Cohomology theory of the kenematical groups

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Thesis submitted to the

UNIVERSITY OF DURHAM

by

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FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

Department of Physics, University of Durham, June 1969.

Simplico: "Concerning natural things we need not always seek the necessity of mathematical demonstration".

Sagredo: "Of course, when you cannot reach it. But if you can, why not?"

Galileo Galilei. Dialogue on the Two Major Systems of the World.

DEDICATED TO MY MOTHER AND FATHER

ABSTRACT

A number of applications of S. Eilenberg and S. Maclane's cohomology theory of groups to the kinematical groups of physics are presented. Within this field, we apply the theory of group extensions by Abelian and non-Abelian kernels to the study of the algebraic structures of the Galilei, Static and Carroll groups, and introduce to physics the mathematical concepts of group enlargements and prolongations.

The global algebraic structures of the kinematical groups are analysed in depth using these tools and a generalisation of kinematical groups is attempted. The use of the methods of homological algebra in classical mechanics is discussed from the new view point of Lagrangian mechanics introduced by Lévy-Leblond. In this direction two advances are made. Homological algebra is introduced to the study of Hamilton's principle and then a reformulation of Levy-Leblond's free Lagrangian mechanics is obtained. Whilst the above author concentrates on a certain second cohomology group, we see that it is a first cohomology group which is more relevant to this approach.

The group theoretic discussion of non-inertial motions is initiated using the theory of the loop prolongations of a group Q by a group K, where a loop is a 'non-associative group'. Our preliminary results enable us to give a cohomological description of constant Newtonian acceleration.

Preface and Acknowledgements.

This thesis presents an algebraic analysis of classical relativity schemes using the powerful tools developed by the algebraists S. Eilenberg and S. Maclane. The results may be regarded as physically trivial by some as no new physical results emerge. Any originality claimed for this thesis by the author is in his choice of weapons to attack some rather simple physics. Thus all he claims is that his applied mathematics is new, offering a new start on some supposedly well understood results. By adopting and developing Lévy-Leblond's group theoretic view of free classical mechanics, a rather deep mathematical insight into the mechanics is obtained. The physical justification of his work, submitted as physics rather than applied mathematics, is summarised by three quotations. The first is due to J.M. Lévy-Leblond and is taken from his paper[1)] on classical mechanics which inspired the work of chapter (4) of this thesis.

'Invariance principles nowadays have become one of the most useful concepts of theoretical physics, mainly due to their importance in quantum theories. Indeed, when the dynamical laws obeyed by physical systems are not known, or poorly understood, invariance principles act as super-laws to restrict the possible forms of these laws as guides to find them. Conversely, it may be argued that the use of invariance principles becomes obviated once these laws are known as is the case at the classical level. However, it seems even then, invariance principles keep a prominent role in that they enable us to reach a deeper understanding of these laws, reduce some of their apparant arbitariness and related previously unrelated concepts. The tighter structure and greater unity thus obtained have both an epistemological and a pedagogical significance......".

The second quotation is due to H. Eckstien:- "Considerable uncertainty concerning the dynamical laws of particle interactions exists and is likely to last. In the mean-time, symmetry principles provide a powerful but incomplete set of predictive statements. It is important that we exploit the reliable symmetry principles to the greatest possible extent. In particular, the space-time symmetry.....".

The last quotation is by L. Michel who is the leading exponent of group-theoretical techniques in theoretical physics after E.P. Wigner. "We physicists have to consider several kinds of invariance: relativistic invariance, guage invariance ..... How are the invariance groups related? This is a fundamental question to answer. Too often physicists consider them separately because they do not know of other solutions".

The above three quotations provide the motivation for the work of this thesis. In writing a thesis in which the physical applications of an abstract algebraic scheme are sought, the author finds himself between two stools. Perhaps he should apologise to the physicists for the lack of new physical results and the partial eclipse of physics by mathematics he presents as physics; and to the mathematicians for the cruel way he abuses their beautiful apparatus.

In this thesis, many original applications of Eilenberg-Maclane's cohomology theory of groups are made. L. Michel was the first physicist to use these tools and it is from his viewpoint that we shall discuss the classical relativity groups. Michel, of course, studied the Poincaré group of Special Relativity , whilst our discussions are of the Galilei and similar groups.

Chapter (1) is an elementary introduction to the algebraic methods used in the thesis. Its inclusion in the first chapter should be taken to imply that its material is basic to the rest of the thesis and hence cannot be relegated to an appendix. It serves also as a list of definitions. Chapter (1) is preceded by a list of the symbols employed in the thesis.

In chapter (2) a semi-axiomatic discussion of relativity models is to be found. Its discussion acts as an introduction to the approach to Newtonian relativity which we shall use in our analysis of the Galilei and related groups. In this chapter, we also discuss Einstein's theory of Special Relativity and the Poincaré and Causality groups, and the Carroll and Static relativity groups introduced by J.M. Lévy-Leblond.

Chapter (3) is an introduction to the cohomology theory of abstract groups developed by Eilenberg and Maclane. In it, the theory of group extensions and group enlargements is discussed, the latter prior to its first applications to theoretical physics in chapter (4). Chapter (4) applies the theory of group <u>extensions</u> to elicit the global algebraic structures of the Galilei, Carroll and Static groups. Two approaches to the theory of the Galilei group are presented. The first approach, capable of greater generalisation, discusses the Galilei group as the subgroup of inertial world automorphisms of the group of world automorphisms of the Newtonian world, in the axiomatic approach outlined in chapter (2). The second approach is more straight-forward and direct and serves to list the various global algebraic structures of the aforementioned groups. Next, the algebraic generalisation of the Galilei and Carroll groups is attempted using the theory of group enlargements. A theorem derived by G.W. Mackey in the discussion of the unitary ray representations of group exten-sions is generalised to a more general algebraic context and enables us to compute, in principle, all the central extensions by an Abelian group of a trivial group extension. Finally in this chapter, Eilenberg-Maclane's 'cup-products' of cochains are introduced and applied to the above theory. It is shown to have wide applicability in this field.

In chapter (5), J.M. Lévy-Leblond's group theoretical scheme of classical mechanics is discussed. A more mathematically precise reformulation of his results is obtained. Whilst Lévy-Leblond concentrated on a certain second cohomology group we show that the appropriate group to discuss is a certain first cohomology group. An algebraic formulation of Hamilton's principle is developed and applied. In our scheme, 'Lagrange functions' are cocycles and 'trivial-

Lagrangians' coboundaries. Similarly, Hamilton's Action functionals are cycles and 'trivial Action functionals' are boundaries. That is Lagrangians are cochains in a certain cohomology complex and Action functionals chains in a homology complex. In this algebraic scheme of classical mechanics, one is able to see in a lucid manner how such concepts as 'inertial mass' and 'kinetic energy' are essentially group theoretical. Motions in Newtonian relativity under the Galilei and Static groups in Special Relativity under the Poincaré and Causal groups and under the Carroll group are discussed in the new context.

