} - \underline{\epsilon}/2) + \psi^+_{\alpha'}(\underline{x}' - \underline{\epsilon}'/2, 0) M'_{\alpha'\beta'} M_{\alpha\beta}$$

$$\psi^+_\alpha (\underline{x} - \underline{\epsilon}/2, 0) \psi_{\beta'}(\underline{x}' + \underline{\epsilon}'/2, 0) \psi_\beta(\underline{x} + \underline{\epsilon}/2, 0) \Big] \qquad (1.32)$$

The 2nd and 4th terms cancel and we can take the limit $\epsilon \to 0$ and write

$$\lim_{\epsilon' \to 0} \Big[\psi^+_\alpha (\underline{x},0)(MM')_{\alpha\beta'} \psi_{\beta'}(\underline{x}' + \epsilon'/_2, 0)\delta^3(\underline{x} - \underline{x}' + \epsilon'/_2)$$

$$- \psi^+_{\alpha'}(\underline{x}' - \epsilon'/_2, 0)(M'M)_{\alpha'\beta} \psi_\beta(\underline{x},0)\delta^3(\underline{x}' + \epsilon/_2 - \underline{x})\Big] \tag{1.33}$$

By virtue of the δ-function we can write this as

$$\lim_{\epsilon' \to 0} \Big[\psi^+_\alpha(\underline{x}' - \epsilon'/_2, 0)(MM')_{\alpha\beta'} \psi_{\beta'}(\underline{x}' + \epsilon'/_2, 0)\delta^3(\underline{x} - \underline{x}' + \epsilon'/_2)$$

$$- \psi^+_{\alpha'} (\underline{x}' - \epsilon'/_2, 0)(M'M)_{\alpha'\beta} \psi_\beta(\underline{x}' + \epsilon'/_2, 0)$$

$$\delta^3(\underline{x} - \underline{x}' - \epsilon'/_2) \Big] \tag{1.34}$$

By adding and subtracting terms we finally rewrite this in a form which involves commutators and anticommutators of M and M'

$$\lim_{\epsilon' \to 0} \Big[\tfrac{1}{2}\psi^+(\underline{x}' - \underline{\epsilon}'/_2, 0)[M,M']\, \psi(\underline{x}' + \underline{\epsilon}'/_2, 0)$$

$$\left(\delta^3(\underline{x} - \underline{x}' + \underline{\epsilon}'/_2) + \delta^3(\underline{x} - \underline{x}' - \underline{\epsilon}'/_2) \right)$$

$$+ \tfrac{1}{2}\psi^+(\underline{x}' - \underline{\epsilon}'/_2, 0)\{M,M'\}\, \psi(\underline{x}' + \underline{\epsilon}'/_2, 0)$$

$$\left(\delta^3(\underline{x} - \underline{x}' + \underline{\epsilon}'/_2) - \delta^3(\underline{x} - \underline{x}' - \underline{\epsilon}'/_2) \right) \Big] \tag{1.35}$$

The 1st term is the ordinary term and the 2nd one is the Schwinger term

$$(\text{Ordinary term}) = \psi^+(\underline{x}', 0)[M,M']\, \psi(\underline{x},0)\delta^3(\underline{x} - \underline{x}') \tag{1.36}$$

$$(\text{Schwinger term}) = \lim_{\epsilon' \to 0} \frac{\underline{\epsilon}}{2} \cdot [\nabla_{\underline{x}}\, \delta^3(\underline{x} - \underline{x}')]$$

$$\times\ \psi^+(\underline{x}' - \underline{\epsilon}/_2, 0)\{M,M'\}\, \psi(\underline{x}' + \underline{\epsilon}/_2, 0) \tag{1.37}$$

Taking, for example, M,M' to be of the form $\gamma_o\gamma_\mu \otimes \lambda^i$ we have by (1.15)

$$[M,M'] = \tfrac{1}{2}[\gamma_0\gamma_\mu, \gamma_0\gamma_\nu]\{\lambda^i,\lambda^j\} + \tfrac{1}{2}\{\gamma_0\gamma_\mu, \gamma_0\gamma_\nu\}[\lambda^i,\lambda^j] \qquad (1.38)$$

which is completely antisymmetric under the simultaneous exchange of the Lorentz and unitary spin indices. In a similar manner

$$\{M,M'\} = \tfrac{1}{2}[\gamma_0\gamma_\mu, \gamma_0\gamma_\nu][\lambda^i,\lambda^j] + \tfrac{1}{2}\{\gamma_0\gamma_\mu, \gamma_0\gamma_\nu\}\{\lambda^i,\lambda^j\} \qquad (1.39)$$

which is completely symmetric. In order to eliminate such contributions some authors(6) start from properly symmetrised commutators before deriving sum rules. We wish to point out that in the quark model these Schwinger terms being proportional to derivatives of δ-functions disappear when the spatial integration is performed and hence do not contribute to the commutation relations between charges.

The PCAC Principle :

This important principle provides a link between the weak and strong interactions. Originally the PCAC hypothesis was put forward by Gell-Mann and Lévy(7) and independently by Nambu(7) to find a simple justification for the Goldberger-Treiman relation(8). The formulation due to Gell-Mann and Lévy is as follows. Consider the isotopic axial vector current $j^i_{5\mu} = \bar{q}\,\gamma_\mu\gamma_5\tau^i q$. Its divergence has the same quantum numbers as the pion field ϕ^i. This is true at least if there are no hidden quantum numbers that can distinguish them. Then we can use $\partial_\mu j^i_{5\mu}$ as the definition of the pion field after proper normalization

$$\partial_\mu j^i_{5\mu}(x) = C_\pi \phi^i(x) \tag{1.40}$$

where C_π is a constant. It is known that there is no unique way of defining a phenomenological field for a particle. Different definitions of the field agree by necessity on the mass-shell but may differ if we go off the mass-shell. Now the pion-nucleon coupling constant g is defined by

$$(m_\pi^2 - q^2) < p_2|\phi^i|p_1 > = g\, K_{NN\pi}(q^2) i\, \bar{u}\,(p_2)\gamma_5\tau^i u(p_1) \tag{1.41}$$

where the form-factor $K_{NN\pi}$ is normalized to unity at the pole, $K_{NN\pi}\,(m_\pi^2) = 1$ and $|p_1>$, $|p_2>$ are single nucleon states, q^2 being the square of momentum transfer. But from (1.40) the L.H.S. of (1.41) can be written as

$$\frac{1}{C_\pi}\, iq_\mu < p_2|j^i_{5\mu}|p_1 > = \frac{1}{C_\pi}\, iq_\mu \bar{u}(p_2) \Big[\gamma_\mu\gamma_5\, G_1(q^2) + q_\mu\gamma_5 G_2(q^2) + \sigma_{\mu\nu}q_\nu\gamma_5 G_3(q^2)\Big]\, \tau^i u(p_1) \tag{1.42}$$

The expansion of the axial-vector matrix element into three form-factors is dictated by Lorentz invariance. Equation (1.42) simplifies to

$$\frac{1}{C_\pi} iq_\mu < p_2 | j^i_{5\mu} | p_1 > = \frac{1}{C_\pi} i\,\bar{u}(p_2)\Big[2M_N G_1(q^2) + q^2 G_2(q^2) \Big] \tau^i \gamma_5\, u(p_1)$$

(1.43)

Comparing this with (1.41) and taking the limit $q^2 \rightarrow 0$ we find

$$g(0)K_{NN\pi}(0) = \frac{1}{C_\pi} 2M_N m_\pi^2\, G_1(0) \qquad (1.44)$$

On the R.H.S. we have extrapolated values for the pion-nucleon coupling constant and the pionic form-factor of the nucleon. Now the pion weak decay amplitude is defined by

$$< 0 | j^i_{5\mu} | \pi^i(q) > = i\, F_\pi(q^2) q_\mu \qquad (1.45)$$

Therefore

$$< 0 | \partial_\mu\, j^i_{5\mu} | \pi^i(q) > = m_\pi^2\, F_\pi \qquad (1.46)$$

By PCAC this is equal to

$$C_\pi < 0 | \phi^i | \pi^i > = C_\pi \qquad (1.47)$$

Therefore

$$C_\pi = m_\pi^2\, F_\pi \qquad (1.48)$$

Substituting into (1.44) we finally get

$$F_\pi = \frac{2M_N\, G_1(0)}{g(0)K_{NN\pi}(0)} \qquad (1.49)$$

This is the celebrated Goldberger-Treiman formula. The original derivation was based on dispersion theory and some simple dynamical assumptions. Thus the PCAC principle gives this remarkable formula with the minimum of labour and fuss. We further assume that the coupling constants and form-factors are smooth functions of q^2 and vary gently in the region $0 < q^2 < m_\pi^2$. Then we can set $K_{NN\pi}(0) \simeq 1$ and $g(0)$ to be equal to g, the experimentally measured pion-nucleon coupling constant. We then find that the relation is satisfied experimentally to within 10% accuracy. This success supports our assumption of smooth behaviour of form-factors and we can use it with confidence in other situations.

The PCAC principle can be generalized in an obvious fashion to the whole octet of axial-vector currents and the corresponding octet of pseudoscalar mesons. In particular we could assume that the divergence of the strangeness-changing axial-vector current is proportional to the K-meson field

$$\partial_\mu\, j_{5\mu}^{i(\Delta s=1)} = C_K\, \phi_K^i \qquad (1.50)$$

We shall make use of this equation in chapter 2 where we show how the PCAC principle leads to consistency conditions on the K-N interaction.

An alternative approach to PCAC is through dispersion theory[9]. One assumes that the matrix element of the divergence of axial-vector current is a highly-convergent quantity obeying an unsubtracted dispersion relation. Calling the quantity inside the bracket of the R.H.S. of equation (1.43), $G(q^2)$ we write

$$G(q^2) = \frac{1}{\pi}\int_{q_0^2}^{\infty} \frac{\mathrm{Im}\, F(q'^2)}{q'^2 - q^2 - i\epsilon}\, dq'^2 \qquad (1.51)$$

We separate the pion-contribution on the R.H.S. The next states having the same quantum numbers are 3π configurations with mass $\geq 3m_\pi$. Hence the threshold on the integral starts at $(3m_\pi)^2$ and we write

$$G(q^2) = \frac{F_\pi g m_\pi^2}{q^2 - m_\pi^2} + \frac{1}{\pi}\int_{9m_\pi^2}^{\infty} \frac{\mathrm{Im}\, F(q'^2)}{q'^2 - q^2 - i\epsilon}\, dq'^2 \qquad (1.52)$$

Now since the mass ratio $(m_\pi/3m_\pi)^2 = 1/9$ is small it may be reasonable to expect that near the pion mass-shell $0 \leq q^2 \leq m_\pi^2$ the pion pole dominates and we have

$$G(0) = F_\pi g \tag{1.53}$$

i.e.

$$F_\pi = \frac{2M_N G_1(0)}{g} \tag{1.54}$$

In this language the PCAC principle i.e. the partically conserved axial-vector current principle is often called PDDAC i.e. pion dominance of the divergence of axial-vector current. According to this interpretation, however, the same idea should not work so well for the K-mesons where the q^2 extrapolation ranges $0 \leq q^2 \leq m_K^2$ with $m_K = 500$ MeV and the next state K-π-π having mass not so far away, $\geqslant$ 780 MeV.

We now proceed to show how these two sets of information, the current commutation rules and the PCAC principle combine to give a useful description of several elementary particle phenomena. In particular we will see how the non-linear relations imposed by current commutators when combined with linear dispersion relations lead to useful results concerning reaction amplitudes and especially concerning strong interaction renormalization effects. There are several approaches to the question of sum rules and we shall begin by describing the dispersion approach.

The Dispersion Theory of Current Algebras:

In a series of papers Fubini and collaborators[(10),(11),(12)] showed how the information contained in current commutators could be exploited. Here we follow the approach discussed by Fubini in reference (12). We define an amplitude $T^{ij}_{\mu\nu}$ by

$$T^{ij}_{\mu\nu} = \int d^4x \; e^{iq'x} < p'|R(j^i_\mu(x)j^j_\nu(0))|p > \qquad (1.55)$$

The currents appearing in the retarded product are taken to be vector or axial-vector. The absorptive part of this amplitude is given by

$$t^{ij}_{\mu\nu} = \tfrac{1}{2}\int d^4x \; e^{iq'x} < p'|[j^i_\mu(x),j^j_\nu(0)]|p > \qquad (1.56)$$

The retarded product is defined by

$$R(A(x)B(y)) = i \,\theta(x_o - y_o)[A(x),B(y)] \qquad (1.57)$$

For (1.55) to be mathematically well-defined we consider the matrix element of the retarded product as a tempered distribution. Contraction with the vector q'^μ then gives

$$q'^\mu T^{ij}_{\mu\nu} = -i\int d^4x(\partial^\mu e^{iq'x}) < p'|R(j^i_\mu(x)j^j_\nu(0))|p > \qquad (1.58)$$

The theory of tempered distributions then tells us that it is permissible to integrate by parts without any 'surface term' appearing

$$q'^{\mu} T^{ij}_{\mu\nu} = i \int d^4x \, e^{iq'.x} \partial^{\mu} < p' | R(j^i_{\mu}(x) j^j_{\nu}(0)) | p > \tag{1.59}$$

Performing the differentiation yields

$$q'^{\mu} T^{ij}_{\mu\nu} = R^{ij}_{\nu} - F^{ij}_{\nu} \tag{1.60}$$

where

$$R^{ij}_{\nu} = i \int d^4x \, e^{iq'.x} < p' | R(\partial_{\mu} \; j^i_{\mu}(x) j^j_{\nu}(0)) | p > \tag{1.61}$$

and

$$F^{ij}_{\nu} = \int d^4x \, e^{iq'.x} < p' | \delta(x_o) [j^i_o(x), j^j_{\nu}(0)] | p > \tag{1.62}$$

We take the external states $|p>$, $|p'>$ to be of the same mass. We re-write equation (1.60) as

$$R^{ij}_{\nu} - q'^{\mu} T^{ij}_{\mu\nu} = F^{ij}_{\nu} \tag{1.63}$$

The amplitudes $T^{ij}_{\mu\nu}$ and R^{ij}_{ν} are functions of 4 four-vectors p, q, p', q' satisfying

$$p + q = p' + q' \tag{1.64}$$

We choose as variables the three Mandelstam variables

$$
\begin{aligned}
s &= (p + q)^2 \\
t &= (p' - p)^2 \\
u &= (p' - q)^2
\end{aligned}
\tag{1.65}
$$

and the four square masses: $p^2 = p'^2 = m^2$, q^2 and q'^2. These seven variables are connected by

$$s + t + u = p^2 + p'^2 + q^2 + q'^2 \tag{1.66}$$

Fixing p^2, p'^2 equal to m^2 leaves only 4 variables which, for instance, can be taken to s, t, q^2 and q'^2. The amplitudes $T^{ij}_{\mu\upsilon}$ and R^{ij}_{υ} (or more precisely the invariant amplitudes obtained in the decomposition of $T^{ij}_{\mu\upsilon}$ and R^{ij}_{υ}) are thus functions of these four variables. On the other hand because of locality of currents the commutator in F^{ij}_{υ} has its support at the point $x = 0$. As a result the q' dependence of F^{ij}_{υ} can come only from the possible Schwinger terms. These terms are polynomials in the space-components of q' as demonstrated before and they tend to zero with q'. Now in most applications of current algebra the limit $q' \to 0$ is taken and we can forget about the Schwinger terms. F^{ij}_{υ} is a function of a single variable t. Thus

$$R^{ij}_{\upsilon}(s,t;q^2,q'^2) - q'^{\mu}\, T^{ij}_{\mu\upsilon}(s,t;q^2,q'^2) = F^{ij}_{\upsilon}(t) \tag{1.67}$$

The relation (1.67) imposes very stringent conditions on the amplitudes R and T. Indeed, the L.H.S. is a function of 4 variables while the R.H.S. depends only on t. This signifies in particular that all the singularities in s, q^2, q'^2 appearing in R^{ij}_{υ} should be found in $q'^{\mu} T^{ij}_{\mu\upsilon}$ with the same weight.

An identical calculation with $t^{ij}_{\mu\upsilon}$ gives

$$q'^{\mu} t^{ij}_{\mu\upsilon} = r^{ij}_{\upsilon} \tag{1.68}$$

where r^{ij}_{υ} is the absorptive part of R^{ij}_{υ} :

$$r^{ij}_{\upsilon} = \frac{i}{2}\int d^4x \, e^{iq'x} < p' | [\partial_{\mu} j^{i}_{\mu}(x), j^{j}_{\upsilon}(0)] | p > \tag{1.69}$$

Equivalently we could compute the quantity $T^{ij}_{\mu\upsilon} q^{\upsilon}$ and obtain an analogous result

$$T^{ij}_{\mu\upsilon} q^{\upsilon} = R'^{ij}_{\mu} - F'^{ij}_{\mu} \tag{1.70}$$

where

$$R'^{ij}_{\mu} = - i \int d^4x \, e^{-iq.x} < p' | R(j^{i}_{\mu}(0) \partial_{\upsilon} j^{j}_{\upsilon}(x)) | p > \tag{1.71}$$

$$F'^{ij}_{\mu} = \int d^4x \, e^{-iq.x} \, \delta(x_o) < p' | [j^{i}_{\mu}(0), j^{j}_{o}(x)] | p > \tag{1.72}$$

For absorptive parts we have:

$$t^{ij}_{\mu\upsilon} q^{\upsilon} = r'^{ij}_{\mu} \tag{1.73}$$

where

$$r'^{ij}_{\mu} = - i \int d^4x \; e^{-iq.x} < p' | [j^i_\mu(0), \partial_\upsilon j^j_\upsilon(x)] | p > \tag{1.74}$$

We can obviously extend the method to evaluate $q'^{\mu} T^{ij}_{\mu\upsilon} q^{\upsilon}$:

$$q'^{\mu} T^{ij}_{\mu\upsilon} q^{\upsilon} = M^{ij} + F^{ij} + D^{ij} \tag{1.75}$$

where

$$M^{ij} = \int d^4x \; e^{iq'.x} < p' | R(\partial^\mu j^i_\mu(x) \; \partial^\upsilon j^j_\upsilon(0)) | p > \tag{1.76}$$

$$F^{ij} = \tfrac{1}{2}(q^{\upsilon} R^{ij}_{\upsilon} + q'^{\mu} R'^{ij}_{\mu}) \tag{1.77}$$

and

$$D^{ij} = \frac{i}{2} \int d^4x \; e^{iq'.x} < p' | [j^i_o(x), \partial^\upsilon j^j_\upsilon(0)] + [j^j_o(0), \partial^\upsilon j^i_\upsilon(x)] | p > \delta(x_o) \tag{1.78}$$

We now consider a second method suggested by Adler's work on neutrino induced reactions.[13],[14] Define

$$P = \tfrac{1}{2}(p + p') \; ; \; Q = \tfrac{1}{2}(q + q')$$

$$\upsilon = Q.P \tag{1.79}$$

We compute $\upsilon\, T^{ij}_{\mu\upsilon}$ using the technique of partial integration as before. We then write

$$\upsilon\, T^{ij}_{\mu\upsilon} = U^{ij}_{\mu\upsilon} + G^{ij}_{\mu\upsilon} \tag{1.80}$$

where

$$U^{ij}_{\mu\upsilon} = \tfrac{1}{2}P_\sigma\, i\int d^4x\, e^{iq'.x} < p'|R(\partial^\sigma j^i_\mu(x) j^j_\upsilon(0)) - R(j^i_\mu(x)\partial^\sigma j^j_\upsilon(0))|p > \tag{1.81}$$

$$G^{ij}_{\mu\upsilon} = -\int d^4x\, e^{iq'.x} < p'|\partial^\sigma \text{ⓤ}\, (x_o)[j^i_\mu(x), j^j_\upsilon(0)]|p > P_\sigma \tag{1.82}$$

Between the absorptive parts of T and U we simply have

$$\upsilon t^{ij}_{\mu\upsilon} = u^{ij}_{\mu\upsilon} \tag{1.83}$$

where

$$u^{ij}_{\mu\upsilon} = \tfrac{1}{4}P_{\sigma} \int d^4x \, e^{iq'.x} < p'|R(\partial^{\sigma} j^i_{\mu}(x) j^j_{\upsilon}(0))$$

$$- R(j^i_{\mu}(x) \, \partial^{\sigma} j^j_{\upsilon}(0))|p > \qquad (1.84)$$

We now discuss the derivation of sum rules from the relations we obtained so far. We consider the simple situation when all the particles involved are spinless. In addition to the variables introduced before we define

$$\Delta = q' - q = p - p' \qquad (1.85)$$

It is convenient to use the scalar variable υ instead of the Mandelstam variable, s. We choose

$$\upsilon = Q.P = q.P = q'.P \qquad (1.86)$$

We introduce the complete basis of 10 tensors constructed from P, q, q' and the metric tensor

$$T^{ij}_{\mu\upsilon} = A^{ij}P_{\mu}P_{\upsilon} + B^{ij}_1 P_{\mu}q_{\upsilon} + B^{ij}_2 P_{\mu}q'_{\upsilon} + B^{ij}_3 q_{\mu}P_{\upsilon}$$

$$+ B^{ij}_4 q'_{\mu}P_{\upsilon} + C^{ij}_1 q_{\mu}q'_{\upsilon} + C^{ij}_2 q_{\mu}q_{\upsilon} + C^{ij}_3 q'_{\mu}q'_{\upsilon}$$

$$+ C^{ij}_4 q'_{\mu} q_{\upsilon} + C^{ij}_5 g_{\mu\upsilon} \qquad (1.87)$$

The ten scalar functions A, B_k, C_k, C_5 (k = 1, ... 4) depend on the four scalar variables ν, t, q^2, q'^2. Under crossing(14)

$$P \Longrightarrow - P, \quad q_1 \Longrightarrow q_1, \quad q_2 \Longrightarrow q_2 \qquad \nu \Longrightarrow - \nu$$

the scalar amplitudes possess the following property

$$H^{ij}(-\nu) = \epsilon_H H^{ji}(\nu) \tag{1.88}$$

where

$$\epsilon_H = +1 \quad \text{for} \quad A, C_1, C_2, C_3, C_4, C_5$$

$$= -1 \quad \text{for} \quad B_1, B_2, B_3, B_4$$

The amplitude $t^{ij}_{\mu\nu}$ can be expanded on the same basis and we shall use the corresponding small letters for the scalar functions. Their crossing properties are simple obtained from $\epsilon_h + \epsilon_H = 0$. For the amplitude R^{ij}_{ν} we write

$$R^{ij}_{\nu} = L^{ij} P_{\nu} + N^{ij}_1 q_{\nu} + N^{ij}_2 q'_{\nu} \tag{1.89}$$

Similar expansions hold for R'^{ij}_{ν}, r^{ij}_{ν} and r'^{ij}_{ν}. The symmetry properties are given by

$$\epsilon_L = -1 \qquad \epsilon_{N_1} = \epsilon_{N_2} = +1$$

We now apply Fubini's method. We write three equalities associated with equation (1.67) and three equalities associated with equation (1.70).

$$\left.\begin{aligned}
- \upsilon A^{ij} - q'.q\, B_3^{ij} - q'^2 B_4^{ij} &= - L^{ij} + 2f_{ijk}\, G^k(t) \\
- \upsilon B_1^{ij} - q'.q\, C_2^{ij} - q'^2 C_4^{ij} &= - N_1^{ij} \\
- \upsilon B_2^{ij} - q'.q\, C_1^{ij} - q'^2 C_3^{ij} - C_5^{ij} &= - N_2^{ij}
\end{aligned}\right\} \quad (1.90)$$

And

$$\left.\begin{aligned}
- \upsilon A^{ij} - q^2 B_1^{ij} - q'.q\, B_2^{ij} &= - L'^{ij} + 2f_{ijk}\, G^k(t) \\
- \upsilon B_3^{ij} - q^2 C_2^{ij} - q'.q\, C_1^{ij} &= - N_1'^{ij} \\
- \upsilon B_4^{ij} - q^2 C_4^{ij} - q'.q\, C_3^{ij} &= - N_2'^{ij}
\end{aligned}\right\} \quad (1.91)$$

We have written the equal-time commutator contribution as

$$\begin{aligned}
F_\upsilon^{ij} &= \int d^3x\, e^{-i\underline{q}'\cdot\underline{x}} < p'|[j_0^i(\underline{x},0), j_\upsilon^j(0)]|p > \\
&= if_{ijk} < p'|j_\upsilon^k(0)|p > \\
&= 2if_{ijk}\, G^k(t) P_\upsilon
\end{aligned} \qquad (1.92)$$

where we have assumed a conserved vector current and hence only one form-factor is involved. The six equalities involving the absorptive parts are immediately deduced from (1.90) and (1.91) by putting formally $f_{ijk} = 0$.