Finally, in chapter (6) the group theoretical treatment of non-inertial motions is initiated. The theory of loops of motions is initiated. The theory of loops or not necessarily associative groups is introduced into theoretical physics. Eilenberg and Maclane's cohomology theory of the loop prolongations of a group Q by a group K is discussed and applied to the analysis of uniform accelerative motion in Newtonian relativity. A loop of 'semi-inertial' world automorphisms of Newtonian relativity is obtained as a natural generalisation of the Galilei group to non-inertial motions.

The work presented in chapters (4), (5) and (6) is original except that due to Lévy-Leblond in the latter part of chapter (5). Here we rederive some of his results in a completely new fashion, and of course, reinterprete his original approach. Some of the work in chapter (2) is also original, particularly our approach to Newtonian relativity, an approach extending that of Noll.

The author feels that this is the appropriate place to thank a long list of teachers and friends for the patient and expert guidance in his education. Firstly then, he wishes to thank the teachers of Roade Secondary Modern School, Roade, Northamptonshire for their reaction to his youthful interest in science. Next, he also expresses his thanks to the lecturers in the Science and Mathematics departments of the Northampton College of Further Education, Northampton, for their interest in him and for the way in which they transmitted their scientific enthusiasm to him. Special thanks are due to Dr. W. Sidall, Mrs. A. Newman and Mr. Mallard in the science department and Mr. J. Boyle in the commerce department, for teaching him to think.

Thanks are due to the staff of the physics department of the University of Exeter and especially to Dr. Klien of the Applied Mathematics department for teaching him physics and in the latter case for introducing him to the subject of Applied Mathematics.

At Durham University, he wishes to thank Professor G.D. Rochester for enabling him to study theoretical physics; the staff of the applied mathematics and theoretical physics research group for their teaching and friendly advice; and his fellow research students for their patient discussive and listening powers. He wishes to acknowledge the friendly guidance of his supervisor Dr. P.D.B. Collins and the academic freedom allowed him.

Many thanks are due to the ratepayers of Northamptonshire, who, through the Northampton County Council provided his undergraduate grant and to the general taxpayers who, through the Scientific Research Council, provided his research grant during the last three years.

The most sincere thanks have been saved until last. These are for my mother and late father who have supported me since I left school ten years ago. This thesis is dedicated to them. I should also like to thank my future wife, Judy, for typing this thesis.

To all the people mentioned above, the author owes an unpayable debt. He hopes that this thesis justifies, to some small extent, their interest in him.

Durham, April, 1969.

CONTENTS

List of Symbols Used In the Text.

Symbol	Meaning
$\in$, $\notin$	Membership of a set; negation
$\exists$, $\nexists$	There exists , negation
$\ni$	'Such that'
$\forall$	'For Every'
$\Rightarrow$	'Implies'
Iff, $\Leftrightarrow$	'If and only If'
$\subset$, $\not\subset$	Subset inclusion; negation
$\supset$, $\not\supset$	'Includes' ; negation
$\subseteq$, $\not\subseteq$	Proper Inclusion; negation
P(S)	Class of subsets of a set S
$\emptyset$	The empty set.
$X_1 \cup X_2$	Union of sets
$X_1 \cap X_2$	Intersection of sets
$X_1 \dot{\cup} X_2$	Union of disjoint sets
$X_1 \times X_2$	Cartesian product of sets
R^{-1}	Inverse of a relation R.
Δ	Identity relation
$R_2 \circ R_1$	Composition of Relations
R: $X \longrightarrow Y$ $X \xrightarrow{R} Y$	Mapping from a set X to a set Y induced by the relation R.
R: $x \longmapsto y$ $x \overset{R}{\longmapsto} y$	$y \in Y$ is the image of $x \in X$ under the mapping R: $X \longrightarrow Y$

$m_2 \circ m_1$	Composition of mappings:- $m_2 \circ m_1 : x \longmapsto m_2(m_1(x)) \forall x \in X$.
X/R	Quotient set of X under an equivalence relation R.
$C(X_1, X_2)$	Set of functions from X_1 into X_2.
$In(X_1, X_2)$	Set of injective functions in $C(X_1, X_2)$
$Sur(X_1, X_2)$	Set of surjective functions in $C(X_1, X_2)$.
$B(X_1, X_2)$	Set of bijective functions in $C(X_1, X_2)$
$X_1 \rightleftharpoons X_2$	Equivalence of objects X_1 and X_2 in a category
$\langle F_i \rangle\ i \in I$	Family of objects indexed by a set I.
$\bigcup \{ F_i \mid i \in I \}$	Union of a family indexed by I.
$\bigcap \{ F_i \mid i \in I \}$	Intersection of a family indexed by I
$\dot{\bigcup} \{ F_i \mid i \in I \}$	Union of a disjoint family of subsets $F_i \cap F_j = \emptyset \ \forall (i,j) \in I \times I - \Delta$
$X \{ F_i \mid i \in I \}$	Cartesian product of a family of sets indexed by I
X^n	Cartesian product where $F_i \rightleftharpoons X \ \forall i \in I, \# (I) = n$.
$C^n(X, Y)$	$C(X^n, Y)$
$\mathbb{R}$	Real line
$\mathbb{R}_{m}$.	Multiplicative subgroup of $\mathbb{R}$.
$\mathbb{R}_{add}$	Group of Additive reals
$\mathbb{R}^+$	Positive Reals ≥ 0
Z	Set of integers
Z^+	Set of positive o integers
$\mathbb{R}^n$	Set of real 'n-tuples'.

Hom(A, B)	Set of semi-group homomorphisms in C(A, B)
Mon(A, B)	Set of injective-semi-group homomorphisms in C(A, B) or monomorphisms in C(A, B)
Sur(A, B)	Set of surjective-semi-group homomorphisms in C(A, B) or epimorphisms in C(A, B).
End(A)	Set of semi-group endomorphisms of A.
$<$	Sub-semi group inclusion
Ker(f)	Kernel of a group homomorphism f.
Im(f)	Image of X_1 under the homomorphism $f \in \mathrm{Hom}(X_1, X_2)$ in X_2.
Aut(X)	Group of semi-group autmorphisms of X.
In.	Canonical mapping of G into Aut(G)
Int(G)	$\mathrm{Im}(\mathrm{In}) < \mathrm{Aut}(G)$
$\mathrm{Hom}_S(X_1, X_2)$	Set of 'S homomorphisms' from a group X_1 into a group X_2.
$\lhd$	Normal inclusion of a subgroup
$\lhd$ (with bar)	Characteristic inclusion
G/H	Quotient [set?] of a group by a [normal?] subgroup $H < G$.
$\mathrm{Ob}(\mathcal{C})$	Collection of objects in a category $\mathcal{C}$.
$\mathrm{Ar}(\mathcal{C})$	Set of all morphisms in a category. $\mathcal{C}$
Mor(A, B)	Set of morphisms between objects $A, B \in \mathrm{Ob}(\mathcal{C})$ in a category $\mathcal{C}$.
Mor	Bi functor from $\mathcal{C} \times \mathcal{C}$ to $\mathrm{Ar}(\mathcal{C})$.
M^A	Contravariant functor from $\mathcal{C}$ to $\mathrm{Ar}(\mathcal{C})$.
M_A	Covariant functor from $\mathcal{C}$ to $\mathrm{Ar}(\mathcal{C})$
$\mathcal{G}$	Category of groups and homomorphisms.
$\mathcal{S}$	Category of sets and functions