Let us discuss the general equations

$$\left.\begin{aligned} -\upsilon H^{ij} &= Q^{ij} + F^{ij} + D^{ij} \\ -\upsilon h^{ij} &= q^{ij} \end{aligned}\right\} \qquad (1.93)$$

where F^{ij}, D^{ij} are equal-time commutator contributions antisymmetric and symmetric in the internal indices respectively. It is easily seen that any member of the sets (1.90) and (1.91) can be written in this form. The function H^{ij} has the crossing property

$$H^{ji}(-\upsilon) = \epsilon_H H^{ij}(\upsilon)$$

We assume for fixed t, q^2, q'^2, an unsubtracted dispersion relation for the scalar functions H^{ij} in υ

$$H^{ij}(\upsilon) = \frac{1}{\pi} \int_R \left[\frac{h^{ij}(\upsilon')}{\upsilon' - \upsilon} + \epsilon_H \frac{h^{ji}(\upsilon')}{\upsilon' + \upsilon} \right] d\upsilon' \qquad (1.94)$$

The integral extends over the R.H. cut, including possible poles. We now make an assumption about high-energy behaviour. We assume the quantity $\upsilon H^{ij}(\upsilon)$ has a limit as $\upsilon \to \infty$ that is given by the integral of the spectral function:

$$\lim_{\upsilon\to\infty} \upsilon\, H^{ij}(\upsilon) = -\frac{1}{\pi}\int_R \left[h^{ij}(\upsilon') - \epsilon_H h^{ji}(\upsilon') \right] d\upsilon' \qquad (1.95)$$

We are now in a position to write down the sum rules. We notice two types of sum rules: those involving the antisymmetric combination $H^{ij}(\upsilon) - H^{ji}(\upsilon)$ - the A-type, and those involving the symmetric combination $H^{ij}(\upsilon) + H^{ji}(\upsilon)$ - the S-type. Taking the crossing properties of the scalar amplitudes into account the only non-trivial sum rules of the A-type have $\epsilon_H = +1$ and those of the S-type have $\epsilon_H = -1$. We thus have:

A-type $\epsilon_H = +1$

$$\frac{1}{\pi}\int \left[h^{ij}(\upsilon) - h^{ji}(\upsilon) \right] d\upsilon = \tfrac{1}{2}\left(Q^{ij}(\infty) - Q^{ji}(\infty) \right) + F^{ij} \qquad (1.96)$$

S-type $\epsilon_H = -1$

$$\frac{1}{\pi}\int \left[h^{ij}(\upsilon) + h^{ji}(\upsilon) \right] d\upsilon = \tfrac{1}{2}\left(Q^{ij}(\infty) + Q^{ji}(\infty) \right) + D^{ij} \qquad (1.97)$$

A discussion of a sum-rule of the S-type arrived at by using Adler's method will be given in Chapter 4. Here we confine ourselves to sum rules of the A-type.

Going back to the two sets (1.90) and (1.91) we write down one sum-rule of the A-type. The physical assumption made by Fubini is that one usually deals with conserved or partially conserved currents and we expect the functions L^{ij} to obey unsubtracted dispersion relations in υ. We can then set for $\upsilon \to \infty$

$$L^{ij}(\infty) - L^{ji}(\infty) = L'^{ij}(\infty) - L'^{ji}(\infty) = 0$$

We then obtain

$$\frac{1}{\pi}\int_R \Big[a^{ij}(\upsilon,t,q^2,q'^2) - a^{ji}(\upsilon,t,q^2,q'^2) \Big] d\upsilon = 2f_{ijk}\, G^k(t) \tag{1.98}$$

This is the sum rule derived by Fubini[12] and by Dashen and Gell-Mann[15]. Notice that the relation (1.75) would again lead to precisely this sum rule and would not give anything new. Notice also that the equations (1.80) and (1.83) could be used to write down a number of sum rules one of which will again be (1.98). However now we are dealing with commutators involving arbitrary components of currents which appear multiplied by a derivative of the step-function. Care must be taken in evaluating such terms[14]. We shall discuss a particular situation in Chapter 4.

Nearby Singularities[16]

We rewrite our basic equation (1.67) as follows

$$t_{\nu}^{ij}(s,t;q^2,q'^2) - q'^{\mu}\, T_{\mu\nu}^{ij}(s,t;q^2,q'^2) = F_{\nu}^{ij}(t) \quad (1.99)$$

where the currents appearing in (1.55) are taken to be axial-vector and where we have changed the notation slightly calling t_{ν}^{ij} what we called R_{ν}^{ij} before. As emphasized before all the singularities in s, q^2, and q'^2 appearing in t_{ν}^{ij} must occur in $q'^{\mu}\, T_{\mu\nu}^{ij}$ with the same weight. The minimum singularities are, in the plane of q^2 and q'^2, a pole at m_{π}^2 and a cut commencing at $9m_{\pi}^2$, and in the s-plane a pole at the square mass of the nucleon, m_N^2 and a cut commencing at $(m_N + m_{\pi})^2$. (We are taking the external states to be proton states in (1.55).)

In the amplitudes t and T we distinguish different terms having, for example, the 3 poles at $q^2 = q'^2 = m_{\pi}^2$ and $s = m_N^2$ or two of these poles or only one of them or none. Thus $T_{\mu\nu}^{ij}$ is decomposed in the manner exhibited in Fig. 1, where, for instance, the first graph (a) has simultaneously 3 poles and the 4th graph (d) has only one pole at $q'^2 = m_{\pi}^2$. In the same manner $t_{\mu\nu}^{ij}$ is decomposed as shown in Fig. 2. In figures (1) and (2) the symbol

$x^{i\mu}$ ----- represents the part of the axial current $j^i_{5\mu}$ which is coupled to the pion field, while x represents the rest of $j^i_{5\mu}$. The symbols $\overset{i}{\bullet}$---- and $\circ$ have the same significance for the divergence $\partial^\mu j^i_{5\mu}$.

PDDAC consists in neglecting the last 4 graphs - e,f,g,h - in Fig. 2, in which the divergence $\partial^\mu j^i_{5\mu}$ is not coupled through the mediation of the pion field. We first separate the poles in q^2 and q'^2. We know from equation (1.45) that

$$< 0 | j^i_{5\mu}(0) | \pi^i(q) > = iF_\pi q_\mu$$

This permits us to write the axial current as a sum of two terms

$$j^i_{5\mu}(x) = - F_\pi \partial_\mu \phi^i_\pi(x) + \tilde{j}^i_{5\mu}(x) \qquad (1.100)$$

where $\tilde{j}^i_{5\mu}(x)$ is the part of the axial-vector current which is not coupled to the pion field. From this we easily deduce that in the expression (1.55) of $T^{ij}_{\mu\upsilon}$ we can make the substitutions

$$j^i_{5\mu}(x) \rightarrow F_\pi iq'_\mu \phi^i_\pi(x) + \tilde{j}^i_{5\mu}(x)$$

$$(1.101)$$

$$j^j_{5\upsilon}(0) \rightarrow -F_\pi iq_\upsilon \phi^j_\pi(0) + \tilde{j}^j_{5\upsilon}(0)$$

The amplitude $T^{ij}_{\mu\upsilon}$ is thus decomposed into 4 terms where the poles in q^2 and q'^2 appear explicitly

$$T^{ij}_{\mu\nu} = F^2_\pi \frac{q'_\mu}{m^2_\pi - q'^2} \frac{q_\nu}{m^2_\pi - q^2} M^{ij}(s,t;q^2q'^2)$$

$$+ F_\pi \frac{q'_\mu}{m^2_\pi - q'^2} \Lambda^{ij}_\nu (s,t;q^2,q'^2)$$

$$+ F_\pi \frac{q_\nu}{m^2_\pi - q^2} \bar{\Lambda}^{ij}_\mu (s,t;q^2,q'^2)$$

$$+ R^{ij}_{\mu\nu} (s,t;q^2,q'^2) \qquad (1.102)$$

We have put by definition

$$M^{ij} (s,t;q^2,q'^2) = (m^2_\pi - q^2)(m^2_\pi - q'^2)$$

$$\int d^4x \; e^{iq'x} < p' | R\left(\phi^i_\pi(x) \; \phi^j_\pi(0)\right) | p > \qquad (1.103)$$

$$\Lambda^{ij}_\nu (s,t;q^2,q'^2) = i(m^2_\pi - q'^2)$$

$$\int d^4x \; e^{iq'x} < p' | R\left(\phi^i_\pi(x) \; \tilde{j}^j_{5\nu}(0)\right) | p > \qquad (1.104)$$

$$\bar{\Lambda}^{ij}_{\nu} (s,t;q^2,q'^2) = - i(m_\pi^2 - q^2) \int d^4x\, e^{iq'x} < p'|R\left(\tilde{j}^{\,i}_{5\mu}(x)\ \phi^j_\pi(0)\right)|p > \qquad (1.105)$$

$$R^{ij}_{\mu\nu} (s,t;q^2,q'^2) = \int d^4x\, e^{iq'x} < p'|R\left(\tilde{j}^{\,i}_{5\mu}(x)\ \tilde{j}^{\,j}_{5\nu}(0)\right)|p > \qquad (1.106)$$

In M^{ij} we recognize the $\pi - N$ scattering amplitude defined for off-mass-shell pions; more precisely the physical amplitude is the limit of $M^{ij}(s,t;q^2,q'^2)$ when $q^2, q'^2 \to m_\pi^2$. Λ^{ij}_{ν} represents the pion production amplitude by the scattering of a nucleon in an external axial field coupled with the current $\tilde{j}_5$. $\bar{\Lambda}^{ij}_{\mu}$ is related to Λ^{ij}_{ν} by crossing

$$\bar{\Lambda}^{ij}_{\mu} (s,t;q^2,q'^2) = \Lambda^{ji}_{\mu} (u,t;q'^2,q^2) \qquad (1.107)$$

The function $R^{ij}_{\mu\nu}$ represents double scattering of a nucleon in the same axial field coupled with a current $\tilde{j}_5$. In the 4 successive terms of (1.102) we recognize contributions from graphs (a and b), (c and d), (e and f) and (g and h) respectively, of Fig. 1.

The separation of poles in q^2 and q'^2 is simpler in the amplitude $t^{ij}_{\mu\nu}$ because of PDDAC

$$\partial^{\mu} j^{i}_{5\mu}(x) = F_{\pi} m^2_{\pi} \phi^{i}_{\pi}(x)$$

It is simply

$$t^{ij}_{\mu\nu} (s,t;q^2q'^2) = F^2_{\pi} \frac{m^2_{\pi}}{m^2_{\pi} - q'^2} \frac{q_{\nu}}{m^2_{\pi} - q^2} M^{ij}(s,t;q^2,q'^2)$$

$$+ F_{\pi} \frac{m^2_{\pi}}{m^2_{\pi} - q'^2} \Lambda^{ij}_{\nu} (s,t;q^2,q'^2) \qquad (1.108)$$

The first term corresponds to graphs (a and b) of Fig. 2 while the second term corresponds to (c and d).

Substituting the expressions (1.102) and (1.108) into equation (1.99) we get

$$F_{\pi} \frac{q_{\nu}}{m^2_{\pi} - q^2} \left[F_{\pi} M^{ij} - q'^{\mu} \bar{\Lambda}^{ij}_{\mu} \right] + F_{\pi} \Lambda^{ij}_{\nu}$$

$$- q'^{\mu} R^{ij}_{\mu\nu} - F^{ij}_{\nu} = 0 \qquad (1.109)$$

The pole in $q'^2 = m_\pi^2$ has automatically disappeared. On the other hand the pole at $q^2 = m_\pi^2$ remains. So we must put the residue of this pole equal to zero and this gives the following relation

$$F_\pi M^{ij}(s,t;m_\pi^2,q'^2) = q'^\mu \bar{\Lambda}_\mu^{ij}(s,t;m_\pi^2,q'^2) \tag{1.110}$$

This relation is written at the point $q^2 = m_\pi^2$. We make the additional assumption as before that it is still valid in some neighbourhood of $q^2 = m_\pi^2$ extending at least as far as $q^2 = 0$:

$$F_\pi M^{ij}(s,t;q^2,q'^2) = q'^\mu \bar{\Lambda}_\mu^{ij}(s,t;q^2,q'^2) \tag{1.111}$$

This hypothesis enables us to split equation (1.109) into two terms, one being (1.111) and the other

$$F_\pi \Lambda_\nu^{ij}(s,t;q^2,q'^2) = F_\nu^{ij}(t) + q'^\mu R_{\mu\nu}^{ij}(s,t;q^2,q'^2) \tag{1.112}$$

Now current algebra gives F_ν^{ij} but leaves $R_{\mu\nu}^{ij}$ unknown. To eliminate it it is necessary to put $q' = 0$. However we must do this with caution, for in this limit, M, Λ, $\bar{\Lambda}$ and R become infinite. This is due to the graphs (a, c, e and g) of Fig. 1, the intermediate nucleon being on its mass shell when $q' \to 0$. So we must isolate the contribution of the one-nucleon intermediate state in M, Λ, $\bar{\Lambda}$ and R. First we make the charge states of the in and out pions precise by putting: $i = \frac{1}{\sqrt{2}}(1 + i2)$ and $j = \frac{1}{\sqrt{2}}(1 - i2)$. With this choice, M^{ij} is the elastic scattering amplitude $M_{\pi^- p}$ of −ve pions on protons. So:

$$M_{\pi^- p}(s,t;q^2,q'^2) = -(g\sqrt{2})^2 K(q^2)K(q'^2)\,\bar{u}(p')\, i\gamma_5 \frac{\not{p} + \not{q} + m_N}{s - m_N^2} i\gamma_5\, u(p) + \tilde{M}_{\pi^- p}(s,t;q^2,q'^2) \qquad (1.113)$$

where $g\sqrt{2}\, i\gamma_5$ is the charged pion-nucleon coupling and $K(q^2)$ is the pion-nucleon form factor previously defined. The factor $-\frac{(\not{p} + \not{q} + m)}{s - m^2}$ is the propagator of the intermediate nucleon and $\tilde{M}_{\pi^- p}$ is the contribution (regular at $s = m_N^2$) of intermediate states other than single nucleon states. In an analogous way we get

$$\bar{\Lambda}^{+-}_{\mu}(s,t;q^2,q'^2) = i\,g\sqrt{2}\,K(q^2)\,\frac{G_A(q'^2)}{\sqrt{2}}\,\bar{u}(p')\gamma_\mu\gamma_5 \frac{\not{p} + \not{q} + m_M}{s - m_N^2}\, i\gamma_5 u(p) + \tilde{\bar{\Lambda}}^{+-}_{\mu}(s,t;q^2,q'^2) \tag{1.114}$$

$$\Lambda^{+-}_{\upsilon}(s,t;q^2,q'^2) = -i\,g\sqrt{2}\,K(q'^2)\,\frac{G_A(q^2)}{\sqrt{2}}\,\bar{u}(p')i\gamma_5 \frac{\not{p}' + \not{q}' + m_N}{s - m_N^2}\,\gamma_\upsilon\gamma_5\, u(p) + \tilde{\Lambda}^{+-}_{\upsilon}(s,t;q^2,q'^2) \tag{1.115}$$

where we have retained only the axial term $\gamma_\mu\gamma_5 G_A(q^2)/\sqrt{2}$ in the coupling of the axial current and neglected the induced term $q_\mu\gamma_5$, which disappears from the final result anyway when we let $q \to 0$. The amplitudes $\tilde{\Lambda}$ and $\tilde{\bar{\Lambda}}$ are regular at $s = m_N^2$ and crossing gives

$$\tilde{\bar{\Lambda}}^{+-}_{\mu}(s,t;q^2,q'^2) = \tilde{\Lambda}^{+-}_{\mu}(u,t;q'^2,q^2) \tag{1.116}$$

Finally

$$R^{+-}_{\mu\nu}(s,t;q^2,q'^2) = -\tfrac{1}{2}G_A(q^2)G_A(q'^2)\bar{u}(p')\gamma_\mu\gamma_5 \frac{\not{p}+\not{q}+m_N}{s-m_M^2}\gamma_\nu\gamma_5\, u(p) + \tilde{R}^{+-}_{\mu\nu}(s,t;q^2,q'^2) \tag{1.117}$$

With the four equations (1.113), (1.114), (1.115) and (1.117) we have performed the decomposition of $T^{ij}_{\mu\nu}$ into 8 terms corresponding to the eight graphs of Fig. 1 and similarly the decomposition of t^{ij}_{ν} into 4 terms corresponding to the first 4 graphs of Fig. 2.

Substituting (1.113) and (1.114) into (1.111) we get after some algebra

$$\frac{2gK(q^2)}{s-m_N^2}\left[F_\pi\, gK(q'^2) - m_N\, G_A(q'^2)\right]\bar{u}(p')\not{q}\, u(p) + gK(q^2)G_A(q'^2)\bar{u}(p')u(p) + q'^\mu\,\tilde{\Lambda}^{+-}_{\mu}(u,t;q'^2,q^2) - F_\pi\,\tilde{M}_{\pi^-p}(s,t;q^2,q'^2) = 0 \tag{1.118}$$

To arrive at this relation we have used

$$\bar{u}(p')\not{q}'\not{q}\, u(p) = \bar{u}(p')(-2m_N\not{q} + s - m_N^2)u(p)$$

Now the residue at the pole $s = m_N^2$ must vanish and this gives the following relation (written at $q'^2 = 0$):

$$F_\pi = \frac{m_N\, G_A(0)}{g\, K(0)} \tag{1.119}$$

We recognize here the Goldberger-Treiman relation appearing as one of the constraints imposed by (1.99). Furthermore (1.118) yields

$$F_\pi\, \tilde{M}_{\pi^- p}(s,t;q^2,q'^2) = g\, K(q^2)G_A(q'^2)\bar{u}(p')u(p)$$

$$+ q'^\mu\, \tilde{\Lambda}_\mu^{+-}\, (u,t;q'^2,q^2) \tag{1.120}$$

In the forward direction ($p = p'$, $q = q'$, $t = 0$) this can be written

$$F_\pi\, \tilde{M}_{\pi^- p}\, (s,0;q^2,q^2) = g\, K(q^2)G_A(q^2)$$

$$+ q^\mu\, \tilde{\Lambda}_\mu^{+-}\, (u,0;q^2,q^2) \tag{1.121}$$

The amplitudes $\tilde{M}$ and $\tilde{\Lambda}$ are regular at $q = 0$. When $q \to 0$ $s \to m_N^2$ and (1.121) becomes

$$F_\pi \tilde{M}_{\pi^- p} (m_N^2,0;0,0) = g\, K(0)\, G_A(0) \qquad (1.122)$$

Taking the Goldberger-Treiman formula into consideration this becomes

$$\tilde{M}_{\pi^- p} (m_N^2,0;0,0) = \frac{g^2}{m_N} [K(0)]^2 \qquad (1.123)$$

This consistency condition was first demonstrated by Adler(17). It refers purely to strong interaction quantities. Furthermore it is independent of current algebra like the Goldberger-Treiman formula. Indeed so far we have not used the explicit form of F_υ^{ij} . We used only the fact that F_υ^{ij} is regular at s, q^2 and q'^2 (which results from the locality of currents) as well as PDDAC and the supplementary assumption of gentle variation. The relation (1.123) is important for it fixes the subtraction constant in the dispersion relations for π-N forward scattering amplitude $M_{\pi^- p}(s,0;0,0)$. According to Adler it is verified to within 10% of experimental result. In chapter 2 we shall demonstrate how 'consistency conditions' for the K-N interaction can be established.

From (1.121) we write for $\tilde{\Lambda}_\mu^{+-}$ $(u,0;q^2,q^2)$ at the point $q = 0 \quad u = m_N^2$

$$\tilde{\Lambda}_\mu^{+-}(m_N^2,0;0,0) = F_\pi \frac{\partial}{\partial q^\mu} \tilde{M}_{\pi^- p}(s,0;0,0)\Bigg|_{s=m_N^2} = 2p_\mu F_\pi \frac{\partial}{\partial s} \tilde{M}_{\pi^- p}(s,0;0,0)\Bigg|_{s=m_N^2}$$

(1.124)

If we now substitute the expressions (1.115) and (1.117) for Λ and R into equation (1.112) we get the following relation

$$\frac{G_A(q^2)}{s - m_N^2} \left[F_\pi\, g\, K(q'^2) - m_N\, G_A(q'^2) \right] \bar{u}(p') \not{q}' \gamma_\nu\, u(p)$$

$$+ F_\pi \tilde{\Lambda}_\nu^{+-}(s,t;q^2,q'^2) + \tfrac{1}{2} G_A(q^2) G_A(q'^2)\, \bar{u}(p')\gamma_\nu u(p)$$

$$- F_\nu^{+-}(t) - q'^\mu\, \tilde{R}_{\mu\nu}^{+-}(s,t;q^2,q'^2) = 0 \qquad (1.125)$$

To arrive at this equation we made use of the relation

$$\bar{u}(p')\not{q}'(\not{p}' + \not{q}' - m_N)\gamma_\nu u(p) = \bar{u}(p')(-2m_N \not{q}' + s - m_N^2)\gamma_\nu u(p) \qquad (1.126)$$

The vanishing of the residue of the pole at $s = m_N^2$ again yields the Goldberger-Treiman formula. This leaves us with the following equation

$$F_\pi \tilde{\Lambda}_\nu^{+-}(s,t;q^2,q'^2) = F_\nu^{+-}(t) - \tfrac{1}{2} G_A(q^2) G_A(q'^2) \bar{u}(p')\gamma_\nu u(p)$$

$$+ q'^\mu\, \tilde{R}_{\mu\nu}^{+-}(s,t;q^2,q'^2) \qquad (1.127)$$

For forward scattering and for $q = q' = 0$ we have

$$F_\pi \tilde{\Lambda}^{+-}_\upsilon (m_N^2,0;0,0) = F^{+-}_\upsilon(0) - \tfrac{1}{2}G_A(0)^2 \frac{p_\upsilon}{m_N} \qquad (1.128)$$

The algebra of currents gives

$$F^{+-}_\upsilon(0) = \langle p | j^3_\upsilon(0) | p \rangle$$

$$= \tfrac{1}{2}\,\bar{u}(p)\gamma_\upsilon\, u(p) = \frac{p_\upsilon}{2m_N} \qquad (1.129)$$

Hence

$$F_\pi \tilde{\Lambda}^{+-}_\upsilon (m^2,0;0,0) = (1 - G_A(0)^2) \frac{p_\upsilon}{2m_N} \qquad (1.130)$$

We use this equation to eliminate $\tilde{\Lambda}$ from (1.124). We then get with the aid of the Goldberger-Treiman formula:

$$1 - \left(\frac{G_V}{G_A}\right)^2 = -\left(\frac{2m_N}{g\,K(0)}\right)^2 m_N \frac{\partial}{\partial s} \tilde{M}_{\pi^- p}(s,0;0,0)\Bigg|_{s=m_N^2}$$

(1.131)

This relation is no more than the Adler-Weisberger[(18),(19)] formula. To cast in the usual form we must use forward dispersion relations for the amplitude $M_{\pi^- p}(s,0;0,0)$ where the pions are off the mass-shell. It may be shown in a general way that the amplitude $M_{\pi^- p}(s,0;q^2,q'^2)$ is an analytic function of s in the plane cut from $s = (m_N + m_\pi)^2$ to $+\infty$ and from $u = (m_N + m_\pi)^2$ to $+\infty$. (s and u are related by $s+u = 2m_N^2 + 2q^2$). The discontinuity across the R.H. cut is given by the optical theorem in the s-channel

$$\mathrm{Im}\, \tilde{M}_{\pi^- p}(s,0;q^2,q^2) \Big|_{s > (m_N + m_\pi)^2} = |\underset{\sim}{q}| \frac{W}{m_N} \sigma^-_{tot}(s,q^2) \tag{1.132}$$

$|\underset{\sim}{q}|$ is the centre-of-mass momentum of the incident pion and $\sigma^-_{tot}(s,q^2)$ is the $\pi^- p$ total cross-section at the C-M total energy $s = W^2$.

In an analogous way the discontinuity across the L.H.cut is given by the optical theorem in the crossed channel u :

$$\mathrm{Im}\, \tilde{M}_{\pi^- p}(s,0;q^2,q^2) \Big|_{u > (m_N + m_\pi)^2} = - |\underset{\sim}{q}|_u \frac{W_u}{m_M} \sigma^+_{tot}(u,q^2) \tag{1.133}$$

With $u = W_u^2$. When $q^2 = 0$

$$|\underset{\sim}{q}| W = \frac{s - m_N^2}{2} \quad \text{and} \quad |\underset{\sim}{q}|_u W_u = \frac{u - m_N^2}{2}$$

Suppose now that $\sigma^+_{tot}\ (s,q^2)$ and $\sigma^-_{tot}\ (s,q^2)$ tend to finite limits as $s \to \infty$ even for off-mass-shell pions. The Pomeranchuk theorem tells us then that these limits are the same for σ^+ and σ^- and this enables us to write down a dispersion relation with one subtraction for $M_{\pi^- p}$. When $q^2 = 0$ it becomes, making the subtraction at $s = u = m_N^2$:

$$m_N\, M_{\pi^- p}(s,0;0,0) = \text{Constant} + \frac{s - m_N^2}{2\pi} \int_{(m_N + m_\pi)^2}^{\infty} \left(\frac{\sigma^-_{tot}(s',0)}{s' - s} - \frac{\sigma^+_{tot}(s',0)}{s' - u} \right) ds' \tag{1.134}$$

Differentiation w.r.t. s gives at $s = m_N^2 = u$

$$m_N \left. \frac{\partial}{\partial s} M_{\pi^- p}\ (s,0;0,0) \right|_{s = m_N^2} = \frac{1}{2\pi} \int_{(m_N + m_\pi)^2}^{\infty} \frac{ds'}{s' - m_N^2} \left[\sigma^-_{tot}(s',0) - \sigma^+_{tot}(s',0) \right] \tag{1.135}$$

Substituting this into (1.131) we finally get

$$1 - \left(\frac{G_V}{G_A} \right)^2 = - \left(\frac{2m_N}{gK(0)} \right)^2 \frac{1}{2\pi} \int_{(m_N + m_\pi)^2}^{\infty} \frac{ds'}{s' - m_N^2}$$

$$\left[\sigma^-_{tot} (s',0) - \sigma^+_{tot} (s',0) \right] \qquad (1.136)$$

This is the celebrated Adler-Weisberger formula for the renormalization of the axial-vector coupling constant. A numerical evaluation by Adler[18] gives

$$\left| \frac{G_A}{G_V} \right| = 1.24 \pm 0.03 \qquad (1.137)$$

while Weisberger[19] finds

$$\left| \frac{G_A}{G_V} \right| = 1.16 \qquad (1.138)$$

Experimentally

$$\left| \frac{G_A}{G_V} \right| = 1.18 \pm 0.02 \qquad (1.139)$$

The striking agreement of this formula with experiment is one of the major successes of current algebra.

The Infinite Momentum Frame:

This is the original Fubini-Furlan method[10]. It was used by Adler[18] to derive the sum rule (1.136) and by Dashen and Gell-Mann[15] to derive the more general type of sum-rules (1.98). We follow here an approach due to Bollini and Giambiagi[20] and Amati, Jengo and Remiddi[21]. We shall show that the sum rules of the type (1.98) emerge even when a Schwinger term proportional to a first order derivative of a δ-function is included. We start from the equal-time commutator of two currents (vector or axial) of the form

$$\left[j_o^i\left(\frac{x}{2}\right), j_\mu^j\left(-\frac{x}{2}\right)\right]\delta(x_o) = \left(if_{ijk}\, j_\mu^k(0) + i\,\Sigma_{ij}(0)\delta_{\mu n}\partial_\mu\right)\delta^4(x) \tag{1.140}$$

where $\Sigma_{ij}(x)$ is the operator occuring in the Schwinger term (if present). We take matrix elements of this commutator between states of momenta p and p'. We define $t_{\mu\nu}^{ij}(\underline{Q},Q_o)$ by the Fourier transformation

$$t_{\mu\nu}^{ij}(\underline{Q},Q_o) = \int d^4x\, e^{iQ.x} < p'|[\, j_\mu^i\left(\frac{x}{2}\right), j_\nu^j\left(-\frac{x}{2}\right)]|p > \tag{1.141}$$

We introduce the following kinematical variables some of which we used before

$$\left.\begin{array}{l} P = \frac{1}{2}(p + p'), \quad \Delta = \frac{1}{2}(p - p'), \quad q = Q - \Delta \\ q' = Q + \Delta, \quad S = (P + Q)^2, \quad t = 4\Delta^2 \\ \upsilon = 2P.Q, \quad v = 2\Delta.Q = \frac{1}{2}(q'^2 - q^2) \\ w = Q^2 + \Delta^2 = \frac{1}{2}(q^2 + q'^2) \end{array}\right\} \quad (1.142)$$

We take the Fourier transform (F.T.) of both sides of equation (1.140). Now the F.T. of a product of two distributions is the convolution of the two F.T.s for each factor separately i.e.

$$\mathcal{F}[f'.f^2] = \frac{1}{(2\pi)^4} \mathcal{F}[f'] * \mathcal{F}[f^2] \qquad (1.143)$$

This relation is written for the 4-dimensional Lorentz space; the symbol $\mathcal{F}$ denotes the F.T. while the * stands for the convolution product. We then obtain

$$\frac{1}{2\pi}\int t^{ij}_{o\mu}(\underline{Q},Q_o)dQ_o = if_{ijk} < p'|j^k_\mu(0)|p >$$

$$- < p'|\Sigma_{ij}(0)|p > Q_\mu \, \delta_{\mu\, n} \qquad (1.144)$$

In this equation the integration runs over Q_o for all values of

$\underline{Q}$ and the result is independent of $\underline{Q}$, except for the linear dependence on $\underline{Q}$ arising from the presence of the Schwinger term in (1.140). (Higher order Schwinger terms would lead to higher powers of $\underline{Q}$).

Equation (1.141) is not Lorentz-invariant as both Q_o and $t_{o\mu}$ depend on the frame of reference. Since it must however hold in any frame and different choices lead to different types of information about the matrix elements of the current. Let us take the currents to be both vector and the states $|\, p >$, $|p' >$ to be spinless and of equal mass. Then we define

$$< p'|j^k_\mu(0)|p > \;=\; F^k_1(t)\; P_\mu + F^k_2(t)\,\Delta_\mu \qquad (1.145)$$

(The second form-factor $F^k_2(t)$ vanishes for a conserved current). We also write

$$< p'|\Sigma_{ij}(0)|p \;=\; S_{ij}(t) \qquad (1.146)$$

We expand $t^{ij}_{\mu\upsilon}$ on a tensor basis as before. We write

$$\begin{aligned} t^{ij}_{\mu\upsilon} = \;& a^{ij}_1 P_\mu P_\upsilon + a^{ij}_2\, P_\mu \Delta_\upsilon + a^{ij}_3\, P_\mu Q_\upsilon \\ & + b^{ij}_1 \Delta_\mu P_\upsilon + b^{ij}_2 \Delta_\mu \Delta_\upsilon + b^{ij}_3 \Delta_\mu Q_\upsilon \\ & + c^{ij}_1 Q_\mu P_\upsilon + c^{ij}_2 Q_\mu \Delta_\upsilon + c^{ij}_3\, Q_\mu Q_\upsilon \\ & + d^{ij}\, \delta_{\mu\upsilon} \end{aligned} \qquad (1.147)$$

This decomposition is slightly different from the one used before (1.87) but the two sets are related by simple linear relationships. As before a_i, b_i, c_i and d are invariant amplitudes that are functions of s,t,q^2 and q'^2 or s,t,v, and w. We begin by taking P to be time-like so that as $P.\Delta = 0$, Δ is space-like. We define the Breit system by $\underline{P} = 0$ and $\therefore \Delta_o = 0$ and $P_o = \sqrt{P^2}$. Any other system can be reached from the Breit system by a Lorentz transformation characterized by a relative velocity β and a direction which we take to be the z-axis. In the Breit system

$$P = (\sqrt{P^2}, 0, 0, 0)$$

$$\Delta = (0, \Delta_x, \Delta_y, \Delta_z)$$

In any other frame

$$\left.\begin{aligned} P &= (\gamma\sqrt{P^2}, 0, 0, -\beta\gamma\sqrt{P^2}) \\ &\text{and} \\ \Delta &= (-\beta\gamma\Delta_z, \Delta_x, \Delta_y, \gamma\Delta_z) \end{aligned}\right\} \qquad (1.148)$$

where $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$. From equation (1.142) we have

$$Q_o = + \frac{\upsilon}{2\gamma\sqrt{P^2}} - \beta Q_z \qquad (1.149)$$

Q_z is a fixed quantity in equation (1.144) and so we can use υ as an integration variable. With the aid of (1.148) and (1.149) we find

$$\left.\begin{aligned} Q^2 &= \frac{\upsilon^2}{4\gamma^2 P^2} - \frac{\beta Q_z \upsilon}{\gamma\sqrt{P^2}} - \frac{Q_z^{\ 2}}{\gamma^2} - Q_x^{\ 2} - Q_y^{\ 2} \\ Q.\Delta &= -\frac{\beta\Delta_z \upsilon}{2\sqrt{P^2}} - \frac{Q_z\Delta_z}{\gamma} - Q_x\Delta_x - Q_y\Delta_y \end{aligned}\right\} \quad (1.150)$$

The integration in (1.144) is performed over υ for fixed values of $\underline{Q}$; it is ∴ clear from equations (1.142) and (1.150) that q^2 and q'^2 as well as s vary along the path of integration. In the Breit system, for instance, $\beta = 0$ and ∴

$$\begin{aligned} q^2 &= -\underline{Q}^2 + 2\underline{Q}.\underline{\Delta} + \frac{\upsilon^2}{4P^2} + \frac{t}{4} \\ q'^2 &= -\underline{Q}^2 - 2\underline{Q}.\underline{\Delta} + \frac{\upsilon^2}{4P^2} + \frac{t}{4} \qquad (1.151) \\ s &= \upsilon + \frac{\upsilon^2}{4P^2} + P^2 - \underline{Q}^2 \end{aligned}$$

Using equation (1.149) equation (1.144) then gives

$$\frac{1}{4\pi}\int d\upsilon\left\{a_1^{ij}+\frac{\upsilon}{2P_o^2}(a_3^{ij}+c_1^{ij})+\frac{\upsilon^2}{4P_o^4}c_3^{ij}+\frac{d^{ij}}{P_o^2}\right\} = if_{ijk}F_1^k(t)$$

(1.152)

$$\frac{1}{4\pi}\int\left\{a_2^{ij}+\frac{\upsilon}{2P_o^2}c_2^{ij}\right\}d\upsilon = i\,f_{ijk}\,F_2^k(t) \qquad (1.153)$$

$$\frac{1}{4\pi}\int\left\{a_3^{ij}+\frac{\upsilon}{2P_o^2}c_3^{ij}\right\}d\upsilon = -\,S_{ij}(t) \qquad (1.154)$$

The scalar functions depend on s, q^2, q'^2 which vary with υ as indicated in (1.151) with $\underline{Q}$ and $\underline{Q}.\underline{A}$ being arbitrary constants.

The $P_o \to \infty$ system is defined by

$$\Delta_z = 0, \qquad \beta \to 1 \qquad (1.155)$$

If the integrals over υ converge, the contribution from high values of υ is negligible so that we can regard υ as bounded by a finite quantity. For finite υ we have as $\beta \to 1$

$$P_o \to \infty, \quad P_z \to -P_o, \quad Q_o \to -Q_z, \quad \Delta_z = \Delta_o = 0$$

(1.156)

and equations (1.150) become

$$\begin{aligned} Q^2 &= - Q_x^{\,2} - Q_y^{\,2} \\ Q.\Delta &= - Q_x\Delta_x - Q_y\Delta_y \\ s &= \upsilon + P^2 + Q_x^{\,2} + Q_y^{\,2} \end{aligned} \qquad (1.157)$$

In this system q^2, q'^2 are independent of υ . The arbitrary finite quantity Q_z no longer enters in the determination of s, q^2 and q'^2. Using now 's' as an integration variable we write

$$\begin{aligned} \frac{1}{4\pi}\int ds\ a_1^{ij}\ (s,t,q^2,q'^2) &= if_{ijk}\ F_1^k(t) \\ \frac{1}{4\pi}\int ds\ a_2^{ij}\ (s,t,q^2,q'^2) &= if_{ijk}\ F_2^k(t) \\ \frac{1}{4\pi}\int ds\ a_3^{ij}\ (s,t,q^2,q'^2) &= - S_{ij}(t) \end{aligned} \qquad (1.158)$$

These equations have been obtained from (1.144) by equating, for every value of the index μ , the leading terms in an expansion in P_o. The leading term for $\mu = 0$ (or 3) gives the 1st equation and is a factor P_o larger than the leading terms for $\mu = 1$ or 2 which give the other two equations. We recognize in (1.158) the Fubini and Dashen-Gell-Mann sum rules derived before.

By taking P now to be space-like and Δ time-like and starting from a "Breit system" $\underline{\Delta} = 0$, Amati, Jengo and Remiddi[21] were able to derive 3 new sum-rules in the $\Delta_o \to \infty$ system. These are

$$\frac{1}{4\pi}\int b_1^{ij}\ (s,t,v,w)dv = i f_{ijk}\ F_1^k(t)$$

$$\frac{1}{4\pi}\int b_2^{ij}\ (s,t,v,w)dv = i f_{ijk}\ F_2^k(t) \qquad (1.159)$$

$$\frac{1}{4\pi}\int b_3^{ij}(s,t,v,w)dv = -S_{ij}(t)$$

The manner of derivation is exactly the same as before.

It is important to note that in the sum rules (1.158) q^2 and q'^2 do not vary along the integration path. Equation (1.156) imposes restrictions on the values of q^2, q'^2 and t for which (1.158) have been obtained. Indeed they require q, q' and Δ to space-like and that $\sqrt{-q^2}$, $\sqrt{-q'^2}$ and $\sqrt{-\Delta^2}$ satisfy triangular inequalities, i.e. any one of them is smaller than (or equal to) the sum of the other two and larger than (or equal to) the modulus of the difference. We can now argue that since the equations (1.158) hold in such a kinematical region they will also hold in all regions that are accessible through the procedure of analytic continuation. Amati, Jengo and Remiddi[21] make the interesting observation that this is analogous to the situation which arises in the derivation of dispersion relations[22].

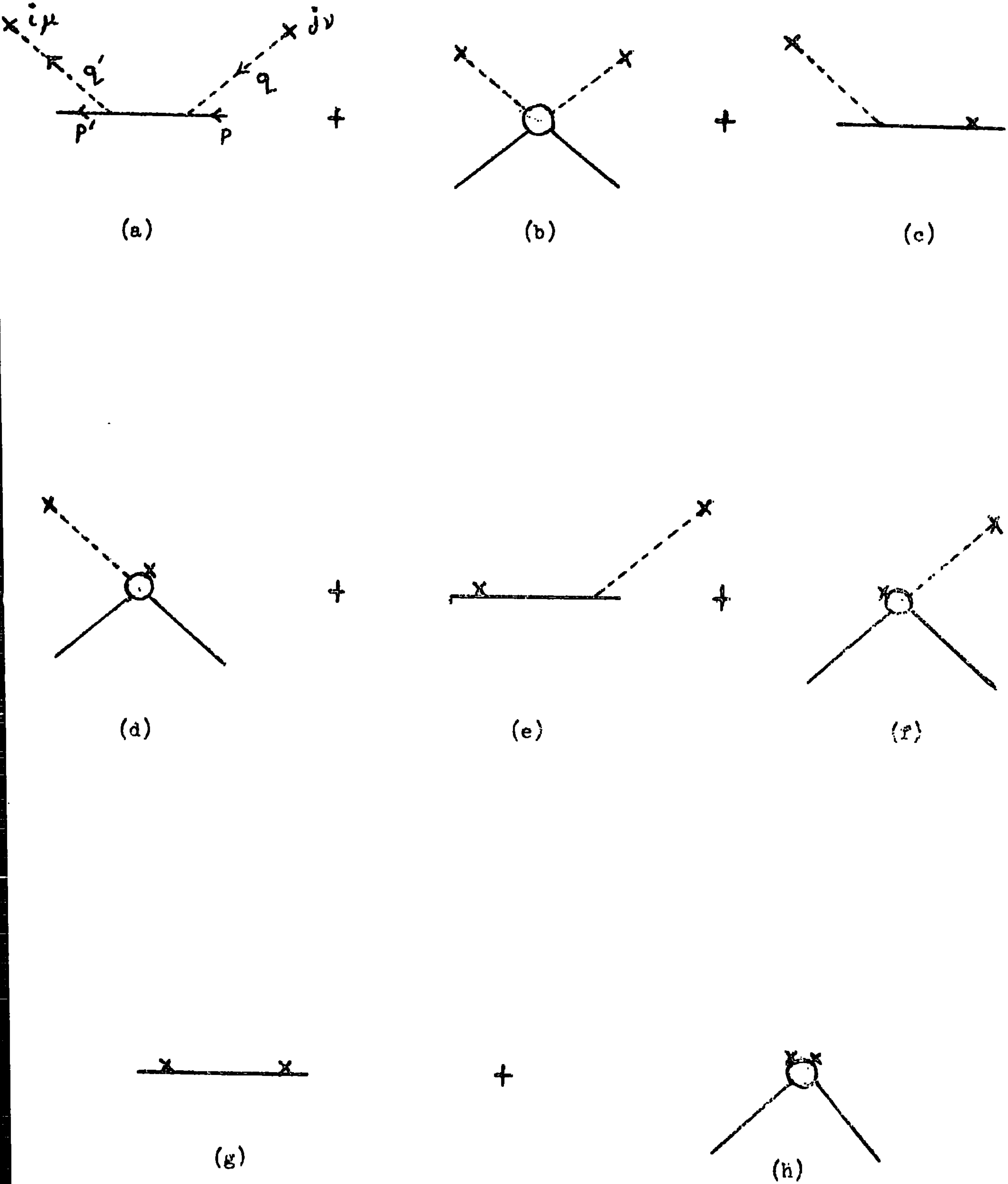
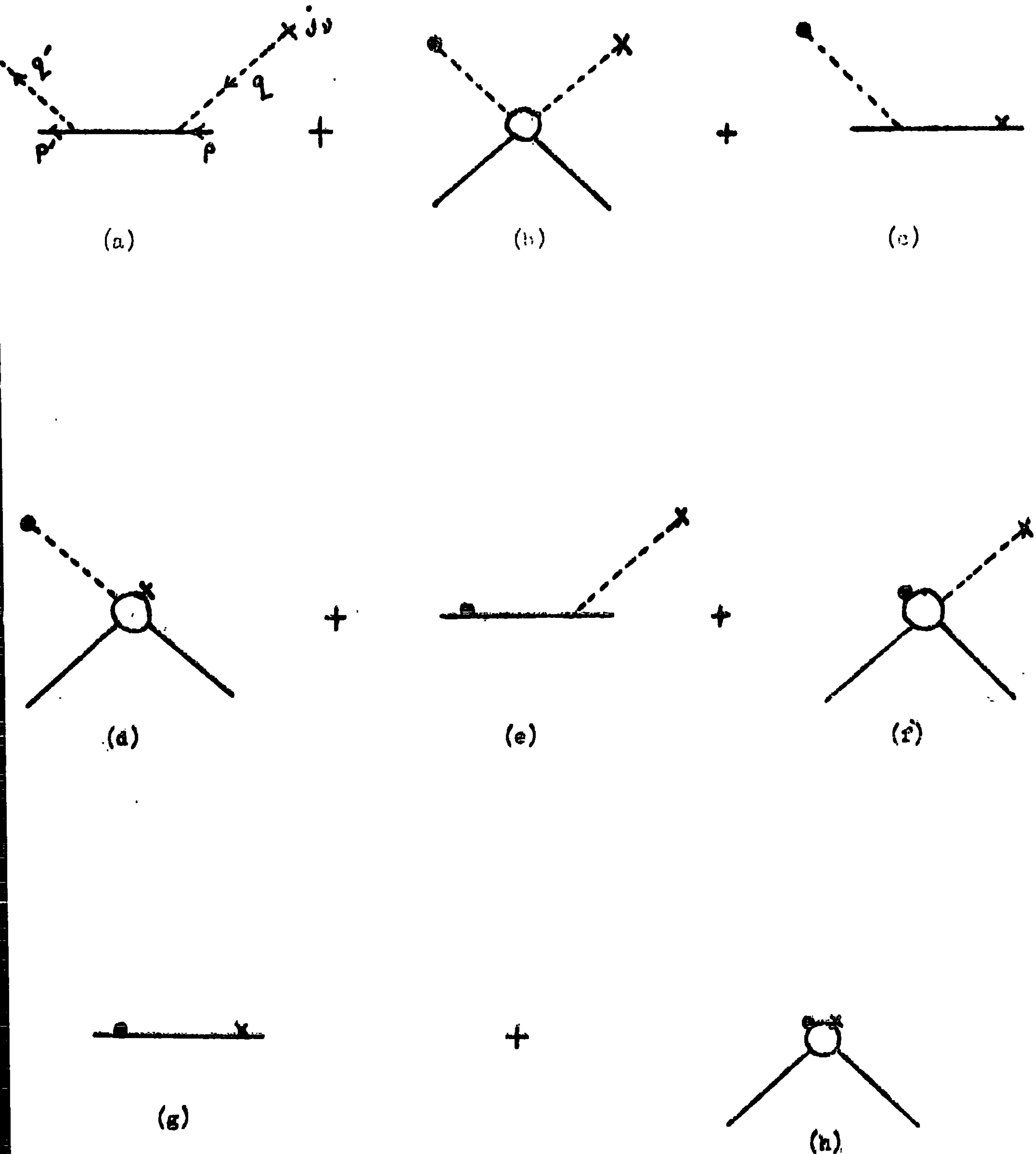