τ	time-lapse function
∂	family of instantaneous Euclidian metrics
$\mathcal{W}$	The event world or space-time
$\mathcal{S}$	Partition of $\mathcal{W}$ into instantaneous spaces
F	Future relation
P	Past relation
S	Simultenaity
$\mathcal{W}$ (N)	The Newtonian world
$\mathfrak{a}(\mathcal{W})$	'Group' of world automorphisms of a world model
$I(\mathcal{W})$	Set of inertial automorphisms of $\mathcal{W}$.
$M(n, \mathbb{R})$	Sfield of n x n real matrices
$GL(n, \mathbb{R})$	Group of units of $M(n, \mathbb{R})$
$O(p, q; \mathbb{R})$	Group of pseudo-orthogonal matrices in $GL(n, \mathbb{R})$ which preserve the inner product in (p+q) dimensional Minkowski space with q space dimensions
σ_0	The present
Q	Minkowski form
τ_0	Proper time
T	Time-tube relation
$\mathcal{E}$	Space-tube relation
L	Light-cone relation
V_+	Zeeman causal order on $\mathcal{W}$
$\mathcal{M}$	4-dimensonal Euclidian Space
$\mathcal{M}(\mathcal{W})$	Minkowski space
$P(\mathbb{R})$	The Poincaré Group
$L(\mathbb{R})$	The Lorentz Group

$Z(2)$	Two element cyclic group
P	Space-reflection
T	Time-reflection
Γ	Space-time reflection
$L_+(\mathbb{R})$	Proper Lorentz group
$L^{\uparrow}(\mathbb{R})$	Orthochronous Lorentz group
$P_+(\mathbb{R})$	Proper Poincaré group
$P^{\uparrow}(\mathbb{R})$	Orthochronous Poincaré group
$Out(K)$	$Aut(K)/Int(K)$
$(K_1,p_1) \wedge (K_2,p_2)$	G product of Q kernels
$(E_1,\phi_1) \wedge (E_2,\phi_2)$	Extension product
Z_n	$Ker(\partial_n)$ for lower semi-exact sequences
Z^n	$Ker(\delta^n)$ for an upper semi-exact sequence
B_n	$Im(\partial_n)$ for lower semi-exact sequences
B^n	$Im(\delta^n)$ for upper semi-exact sequences
H_n	'n'th homology group
H^n	'n'th cohomology group
Ext	Extension functor
$A \otimes B$	Direct product of groups
$A \boxtimes_g B$	Semi-direct product of A by B with $g \in Hom(B,Aut(A))$
$A \boxtimes_\xi B$	Abelian extension of B by A with factor system $\xi \in Z^2{}_p(B, A)$.
$A \otimes_\xi B$	Abelian central extension of B by A with $\xi \in Z^2{}_o(B, A)$
$Enl(G,(Q,K))$	Group of G enlargements of Q by K

$G(3,\mathbb{R})$	Galilei group
$C(3,\mathbb{R})$	Carroll group
$S(3,\mathbb{R})$	Static group
$\mathbb{R}^1$	Group of temporal translations
$\mathbb{R}^3$	Group of spatial translations
$\mathbb{R}^3_T$	Group of pure Galilei velocity boosts
$\mathbb{R}^3_{TT}$	Group of Galilei acceleration boosts
$E(3,\mathbb{R})$	Three-dimensional Euclidian group
$f_1 \cup f_2$	Cup product of cochains
$\pi_1 \cup \pi_2$	Pairing of elements $\pi_1 \in \Pi_1, \pi_2 \in \Pi_2$ to an element in a group
$P_1 \wedge P_2$	Prolongation product
$K \boxminus_{\zeta} Q$	Loop prolongation of Q by K specified by $\zeta \in B^3_p(Q, K)$
$V_1 \otimes_{\Lambda} V_2$	Tensor product of real linear spaces
$x_1 \otimes x_2$	Element of $V_1 \otimes_{\Lambda} V_2$.
V^*	Dual of a linear space.
∇	Gradient linear operator*.
∇^n	$\partial/\partial(d^n/dt^n(x))$, $\nabla^0 = \nabla$.
${}^*\nabla$	or Antidiagonal of a group $G \otimes G$.

CHAPTER (1)

A L G E B R A I C   I N T R O D U C T I O N

ALGEBRAIC INTRODUCTION

Section (1) Sets and Relations.

The intuitive definitions of a set will be used. Firstly we can define a set by tabulating all its members e.g.:- $S \equiv \{x, y, \ldots \ldots, \}$. If x is an element or member of a set S we write $x \in S$. We define subsets of S by making some propositions about its members. Let $\phi(x)$ be such a proposition about $x \in S$, we define a subset Φ of S by $\Phi \equiv \{x \in S \ni \phi(x) \text{ is true}\}$. If X is a subset of S we write $X \subset S$. Two sets are said to be equal if they are subsets of each other. I.e.:- if $X_1, X_2 \subset S$ then $X_1 = X_2$ iff $\forall x \in X_1$, $x \in X_1 \Leftrightarrow x \in X_2$. If $X \subset S$ and $X \neq S$ we write $X \subsetneq S$ and call X a proper subset of S. Call P(s) the class on set of subsets of S, and $\emptyset$ the empty set, then $\emptyset$, $S \in P(S)$ at least.

Given two elements $X_1, X_2 \in P(S)$ we can obtain two others. More about this type of process will be said later. Define then :-
$X_1 \cap X_2 \in P(S)$ via:- $X_1 \cap X_2 \equiv \{ x \in S \ni x \in X_1; x \in X_2 \}$, also
$X_1 \cup X_2 \in P(S)$ via:- $X_1 \cup X_2 \equiv \{ x \in S \ni x \in X_1 \text{ or } x \in X_2 \text{ or } x \in X_1 \cap X_2 \}$. Evidently $X_1 \cap X_2 \subset X_1 \cup X_2$. If $X_1 \cap X_2 = \emptyset$, X_1 and X_2 are said to be disjoint. If $X_1 \cap X_2 = \emptyset$ we write $X_1 \dot{\cup} X_2$ for $X_1 \cup X_2$. We define the Cartesian product of two elements $X_1, X_2 \in P(S)$ via $X_1 \times X_2 = \{ (x_1, x_2) \ni x_1 \in X_1, x_2 \in X_2. \}$

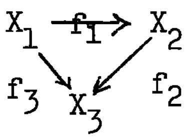
One calls $P(X_1 \times X_2)$ the set of relations from X_1 into X_2. Let $R \in P(X_1 \times X_2)$, if $(x_1, x_2) \in R$ we say $x_1 \, R \, x_2$. If $X_1 = X_2 \equiv X$ we call $R \in P(X \times X)$ a relation on X. Given a relation $R \in P(X_1 \times X_2)$ we define a relation from X_2 into X_1 via $R^{-1} \equiv \{(x_2, x_1), x_1 \in X_1 \; x_2 \in X_2 \ni (x_1, x_2) \in R\}$. We define a composition of relations as follows. Let $X_1, X_2, X_3 \in P(S)$ and consider the relations $R_1 \in P(X_1 \times X_2)$, $R_2 \in P(X_2 \times X_3)$, one defines the relation $R_1 o R_2 \in P(X_1 \times X_3)$ via $R_1 o R_2 \equiv \{(x_1, x_3) \; x_1 \in X_1, x_3 \in X_3 \ni \exists \, x_2 \in X_2, (x_1, x_2) \in R_1, (x_2, x_3) \in R_2\}$. Let $X \in P(S)$ we call the identity relation on X the element $\Delta(X \times X) \in P(X \times X)$, $\Delta(X \times X) = \{(x_1, x_2) \in X \times X \ni x_1 = x_2\}$. A relation is said to be symmetric iff $R = R^{-1}$. It is said to be reflexive of $\Delta(X \times X) \subset R \in P(X \times X)$ and transitive if $RoR \subset R$. We define certain special types of relation as follows. Let $S \subset P(X \times X)$ be the set of symmetric relations on X, $\mathcal{R}$ the subset of reflexitive relations on X and $\mathcal{T}$ the set of transitive relations. Then, if $R \in S \cap \mathcal{R} \cap \mathcal{T}$ one calls R an equivalence relation. Consider a relation $R \in \mathcal{T} \cap \mathcal{R}$ such that $R \cap R^{-1} = \Delta(X \times X)$; one calls such a relation a partial order on X. One calls the pair (X, R) a partially ordered set. An ordered set is a pair (X, R) $X \in P(S)$, $R \in P(X \times X)$ where R is a partial order and $X \times X = R \cup R^{-1}$. Let R be a relation on X. Define a subset $R(x) \in P(X) \; \forall \, x \in X$ by $R(x) \equiv \{y \in X \ni (x, y) \in R\}$. If R is an equivalence relation, one can partion X into a set of equivalence classes which are mutually disjoint. We call the set of equivalence classes of X under the equivalence relation R, X/R. (The map $\pi : R \longrightarrow X/R \ni \pi : x \longmapsto R(x)$ is called the canonical map from X to X/R. (It is a surjective function)). We have $R(x_1) \cap R(x_2) = \emptyset$ if $(x_1, x_2) \notin R$, $R(x_1) \cap R(x_2) = R(x_1)$ iff $(x_1, x_2) \in R$.

Since $R = R^{-1}$, $R(x_1) \cap R(x_2) = R(x_1) = R(x_2)$ iff $(x_1, x_2) \in R$.

Let $R \in P(X \times X)$ if $(x, y) \in R$ we write $x R y$ or $y = R(x)$. The latter concept introduces the concept of a mapping, we say that R gives rise to a mapping $R: X \longrightarrow X$. Under the mapping R an element $x \in X$ is transformed to an element $y = R(x)$. We write this as $R: x \longmapsto y$. Our interest will mainly be in a class of mappings called single-valued or functions. A function is a mapping or a relation 'f' such that if $(x, y), (x, z) \in f$ then $y = z$. It is clear that if f_1 and f_2 are two functions on X then $f_1 \circ f_2$ is also a function. Let R be a relation from X_1 into X_2, it gives rise to a mapping $R: X_1 \longrightarrow X_2$, similarly let R be a relation from X_1 into X_2 such that $(x_1, x_2), (x_1, x_2') \in R \Leftrightarrow x_2 = x_2'$; R is then a function from X_1 into X_2. Let $C(X_1, X_2)$ be the set of functions from X_1 into X_2. Define $In(X_1, X_2)$ as the set of injective functions from X_1 into X_2, an injective function being one such f^{-1} is also a function. Define $\mathcal{S}(X_1, X_2)$ as the class of functions from X_1 into X_2 such $\forall\, y \in X_2 \exists\; x \in X_1 \ni y = f(x)$. The elements of $Sur(X_1, X_2)$ are called surjective functions. We have also a set $B(X_1, X_2) = In(X_1, X_2) \cap Sur(X_1, X_2)$ of functions which we call bijective. When $B(X_1, X_2) \neq \emptyset$, the sets X_1 and X_2 are said to be isomorphic sets. The relation or being isomorphic sets is an equivalence relation written $X_1 \gtreqless X_2$. We note that if $f_1, f_2 \in In(X\ , X\ )$ then $f_1 \circ f_2 \in In(X, X)$ and that if $f_1, f_2 \in Sur(X, X)$ then $f_1 \circ f_2 \in Sur(X, X)$, More generally if $(f_1, f_2) \in In(X_1, X_2) \times In(X_2, X_3)$ then $f_2 \circ f_1 \in In(X_1, X_3)$ and if $f_1, f_2 \in Sur(X_1, X_2) \times Sur(X_2, X_3)$, $f_2 \circ f_1 \in Sur(X_1, X_3)$. Similarly if $(f_1, f_2) \in B(X_1, X_2) \times B(X_2, X_3)$ then

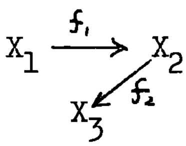
$f_1 \circ f_2 \in B(X_1, X_3)$. Moreover, if $f \in B(X_1, X_2) \exists f' \in B(X_2, X_3) \ni$ $f' \circ f = 1\!\!1_1$ where $1\!\!1_1$ is the identity function on X_1 and $f \circ f' = 1\!\!1_2$, where $1\!\!1_2$ is the identity function on X_2. Thus $\forall f \in B(X, X) \exists f' \in B(X, X)$ $\ni f' \circ f = f \circ f' = 1\!\!1$. We shall soon give another interpretation of $B(X, X)$ for $X \in P(S)$.