Figure 1



<u>Figure 2</u>

CHAPTER II

In the previous chapter we derived a 'consistency condition' for the pion-nucleon amplitude, equation (1.123), the pions involved being off-mass-shell. In this chapter we investigate such conditions for the K-meson nucleon scattering amplitude. We follow the approach described in reference (23) which parallels Adler's original derivation[17] of the π-N condition.

We assume PCAC for the strangeness-changing axial vector current in the form

$$\partial_\mu \, j_{5\mu}^{(\Delta S = 1)} = C_K \, \phi_K \tag{2.1}$$

where ϕ_K is the renormalized field operator of the K-meson. To determine C_K we take matrix elements of $\partial_\mu \, j_{5\mu}$ between a Σ state and a nucleon state. We expand the matrix element of $j_{5\mu}$ in terms of three form-factors as usual

$$< N(p')|\, j_{5\mu}^{(\Delta S = 1)} \,|\Sigma(p) > = \sqrt{\frac{m_\Sigma \, m_N}{p_o' \, p_o}} \, \bar{u}(p') \Big[g_A^{N\Sigma}(q^2)$$

$$+ f_A^{N\Sigma}(q^2) \sigma_{\mu\upsilon} q_\upsilon + i \, h_A^{N\Sigma}(q^2) q_\mu \Big] \gamma_5 \, u(p)$$

(2.2)

where $q = p' - p$. We have labelled the form-factors by the external states Σ and N since we shall be concerned with several matrix elements of $j_{5\mu}$ between different states. Now

$$< N(p')|\partial_\mu\, j_{5\mu}^{(\Delta S=1)}|\Sigma(p)> \Big|_{q^2=0} = -iq_\mu < N(p')|j_{5\mu}^{(\Delta S=1)}|\Sigma(p)>$$

$$= -iq_\mu \sqrt{\frac{m_\Sigma m_N}{p_o' p_o}}\, \bar{u}(p') g_A^{N\Sigma}(0) \gamma_\mu \gamma_5 u(p) \tag{2.3}$$

With the help of the Dirac equations

$$\left.\begin{aligned} \bar{u}(p)(\not{p} - im) &= 0 \\ (\not{p} - im)u(p) &= 0 \end{aligned}\right\} \tag{2.4}$$

We obtain

$$< N(p')|\partial_\mu\, j_{5\mu}^{(\Delta S=1)}|\Sigma(p)> = (m_\Sigma + m_N) g_A^{N\Sigma}(0) \sqrt{\frac{m_\Sigma\, m_N}{p_o'\, p_o}}\, \bar{u}(p') \gamma_5 u(p) \tag{2.5}$$

However by (2.1) the L.H.S. of (2.5) is equal to

$$C_K < N(p')|\phi_K|\Sigma(p) > = \frac{C_K}{m_K^2 + q^2} < N(p')|(-\Box + m_K^2)\phi_K|\Sigma(p) >$$

$$= \frac{C_K}{m_K^2 + q^2} < N(p')|j_K|\Sigma(p) >$$

$$= \frac{C_K}{m_K^2 + q^2}\, i\, g_{\Sigma NK}\, K^{\Sigma NK}(q^2)\sqrt{\frac{m_\Sigma m_N}{p_0' \, p_0}}$$

$$\times \; \bar{u}(p')\, \gamma_5 u(p) \qquad (2.6)$$

$g_{\Sigma NK}$ is the coupling constant and $K^{\Sigma NK}(q^2)$ is the ΣNK-vertex form-factor normalized so that $K^{\Sigma NK}(-m_K^2) = 1$. At $q^2 = 0$ we compare (2.5) and (2.6) and get

$$C_K = \frac{-\, i(m_\Sigma + m_N) g_A^{N\Sigma}(0) m_K^2}{g_{\Sigma NK} K^{\Sigma NK}(0)} \qquad (2.7)$$

Alternatively we could have started with the vertex $< N|j_{5\mu}|\Lambda >$ and arrived at

$$C_K = \frac{-\, i(m_\Lambda + m_N) g_A^{N\Lambda}(0) m_K^2}{g_{\Lambda NK} K^{\Lambda NK}(0)} \qquad (2.8)$$

These two determinantions of C_K must agree and hence

$$\frac{(m_\Sigma + m_N) g_A^{N\Sigma}(0)}{g_{\Sigma NK}\ K^{\Sigma NK}(0)} = \frac{(m_\Lambda + m_N) g_A^{N\Lambda}(0)}{g_{\Lambda NK}\ K^{\Lambda NK}(0)} \tag{2.9}$$

It could be that $\partial_\mu\, j_{5\mu}^{(\Delta S = 1)}$ is not coupled entirely to the K-meson field. There could be another operator R such that

$$\partial_\mu\, j_{5\mu}^{(\Delta S = 1)} = \frac{-\,i(m_\Sigma + m_N) g_A^{N\Sigma}(0) m_K^2}{g_{\Sigma NK}\ K^{\Sigma NK}(0)}\, \phi_K + R \tag{2.10}$$

Then our subsequent analysis is valid provided the residual operator R satisfies the following condition: for states A, B such that $< A|\phi_K|B > \neq 0$ and for momentum transfers near the K-pole (e.g. $-m_K^2 < q^2 < m_K^2$) then

$$\frac{|< B|R|A >|}{\left| \left[(m_\Sigma + m_N) g_A^{N\Sigma}(0) m_K^2 \Big/ g_{\Sigma NK}\ K^{\Sigma NK}(0) \right] < B|\phi_K|A > \right|} \ll 1 \tag{2.11}$$

To derive the desired consistency conditions consider the matrix element $< KN|j_{5\mu}^{(\Delta S = 1)}|N >$. This may be decomposed into 8 invariant amplitudes A_j given by[24]

$$\sqrt{\frac{p_o' \, p_o}{m_N m_N}}\, 2q'_o < KN|j_{5\mu}^{(\Delta S = 1)}|N > = \bar{u}(p')\, i \sum_{j=1}^{8} O_\mu^j A_j(\upsilon, \upsilon_B, q^2) u(p) \tag{2.12}$$

where $p(p')$ is the four-momentum of the initial (final) nucleon; q' that of the outgoing K-meson and q is the momentum transfer. The variables υ, υ_B are defined by

$$\left.\begin{aligned} \upsilon &= -(p + p').q/2m_N \\ \upsilon_B &= q'.q/2m_N \end{aligned}\right\} \qquad (2.13)$$

The operators O^j_μ are given by

$$\begin{aligned} O^1_\mu &= \tfrac{1}{2}(\not{q}'\gamma_\mu - \gamma_\mu\not{q}') & O^5_\mu &= i\not{q}(p + p')_\mu \\ O^2_\mu &= (p + p')_\mu & O^6_\mu &= i\not{q}q'_\mu \\ O^3_\mu &= q'_\mu & O^7_\mu &= q_\mu \\ O^4_\mu &= im_N\gamma_\mu & O^8_\mu &= i\not{q}q_\mu \end{aligned} \qquad (2.14)$$

We write for the isotopic-spin structure of the amplitude A_j the following

$$A_j(\upsilon,\upsilon_B,q^2) = \chi^*_f\, \eta^*_\alpha\, A_j(\upsilon,\upsilon_B,q^2)_{\alpha\beta}\, \eta^+_\beta\, \chi_i \qquad (2.15)$$

where χ_i, χ_f are the isotopic spinors of the initial and final nucleons respectively and η_α is the isotopic spin wave function for the final K-meson. The object η^+_β must be there to saturate the isotopic indices β on $A_{j,\alpha\beta}$. It is a consequence of the fact that $j_{5\mu}^{(\Delta S=1)}$ transforms like the component λ^{4+i5} of the unitary-spin current. We can think of it as the unitary-spin wave function of a spurion which carries off $I = \frac{1}{2}$.

We split each A_j into

$$A_j(\upsilon,\upsilon_B,q^2)_{\alpha\beta} = A^p_j(\upsilon,\upsilon_B,q^2)_{\alpha\beta} + \bar{A}_j(\upsilon,\upsilon_B,q^2)_{\alpha\beta} \tag{2.16}$$

where A^p_j is the sum of Born term contributions (i.e. the single Λ, Σ states) and $\bar{A}_j$ is simply the residual amplitude.

Evaluating the Born terms with the aid of figures 3 and 4 gives

$$\bar{u}(p')i\sum_{j=1}^{8} O^j_\mu \, X^*_f \, \eta^*_\alpha \, A^p_{j\alpha\beta} \, X_i \, \eta^+_\beta \, u(p)$$

$$= \bar{u}(p)X^*_f \, \eta^*_\alpha \Big[\Big\{ i\gamma_5 \, g_{\Lambda NK} \frac{1}{\not{p}' + \not{q}' - im_\Lambda} \, \mathcal{P}^o_{\alpha\beta} \Big[g^{N\Lambda}_A(q^2)\gamma_\mu\gamma_5$$

$$+ f^{N\Lambda}_A(q^2)\sigma_{\mu\upsilon}q_\upsilon\gamma_5 + i h^{N\Lambda}_A(q^2)q_\mu\gamma_5 \Big] + \mathcal{P}^o_{\alpha\beta} \Big[g^{N\Lambda}_A(q^2)\gamma_\mu\gamma_5$$

$$+ f_A^{N\Lambda}(q^2)\,\sigma_{\mu\upsilon}q_\upsilon\gamma_5 + i\,h_A^{N\Lambda}(q^2)q_\mu\gamma_5 \Big]\frac{1}{\not p - \not q' - im_\Lambda}\, i\gamma_5\, g_{\Lambda NK}\Big\}$$

$$+ \Big\{ i\gamma_5\, g_{\Sigma NK}\frac{1}{\not p' + \not q' - im_\Sigma}\,\mathcal{P}^1_{\alpha\beta}\Big[g_A^{N\Sigma}(q^2)\gamma_\mu\gamma_5 + f_A^{N\Sigma}(q^2)\sigma_{\mu\upsilon}q_\upsilon\gamma_5$$

$$+ i\,h_A^{N\Sigma}(q^2)q_\mu\gamma_5\Big] + \mathcal{P}^1_{\alpha\beta}\Big[g_A^{N\Sigma}(q^2)\gamma_\mu\gamma_5 + f_A^{N\Sigma}(q^2)\sigma_{\mu\upsilon}q_\upsilon\gamma_5$$

$$+ i\,h_A^{N\Sigma}(q^2)q_\mu\gamma_5\Big]\frac{1}{\not p - \not q' - im_\Sigma}\, i\gamma_5\, g_{\Sigma NK}\Big\}\Big]\,\chi_i\,\eta_\beta^+\,u(p) \tag{2.17}$$

where $\mathcal{P}^0$, $\mathcal{P}^1$ are the projection operators for isotopic singlet and triplet states. Specifically

$$\left.\begin{aligned}\mathcal{P}^0 &= \tfrac{1}{4}(1 - \underline{\tau}_N\cdot\underline{\tau}_K)\\ \mathcal{P}^1 &= \tfrac{1}{4}(3 + \underline{\tau}_N\cdot\underline{\tau}_K)\end{aligned}\right\} \tag{2.18}$$

where $\underline{\tau}_N$, $\underline{\tau}_K$ are the isotopic-spin operators for the nucleon and K-meson respectively. We treat the amplitude as a matrix in the nucleon sub-space. We are interested in the divergence $\partial_\mu\, j_{\mu 5}$ for zero momentum transfer, $q^2 = 0$. This simplifies

(2.17) enormously as contributions from the factors involving f_A and h_A drop out. We write

$$\bar{u}(p')i\sum_{j=1}^{8} O_\mu^j \chi_f^* \eta_\alpha^* A_{j\alpha\beta}^P \chi_i \eta_\beta^+ u(p)$$

$$= \bar{u}(p')\chi_f^* \eta_\alpha^* \Big\{ i\gamma_5 g_{\Lambda NK} \frac{\not{p}' + \not{q}' + im_\Lambda}{p'^2 + q'^2 + 2q'.p' + m_\Lambda^2} \mathcal{P}_{\alpha\beta}^0 g_A^{N\Lambda}(q^2)\gamma_\mu\gamma_5$$

$$+ \mathcal{P}_{\alpha\beta}^0 g_A^{N\Lambda}(q^2)\gamma_\mu\gamma_5 \frac{\not{p} - \not{q}' + im_\Lambda}{p^2 + q'^2 - 2q.p + m_\Lambda^2} i\gamma_5 g_{\Lambda NK}$$

$$+ i\gamma_5 g_{\Sigma NK} \frac{\not{p}' + \not{q}' + im_\Sigma}{p'^2 + q'^2 + 2q'.p' + m_\Sigma^2} \mathcal{P}_{\alpha\beta}^1 g_A^{N\Sigma}(q^2)\gamma_\mu\gamma_5$$

$$+ \mathcal{P}_{\alpha\beta}^1 g_A^{N\Sigma}(q^2)\gamma_\mu\gamma_5 \frac{\not{p} - \not{q}' + im_\Sigma}{p^2 + q'^2 - 2q'.p + m_\Sigma^2} i\gamma_5 g_{\Sigma NK} \Big\} \chi_i \eta_\beta u(p)$$

(2.19)

Further simplification is made possible through the Dirac equations (2.4). Eg. a term like

$$\bar{u}(p')\gamma_5(\not{p}' + \not{q}' + im_\Lambda)\gamma_\mu\gamma_5 u(p)$$

becomes

$$\bar{u}(p') \left[i(m_N - m_\Lambda) + \not{q}' \right] \gamma_\mu \gamma_5 \, u(p)$$

and so on. Now (2.19) becomes

$$\text{L.H.S.} = \bar{u}(p') \, \chi_f^* \, \eta_\alpha^* \Bigg\{ i g_{\Lambda NK} \frac{i(m_N - m_\Lambda) + \not{q}'}{m_\Lambda^2 - m_N^2 + q'^2 + 2q'.p'} \, \mathcal{P}^0_{\alpha\beta} g_A^{N\Lambda}(q^2) \gamma_\mu$$

$$+ \mathcal{P}^0_{\alpha\beta} \, i \, g_{\Lambda NK} \gamma_\mu \frac{i(m_\Lambda - m_N) + \not{q}'}{m_\Lambda^2 - m_N^2 + q'^2 - 2q'.p'} \, g_A^{N\Lambda} (q^2)$$

$$+ i \, g_{\Sigma NK} \frac{i(m_N - m_\Sigma) + q'}{m_\Sigma^2 - m_N^2 + q'^2 + 2q'.p'} \, \mathcal{P}^1_{\alpha\beta} \, g_A^{N\Sigma}(q^2) \gamma_\mu$$

$$+ i \, g_{\Sigma NK} \gamma_\mu \frac{i(m_\Sigma - m_N) + \not{q}'}{m_\Sigma^2 - m_N^2 + q'^2 - 2q'.p} \, \mathcal{P}^1_{\alpha\beta} \, g_A^{N\Sigma}(q^2) \Bigg\} \chi_i \, \eta_\beta^+ \, u(p)$$

(2.20)

From the kinematical relations (2.13) we have

$$\left. \begin{aligned} 2m_N(\nu + \nu_B) &= q'^2 - 2q'.p \\ 2m_N(\nu_B - \nu) &= q'^2 + 2q'.p' \end{aligned} \right\} \qquad (2.21)$$

We define

$$\Delta = \frac{m_\Lambda^2 - m_N^2}{2m_N} \; ; \qquad \Delta' = \frac{m_\Sigma^2 - m_N^2}{2m_N} \tag{2.22}$$

From the anticommutation property of Dirac matrices

$$q^\mu \gamma_\mu \gamma_\upsilon = \tfrac{1}{2} q^\mu \gamma_\mu \gamma_\upsilon + q^\mu \delta_{\mu\upsilon} - \tfrac{1}{2} q^\mu \gamma_\upsilon \gamma_\mu$$

we now rewrite (2.20) as

$$\text{L.H.S.} = \bar{u}(p')\chi_f^* \, \eta_\alpha^* \Big\{ i\, g_{\Lambda NK} \mathcal{P}^0_{\alpha\beta} \, g_A^{N\Lambda}(q^2) \frac{1}{2m_N} \Big[\frac{i(m_N - m_\Lambda)\gamma_\mu}{\upsilon_B - \upsilon + \Delta}$$

$$+ \frac{\frac{1}{2} q^\upsilon \gamma_\upsilon \gamma_\mu + q^\upsilon \delta_{\upsilon\mu} - \frac{1}{2} q^\upsilon \gamma_\mu \gamma_\upsilon}{\upsilon_B - \upsilon + \Delta} + \frac{i(m_\Lambda - m_N)\gamma_\mu}{\upsilon_B + \upsilon + \Delta}$$

$$+ \frac{\frac{1}{2} q^\upsilon \gamma_\mu \gamma_\upsilon + q^\upsilon \delta_{\upsilon\mu} - \frac{1}{2} q^\upsilon \gamma_\upsilon \gamma_\mu}{\upsilon_B + \upsilon + \Delta} \Big] + i\, g_{\Sigma NK} \mathcal{P}^1_{\alpha\beta} \, g_A^{N\Sigma}(q^2)$$

$$\times \frac{1}{2m_N} \Big[\frac{i(m_N - m_\Sigma)\gamma_\mu}{\upsilon_B - \upsilon + \Delta'} + \frac{\frac{1}{2} q^\upsilon \gamma_\upsilon \gamma_\mu + q^\upsilon \delta_{\upsilon\mu} - \frac{1}{2} q^\upsilon \gamma_\mu \gamma_\upsilon}{\upsilon_B - \upsilon + \Delta'}$$

$$+ \frac{i(m_\Sigma - m_N)\gamma_\mu}{\upsilon_B + \upsilon + \Delta'} + \frac{\frac{1}{2}q^\upsilon \gamma_\mu \gamma_\upsilon + q^\upsilon \delta_{\mu\upsilon} - \frac{1}{2}q^\upsilon \gamma_\upsilon \gamma_\mu}{\upsilon_B + \upsilon + \Delta'} \Big] \Big\} \chi_i \, \eta_\beta^+ \, u(p)$$

(2.23)

Comparing the coefficients of the operators O_μ^j on both sides of equation (2.23) we finally extract the following expressions for the pole terms

$$A^P_{1\alpha\beta} = \frac{1}{2m_N} \Big[g_{\Lambda NK} g_A^{N\Lambda}(q^2) \mathcal{P}^0_{\alpha\beta} \left(\frac{1}{\upsilon_B - \upsilon + \Delta} - \frac{1}{\upsilon_B + \upsilon + \Delta} \right)$$

$$+ g_{\Sigma NK} \, g_A^{N\Sigma}(q^2) \mathcal{P}^1_{\alpha\beta} \left(\frac{1}{\upsilon_B - \upsilon + \Delta'} - \frac{1}{\upsilon_B + \upsilon + \Delta} \right) \Big]$$

(2.24)

$$A^P_{3\alpha\beta} = \frac{1}{2m_N} \Big[g_{\Lambda NK} \, g_A^{N\Lambda}(q^2) \mathcal{P}^0_{\alpha\beta} \left(\frac{1}{\upsilon_B - \upsilon + \Delta} + \frac{1}{\upsilon_B + \upsilon + \Delta} \right)$$

$$+ g_{\Sigma NK} \, g_A^{N\Sigma}(q^2) \mathcal{P}^1_{\alpha\beta} \left(\frac{1}{\upsilon_B - \upsilon + \Delta'} + \frac{1}{\upsilon_B + \upsilon + \Delta'} \right) \Big]$$

(2.25)

$$A^{P}_{4\alpha\beta} = \frac{1}{2m_N^2}\Big[g_{\Lambda NK}\, g_A^{N\Lambda}(q^2)(m_\Lambda - m_N)\, \mathcal{P}^{o}_{\alpha\beta} \left(\frac{1}{\upsilon_B + \upsilon + \Delta} - \frac{1}{\upsilon_B - \upsilon + \Delta} \right)$$

$$+ g_{\Sigma NK}\, g_A^{N\Sigma}(q^2)(m_\Sigma - m_N)\, \mathcal{P}^{1}_{\alpha\beta} \left(\frac{1}{\upsilon_B + \upsilon + \Delta'} - \frac{1}{\upsilon_B + \upsilon + \Delta'} \right) \Big]$$

(2.26)

Let us now evaluate $< KN|\partial_\mu\, j_{5\mu}|N >$ at $q^2 = 0$.

We write

$$\sqrt{\frac{p_o p'_o}{m_N^2}\, 2q'_o}\; < KN|\partial_\mu\, j_{5\mu}^{(\Delta S = 1)}|N > \Big|_{q^2 = 0} = \bar{u}(p')\chi_f^*\, \eta_\alpha^*\, M_{\alpha\beta}\sqrt{2}\, \eta_\beta^+\, u(p)$$

(2.27)

where $M_{\alpha\beta}$ is defined by

$$M_{\alpha\beta} = A(\upsilon,\upsilon_B)_{\alpha\beta} - i\not{q}\, B(\upsilon,\upsilon_B)_{\alpha\beta} \qquad (2.28)$$

To determine A and B we consider the expression

$$\bar{u}(p')q_\mu \sum_{j=1}^{8} A_{j\alpha\beta}u(p) = \bar{u}(p')\Big[\tfrac{1}{2}(\not{q}'\not{q} - \not{q}\not{q}')A_{1\alpha\beta} + (p + p').q\, A_{2\alpha\beta}$$

$$+ q'.q\, A_{3\alpha\beta} + im_N\not{q}\, A_{4\alpha\beta} + i\not{q}(p + p').q\, A_{5\alpha\beta}$$

$$+ i \not{q} \not{q}' A_{6\alpha\beta} + q^2 A_{7\alpha\beta} + i \not{q} q^2 A_{8\alpha\beta} \Big] u(p)$$

(2.29)

The last two terms drop out at $q^2 = 0$. Consider the non-pole terms first. Then

$$\bar{M}_{\alpha\beta} = \bar{u}(p') \Big[\tfrac{1}{2}(\not{q}'\not{q} - \not{q}\not{q}')\bar{A}_{1\alpha\beta} - 2\upsilon m_N \bar{A}_{2\alpha\beta} + 2\upsilon_B \bar{A}_{3\alpha\beta}$$

$$+ i m_N \not{q} \bar{A}_{4\alpha\beta} - i \not{q} 2m_N\upsilon \bar{A}_{5\alpha\beta} + i \not{q} 2\upsilon_B m_N \bar{A}_{6\alpha\beta} \Big] \frac{1}{\sqrt{2}} u(p)$$

(2.30)

Further reduction is afforded by the use of

$$\not{q}'\not{q} + \not{q}\not{q}' = 2m_N\upsilon_B \quad \text{and } q' = p + q - p'$$

We can therefore write equation (2.30) as

$$\bar{M}_{\alpha\beta} = \bar{u}(p) \Big[-2m_N\upsilon \bar{A}_{1\alpha\beta} - 2m_N\upsilon \bar{A}_{2\alpha\beta} + 2m_N\upsilon_B \bar{A}_{3\alpha\beta}$$

$$- i \not{q} (- m_N \bar{A}_{4\alpha\beta} + 2m_N\upsilon \bar{A}_{5\alpha\beta} - 2m_N\upsilon_B \bar{A}_{6\alpha\beta}) \Big] \frac{1}{\sqrt{2}} u(p)$$

(2.31)

Hence

$$\bar{A}_{\alpha\beta} = \frac{1}{\sqrt{2}}\left(- 2m_N\upsilon\, \bar{A}_{1\alpha\beta} - 2m_N\upsilon\, \bar{A}_{2\alpha\beta} + 2m_N\upsilon_B\, \bar{A}_{3\alpha\beta} \right)$$

(2.32)

$$\bar{B}_{\alpha\beta} = \frac{1}{\sqrt{2}}\left(- m_N\, \bar{A}_{4\alpha\beta} + 2m_N\upsilon\, \bar{A}_{5\alpha\beta} - 2m_N\upsilon_B\, \bar{A}_{6\alpha\beta} \right)$$

We now write down the pole terms

$$\bar{u}(p')\sum_{j=1}^{8} O^j_\mu\, A^P_{j\alpha\beta} u(p) q_\mu = \bar{u}(p')\Big[- 2m_N\upsilon\, A^P_{1\alpha\beta} + 2m_N\upsilon_B\, A^P_{3\alpha\beta}$$

$$- i\not{q}\,(2m_N\, A^P_{1\alpha\beta} - m_N\, A^P_{4\alpha\beta}) \Big] \frac{1}{\sqrt{2}}\, u(p)$$

(2.33)

We can now identify the pole parts of the A and B amplitudes appearing in (2.28) as

$$\left.\begin{aligned} A^P_{\alpha\beta} &= \frac{1}{\sqrt{2}}\left(- 2m_N\upsilon\, A^P_{1\alpha\beta} + 2m_N\upsilon_B\, A^P_{3\alpha\beta} \right) \\ B^P_{\alpha\beta} &= \frac{1}{\sqrt{2}}\left(2m_N\, A^P_{1\alpha\beta} - m_N\, A^P_{4\alpha\beta} \right) \end{aligned}\right\} \quad (2.34)$$

However according to PCAC $< KN|\partial_\mu j_{5\mu}^{(\Delta S=1)}|N >$ is equal to

$$C_K < KN|\phi_K|N > = \frac{C_K}{m_K^2 + q^2} < KN|j_K|N > \qquad (2.35)$$

By the LSZ formalism we recognize in (2.35) the K-N scattering amplitude. This gives for $M_{\alpha\beta}$

$$M_{\alpha\beta} = \frac{C_K}{\sqrt{2}\, m_K^2}\left[A^{KN}(\upsilon,\upsilon_B,q^2=0)_{\alpha\beta} - i\,\not{q}\,B^{KN}(\upsilon,\upsilon_B,q^2=0)_{\alpha\beta}\right] \qquad (2.36)$$

Adding equations (2.32) and (2.34) for the A-amplitude and comparing with (2.36) we find

$$(-2m_N\upsilon\,\bar{A}_{1\alpha\beta} - 2m_N\upsilon\,\bar{A}_{2\alpha\beta} + 2m_N\upsilon_B\,\bar{A}_{3\alpha\beta} - 2m_N\upsilon\,A^P_{1\alpha\beta} + 2m_N\upsilon_B\,A^P_{3\alpha\beta})$$

$$= \frac{(m_N + m_\Sigma)}{g^{\Sigma NK}\,K^{\Sigma NK}(0)}\,A^{KN}_{\alpha\beta}\,(\upsilon,\upsilon_B,q^2=0) \qquad (2.37)$$

Finally since the amplitudes $\bar{A}_j$ have no kinematical singularities[(25)]

$$\lim_{\upsilon\to 0}\upsilon(\bar{A}_1 + \bar{A}_2)_{\alpha\beta} = \lim_{\upsilon_B\to 0}\upsilon_B\,\bar{A}_{3\alpha\beta} = 0 \qquad (2.38)$$

We simplify the pole term contribution by making the customary decomposition

$$A^{KN}_{\alpha\beta} = A^{KN(+)}\,\delta_{\alpha\beta} + A^{KN(-)}\,\underline{\tau}_N \cdot \underline{\tau}_{K\alpha\beta} \qquad (2.39)$$

We then find that the coefficient of $g_{\Lambda NK}\, g_A^{N\Lambda}(0)\, \delta_{\alpha\beta}$ for example, is proportional to

$$\frac{\upsilon_B^2 - \upsilon^2 + 2\Delta\upsilon_B}{(\upsilon_B - \upsilon + \Delta)(\upsilon_B + \upsilon + \Delta)}$$

Similar expressions hold for the coefficient of $g_{\Lambda NK} g_A^{N\Lambda}(0)\ \underline{\tau}_N \cdot \underline{\tau}_{K\alpha\beta}$ and for the Σ-terms. If we neglect the mass difference term Δ and set $\upsilon_B = \upsilon = 0$ then the above terms are equal to unity and equation (2.37) leads to the following two relations

$$\frac{g_{\Sigma NK}\, g_{\Lambda NK}\, g_A^{N\Lambda}(0) + 3 g^2_{\Sigma NK}\, g_A^{N\Sigma}(0)}{2(m_\Sigma + m_N)\, g_A^{N\Sigma}(0)} = \frac{A^{KN(+)}\ (\upsilon = 0, \upsilon_B = 0, q^2 = 0)}{K^{\Sigma NK}(0)}$$

$$(2.40)$$

$$\frac{g^2_{\Sigma NK}\, g_A^{N\Sigma}(0) - g_{\Sigma NK}\, g_{\Lambda NK}\, g_A^{N\Lambda}(0)}{2(m_\Sigma + m_N) g_A^{N\Sigma}(0)} = \frac{A^{KN(-)}\ (\upsilon = 0, \upsilon_B = 0, q^2 = 0)}{K^{\Sigma NK}(0)}$$

However if we maintain the mass differences then we get instead the null relations

$$A^{KN(+)}(\upsilon = 0, \upsilon_B = 0, q^2 = 0) = 0$$

$$A^{KN(-)}(\upsilon = 0, \upsilon_B = 0, q^2 = 0) = 0 \qquad (2.41)$$

because

$$\lim_{\upsilon \to 0} \upsilon\, A^P_{1\alpha\beta} = \lim_{\upsilon_B \to 0} \upsilon_B\, A^P_{3\alpha\beta} = 0 \qquad \text{when } \Delta, \Delta' \neq 0$$

Thus in the symmetry limit we obtain equations which link weak and strong interaction quantities while for the broken symmetry we get a condition on the K-N scattering amplitude alone. These results are to be compared with Adler's equations

$$\frac{g^2_{\pi NN}}{m_N} = \frac{A^{\pi N(+)}(\upsilon = 0, \upsilon_B = 0, q^2 = 0)}{K^{NN\pi}(0)}$$

$$0 = A^{\pi N(-)}(\upsilon = 0, \upsilon_B = 0, q^2 = 0) \qquad (2.42)$$

Adler obtained his results by neglecting the neutron-proton mass difference, i.e. a symmetry result and corresponds to our results (2.40).

We have a rather puzzling situation in that switching the mass differences on or off gives such widely different conclusions.

The resolution of this difficulty was made by Fuchs[26] and was drawn to the author's attention by Adler[27]. The argument is a simple one and was suggested by the polology approach to current algebras described in some detail in Chapter I. We consider the amplitude

$$M_\mu = \int d^4x \, e^{iq.x} < KN| j_\mu^{(\Delta S = 1)} |N > \qquad (2.43)$$

Integration by parts gives

$$q^\mu M_\mu = i \int d^4x \, e^{iq.x} < KN| \partial_\mu \, j_\mu^{(\Delta S = 1)}(x) |N > \qquad (2.44)$$

M_μ is related to the amplitude for K-meson production by the scattering of a nucleon in an external strangeness-bearing axial-vector field. For $\upsilon = 0$, $q^2 = 0$ we assume that $q^\mu M_\mu$ is dominated by the nearby singularities which are the Λ, Σ Born terms and the K-pole at $q^2 = -m_K^2$. If we keep the masses different then the Λ, Σ poles offer no problem and $q^\mu M_\mu \rightarrow 0$ as $q^\mu \rightarrow 0$. The residue of the pole in q^2 at $-m_K^2$ is just the on-mass-shell K-N forward scattering amplitude. This residue must vanish and hence we obtain the null condition (2.41). If however the masses are degenerate then a different situation arises. We must remember that when we want to test either of the conditions (2.40) or (2.41) we must write a dispersion relation for A^{KN}. In the equal mass limit M_μ is no longer regular as $q^\mu \to 0$ and we must separate the Born terms from it before taking the limit and this procedure would then give rise to the non-null relations (2.40). In the dispersion relation we eventually write for A^{KN} the Born terms contribute when the masses are distinct and do not contribute when the masses are equal. Thus in (2.41) A^{KN} includes Born terms while in (2.40) these are absent. In the limit of SU(3) all relations are non-null while for the broken symmetry all relations are null.

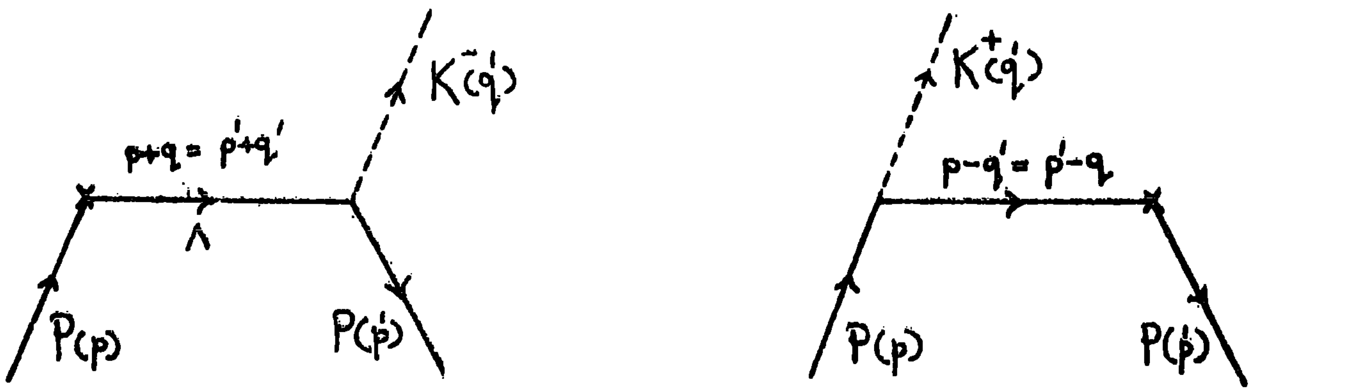


Figure 3

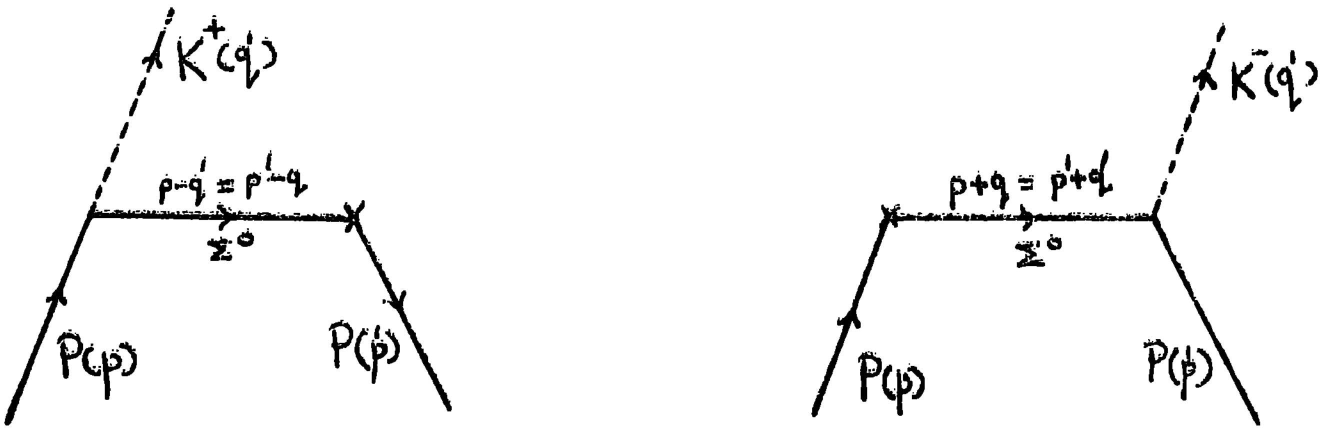


Figure 4

CHAPTER III

In reference (11), Fubini et al. have discussed a number of interesting applications of the theory of current algebra. One particularly interesting example is the derivation of the Gell-Mann - Okubo mass formula. The starting point is the commutator

$$\left[Q^{4+i5}(t), D^{4+i5}(x) \right]_{x_0 = t} = 0 \qquad (3.1)$$

where both the charge Q and the divergence D refer to the 4 + i5 component of the octet of vector currents. The interesting point is that this commutator, when sandwiched between appropriate states, leads to a linear mass formula for fermions and to a quadratic mass formula for bosons. This result was known for some time from group theory but no reason is given by the group-theoretical method as to why this distinction should occur between bosons and fermions. In current algebra this arises quite naturally. The usual assumption made in group theory about the symmetry-breaking Hamiltonian, namely that it transforms like the 8^{th} member of an octet certainly implies (3.1). On the other hand we could start from (3.1), which is implied by the quark model, as the basic object. (3.1) can be generalized to other values of the unitary spin indices,

$$\left[Q^A(t),\ D^B(x) \right]_{x_0 = t} = 0 \qquad (3.2)$$

where A, B are V-spin, U-spin or I-spin translation operators taken in suitable combinations. This was indeed done by Faustov[(28)] who was able to obtain various relations among the electromagnetic mass differences including the Coleman-Glashow[(29)] formula as well as two new formulae for the baryon decuplet.

In this chapter we investigate the consequences of the hypothesis that[(30)]

$$\left[Q_5^{4+i5}(t),\ D_5^{4+i5}(x) \right]_{x_0 = t} = 0 \qquad (3.3)$$

where both the charge Q_5 and the divergence D_5 refer to the component $4 + i5$ of the axial-vector octet, i.e. the strangeness-changing axial-vector weak current. Clearly this is a stronger assumption than the one made for the vector currents, (3.1). The commutator (3.3) is satisfied in the quark model and so in accordance with the basic philosophy of the current algebra theory it is possible to abstract it from the model and assume its validity in general. We shall discuss this further at the end of the chapter. Meanwhile we can give other arguments to support our hypothesis. The commutator (3.3) corresponds to a double strangeness exchange and we can imagine the trajectory of a particle in the exchange channel to be very depressed[(31)] and therefore hope that the sum rule which results from it converges. Notice also that it is a commutator between a good and a bad operator[(32)] and hence it is very interesting to examine its consequences. The theorem of Ademollo and Gatto[(33)] which says that the octet of vector currents is unrenormalized up to first order in the symmetry breaking interactions, holds for (3.1) and enables us to set all the renormalization ratios equal to one and end up with a function involving the masses only, i.e. a mass formula. However there is no such theorem for the axial octet and there is no hope of getting an SU(6)-type mass formula out of (3.3). We are led to view the sum rule arising from (3.3) as a relation between the F/D and F + D quantities of weak semi-leptonic decays or as a determination of the F/D ratio if one regards F + D as known from experiment (equation 1.139) or from current algebra[(18),(19)].

We now turn to a discussion of the sum rule arising from (3.3). The axial-vector current $j^i_{5\mu}$ is an octet operator and we can write the following expansion for its matrix element between any two baryon octet states Bi, Bj

$$< Bj | j^{K}_{5\mu} | Bi > = \bar{u}_j \left\{ \left[G^{s}_{1} (q^2) d_{kij} + G^{a}_{1} (q^2)\, i\, f_{kij} \right] \gamma_\mu \gamma_5 \right.$$

$$\left. - \left[G^{s}_{2} (q^2) d_{kij} + G^{s}_{2} (q^2)\, i\, f_{kij} \right] i q_\mu \gamma_5 \right\} u_i \qquad (3.4)$$

We have not included the term proportional to $\sigma_{\mu\upsilon} q_\upsilon \gamma_5$ - the 2nd class covariant; i.e. only terms with $C = +1$ have been written down. This is not really an assumption about the charge conjugation properties of the axial-vector octet because this term is not going to contribute to the divergence anyway. We may transcribe the above equation from the Cartesian to the 'spherical' non-Hermitian basis with the aid of the spherical vectors $e_i^{(\alpha)}$ (34) where α denotes the 'magnetic' quantum numbers (I, I_3, Y). Using

$$e_i^{(\lambda)}\, e_j^{(\mu)}\, e_k^{(\upsilon)*}\, f_{ijk} = i\sqrt{3} \begin{pmatrix} 8 & 8 & 8a \\ \lambda & \mu & \upsilon \end{pmatrix}$$

$$e_i^{(\lambda)}\, e_j^{(\mu)}\, e_k^{(\upsilon)*}\, d_{ijk} = \sqrt{\frac{5}{3}} \begin{pmatrix} 8 & 8 & 8s \\ & & \\ \lambda & \mu & \upsilon \end{pmatrix}$$

$$e^{(\lambda)} \cdot e^{(\mu)} = -\sqrt{8} \begin{pmatrix} 8 & 8 & 1 \\ & & \\ \lambda & \mu & 0 \end{pmatrix}$$

(3.5)

where $\begin{pmatrix} N & N' & N''_{\xi} \\ & & \\ \mu & \upsilon & \lambda \end{pmatrix}$ is the SU(3) Clebsch-Gordan coefficient as defined by de Swart[35]. Remembering that a physical baryon state $B^{(\alpha)}$ is related to the Cartesian state B_i by

$$B^{(\alpha)} = e_i^{(\alpha)}\, B_i \qquad (3.6)$$

we readily obtain

$$< B^{(\alpha)} | j_{5\mu}^{(\alpha')} | B^{(\alpha'')} > = \bar{u}_{(\alpha)} \left\{ \left[G_1^s(q^2) \sqrt{\frac{5}{3}} \begin{pmatrix} 8 & 8 & 8_s \\ & & \\ \alpha'' & \alpha' & \alpha \end{pmatrix} \right.\right.$$

$$\left. + G_1^a(q^2)\sqrt{3} \begin{pmatrix} 8 & 8 & 8_a \\ & & \\ \alpha'' & \alpha' & \alpha \end{pmatrix} \right] \gamma_\mu \gamma_5$$

$$- \left[G_2^s(q^2) \sqrt{\frac{5}{3}} \begin{pmatrix} 8 & 8 & 8_s \\ \alpha'' & \alpha' & \alpha \end{pmatrix} \right.$$

$$\left. + G_2^a(q^2) \sqrt{3} \begin{pmatrix} 8 & 8 & 8_a \\ \alpha'' & \alpha' & \alpha \end{pmatrix} \right] i q_\mu \gamma_5 \Big\} u_{(\alpha'')}$$