Consider the two functions $f_1: X_1 \longrightarrow X_2$ and $f_2\ X_2 \longrightarrow X_3$ which defines $f_2 \circ f_1: X_1 \longrightarrow X_3$. We can draw the diagram:-



Such a diagram is said to be commutative iff $f_3 = f_2 o f_1$.

Larger diagrams including more than 3 sets are said to be commutative iff all sub diagrams included in it are commutative. The diagram



is said to imply the function $f_2 \circ f_1$, implied functions are denoted by a curly arrow e.g.:- $X_1 \overset{f_2 \circ f_1}{\rightsquigarrow} X_3$.

Given a set S, a family of elements of S is a function $F: I \longrightarrow S$ where I is some index set. We will call a family 'F' the object $\langle F(i) \rangle\ i \in I$. If $I \subseteq Z$, we call a family a sequence. Consider a family $\langle X(i) \rangle\ i \in I$ of subsets of S i.e.:- $X : I \longrightarrow P(S)$. One can form the subsets $\bigcup\{X(i) \mid i \in I\}$ when $I \subseteq Z$ and $\bigcap\{X(i) \mid i \in I\}$ the definitions of these subsets follow by induction for the definitions of $X_1 \cup X_2$ and $X_1 \cap X_2$. Similarly, when $I \subseteq Z$ the definition of the Cartesian product set $X_{i=1}^{n}\{A_i\}$, where $\#(I) = n$, also follows. When A_i is equivalent to $Z\ \forall i \in I$, the notation $X_{i=1}^{n}\{A_i\} \equiv A^n$ will be used. The class of functions $f: A^n \longrightarrow B$ will be written as $\mathcal{C}^n(A, B)$, with

$C^o(A, B)$ interpreted as B and $C^1(A, B)$ as $C(A, B)$. We generalise the definition also for cases when I is an arbitrary set. Given $f \in C(X_1, X_2)$ we define the set $f(X_1) \subset X_2$ by $f(X_1) \equiv \{y \in X_2 \ni \exists x \in X_1, y = f(x)\}$, and the set $f^{-1}(X_2) \subset X_1$ via $f^{-1}(X_2) \equiv \{x \in X_1 f(x) \in X_2\}$. Consider a set $X \in P(S)$ we call a function $f \in C(X \times X, X)$ a binary operation. One writes $f(x_1, x_i) = x_1 \cdot x_2$. A binary operation on a set is associative iff $\forall x_1, x_2, x_3 \in X$, $(x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3)$. Binary operations may be required to satisfy certain conditions including the above. The algebraic objects we now consider consist of underlying sets with one or more binary operations, which may or may not be associative. Firstly, we consider sets with 1 binary operation and gradually impose more conditions on the operation obtaining algebraic objects of increasing complexity. We will consider later objects with two binary operations.

Section (2). Semigroups.

Consider a pair (S, f) where S is a set and f is a binary operation on S. Iff f is assosiative, we call the pair a semi-group. We have already met an example, this is the pair $(C(X, X), O)$, $X \in P(S)$ where O is map composition which is associative. Let us define a sub semi-group as a pair (X, f') where $X \subset S$ and f' is the restriction of f to the subset $X \times X \subset S \times S$. Iff $f'(x_1, x_2) \in X \ \forall (x_1, x_2) \in X \times X$ we say that (X, f') is a sub semi-group of (S, f) and write $(X, f') < (S, f)$. Consider two semi-groups (X_1, f_1) and (X_2, f_2). Define as $Hom(X_1, X_2)$ the subset of functions $\{\phi \in C(X_1, X_2) \ni \phi \circ f_1(x_1, x_2) = f_2(\phi(x_1), \phi(x_2))\}$

or $\{ \phi \in C(X_1, X_2) \ni \phi(x_1 \cdot x_2) = \phi(x_1) \cdot \phi(x_2) \forall (x_2, x_2) \in X_1 \}$ We call $Hom(X_1, X_2)$ the set of semi-group homomorphisms from X_1 into X_2. A semi-group (X, f) is said to be Abelian or commutative iff $x_1 \cdot x_2 = x_2 \cdot x_1 \forall x_1, x_2 \in X$. We call $Hom(X_1, X_2) \cap In(X_1, X_2) \equiv Mon(X_1, X_2)$ the set of semi-group monomorphisms from X_1 into X_2, on the set of injective semi-groups homomorphisms. Similarly the subset $Ep(X_1, X_2) = Hom(X_1, X_2) \cap Sur(X_1, X_2\}$ is called the set of semi-group epimorphisms. Finally $Mon(X_1, X_2) \cap Ep(X_1, X_2) = Hom(X_1, X_2) \cap B(X_1, X_2)$ is called the set of semi-group isomorphisms. We recall that C(X , X) is a semi-group when X is a set, Sur(X, X) and In(X, X) are sub-semi-groups. So is $Hom(X, X) \equiv End(X)$, the set of semi-group endomorphisms of a semi-group X. B(X, X) is also a sub-semi-group. Given a semi-group S and sub-semi-groups $X_1, X_2 < S$, we readily observe that $X_1 \cap X_2 < S$, which implies that $B(X_1, X_2)$ is a sub-semi-group of $C(X_1, X_2)$ and M on $(X_1, X_2) \cap Ep(X_1, X_2)$ of $Hom(X_1, X_2)$. Let $X_1, X_2 \subset S$ a semi-group. One defines the subset $X_1 \cdot X_2 \subset S$ as $\{ x_1 \cdot x_2 \ni (x_1, x_2) \in X_1 \times X_2 \}$.

Section (3) Monoids.

We call a monoid a semi-group with an identity element. An element $e \in X$ a semi-group is an identity element iff $\forall x \in X$ $e \cdot X = x \cdot e = e$. By its definition iff $e \in X$ is an identity then it is unique. Let M_i be monoids i = 1, 2. Let $f \in Hom(M_1, M_2)$ be a semi-group homomorphism, we must have $f(e_1) = e_2$ where e_i are the identities of M_i respectively. Define $Ker(f) \equiv \{ x \in M_1 \ni f(x) = e_2 \}$ and $Im(f) = f(M_1)$. We then see that $Ker(f) < M_1$ and $Im(f) < M_2$ i.e.:- are submonoids of M_1 and M_2. As above, if M_1 and M_2 are submonides or M then $M_1 \cap M_2$ is a submonida of M. If X is a semi-group we call $End(X) \cap B(X,X) \equiv Aut(X)$ the semi-group of semi-group automorphisms of X. Similarly if M is a monoid, Aut(M) is the semi-group of monoid automorphisms of M. Recall that $\mathbb{1} \in C(X,X)$ the identity function is also trivially a semi-group endomorphism. This means that $\forall$ X_1, X_1, semi-group, that $C(X_1, X_1)$ is monoid, and also that In(X, X), Sur(X, X), Ep(X, X), Mon(X,X), Hom(X, X), $Ep(X, X) \cap Mon(X, X)$, $Sur(X, X) \cap Im(X,X)$ are also monoids. A monoid which is an Abelian semi-group is said to be an Abelian monoid.