(3.7)

We shall need the formula for the divergence which we write as

$$< B^{(\alpha)}(p_1) | D_5^{(\alpha')} | B^{(\alpha'')}(p_2) >$$

$$= \left(m_{B(\alpha)} + m_{B(\alpha'')} \right) \bar{u}_{(\alpha)}(p_1) G_{\alpha'}^{\alpha\alpha''}(q^2) \, \gamma_5 \, u_{(\alpha'')}(p_2)$$

(3.8)

where $G_{\alpha'}^{\alpha\alpha''}(q^2)$ is defined by

$$G_{\alpha'}^{\alpha\alpha''}(q^2) = \sqrt{\frac{5}{3}} \begin{pmatrix} 8 & 8 & 8_s \\ \alpha'' & \alpha' & \alpha \end{pmatrix} \left[G_1^S(q^2) - \frac{q^2}{m_{B(\alpha)} + m_{B(\alpha'')}} G_2^S(q^2) \right]$$

$$+ \sqrt{3} \begin{pmatrix} 8 & 8 & 8_a \\ \alpha'' & \alpha' & \alpha \end{pmatrix} \left[G_1^a(q^2) - \frac{q^2}{m_{B(\alpha)} + m_{B(\alpha'')}} G_2^a(q^2) \right]$$

(3.9)

We see that in the limit of zero momentum transfer only two constants $G_1^s(0)$ and $G_1^a(0)$ remain. Following reference (11) we write the charge as a 4-dimensional integral of a divergence by introducing an exponential in some four vector, k, and a Ⓤ-function. Specifically (3.3) becomes

$$\lim_{k_\mu \to 0} \int d^4x \, Ⓤ(-x_0) e^{-ik.x} < f | [D_5^{4+i5}(x), D_5^{4+i5}(0)] | i > \;=\; 0 \tag{3.10}$$

where $|\,i>$, $|\,f>$ are suitable initial and final baryon states. We define an amplitude T_{if} by

$$T_{if} = \int d^4x \; Ⓤ(-x_0) e^{-ik.x} < f | [D_5^{4+i5}(x), \; D_5^{4+i5}(0)] | i > \tag{3.11}$$

Defining the usual scalar invariants by

$$\begin{aligned} s &= -(k + p_f)^2 \\ t &= -(p_f - p_i)^2 \; ; \quad p_f^2 = -m_f^2 \, , \quad p_i^2 = -m_i^2 \\ u &= -(p_i + k)^2 \end{aligned} \tag{3.12}$$

We make use of the arbitrariness in the vector k and choose it to satisfy

$$k^2 = 0 \; ; \quad s - m_f^2 = u - m_i^2 = \upsilon \quad \text{i.e.} \quad k.p_i = k.p_f \tag{3.13}$$

We now assume for T_{if} a dispersion relation in υ at constant t

$$T_{if}(\upsilon,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\phi_{if}(\upsilon',t)}{\upsilon' - \upsilon}\, d\upsilon' \qquad (3.14)$$

The absorptive part is given by

$$\phi_{if}(\ ,t) = \tfrac{1}{2}(2\pi)^4 \Big\{ \sum_n \delta^4(p_f + k - p_n) < f|D_5^{4+i5}(0)|n >$$

$$< n|D_5^{4+i5}(0)|i > - \sum_n \delta^4(p_i - k - p_{n'})$$

$$< f|D_5^{4+i5}(0)|n' > < n'|D_5^{4+i5}(0)|i > \Big\} \qquad (3.15)$$

We shall refer to the coefficient of the 1st δ -function as the direct term and the coefficient of the 2nd δ-function as the crossed term. The contribution to T_{if} of a single particle (S.P) intermediate state is given by

$$T_{if}^{S.P} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\upsilon'}{\upsilon' - \upsilon}\, \tfrac{1}{2}(2\pi)^4 \Big\{ \frac{1}{(2\pi)^3} \int \frac{m}{E}\, d^3p \Big[\delta^4(p_f + k - p)$$

$$< p_f|D_5^{4+i5}(0)|p > < p|D_5^{4+i5}(0)|p_i > - \delta^4(p_i - k - p)$$

$$< p_f|D_5^{4+i5}(0)|p > < p|D_5^{4+i5}(0)|p_i > \Big]\Big\} \qquad (3.16)$$

Make the replacement

$$\int \frac{d^3p}{2E} \rightarrow \int d^4p\ \delta(p^2 + m^2) = \int d^4p\ \delta(m^2 - m_f^2 - \upsilon)$$

where p, m are the momentum and mass of the S.P. intermediate state. If we write

$$T_{if}^{S.P} = C_1 + C_2 \tag{3.17}$$

where C_1 comes from the direct term and C_2 from the crossed term then we immediately find

$$C_1 = \frac{< f|D_5^{4+i5}(0)|p > < p|D_5^{4+i5}(0)|i >}{m^2 - m_f^2 - \upsilon} \cdot 2m \tag{3.18}$$

We now make specific choices for the states $|f>$, $|i>$ as proton and Ξ^- states respectively

$$< f| = < p| \ ; \quad |i> = |\Xi^-> \tag{3.19}$$

The possible single-particle intermediate states are then Σ^0 and Λ. The numerator of (3.18) then reads for Σ^0, for example

$$< P|D^5_{\frac{1}{2}\frac{1}{2}1}(0)|\Sigma^0 > < \Sigma^0|D^5_{\frac{1}{2}\frac{1}{2}1}(0)|\Xi^- > = (m_p + m_{\Sigma^0})(m_{\Sigma^0} + m_{\Xi^-})$$

$$\times\ G^{P\Sigma^0}_{\frac{1}{2}\frac{1}{2}1}(0)\ G^{\Sigma^0\Xi^-}_{\frac{1}{2}\frac{1}{2}1}(t)\bar{u}(p_f)\gamma_5 u(p_f + k)\bar{u}(p_f + k)\gamma_5 u(p_i)$$

(3.20)

The label $\frac{1}{2}\frac{1}{2}1$ on the form-factor stands for the 'magnetic' quantum numbers of the divergence. We are interested in the limit $k \to 0$ and the R.H.S. of (3.20) then simplifies to

$$\text{R.H.S.} = (m^2_{\Sigma^0} - m^2_p)(m_{\Sigma^0} + m_{\Xi^-})\, G^{p\Sigma^0}_{\frac{1}{2}\frac{1}{2}1}(0)\, G^{\Sigma^0\Xi^-}_{\frac{1}{2}\frac{1}{2}1}(t)\bar{u}(p_f)u(p_i) \tag{3.21}$$

We thus see that in the limit $k \to 0$ (i.e. $\nu \to 0$) the 1st factor in (3.21) cancels the denominator term in (3.18). The factor 2m in the numerator of (3.18) is also cancelled by the term 2m arising from the propagator for the intermediate state. The term C_2 is reduced in a similar manner. We then find that Σ^0 contribution to T_{if} is

$$T^{\Sigma^0}_{if} = \left[(m_{\Sigma^0} + m_{\Xi^-})\, G^{p\Sigma^0}_{\frac{1}{2}\frac{1}{2}1}(0)\, G^{\Sigma^0\Xi^-}_{\frac{1}{2}\frac{1}{2}1}(t) + (m_{\Sigma^0} + m_p) G^{p\Sigma^0}_{\frac{1}{2}\frac{1}{2}1}(t) G^{\Sigma^0\Xi^-}_{\frac{1}{2}\frac{1}{2}1}(0) \right]$$

$$\times \quad \bar{u}_p(p_f)u_{\Xi^-}(p_i) \tag{3.22}$$

If we now evaluate the sum-rule for zero momentum transfer, $t = 0$, this simplifies to

$$(2m_{\Sigma^0} + m_p + m_{\Xi^-})\, G^{p\Sigma^0}_{\frac{1}{2}\frac{1}{2}1}(0)\, G^{\Sigma^0\Xi^-}_{\frac{1}{2}\frac{1}{2}1}(0)\, \bar{u}_p(p_f)u_{\Xi^-}(p_i) \tag{3.23}$$

where

$$G^{p\Sigma^0}_{\frac{1}{2}\frac{1}{2}1}(0) = \sqrt{\frac{5}{3}} \begin{pmatrix} 8 & 8 & 8_s \\ 100 & \frac{1}{2}\frac{1}{2}1 & \frac{1}{2}\frac{1}{2}1 \end{pmatrix} G^s_1(0) + \sqrt{3} \begin{pmatrix} 8 & 8 & 8_a \\ 100 & \frac{1}{2}\frac{1}{2}1 & \frac{1}{2}\frac{1}{2}1 \end{pmatrix} G^a_1(0) \tag{3.24}$$

$$G^{\Sigma^0\Xi^-}_{\frac{1}{2}\frac{1}{2}1}(0) = \sqrt{\frac{5}{3}} \begin{pmatrix} 8 & 8 & 8 \\ \frac{1}{2}-\frac{1}{2}-1 & \frac{1}{2}\frac{1}{2}1 & 100 \end{pmatrix} G^s_1(0) + \sqrt{3} \begin{pmatrix} 8 & 8 & 8_a \\ \frac{1}{2}-\frac{1}{2}-1 & \frac{1}{2}\frac{1}{2}1 & 100 \end{pmatrix} \tag{3.25}$$

Similarly, the Λ-contribution to the sum rule is

$$T_{if}^{\Lambda} = (2m_\Lambda + m_p + m_{\Xi^-}) \, G^{p\Lambda}_{\frac{1}{2}\frac{1}{2}1}(0) \, G^{\Lambda\Xi^-}_{\frac{1}{2}\frac{1}{2}1}(0) \, \bar{u}_p(p_f) \, u_{\Xi^-}(p_i) \tag{3.26}$$

where

$$G^{p\Lambda}_{\frac{1}{2}\frac{1}{2}1}(0) = \sqrt{\frac{5}{3}} \begin{pmatrix} 8 & 8 & 8_s \\ 000 & \frac{1}{2}\frac{1}{2}1 & \frac{1}{2}\frac{1}{2}1 \end{pmatrix} G^s_1(0) + \sqrt{3} \begin{pmatrix} 8 & 8 & 8_a \\ 000 & \frac{1}{2}\frac{1}{2}1 & \frac{1}{2}\frac{1}{2}1 \end{pmatrix} G^a_1(0) \tag{3.27}$$

$$G^{\Lambda\Xi^-}_{\frac{1}{2}\frac{1}{2}1}(0) = \sqrt{\frac{5}{3}} \begin{pmatrix} 8 & 8 & 8_s \\ \frac{1}{2}\text{-}\frac{1}{2}\text{-}1 & \frac{1}{2}\frac{1}{2}1 & 000 \end{pmatrix} G^s_1(0) + \sqrt{3} \begin{pmatrix} 8 & 8 & 8_a \\ \frac{1}{2}\text{-}\frac{1}{2}\text{-}1 & \frac{1}{2}\frac{1}{2}1 & 000 \end{pmatrix} G^a_1(0) \qquad (3.28)$$

So our sum-rule (3.10) now becomes

$$\Big[(2m_{\Sigma^0} + m_p + m_{\Xi^-})\, G^{p\Sigma^0}_{\frac{1}{2}\frac{1}{2}1}(0)\, G^{\Sigma^0\Xi^-}_{\frac{1}{2}\frac{1}{2}1}(0) + (2m_\Lambda + m_p + m_{\Xi^-})\, G^{p\Lambda}_{\frac{1}{2}\frac{1}{2}1}(0)\, G^{p\Lambda\Xi^-}_{\frac{1}{2}\frac{1}{2}1}(0) \Big]\, \bar{u}_p(p_f) u_{\Xi^-}(p_i)$$

$$= -\frac{1}{\pi} \int_{\text{Threshold}} d\upsilon' \; \frac{A^{I}(\upsilon')}{\upsilon' - \upsilon} + \frac{A^{II}(-\upsilon')}{\upsilon' + \upsilon} \qquad (3.29)$$

where

$$\left.\begin{aligned} A^{I}(\upsilon) &= \tfrac{1}{2}(2\pi)^4 \sum_{n \neq \Lambda,\Sigma^0} \delta^4(p_f + k - p_n) < p|D^5_{\frac{1}{2}\frac{1}{2}1}|n >< n|D^5_{\frac{1}{2}\frac{1}{2}1}|\Xi^- > \\ A^{II}(\upsilon) &= \tfrac{1}{2}(2\pi)^4 \sum_{n \neq \Lambda,\Sigma^0} \delta^4(p_i - k - p_n) < p|D^5_{\frac{1}{2}\frac{1}{2}1}|n >< n|D^5_{\frac{1}{2}\frac{1}{2}1}|\Xi^- > \end{aligned}\right\} \qquad (3.30)$$

We can now use PCAC in the operator form

$$\partial^{\mu}\, j_{5\mu}^{(\Delta S=1)} = m_K^2\, F_K\, \phi_K \tag{3.31}$$

to convert the absorptive parts A^I and A^{II} into products of scattering amplitudes. e.g.

$$< p|D^5_{\frac{1}{2}\frac{1}{2}1}|n > = \frac{m_K^2\, F_K}{m_K^2 + (p_f - p_n)^2} < p|j_K|n >$$

$$= \frac{m_K^2\, F_K}{m_K^2 + (p_f - p_n)^2}\, T_{n\to PK^-} \tag{3.32}$$

where F_K is the decay constant of the K-mesons into a lepton pair. We can therefore rewrite the absorptive parts as

$$\left.\begin{aligned} A^I &= \tfrac{1}{2}(2\pi)^4 F_K^2 \sum_{n \neq \Lambda,\Sigma^0} \delta^4(p_f + k - p_n) T(n \to PK^-)\, T(\Xi^- K^+ \to n) \\ A^{II} &= \tfrac{1}{2}(2\pi)^4 F_K^2 \sum_{n \neq \Lambda,\Sigma^0} \delta^4(p_i - k - p_n)\, T(n \to PK^-)\, T(\Xi^- K^+ \to n) \end{aligned}\right\} \tag{3.33}$$

Unlike the situation arising in the Adler-Weisberger sum rule here we are dealing with the non-diagonal elements of the

T-matrix and we cannot convert this into cross-sections. We shall assume that the contribution of the integral on the R.H.S. of (3.29) comes mostly from the Y_1^{*0} resonance. There is no real justification for this resonance approximation which is frequently used in current algebras except perhaps the phenomenological one, namely that the high mass states should not make significant contributions to the sum-rules. Even this is not enough, for to evaluate the contribution of the Y_1^{*0} we need to know the form factors occuring in the matrix element of the axial-vector current between baryon and decuplet states, for zero momentum transfer. The expansion involves 4 form-factors and one way of writing it is the following

$$< B^{(\alpha)}(p_1)|j_{5\mu}^{(\alpha')}|\Delta^{(\alpha'')}(p_2) > = i\bar{u}_{(\alpha)}(p_1)\Big[-H_1^{\alpha'}(q^2)\delta_{\mu\upsilon} - \frac{i}{m}H_2^{\alpha'}(q^2)p_{1\upsilon}\gamma_\mu + \frac{1}{m^2}H_3^{\alpha'}(q^2)p_{1\upsilon}(p_1+p_2)_\mu + \frac{1}{m^2}H_4^{\alpha'}(q^2)p_{1\upsilon}(p_1-p_2)_\mu\Big]U^{\upsilon}_{(\alpha'')}(p_2)\begin{pmatrix} 8 & 8 & 10 \\ \alpha & \alpha' & \alpha'' \end{pmatrix} \tag{3.34}$$

No γ_5 appears on the R.H.S. of (3.34) since the Ravita-Schwinger spinor $U^{\upsilon}(p)$ have -ve intrinsic parity and we are using it to describe a +ve parity state. Knowledge of the form factors H_1, H_2, H_3, H_4 allows one to calculate the amplitude for decuplet production by neutrinos on the baryon octet

$$\nu + B \rightarrow \Delta + \mu$$
$$\quad\quad\quad\quad \hookrightarrow B' + \text{meson}$$

For the matrix element of the divergence we have

$$< B^{(\alpha)}(p_1)|D_5^{(\alpha')}|\Delta^{(\alpha'')}(p_2) >$$

$$= \bar{u}_{(\alpha)}(p_1) H^{\alpha\alpha''}_{\alpha'}(q^2) p_{1\nu}\, U^{\nu}_{(\alpha'')}(p_2) \begin{pmatrix} 8 & 8 & 10 \\ \alpha & \alpha' & \alpha'' \end{pmatrix} \tag{3.35}$$

where

$$H^{\alpha\alpha''}_{\alpha'}(q^2) = \Big[-H_1^{\alpha'}(q^2) + \frac{1}{m} H_2^{\alpha'}(q^2)(m_{B\alpha} - m_{\Delta\alpha''})$$

$$+ \frac{1}{m^2} H_3^{\alpha'}(q^2)\; (m^2_{\Delta\alpha''} - m^2_{B\alpha}) + \frac{1}{m^2} H_4^{\alpha'}(q^2) q^2 \Big] \tag{3.36}$$

In the above equations m is the mass of the appropriate pseudoscalar meson which has the same quantum numbers as $D^{\alpha'}_5$. Ideally one would like to be able to use the numerical values of H_1, H_2, H_3, H_4 for zero momentum transfer in evaluating the contribution of the

Y_1^{*o} to the sum-rule. This presumably can be done by current algebra methods. However for our purposes we use the pole dominance model for the divergence of the axial current and approximate the function $H_{\alpha'}^{\alpha\alpha''}(q^2)$ as follows: e.g. for the $PY_1^{*o}K$ vertex we write

$$< P|D^5_{\frac{1}{2}\frac{1}{2}1}|Y_1^{*o} > = \frac{g_{pY_1^{*o}K}}{m_K} \cdot \frac{m_K^2 F_K}{m_K^2 + q^2} \bar{u}_p(p_1)p_{1\upsilon} U^{\upsilon}(p_2) \tag{3.37}$$

This sort of approximation leaves us with only 1 parameter to evaluate instead of four (H_1, H_2, H_3, H_4) namely the coupling constant for the $PY_1^{*o}K$ vertex. Similarly we need to know the coupling constant for the $\Xi^- Y_1^{*o}K$ vertex. Due to the lack of reliable experimental data on these numbers we turn to SU(3) symmetry for their determination. In the SU(3) symmetry scheme the Y_1^{*o} is written as

$$|Y_1^{*o}> = \frac{1}{\sqrt{12}}\Big\{ |\Sigma^-\pi^+> - |N\bar{K}_0> + |\Xi^-K^+> + \sqrt{3}|\Sigma^o N>$$

$$- \sqrt{3}|\Lambda\pi^o> + |PK^-> + |\Xi^o K^o> - |\Sigma^+\pi^-> \Big\} \tag{3.38}$$

The states $|PK^- >$ and $|\Xi^- K^+ >$ enter with equal weights $\frac{1}{\sqrt{12}}$.

The decays of the decuplet resonances are determined by one coupling constant g, apart from Clebsch-Gordon coefficients. To find g we look at the best experimentally measured width, namely that of the N^{*++}. This state is written

$$|N^{*++} > = \frac{1}{\sqrt{2}} |P\pi^+ > \tag{3.39}$$

The transition rate is given by

$$\Gamma_{N^{*++} \to P\pi^+} = \frac{g^2}{m_{N^*}^2} \frac{q_{P\pi}^3}{4} \cdot \frac{E_P + m_P}{m_{N^*}} \tag{3.40}$$

where $q_{P\pi}$ is the centre of mass momentum of the π^+ and P. The factor $\frac{E_P + m_P}{m_{N^*}}$ varies very little among the various decays and is usually dropped. The experimental width of the N* then leads to $g^2 = 49.7$. In terms of g we have

$$g_{PY_1^{*o}K} = g_{\Xi^- Y_1^{*o}K} = \frac{g}{2\sqrt{3}} \tag{3.41}$$

We now substitute th:e expansions (3.37) into the sum rule. The reduction of the Dirac algebra is performed with the help of the subsidiary conditions satisfied by the Rarita-Schwinger spinor

$$(\not{p} - im^*) U^{\upsilon}(p) = 0$$

$$\gamma_{\upsilon} U^{\upsilon}(p) = 0 \qquad (3.42)$$

$$P_{\upsilon} U^{\upsilon}(p) = 0$$

We also need the propagator for spin $^{3}/_{2}$:

$$P_{\mu\upsilon} = \left\{ g_{\mu\upsilon} - \tfrac{1}{3}\gamma_{\mu}\gamma_{\upsilon} + \frac{i}{3m^*}(P_{\mu}\gamma_{\upsilon} - \gamma_{\mu}P_{\upsilon}) + \frac{2}{3m^{*2}} P_{\mu}P_{\upsilon} \right\} \frac{-i\not{p} + m^*}{2m^*}$$

(3.43)

Here of course we treat the resonance as a stable state and compute its contribution in precisely the same way we did for the Σ and Λ states. We can now write the Y_1^{*o} contribution as

$$N \times \left(\frac{H^{PY_1^{*o}} \, H^{Y_1^{*o}\Xi^-}}{m_K^2} \right) \times \begin{pmatrix} 10 & 8 & 8 \\ 100 & \frac{1}{2}\frac{1}{2}1 & \frac{1}{2}\frac{1}{2}1 \end{pmatrix} \begin{pmatrix} 8 & 8 & 10 \\ \frac{1}{2}\text{-}\frac{1}{2}\text{-}1 & \frac{1}{2}\frac{1}{2}1 & 100 \end{pmatrix}$$

(3.44)

where N is a numerical factor containing the masses and arising from the Dirac algebra. The functions $H^{PY_1^{*o}}$ and $H^{Y_1^{*o}\Xi^-}$ are given for zero momentum transfer by

$$H^{PY_1^{*o}} = g_{PKY_1^{*o}} F_K = H^{Y_1^{*o}\Xi^-} \qquad (3.45)$$

We now go back to equation (3.29) and read off the numerical values for the Clebsch-Gordon coefficients from the tables of de-Swart[35]. We find simply

$$G^{P\Sigma^0}_{\frac{1}{2}\frac{1}{2}1}(0) = \tfrac{1}{2}G^s_1(0) - \tfrac{1}{2}G^a_1(0) \tag{3.46}$$

$$G^{\Sigma^0\Xi^-}_{\frac{1}{2}\frac{1}{2}1}(0) = -\tfrac{1}{2}G^s_1(0) - \tfrac{1}{2}G^a_1(0) \tag{3.47}$$

$$G^{P\Lambda}_{\frac{1}{2}\frac{1}{2}1}(0) = -\frac{1}{2\sqrt{3}}G^s_1(0) - \frac{\sqrt{3}}{2}G^a_1(0) \tag{3.48}$$

$$G^{\Lambda\Xi^-}_{\frac{1}{2}\frac{1}{2}1}(0) = \frac{1}{2\sqrt{3}}G^s_1(0) - \frac{\sqrt{3}}{2}G^a_1(0) \tag{3.49}$$

Define

$$G = G^s_1(0) + G^a_1(0) \quad \text{and} \quad a = G^a_1(0)/G \tag{3.50}$$

Hence

$$G^s_1(0) = G(1 - a) \quad \text{and} \quad G^a_1(0) = Ga \tag{3.51}$$

The F/D ratio is by definition $\frac{a}{1 - a}$. We now substitute the values (3.46) - (3.49) into the sum-rule (3.29) and use the relations given by (3.51). It now reads

$$\frac{G^2}{4}\left\{ (2m_{\Sigma^o} + m_P + m_{\Xi^-})(2a - 1) + \tfrac{1}{3}(2m_\Lambda + m_P + m_{\Xi^-})(8a^2 + 2a - 1)\right\}$$

$$= (Y_1^{*o} \text{ contribution}) \qquad (3.52)$$

where the R.H.S. is given by (3.44). In (3.44) N has the numerical value

$$N = 4.06 \times 10^3$$

The value 1.09×10^3 for F_K^2 (36) as well as the value $49.7/2\sqrt{3}$ for $g_{PY_1^{*o}K}$ and $g_{\Xi^-Y_1^{*o}K}$ give for (3.)

$$4.06 \times 10^3 \times 1.8 \times 10^{-1} \times \frac{1}{2\sqrt{3}} \times -\frac{1}{\sqrt{15}} \qquad (3.53)$$

where the last two numbers are the Clebsch-Gordon coefficients. Putting all this as well as the known masses into equation (3.52) we get a quadratic equation for the parameter a;

$$(1496.7)8a^2 + (6140.4)(2a - 1) + \frac{217.6}{G^2} = 0 \qquad (3.54)$$

We now regard G either as an experimental input or as a result from the Adler-Weisberger formula. It has the numerical value $|G| = 1.18$. This now gives two solutions for (3.54)

$$a = -1.39 \quad \text{or} \quad 0.37 \qquad (3.55)$$

Remembering that the $^F/_D$ ratio is $^a/_{1-a}$ we finally obtain

$$^F/_D = -0.58 \quad \text{or} \quad 0.59 \tag{3.56}$$

It is amusing to note that these two solutions closely resemble the solution one obtains for the homogeneous equation $(F/D)^2 = \text{constant}$[(37)]. This similarity might not be purely accidental.