<u>Section (4). Groups</u>.

Let M be a monoid, define the submonoid $G < M$ by $G \equiv \{x \in M \ni \exists \; x' \in M, xx' = x'x = e\}$, one says that G is a group. Conversely we call a semi-group S a group iff (i) $\exists \; e \in S \ni x \in S \Rightarrow e \cdot x = x \cdot e = x$ (ii) $\forall x \in S \; \exists \; x' \in S \ni xx' = x'x = e$. In our definitions so far we have already met two groups. These are B(X,X)for a set X and Aut(S) where S is a semi-group. Groups will, of course, be of central interest in this thesis. Given the set X we will call Sym(X) the group (B(X,X),O), O being map composition. We now present a few definitions specialised to groups.

Consider x_1, $x_2 \in G$ a group; $\forall$ s_1, x_2 the mapping $x_2 \longmapsto x_1x_2x_1^{-1}$ is a bijective function. Also $x_2x'_2 \longmapsto x_1x_2x_2'x_1^{-1}$ under the function However $x_1x_2x_2'x_1^{-1} \equiv (x_1x_2x_1^{-1})(x_1x_2'x_1^{-1})$. Thus the mapping is an automorphism. One calls such a mapping an inner automorphism of G.

More precisely, define a map In: $G \longrightarrow Aut(G)$, $In(x): y \longmapsto xyx^{-1} \forall$ $x, y \in G$. We call $Im(Id) = Int(G) < Aut(G)$. $Ker(Id) = \{x \in G \mid In(g) = 1\} = \{x \in G \ni sys^{-1} = y \forall y \in G\} = \{x \in G \ni xy = yx \forall y \in G\}$. Ker(In) is called the centre of G written $\mathcal{C}(G)$. More generally consider a subset $S \subset G$. Define a subgroup $N(G)(S) < G$ by $N(G)(S) \equiv \{x \in G \ni In(x)[S] = S\}$. One calls N(G)(S) the normaliser of S in G. Similarly, define $\mathcal{C}(G)(S) \equiv \{x \in G \ni xyx^{-1} = y \forall y \in S\}$. $\mathcal{C}(G)(S)$ is called the centraliser of S in G. When G = S we call $\mathcal{C}(G)(G)$ the centre of $G :- \mathcal{C}(G)$. Subgroups of G such that N(G)(X) = G are called normal. We write $X \lhd G$. We have $\mathcal{C}(G) \lhd G$ and $\forall S \subset G$, $\mathcal{C}(G)(S) \lhd N(G)(S)$.

Operator Groups on a set

Let G be a group and S a set. G i s an operator group on S iff $\exists$ a function $\theta : G \times S \longrightarrow S$ such that if $\theta(g,s) \equiv g \cdot s$ then (i) $g_1 \cdot (g_2 \cdot s) = g_1 g_2 \cdot s$ (ii) $e \cdot s = s \forall s \in S$, $(g_1, g_2) \in G$. Given an $s \in S$, one calls $\{g \cdot s \mid g \in G\}$ the orbit of s under G, we denote the orbit of s under G by Ob(s). The relation $s_1 \approx s_2$ iff $s_1 \in Ob(s_2)$ is an equivalence relation on S splitting it into disjoint orbits. If Ob(s) = S for $s \in S$, one says that G acts transitively on S or that the 'action' of G on S is transitive. If $g \cdot s = s \forall (s, g) \in S \times G$ shall say that G acts trivially on S. When S is also a group and $\theta : G \times S \longrightarrow S$ also satisfied $g \cdot (s_1 s_2) = g \cdot s_1 g \cdot s_2$ we say that the group G is a group of left operators for the group S. In this case $\exists$ $p \in Hom(G, Aut(S))$ with $p(g) : s \longmapsto g \cdot s$.

Consider a set S. If there exists a map $f: S \times G \longrightarrow G$ such that (i) $f(\alpha, g) \equiv \alpha \cdot g$, $\alpha \cdot (g_1 g_2) = \alpha \cdot g_1 \, \alpha \cdot g_2$ we say that G is an S group. An S subgroup of G is a subgroup $X < G$ such that $\alpha.X \subset X \; \forall \alpha \in S$. An S homomorphism from an S group G_1 into an S group G_2 is a homomorphism $f \in Hom(G_1, G_2)$ such that $f(\alpha \cdot g) = \alpha \cdot f(g) \; \forall (\alpha, g) \in S \times G_1$. Define the subset $Hom_S(G_1, G_2) \subset Hom(G_1, G_2)$ of S homomorphisms. We can easily see that $Hom_S(G_1, G_2)$ is a sub-semi-group of End(G) for a group G. If $X < G$ is a normal subgroup and an S subgroup we call it a normal S subgroup. Let us consider some examples. If $S = \emptyset$, any group is a $\emptyset$ group. Again let G be any group and S = Int(G). Each normal subgroup of G is an S subgroup. Lastly let S = Aut(G). An S subgroup of G is called a characteristic subgroup of G, we write $X \lhd G$ iff X is an Aut(G) subgroup of G. Consider a group G with $H < G$, then there exists an equivalence relation in G, viz $g_1 \approx_1 g_2$ iff $\exists\, h \in H$ such that $g_1^{-1} g_2 = h$. The equivalence classes of the quotient set $G/(\approx_1)$ are of the form $\pi_1(g) = g \cdot H$ where $\pi_1 : G \longrightarrow G/(\approx_1)$. Similarly there exists an equivalence relation $(\approx_2)$ in G such that $g_1 \approx_2 g_2$ iff $\exists\, h \in H \ni g_1 g_2^{-1} = h$. The canonical map $\pi_2 : G \longrightarrow G/(\approx_2)$ is $\pi_2 : g \longmapsto H \cdot g$. The set $\pi_1(g)$ is called a right H coset and $\pi_2(g)$ a left H coset. When $H \lhd G$, then $H \cdot g = g \cdot H \; \forall g \in G$ and $G/H = G/(\approx)$ where $(\approx_1) = (\approx_2) = (\approx)$ has the structure of a group via the composition $\pi(g_1) \cdot \pi(g_2) = \pi(g_1 g_2)$. Thus the canonical map H: $G \longrightarrow G/H$ is an epimorphism of G onto G/H. Consider Ker(f) where $f \in Hom(G, G_1)$ where G_1 is any group. We have $Ker(f) \lhd G$, whence the natural mapping of G onto G/Ker(f) is an epimorphism whose kernel is Ker(f). Clearly Im(f) is isomorphic to G/Ker(f). Consider the map Im: $G \longrightarrow Aut(G)$. Im(In) = Int(G) and

Ker(In) = $\mathcal{C}(G)$ so that Int(G) $\cong G/\mathcal{C}(G)$. Note that $\mathcal{C}(G) \triangleleft G$.
We say that Int(G) $\triangleleft$ Aut(G) defines Aut(G)/Int(G) $\equiv$ Out(G). One calls Out(G) the group of outer automorphisms of G.