The value 0.59 we obtain for the $^F/_D$ ratio is in good agreement with experiment. Two determinations of this number by two different groups are:[(38)]

$$\begin{array}{ll} & 0.45 \quad \text{(Brene et al)} \\ \text{and} & \\ & 0.59 \quad \text{(Willis et al)} \end{array} \tag{3.57}$$

Returning now to our basic commutator (3.3) we examine it within the framework of a free quark model. We have

$$j^i_{5\mu} = \bar{q}(x)\, \gamma_\mu \gamma_5 \lambda^i\, q(x) \tag{3.58}$$

$$\partial_\mu\, j^i_{5\mu} = 2m_q\, \bar{q}(x)\, \gamma_5 \lambda^i\, q(x) \tag{3.59}$$

Hence at equal times

$$\left[Q_5^i(t), D_5^j(x) \right] = \int d^3x \left[\bar{q}(x)\, \gamma_0 \gamma_5 \lambda^i\, q(x), 2m_q\, \bar{q}(x)\, \gamma_5 \lambda^j\, q(x) \right]$$

$$= m_q\, q^+(\underline{x},t) \left\{ [\gamma_5, \gamma_0\gamma_5]\, \{\lambda^i, \lambda^j\} + \{\gamma_5\, \gamma_0\gamma_5\} [\lambda^i, \lambda^j] \right\} \times q(\underline{x},t)$$

$$= -m_q\, \bar{q}(\underline{x},t) d_{ijk}\, \lambda^k q(\underline{x},t) \qquad (3.60)$$

Thus the R.H.S. is non-vanishing in general, unlike the case of vector currents, and is proportional to a scalar density. However for the particular choice $i = j = 4 + i5$ the number $d_{K^+K^+K} = 0$, where K^+ signifies $4 + i5$. Hence the commutator (3.3) is satisfied in a quark model. Alternatively the commutation (3.3) should vanish since it is a $\Delta S = 2$ operator which cannot be constructed as a bilinear quantity in quark fields[39].

CHAPTER IV

Recently there has been some interest in the algebra of scalar and pseudoscalar densities[40],[41],[42]. In reference (42), for example, the coupling constant of the so-called σ-meson to two pions is calculated under the assumption that 2π is the dominant decay mode. A value for its width is also obtained which is quite narrow. At the moment there is no conclusive evidence experimentally for such a narrow s-wave $I = 0$, π-π resonance. However people have attempted to fit nucleon-nucleon scattering data with such a scalar particle with a certain coupling to nucleon states[43]. In this chapter [44] we attempt to calculate this coupling constant from the algebra of scalar and pseudoscalar densities proposed by Gell-Mann[1].

Our starting point is the retarded amplitude involving the commutator of the axial-vector current with a pseudoscalar density between two spin $\frac{1}{2}$ states. Specifically we define

$$T^{ij}_{\mu} = \int d^4x \, e^{-iq_2 \cdot x} < p_2 | R(j^i_{5\mu}(x) p^j(0) | p_1 > \qquad (4.1)$$

In terms of quark fields the pseudoscalar and scalar densities are given by[1]

$$\left.\begin{aligned} p^i(x) &= -i\,\bar{q}(x)\,\gamma_5\lambda^i\,q(x) \\ s^i(x) &= \bar{q}(x)\,\lambda^i\,q(x) \end{aligned}\right\}\quad i = 0, 1, \ldots, 8 \tag{4.2}$$

The absorptive part of T^{ij}_μ is given by

$$t^{ij}_\mu = \frac{1}{2}\int d^4x\, e^{-iq_2\cdot x} < p_2|[j^i_{5\mu}(x), p^j(0)]|p_1 > \tag{4.3}$$

We evaluate $q_2^\mu T^{ij}_\mu$ by partial integration and write

$$q_2^\mu T^{ij}_\mu = V^{ij} + F^{ij} \tag{4.4}$$

where

$$V^{ij} = \int d^4x\, e^{-iq_2\cdot x}\, \textcircled{u}(x_0) < p_2|[D^i_5(x), p^j(0)]|p_1 > \tag{4.5}$$

and

$$F^{ij} = \int d^4x\, e^{-iq_2\cdot x}\, \delta(x_0) < p_2|[j^i_{50}(x), p^j(0)]|p_1> \tag{4.6}$$

The absorptive part satisfies

$$q_2^\mu t^{ij}_\mu = v^{ij} \tag{4.7}$$

where

$$v^{ij} = \frac{1}{2i} \int d^4x \, e^{-iq_2 \cdot x} < p_2 | [D_5^i(x), \, p^j(0)] | p_1 > \qquad (4.8)$$

The following commutation relations hold at equal times(1)

$$\left. \begin{aligned} [\, Q_5^i(t), \, p^j(\underline{x},t) \,] &= i \, d_{ijk} \, s^k(\underline{x},t) \\ [\, Q_5^i(t), \, s^j(\underline{x},t) \,] &= -i \, d_{ijk} \, p^k(\underline{x},t) \end{aligned} \right\} \qquad (4.9)$$

We shall deal only with pions, i,j,k = 1,2,3. We therefore simplify (4.9) by first defining the scalar density $\sigma(x)$ by

$$\sigma(x) = \sqrt{\frac{2}{3}} \, s^0 + \sqrt{\frac{1}{3}} \, s^8 \qquad (4.10)$$

where $\sigma(x)$ is the field for a so far hypothetical scalar meson which may have something to do with a real 0^+ π-π resonance. We therefore have

$$\left. \begin{aligned} [\, Q_5^\alpha(t), \, p^\beta(\underline{x},t) \,] &= i \, \delta_{\alpha\beta} \, \sigma(\underline{x},t) \\ [\, Q_5^\alpha(t), \, \sigma(\underline{x},t) \,] &= -i \, \delta_{\alpha\beta} \, p^\beta(\underline{x},t) \end{aligned} \right\} \quad \alpha, \beta = 1, 2, 3 \qquad (4.11)$$

We use $p^{\alpha}(\underline{x},t)$ as an interpolating field for the pion and define its normalization constant by

$$< 0|p^{\alpha}(0)|\pi^{\beta}(q) > = a_p \, \delta_{\alpha\beta} \qquad (4.12)$$

Later on we are going to identify $p^{\alpha}(x)$ with the divergence of axial-vector current and this determines a_p uniquely in terms of the pion decay constant. Equation (4.11) implies the following commutator for the densities

$$[\ j^{\alpha}_{50}(\underline{x},0),\ p^{\beta}(0)\ ] = i\,\delta_{\alpha\beta}\,\delta^{3}(\underline{x})\,\sigma(\underline{x},0) + \text{S.T} \qquad (4.13)$$

where S.T. denotes the possible Schwinger terms. If assumed proportional to a finite number of derivatives of δ-functions, the S.T. will integrate to zero in (4.11). With the help of (4.13) we now write for $F^{\alpha\beta}$ (equation 4.6)

$$F^{\alpha\beta} = i\,\delta_{\alpha\beta} < p_2|\sigma(0)|p_1 > + \text{Polynomial in } q_2 \qquad (4.14)$$

As mentioned in Chapter one, the hypothesis is frequently made that the S.T. contribute only to amplitudes symmetric under the simultaneous exchange of Lorentz and unitary spin indices[6]. Here, however, we are dealing only with one vector index and we therefore keep track of the S.T. which will contribute to $F^{\alpha\beta}$

with a finite polynomial in q_2 .

We expand $T_\mu^{\alpha\beta}$ on a convenient basis as follows

$$T_\mu^{\alpha\beta} = \bar{u}(p_2)\Big[i\gamma_\mu A_1^{\alpha\beta} + [\gamma_\mu, \not{Q}]\bar{A}_1^{\alpha\beta} + P_\mu(A_2^{\alpha\beta} + i\not{Q}\bar{A}_2^{\alpha\beta})$$

$$+ Q_\mu(A_3^{\alpha\beta} + i\not{Q}\bar{A}_3^{\alpha\beta}) + \Delta_\mu(A_4^{\alpha\beta} + i\not{Q}\bar{A}_4^{\alpha\beta})\Big] u(p_1) \tag{4.15}$$

We define the usual kinematical quantities

$$\left.\begin{aligned} P &= \tfrac{1}{2}(p_1 + p_2)\ ;\quad Q = \tfrac{1}{2}(q_1 + q_2)\ ;\quad \Delta = p_1 - p_2 = q_2 - q_1 \\ \upsilon &= -q_1.P = -q_2.P = -Q.P\ ;\quad t = \Delta^2 \end{aligned}\right\} \tag{4.16}$$

The scalar amplitudes $A_i^{\alpha\beta}$, $\bar{A}_i^{\alpha\beta}$ are in general functions of υ, t, q_1^2, q_2^2. Having an explicit representation of $T_\mu^{\alpha\beta}$ we now calculate $q_2^\mu\, T_\mu^{\alpha\beta}$. We are going to be interested in the limit $\upsilon \to \infty$ of this quantity. Decomposing the scalar amplitudes A_i, $\bar{A}_i$ into isotopic-spin symmetric and antisymmetric parts we arrive at the following relations

$$-2\upsilon\bar{A}_1^{(+)} - \upsilon A_2^{(+)} + \tfrac{1}{2}(q_2^2 + Q^2 - \tfrac{1}{4}t)A_3^{(+)} + (q_2^2 - Q^2 + \tfrac{1}{4}t)A_4^{(+)}$$

$$= -A^{(+)} + i\,\frac{G_{NN\sigma}\,K_{NN\sigma}(t)}{m_\sigma^2 + t} \tag{4.17}$$

$$-2\nu\bar{A}_1^{(-)} - \nu A_2^{(-)} + \tfrac{1}{2}(q_2^2 + Q^2 - \tfrac{1}{4}t)A_3^{(-)} + (q_2^2 - Q^2 + \tfrac{1}{4}t)A_4^{(-)}$$

$$= -A^{(-)} \qquad (4.18)$$

and

$$A_1^{(\pm)} - 2m_N\bar{A}_1^{(\pm)} - \nu\bar{A}_2^{(\pm)} + \tfrac{1}{2}(q_2^2 + Q^2 - \tfrac{1}{4}t)\bar{A}_3^{(\pm)}$$

$$+ (q_2^2 - Q^2 + \tfrac{1}{4}t)\bar{A}_4^{(\pm)} = \bar{A}^{(\pm)} \qquad (4.19)$$

where we have introduced the following notation

$$< p_2|\sigma(0)|p_1 > = \bar{u}(p_2)\frac{G_{NN\sigma}K_{NN\sigma}(t)}{m_\sigma^2 + t}u(p_1) \qquad (4.20)$$

$$V^{\alpha\beta} = \bar{u}(p_2)\left[-A^{\alpha\beta} + i\not{Q}\bar{A}^{\alpha\beta}\right]u(p_1) \qquad (4.21)$$

Equation (4.20) defines the coupling $G_{NN\sigma}$ of the unitary-spin σ to nucleons. For the absorptive parts we have the relations

$$-2\nu\bar{a}_1^{(\pm)} - \nu a_2^{(\pm)} + \tfrac{1}{2}(q_2^2 + Q^2 - \tfrac{1}{4}t)a_3^{(\pm)}$$

$$+ (q_2^2 - Q^2 + \tfrac{1}{4}t)a_4^{(\pm)} = -a^{(\pm)} \qquad (4.22)$$

$$a_1^{(\pm)} - 2m_N \bar{a}_1^{(\pm)} - \upsilon \bar{a}_2^{(\pm)} + \tfrac{1}{2}(q_2^2 + Q^2 - \tfrac{1}{4}t)\bar{a}_3^{(\pm)}$$

$$+ (q_2^2 - Q^2 + \tfrac{1}{4}t)\bar{a}_4^{(\pm)} = \bar{a}^{(\pm)} \qquad (4.23)$$

where

$$V^{\alpha\beta} = \bar{u}(p_2)[-a^{\alpha\beta} + i\not{Q}\,\bar{a}^{\alpha\beta}]\,u(p_1) \qquad (4.24)$$

The small letters denote the components of $t_\mu^{\alpha\beta}$ on the same basis (4.15). This technique of writing sum rules is due to Fubini[12] and is described in detail in Chapter 1. We recall that the physical assumption made by Fubini is that one usually deals with conserved or partially-conserved currents and we can therefore assume that the scalar amplitudes $A^{\alpha\beta}$, $\bar{A}^{\alpha\beta}$ which define the components of $V^{\alpha\beta}$ in the tensor basis (4.15) tend to zero as $\upsilon \to \infty$ i.e.

$$\lim_{\upsilon\to\infty} A^{\alpha\beta} = \lim_{\upsilon\to\infty} \bar{A}^{\alpha\beta} = 0 \qquad (4.25)$$

If $A^{\alpha\beta}$, $\bar{A}^{\alpha\beta}$ obey unsubtracted dispersion relations then this certainly holds. The physical meaning of this assumption is quite clear in our case since $V^{\alpha\beta}$ is proportional to an off-mass-shell pion nucleon scattering amplitude when we identify both the pseudoscalar density $P^\beta(x)$ and the divergence $D_5^\alpha(x)$ with interpolating pion fields.

The absorptive parts a_i, $\bar{a}_i$ are determined from the coefficients of $i\gamma_\mu$, $[\gamma_\mu, \not{q}] \ldots \Delta_\mu$, in the expansion of $t_\mu^{\alpha\beta}$ i.e. from

$$t_\mu^{\alpha\beta}(p_2, q_2; p_1, q_1) = \tfrac{1}{2}(2\pi)^4 \Big\{ \sum_n \delta^4 (p_n - p_2 - q_2)$$

$$< p_2 | j_{5\mu}^{\alpha}(0) | n > < n | p^{\beta}(0) | p_1 > - \sum_{n'} \delta^4 (p_{n'} - p_1 + q_2)$$

$$< p_2 | p^{\beta}(0) | n > < n | j_{5\mu}^{\alpha}(0) | p_1 > \Big\} \qquad (4.26)$$

We are taking the external states to be nucleon states.

We now make the assumption of unsubtracted dispersion relations for the scalar amplitudes $A_i^{(\pm)}$, $\bar{A}_i^{(\pm)}$ i.e.

$$A_i^{(\pm)} (\nu, t, q_1^2, q_2^2) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a_i^{(\pm)}(\nu', t, q_1^2, q_2^2)}{\nu' - \nu} d\nu' \qquad (4.27)$$

and

$$\bar{A}_i^{(\pm)} (\nu, t, q_1^2, q_2^2) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\bar{a}_i^{(\pm)}(\nu', t, q_1^2, q_2^2)}{\nu' - \nu} d\nu' \qquad (4.28)$$

The dispersion representation has been assumed for amplitudes symmetric and antisymmetric in isospin indices[14]. We are aware of the fact that the symmetric amplitudes might pick up contributions from the Schwinger terms. However we are going to be interested in a particular kinematical configuration in which the S.T. are harmless. Taking the crossing properties of the functions A_i, $\bar{A}_i$ into account we arrive at the following non-trivial sum rule:

$$\frac{1}{\pi} \int d\upsilon \; [2\bar{a}_1^{(+)}(\upsilon,t,q_1^2,q_2^2) + a_2^{(+)} \; (\upsilon,t,q_1^2,q_2^2)]$$

$$= i \; \frac{G_{NN\sigma} \; K_{NN\sigma}(t)}{m_\sigma^2 + t} \qquad (4.29)$$

There is another sum-rule involving $\bar{a}_2^{(-)}$ but it is not interesting because, as we shall see later, it does not have a single-nucleon contribution. We now calculate the contribution of the single-nucleon state to the sum rule. For this purpose we need the following vertices

$$< p|j_{5\mu}^{\alpha}(0)|p' > = i \; \bar{u}(p) \tau^{\alpha} \; [\gamma_\mu \gamma_5 \; G_1(q^2) + i(p' - p)_\mu \gamma_5$$

$$\times \; G_2(q^2)] \; u(p') \qquad (4.30)$$

where $q^2 = (p' - p)^2$ and we have neglected the 2nd class covariant proportional to $\sigma_{\mu\upsilon} \, q_\upsilon$. We also write

$$< p|j_{\pi}^{\alpha}(0)|p' > = i \, g\bar{u}(p)\tau^{\alpha}\gamma_5 \; u(p')K_{NN\pi}(q^2) \qquad (4.31)$$

where $K_{NN\pi}(-m_\pi^2) = 1$. We now substitute these two expressions into equation (4.26). We then find the following single-nucleon terms

$$a_{1,N}^{\alpha\beta} = \tfrac{1}{4}(\delta_{\alpha\beta} + \tfrac{1}{2}[\tau^\alpha, \tau^\beta]) \frac{\pi}{2} \frac{g m_N a_p K_{NN\pi}(q_1^2) G_1(q_2^2)}{m_\pi^2 + q_1^2} \left\{ \delta(\upsilon - \upsilon_o) + \delta(\upsilon + \upsilon_o) \right\} \qquad (4.32)$$

$$a_{2,N}^{\alpha\beta} = \tfrac{1}{4}(\delta_{\alpha\beta} + \tfrac{1}{2}[\tau^\alpha, \tau^\beta]) \frac{\pi}{2} \cdot \frac{g a_p K_{NN\pi}(q_1^2) G_1(q_2^2)}{m_\pi^2 + q_1^2} \left\{ -\delta(\upsilon - \upsilon_o) - \delta(\upsilon + \upsilon_o) \right\} \qquad (4.33)$$

$$a_{3,N}^{\alpha\beta} = 0$$

$$a_{4,N}^{\alpha\beta} = \tfrac{1}{4}(\delta_{\alpha\beta} + \tfrac{1}{2}[\tau^\alpha, \tau^\beta] \frac{\pi}{4} \cdot \frac{g a_p K_{NN\pi}(q_1^2) G_1(q_2^2)}{m_\pi^2 + q_1^2} \left\{ \delta(\upsilon - \upsilon_o) - \delta(\upsilon + \upsilon_o) \right\} \qquad (4.34)$$

We also have a term

$$\frac{1}{4}(\delta_{\alpha\beta} + \frac{1}{2}[\tau^{\alpha},\tau^{\beta}])\ \frac{\pi}{2} \cdot \frac{g a_p K_{NN\pi}(q_1^2) G_1(q_2^2)}{m_\pi^2 + q_1^2}$$

$$\left\{ -\delta(\nu - \nu_o)\,\gamma_\mu\ \not{Q} + \delta(\nu + \nu_o)\not{Q}\,\gamma_\mu \right\} \tag{4.35}$$

Since in the sum rules the integration over ν is carried out it is easy to see that this term will contribute to the coefficient of $[\gamma_\mu\ , \not{Q}]$ i.e. $\bar{a}^{\alpha\beta}_{1,N}$ and to the coefficient of Q_μ , i.e. $a_s^{(-)}$. For the remaining terms we have

$$\bar{a}^{\alpha\beta}_{2,N} = 0 \tag{4.36}$$

$$\bar{a}^{\alpha\beta}_{3,N} = \frac{1}{4}(\delta_{\alpha\beta} + \frac{1}{2}[\tau^{\alpha},\tau^{\beta}])\ \frac{\pi}{2} \cdot \frac{g a_p K_{NN\pi}(q_1^2) G_2(q_2^2)}{m_\pi^2 + q_1^2}$$

$$\left\{ -\delta(\nu - \nu_o) - \delta(\nu + \nu_o) \right\} \tag{4.37}$$

$$\bar{a}^{\alpha\beta}_{4,N} = \frac{1}{4}(\delta_{\alpha\beta} + \frac{1}{2}[\tau^{\alpha},\tau^{\beta}])\ \frac{\pi}{4} \cdot \frac{g a_p K_{NN\pi}(q_1^2) G_2(q_2^2)}{m_\pi^2 + q_1^2}$$

$$\left\{ -\delta(\nu - \nu_o) - \delta(\nu + \nu_o) \right\} \tag{4.38}$$

In the above formulae $\upsilon_o = \frac{1}{2}(P^2 + Q^2 + m_N^2)$.