Let G be an Abelian group. Now be definition of In, $G/\mathcal{C}(G)$ is isomorphic to Int(G). If G is Abelian $\mathcal{C}(G) = G$ whence Int(G) = $\{e\}$. Thus Out(G) is isomorphic to Aut(G) that is, all automorphisms of Abelian groups are outer. If $\mathcal{C}(G) = \{e\}$ and Out(G) = $\{e\}$ we call G a complete group. Thus a group is complete iff Aut(G) is isomorphic to Int(G) which is isomorphic to G If a group G has no Abelian invariant subgroups we say it is semi-simple. If it has no invariant subgroups at all we say it is simple.

Consider a sequence $\langle G_i \rangle$ $i \in I$ of groups and a sequence $\langle f_i \rangle$ $i \in I$ of homomorphisms $f_i \in \mathrm{Hom}(G_i, G_{i+1})$ such that $\mathrm{Im}(f_{i-1}) = \mathrm{Ker}(f_i)$. The sequence $\langle (G_i, f_i) \rangle$ $i \in I$ is called exact. We have already obtained the exact sequences $S_1 : e \longrightarrow \mathrm{Int}(G) \xrightarrow{i_1} \mathrm{Aut}(G) \xrightarrow{\pi} \mathrm{Out}(G) \longrightarrow e$

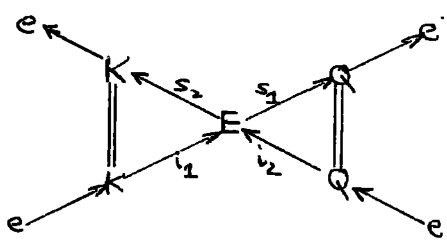
$S_2 : e \longrightarrow e(G) \xrightarrow{i_2} G \xrightarrow{In} \mathrm{Int}(G) \longrightarrow e$

$S_3 : e \longrightarrow \mathrm{Ker}(f) \xrightarrow{i_3} G \xrightarrow{\pi} \mathrm{Im}(f) \longrightarrow e$ where i_1, i_2 and i_3 are identity momomorphisms. (It is trivial that if:- $e \longrightarrow A \xrightarrow{f_1} B$ and $A \xrightarrow{f_2} B \longrightarrow e$ are exact, then f_1 is a monomorphism and f_2 an epimorphism). Exact sequences such as the above are called short exact. In commutative diagrams all exact sequences will be in straight lines.

Given two groups K and Q which satisfy the short exact sequence:-

$$e \longrightarrow K \xrightarrow{i} E \xrightarrow{\Phi} Q \longrightarrow e$$

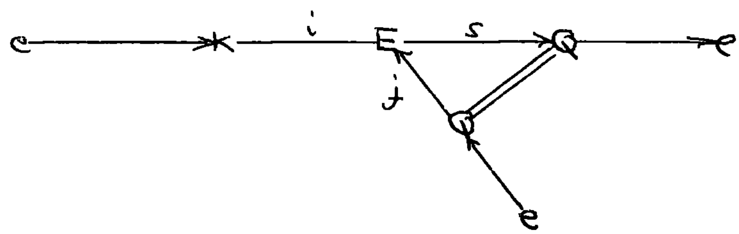
for a group E, then E is said to be a group extension of K by Q. A short exact sequence is said to be split iff (i) $K \rightleftharpoons e$ (ii) $Q \rightleftharpoons e$ or (iii) $\exists$ a monomorphism $j : Q < E \ni \phi \circ j = 1\!\!1_Q$. Here, we shall only be interested in split extensions. When $j : Q \lhd E$ and $E/Q \rightleftharpoons K$ the group E is said to be the direct product of Q by K (or vice-versa). When $Q < E$, E is said to be the semi-direct product of Q by K iff it is also an extension of K by Q. If E is the direct product of Q by K, the below diagram must be true:-



In this case, E is written K x Q and is isomorphic to the group consisting of K x Q with underlying set and composition $(k_1, q_1)(k_2, q_2) = (k_1k_2, q_1q_2)$. One can prove the lemma that E is isomorphic to K x Q when $K, Q \lhd E$, $K \cap Q = e$ ad $E = K \cdot Q = Q \cdot K$. For, in our diagram consider the homomorphism $s_1 \times s_2$ from E into K x Q defined by $s_1 \times s_2 : x \longmapsto (s_2(x), s_1(x))$. Now $x \in Ker(s_1 x s_2) \Rightarrow s_2(x) = e = s_1(x)$ so that $Ker(s_1 x s_2) \subset Ker(s_1) \cap Ker(s_2)$, we readily see that $Ker(s_1 x s_2) = Ker(s_1) \cap Ker(s_2)$ so that by exactness $Ker(s_1 \times s_2) = Im(i_1) \cap Im(i_2)$ i.e.:- $\forall\, x \in Ker(s_1 \times s_2) \exists\, (k_1, k_2) \in K_1 \times K_2$ $x = i_1(k_1) = i_2(k_2)$ whence $s_1(x) = s_1 \circ i_1(k_1) = 1\!\!1(k) = k_1 = e$ so that $k_1 = e$; similarly $k_2 = e$ whence $Ker(s_1 \times s_2) = e$ and $K \cap Q = e$. With the result that $s_1 \times s_2$ is a monomorphism. It is also an epimorphism; given $k \in K_1$ $q \in Q$ then for $x = i_1(k)\, i_2(q)$ satisfies $s_1 \times s_2 : x \longmapsto$

$\longrightarrow (k, q)$ since $s_1 \circ i_1 = o$, $s_2 \circ \phi_2 = o$. So that $E = K \cdot Q$ and $s_1 \times s_2$ is an isomorphism.