Separating the nucleon contribution the sum rule (4.29) now reads

$$\frac{F_\pi m_\pi^2 g\, K_{NN\pi}(q_1^2) G_1(q_2^2)}{4(m_\pi^2 + q_1^2)} + \frac{1}{\pi i}\int d\upsilon \;(2\bar{a}_1^{(+)}\;(\upsilon,t,q_1^2,q_2^2) +$$

$$a_2^{(+)}(\upsilon,t,q_1^2,q_2^2)) = \frac{G_{NN\sigma}\, K_{NN\sigma}(t)}{m_\sigma^2 + t} \qquad (4.39)$$

where we have identified $p^\alpha(x)$ with the divergence of the axial-vector current and ∴

$$a_p = -\frac{F_\pi m_\pi^2}{2} .$$

Before discussing equation (4.39) any further, we wish to turn to the other technique of writing sum-rules namely Adler's method, described in Chapter 1. In this method we evaluate $\upsilon T_\mu^{\alpha\beta}$ by first computing $q_2^\rho\, T_\mu^{\alpha\beta}$. As argued in references (14) and (45) because of Lorentz invariance we can replace $\text{ⓤ}(x_o)$ in $T_\mu^{\alpha\beta}$ by $\text{ⓤ}(-n.x)$ where n is an arbitrary unit time-like vector ($n^2 = -1$). Then

$$q_2^\rho\, T_\mu^{\alpha\beta} = \int d^4x\, e^{-iq_2.x}\; \text{ⓤ}(-n.x) < p_2 | [\, \partial^\rho\, j_{5\mu}^\alpha(x),\; p^\beta(0)] | p_1 >$$

$$+ \int d^4x\, e^{-iq_2.x} < p_2 | [\, j_{5\mu}^\alpha(x),\; p^\beta(0)] | p_1 > \partial^\rho \text{ⓤ}(-n.x)$$

(4.40)

We are going to be interested in the isotopic spin symmetric part of (4.40). We evaluate the commutator in the 2nd term using the quark form of the current and the pseudoscalar density. The isotopic-spin symmetric piece of the commutator is

$$\left[j^{\alpha}_{5\mu}(x),\ p^{\beta}(0)\right] \partial^{\rho}\ \text{Ⓤ}(-n.x) = i\, n^{\rho} n_{\mu}\, d_{\alpha\beta\gamma}\, s^{\gamma}(0)\, \delta^{4}(x) \tag{4.41}$$

Following Gourdin[14] we make the natural choice

$$n_{\mu} = \frac{P_{\mu}}{\sqrt{-P^{2}}} \tag{4.42}$$

Multiplying both sides of equation (4.40) with P_{ρ} we get the following relation

$$-\nu T^{\alpha\beta}_{\mu} = -U^{\alpha\beta}_{\mu} + F^{\alpha\beta}_{\mu} \tag{4.43}$$

where

$$U^{\alpha\beta}_{\mu} = -P_{\rho}\int d^{4}x\, e^{-iq_{2}.x}\ \text{Ⓤ}(-n.x) < p_{2}|[\,\partial^{\rho}\, j^{\alpha}_{5\mu}(x),\ p^{\beta}(0)]|p_{1} > \tag{4.44}$$

and

$$F^{\alpha\beta}_{\mu} = i\, d_{\alpha\beta\gamma} < p_{2}|P_{\mu}\, s^{\gamma}(0)|p_{1} > \tag{4.45}$$

We expand $U_{\mu}^{\alpha\beta}$ on the same tensor basis as before and use capital manuscript letters to designate the scalar components in the expansion. We thus obtain a set of eight equations but only five of them lead to non-trivial sum rules after taking the crossing properties into account. Remembering that we are concerned only with amplitudes symmetric in the internal indices α, β we thus write

$$\left.\begin{aligned} -\upsilon A_1^{\alpha\beta} &= -\mathcal{A}_1^{\alpha\beta} \\ -\upsilon \bar{A}_1^{\alpha\beta} &= -\bar{\mathcal{A}}_1^{\alpha\beta} \\ -\upsilon A_2^{\alpha\beta} &= -\mathcal{A}_2^{\alpha\beta} + i\, d_{\alpha\beta\gamma} R^{\gamma} \\ -\upsilon A_3^{\alpha\beta} &= -\mathcal{A}_3^{\alpha\beta} \\ -\upsilon \bar{A}_4^{\alpha\beta} &= -\mathcal{A}_4^{\alpha\beta} \end{aligned}\right\} \qquad (4.46)$$

where R^{γ} is defined by

$$< p_2 | s^{\gamma}(0) | p_1 > = \bar{u}(p_2) R^{\gamma} u(p_1) \qquad (4.47)$$

We again make the hypothesis of unsubtracted dispersion representation for the scalar amplitudes A_i, $\bar{A}_i$. However we cannot in general demand that the scalar components of $V^{\alpha\beta}$ in Fubini's method and the scalar functions $\mathcal{A}_i^{\alpha\beta}$, $\bar{\mathcal{A}}_i^{\alpha\beta}$ which arise in

Adler's method should vanish simultaneously as $\upsilon \to \infty$ (14). To start with we must make sure that Adler's technique reproduces the sum-rule (4.29) arrived at earlier. For this to happen we consider

$$\lim_{\upsilon\to\infty} (-2\upsilon \bar{A}_1^{\alpha\beta} - \upsilon A_2^{\alpha\beta}) = \lim_{\upsilon\to\infty} (-2\bar{\mathcal{A}}_1^{\alpha\beta} - \mathcal{A}_2^{\alpha\beta}) + i\, d_{\alpha\beta\gamma} R^{\gamma} \tag{4.48}$$

This equation we reproduce (4.29) provided

$$\lim_{\upsilon\to\infty} (2\bar{\mathcal{A}}_1^{\alpha\beta} + \mathcal{A}_2^{\alpha\beta}) = 0 \tag{4.49}$$

It is clear that we only need to assume that such a linear combination vanishes as $\upsilon \to \infty$. Equation (4.49) is of course also satisfied if both $\mathcal{A}_1^{\alpha\beta}$ and $\mathcal{A}_2^{\alpha\beta}$ vanish separately in the limit. This is encouraging and makes it possible for us to extend the assumption for all values of i and in particular for $\bar{\mathcal{A}}_i$ with $i = 3,4$ since these lead to interesting sum rules. Specifically we are going to assume that

$$\lim_{\upsilon\to\infty} \mathcal{A}_3^{\alpha\beta} = \lim_{\upsilon\to\infty} \bar{\mathcal{A}}_4^{\alpha\beta} = 0 \tag{4.50}$$

Then the last two equations in the set (4.46) lead to the following two sum rules with the nucleon contribution separated explicitly (see equations (4.37) and (4.38)).

$$\tfrac{1}{4}\, g\, K_{NN\pi}(q_1^2) G_2(q_2^2) + \int \bar{a}_3^{(+)}(\upsilon, t, q_1^2, q_2^2) d\upsilon = 0 \qquad (4.51)$$

$$\frac{1}{8}\, g\, K_{NN\pi}(q_1^2) G_2(q_2^2) + \int \bar{a}_4^{(+)}(\upsilon, t, q_1^2, q_2^2) d\upsilon = 0 \qquad (4.52)$$

The factor $\frac{1}{2} F_\pi m_\pi^2 (m_\pi^2 + q_1^2)^{-1}$ which occurs also in the integral has been cancelled through. We thus have sum-rules involving the induced pseudoscalar form-factor in the weak axial-vector vertex. We shall return to a further discussion of these relations later.

We now return to our sum-rule (4.39). As is usual in current algebras useful information is obtained in a certain kinematical configuration which invariably involves off-mass-shell quantities. We are going to evaluate (4.39) in the limit $q_1^2 = q_2^2 = t = 0$. Furthermore we are going to assume that the integral is saturated by the contribution of the N_{33}^* resonance. To do this we need the vertex

$$< N(p) | j_{5\mu}^\alpha(0) | N_{33}^*(p') > = i \bar{u}(p) \Big[-H_1^\alpha(q^2)\, \delta_{\mu\upsilon} - \frac{1}{m_\pi} H_2^\alpha(q^2) p_\upsilon \gamma_\mu$$

$$+ \frac{1}{m_\pi^2} H_3^\alpha(q^2) p_\upsilon (p + p')_\mu + \frac{1}{m_\pi^2} H_4^\alpha(q^2) p_\upsilon (p - p')_\mu \Big] U^\upsilon(p')$$

(4.53)

Unlike the situation in Chapter III we know the numerical values of the form-factors $H_1, \ldots H_4$ at zero momentum-transfer. For the particular component $j_{5\mu}^{(-)}$ sandwiched between N^{*+} and neutron states these form-factors were evaluated by Furlan et al[46] using the algebra of currents and the saturation assumption. At zero momentum transfer they obtain

$$\left.\begin{aligned} H_1(0) &= -0.41 \\ H_2(0) &= -1.13 \\ H_3(0) &= -0.088 \\ H_4(0) &= 0.86 \end{aligned}\right\} \qquad (4.54)$$

We also need the following vertex

$$< P(p)|j_\pi^3|N^{*+}(p') > = \sqrt{\frac{2}{3}}\ \frac{\lambda}{m_\pi}\ \bar{u}(p)U^\nu(p')(p'-p)_\nu \qquad (4.55)$$

where $\lambda = 2.2$ is the coupling leading to an isobar width of 125 MeV. We have taken this parameter from Fubini et al[47] who re-evaluated it following the analysis of Gourdin and Salin[48]. The N^*_{33} contribution to the sum-rule is evaluated with the help of the subsidiary conditions and the spin $^3/_2$ -propagator given in Chapter III. Calling, as before, the coefficient of the 1st δ-function in (4.26) the direct term and that of the 2nd δ-function the crossed term we find, e.g., that the vertex in the direct term reads

$$\bar{u}(p_2)\left[-H_1\delta_{\mu\rho} - \frac{i}{m_\pi} H_2 P_{2\rho}\gamma_\mu + \frac{1}{m_\pi^2} H_3 P_{2\rho}(2p_2 + q_2)_\mu\right.$$

$$\left. + \frac{1}{m_\pi^2} H_4 P_{2\rho} q_{2\mu}\right]\left\{ g_{\rho\sigma} - \tfrac{1}{3}\gamma_\rho\gamma_\sigma + \frac{i}{3m^*}\left[(p_2 + q_2)_\rho\gamma_\sigma\right.\right.$$

$$\left.\left. - \gamma_\rho(p_2 + q_2)_\sigma\right] + \frac{2}{3m^{*2}}(p_2 + q_2)_\rho(p_2 + q_2)_\sigma\right\}$$

$$\times \quad \frac{-i(\not{p}_2 + \not{q}_2) + m^*}{2m^*} \times -\sqrt{\frac{2}{3}}\,\frac{\lambda}{m_\pi}\, q_{1\sigma}\, u(p) \qquad (4.56)$$

Simplification of this expression is made possible by the use of the algebra of γ-matrices and the Dirac equation on the final spinors. The algebra is lengthy and tedious but quite straightforward. To illustrate our point we consider a typical term from the above expression:

$$\bar{u}(p_2)\delta_{\mu\rho}\tfrac{1}{3}\gamma_\rho\gamma_\sigma(-i(\not{p}_2 + \not{q}_2) + m^*) - q_{1\sigma}\, u(p_1) \times H_1(q_2^2)\sqrt{\frac{2}{3}}\,\frac{\lambda}{m_\pi}\,\frac{1}{2m^*}$$

$$= \bar{u}(p_2) - \tfrac{1}{3}\gamma_\mu\gamma_\sigma(-i\not{Q} - \tfrac{i}{2}\not{p}_1 - \tfrac{i}{2}\not{p}_2 + m^*)(Q_\sigma - \tfrac{1}{2}\Delta_\sigma)$$

$$\times \; u(p_1) \times H_1(q_2^2)\sqrt{\frac{2}{3}}\,\frac{\lambda}{m_\pi}\cdot\frac{1}{2m^*}$$

(4.57)

where we have used $q_2 = Q + \frac{1}{2}\Delta$ and $q_1 = Q - \frac{1}{2}\Delta$. This simplifies further to

$$\bar{u}(p_2)\left[\frac{1}{3} i\gamma_\mu Q^2 - \frac{i}{3} p_1\cdot Q\,\gamma_\mu - \frac{m_N}{3}\gamma_\mu \not{Q} + \frac{m_N^2}{6} i\gamma_\mu - \frac{m_N}{3} p_{2\mu} + \frac{1}{3} p_2\cdot Q\, i\gamma_\mu\right.$$

$$\left. - \frac{i}{6} p_1\cdot p_2\gamma_\mu - \frac{m^*}{3}\gamma_\mu \not{Q} + \frac{m_N m^*}{3} i\gamma_\mu - \frac{m^*}{3} p_{2\mu}\right] H_1(q_2^2)\sqrt{\frac{2}{3}}\ \frac{\lambda}{m_\pi}\ \frac{1}{2m^*} \tag{4.58}$$

The values (4.54) were quoted for the component $j_{5\mu}^{(-)}$ between N^{*+} and neutron states. Here we are taking the external states to be proton states and we take $\alpha = \beta = 3$. Then we need the following conversion factor

$$\frac{<n|j_{5\mu}^{(-)}|N^{*+}>}{<p|j_{5\mu}^{3}\ |N^{*+}>} = \frac{\begin{pmatrix} 10 & 8 & 8 \\ \frac{3}{2}\frac{1}{2}1 & 1\text{-}10 & \frac{1}{2}\text{-}\frac{1}{2}1 \end{pmatrix}}{\begin{pmatrix} 10 & 8 & 8 \\ \frac{3}{2}\frac{1}{2}1 & 100 & \frac{1}{2}\frac{1}{2}1 \end{pmatrix}} = -\frac{1}{\sqrt{2}} \tag{4.59}$$

Also from the experimental value 1.18 for the β-decay axial coupling constant we calculate the coupling constant occuring in $<p|j_{5\mu}^{3}|p>$ by

$$\frac{<p|j_{5\mu}^{+}|n>}{<p|j_{5\mu}^{3}|\,p>} = -\sqrt{2} \tag{4.60}$$

We are now in a position to write the contribution of the N* resonance as

$$\frac{1}{\pi i}\int d\upsilon \left[2\bar{a}_1^{(+)}(\upsilon,0,0,0) + a_2^{(+)}(\upsilon.0,0,0) \right]_{N^*}$$

$$= \frac{F_\pi}{8}\sqrt{\frac{2}{3}}\lambda \left[\left(\frac{2m_N}{3} + \frac{2m^*}{3} - \frac{m_N^2}{6m^*}\right)\frac{H_1(0)}{m_\pi}\right.$$

$$+ \left(\frac{2m_N m^*}{3} - \frac{2m_N^3}{3m^*} - \frac{4}{3m^{*2}}\upsilon^2_{pole}\right)\frac{H_2(0)}{m_\pi^2}$$

$$\left. + \left(\frac{4}{3m^*} + \frac{4m_N}{3m^{*2}}\right)\upsilon^2_{pole}\frac{H_3(0)}{m_\pi^3}\right] \tag{4.61}$$

where υ_{pole} is the position of the N*-pole and is given by

$$\upsilon_{pole} = \tfrac{1}{2}(m^{*2} - m_N^2) \tag{4.62}$$

The right hand side of equation (4.39) reads $\dfrac{G_{NN\sigma}\,K_{NN\sigma}(0)}{m_\sigma^2}$. Now $K_{NN\sigma}(-m_\sigma^2) = 1$ but we are going to assume that $K_{NN\sigma}(0) \simeq 1$. We know from our experience with the Goldberger-Treiman relation that such extrapolations could lead to variations of about 15% for a not too massive σ. For consistency we are going to use the parameters of the σ-meson calculated within the current algebraic framework. As mentioned at the beginning of this chapter such a calculation was performed in reference (42) and led to a value of 385 MeV for the mass and a width of approximately 73 MeV. Then such an approximation is not too dangerous. We also put $K_{NN\pi}(0) \simeq 1$.

Feeding in the experimental masses into the equation together with the values of pion-nulceon coupling constant and β-decay coupling constant, we arrive at the following number:

$$\frac{G_{NN\sigma}}{m_\sigma^2} = 1.18 \times 10^{-4} \times m_\pi^2 F_\pi \qquad (4.63)$$

We would like to mention that it is the quantity $\frac{G_{NN\sigma}}{m_\sigma^2}$ which appears in current-algebraic calculations. But we know that the divergence of axial-vector current couples with a strength $m_\pi^2 F_\pi$ to the interpolating field. We now propose that the scalar density couples in an analogous manner with a strength $m_\sigma^2 F_\sigma$ to the 'true σ-meson' field. We are therefore led to define a coupling constant of the σ-meson to nucleons by

$$g_{NN\sigma} = \frac{G_{NN\sigma}}{m_\sigma^2 F_\sigma} \qquad (4.64)$$

Therefore

$$g_{NN\sigma} = \frac{1.18 \times 10^{-4} \times m_\pi^2 F_\pi}{F_\sigma} \qquad (4.65)$$

To proceed further we can make one of two possible assumptions. We could, for example, assume that in fact the scalar densities $S^\alpha(x)$ couple with the same strength to the corresponding particle fields as that by which the pseudoscalar densities $p^\alpha(x)$ couple to the pseudoscalar particle fields, i.e. one could assume that the algebra (4.9) holds for the phenomenological particle fields as well. This sort of assumption implies

$$m_\sigma^2 F_\sigma = m_\pi^2 F_\pi \qquad (4.66)$$

and leads to the following numerical value for $g_{NN\sigma}$:

$$g_{NN\sigma} = 16.8 \qquad (4.67)$$

This value is much higher than one would expect. Later on we shall give arguments that in fact the saturation assumption is not very adequate for this type of sum rules and in fact inclusion of higher states should bring the value (4.67) to a lower value. However it will still be too high in our opinion. Before discussing the 2nd possibility we would like to mention that the authors in reference (43) attempt to fit elastic nucleon-nucleon scattering due to the exchange of π, η, ρ, ω, Φ and an effective

I = 0 scalar σ-meson by using unsubtracted partial wave dispersion relations with a cutt-off. The value they use for $g_{NN\sigma}$ depends on the cut-off energy and decreases with increasing energy. E.g. at 800 MeV cut-off they use $g^2_{NN\sigma} = 4.15$. Although they use a more massive σ-meson it is clear that their results favour a low value for $g_{\sigma NN}$. Now we would like to point out the sort of assumption which will do this. It is simply that F_σ is equal or approximately equal to F_π, i.e. the pion decay constant plays a rather fundamental roll in the relationships between densities and particle fields in current algebra. With this assumption the value (4.67) for $g_{NN\sigma}$ is brought down by a factor $\left(\frac{m_\sigma}{m_\pi}\right)^2$ and reads

$$g_{NN\sigma} \simeq 2.3 \tag{4.68}$$

We now turn to a further discussion of the sum-rules (4.51) and (4.52). They contain essentially the same information and we shall content ourselves by investigating (4.51) only. Again we keep only the resonance contribution to the integral and write

$$\int (\bar{a}_3^{(+)})_{N^*} d\upsilon = -\frac{F_\pi m_\pi^2}{2} \sqrt{\frac{2}{3}}\, \lambda \left[\left(-2 + \frac{2m_N}{m^*} + \frac{4m_\pi^2}{3m^{*2}}\right) \frac{H_1(q_2^2)}{m_\pi}\right.$$

$$+ \left(\frac{2m_N m^*}{3} + \frac{4m_N^2}{3} + \frac{2m_N^3}{3m^*} - \frac{2m_N m_\pi^2}{3m^*} + \frac{2m_N^2 m_\pi^2}{3m^{*2}}\right.$$

$$+ \frac{4}{3m^{*2}} \upsilon^2_{pole} \Bigg) \frac{H_3(q_2^2)}{m_\pi^3} + \frac{4}{3}\left(1 - \frac{m_N^2}{m^{*2}} - \frac{m_\pi^2}{m^{*2}}\right)$$

$$\times \quad \upsilon_{pole} \frac{H_4(q_2^2)}{m_\pi^3} \Bigg] \tag{4.69}$$

The factor $(m_\pi^2 + q_1^2)^{-1}$ which also occurs in the nucleon term has been dropped. In this case we have calculated the resonance contribution at $t = 0$ i.e. we set $p_1 \cdot p_2 = -m_N^2$ and $q_1 \cdot q_2 = -m_\pi^2$. In the arguments of scalar form-factors we let $q_2^2 \rightarrow -m_\pi^2$. It is well known that the dominant contributions to $G_2(q_2^2)$ and $H_4(q_2^2)$ come from diagrams where the proper vertex of $j_{5\mu}$ is attached to a terminating external pion line. In the limit $q_2^2 \rightarrow -m_\pi^2$, $G_2(q_2^2)$ and $H_4(q_2^2)$ dominate over the rest of the terms and we write specifically

$$G_2(q_2^2) = \frac{F_\pi}{m_\pi^2 + q_2^2} \, g K_{NN\pi}(-m_\pi^2) \tag{4.70}$$

and

$$\frac{1}{m_\pi^2} H_4(q_2^2) = \frac{F_\pi}{m_\pi^2 + q_2^2} \sqrt{\frac{2}{3}} \frac{\lambda}{m_\pi} \tag{4.71}$$

Substituting these into the sum-rule and passing to the limit $q_2^2 \rightarrow -m_\pi^2$ we get

$$\frac{g^2}{\lambda^2} = \frac{2}{3} \frac{1}{m_\pi^2} \left(1 + \frac{m_N^2}{m^{*2}} + \frac{m_\pi^2}{m^{*2}}\right) \frac{4}{3} \upsilon_{pole} \qquad (4.72)$$

where this time υ_{pole} is given by

$$\upsilon_{pole} = \tfrac{1}{2}(m^{*2} - m_N^2 - m_\pi^2) \qquad (4.73)$$

Substitution of the experimental masses into this equation together with $g = 13.5$ predicts the value 2.8 for λ to be compared with the experimental value of 2.2. The agreement is fair and furthermore shows that higher states do make a small but definite contribution which would optimistically bring the numerical value for λ down. In this connection we would like to mention that Gasiorowicz[49] starting from the retarded amplitude involving the commutator of the vector current and the pion field calculated a value for the nucleon isovector magnetic moment which turned out to be rather high compared with the experimental number. We are thus led to believe that our estimates of $G_{NN\sigma}$ or $g_{NN\sigma}$ are probably also higher than the true values. It seems that sum rules obtained from retarded amplitudes involving a current and a scalar or pseudoscalar density converge more slowly than sum rules involving currents only.

Finally we would like to cite some of the experimental evidence for a 'σ' meson. Recently Lovelace and coworkers[50] from a careful analysis of backward π-N scattering were able to conclude that there is a clear evidence for a rather broad "new elementary particle $I = 0$, $J^P = 0^+$."

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