Similarly, a group E is called a semi-direct product if the below diagram is commutative



Or, given a $g \in \text{Hom}(Q, \text{Aut}(K))$ we call the group $K \boxtimes g\, Q$ whose underlying set is $K \times Q$ and whose composition is $(k_1, q_1)(k_2, q_2) = (k_1 g(q_1)(k_2), q_1 q_2)$ the semi-direct product of Q by K. Both definitions are equivalent and imply $K \lhd K \boxtimes gQ$, $Q < K \boxtimes gQ$ and $K \cap Q = e$, with $K \boxtimes gQ = K \cdot Q$. We must have $\text{Im}(i) \cap \text{Im}(j) = e$ for if $x = i(k) = j(q)$ then so $i = o$ implies $k = e$ and $q = e$ since i is injective. So $\forall\, x \in K \boxtimes gQ$ there exists $(k, q) \in K \times Q$ such that $x = i(k)j(q)$ is a unique member of $K \times gQ$ when k and q are fixed. This $K \boxtimes gQ$ has underlying set $K \times Q \gneqq K \cdot Q$. Since $K \lhd K \boxtimes gQ$, $(\text{Ker}(s) = \text{Im}(i))$, K must be stable under $\text{Int}(K \boxtimes gQ)$, thus $\exists\, g \in \text{Hom}(Q, \text{Aut}(K))$ with $\text{In}(j(q)) \circ i = i \circ g(q)\ \forall\, q \in Q$. We must then have $i(k_1)j(q_1)ik_2)j(q_2) = i(k_1)j(q_1)i(k_2)j(q_1)^{-1}j(q_1)j(q_2) = i(k_1)$. $\text{In}(j(q_1)(i(k_1))j)q_1q_2) = i(k_1)i(g(q_1)(k_2)j(q_1q_2) = i(k_1 g(q_1)(k_2)j(q_1q_2)$, which is the semi-direct product group law. When K is an Abelian group and Q a subgroup of the set of units of a ring with K a natural module, we immediately see that there exists a group $K \boxtimes nQ$, $n \in Hom(Q, Aut(K))$, is called the natural homomorphism.

Choose a new monomorphism $j: Q < K \boxtimes gQ$ such that $soj' = 0$. We must then have $j(q)\, j'(q)^{-1} \in Ker(S) = Im(i)$, so write $j(q)' = i(\Phi(q))j(q)$ where $\Phi \in C^{1}(Q, K)$. Now $Im(j) \rightleftharpoons Im(j')$ which means that $\exists\ k \in K$ with $In(i(k))(Im(j)) = Im(j'))$ viz $i(k)j(q)i(k)^{-1} = i(\Phi(q))j(q)$ or $\Phi(q) = i(k)j(q)i(k)^{-1}j(q)^{-1} = i(k)i(g(q)(k^{-1})) = i(kg(q)(k^{-1}))$. When K is Abelian, we can write $\Phi(q) = k-g(q)(k)$, this result will be elaborated on in chapter (2). When $H < G$ the class $\langle In(g)(H) \rangle$ $g \in G$ are called conjugate subgroups of H in G and are all isomorphic to H.

Section (5) Rings, Sfields and Fields.

We will discuss Rings, Sfields and Fields only very briefly. A ring R is a triple $(S, \odot_1, \odot_2)$ where $(S, \odot_1)$ is an Abelian group and $(S, \odot_2)$ is an arbitary semi-group. The binary operations $(\odot_1, \odot_2)$ have to satisfy the distributative axioms $(x_1 \odot_1 x_2) \odot_2 x_3 = x_1 \odot_2 x_3 \odot_1 x_2 \odot_2 x_3$ and $(x_1 \odot_2 (x_2) \odot_1 x_3) = x_1 \odot_2 x_2 \odot_1 x_2 \odot_2 x_3$. We call the set of units of R the maximal subgroup of the semi-group $(S, \odot_2)$. If $(S, \odot_2)$ is a group, we call $(S, \odot_1, \odot_2)$ a sfield. If $(S, \odot_2)$ is an Abelian group also, one calls $(S, \odot_1, \odot_2)$ a field.

Section (6) Modules and Vector Spaces.

Consider a ring Λ and an Abelian group M. If there exists a map $k_L: \Lambda \times M \rightarrow M$ such that, defining:- $k_2(\lambda, x) \equiv \lambda \cdot x\ \forall\ (\lambda, x) \in \Lambda \times M$; we have $\lambda.(x_1 + x_2) = \lambda \cdot x_1 + \lambda \cdot x_2$; $(\lambda_1 + \lambda_2) \cdot x =$

$\lambda_1 \cdot x + \lambda_2 \cdot x$, $\lambda_1 \cdot \lambda_2 \cdot x = (\lambda_1 \lambda_2) \cdot x$, $e \cdot e = x$; we call the group M a left Λ module. In a similar way we can define the notion of right Λ modules. If Λ is a commutative ring, that is $(\Lambda, 0_2)$ is an Abelian monoid, then every left Λ module is also a right Λ module. Give two left Λ modules, one defines a Λ homomorphisms as a homomorphism $f \in \text{Hom}(M_1, M_2)$ such that $f(\lambda \cdot x) = \lambda \cdot f(x) \; \forall \; x \in M_1$. If M_1, M_2, M_3 are modules and f_1 and f_2 are Λ homomorphisms $f_1 \in \text{Hom}_\Lambda (M_1, M_2)$, $f_2 \in \text{Hom}_\Lambda (M_2, M_3)$ then $f_2 \circ f_1 \in \text{Hom}(M_1, M_3)$. Thus the subset of Λ endomorphisms of End(M), where M is a Λ module, is a semi-group. Since the identity function is also a Λ homomorphism $\text{End}_\Lambda(M)$ is a submonoid of End(M). Because End(M) is a ring when M is Abelian, we see that $\text{End}_\Lambda(M)$ is a subring of End(M). If Λ is commutative, one can also endow $\text{Hom}_\Lambda(M_1, M_2)$ as a Λ module. $\text{Hom}(M_1, M_3)$ being an Abelian group. Every Abelian group is a Z module with $n \cdot Z \equiv Z + Z + \ldots$ to n factors $\forall \; (n, Z) \in Z \times M$.

If Λ is a field, we call a Λ module M a vector space over Λ. If M is a vector space over Λ we call the set $\text{Hom}_\Lambda(M, \Lambda)$ the vector dual space, denoted by M^*. M^* is a Λ linear space via the definitions $(\lambda f_1 + \lambda_2 f_2)(x) \equiv \lambda_1 f_1(x) + \lambda_2 f_2(x) = f_1(\lambda_1 x) + f_2(\lambda_2 x)$. An inner product space is a linear space over Λ and a function f: $M \times M \longrightarrow \Lambda$ with f linear in both variables seperated.

$f(\lambda_1 x_1 + \lambda_2 x_2, x_3) = \lambda_1 f(x_1, x_3) + \lambda_2 f(x_2, x_3)$

$f(x_1, \lambda_2 x_2 + \lambda_3 x_3) = \lambda_2 f(x_1, x_2) + \lambda_3 f(x_1, x_3)$

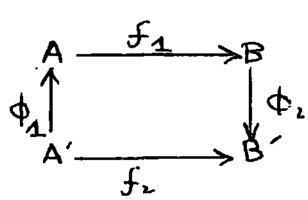
In this case, there exists a Λ isomorphism from M onto M* defined by $x \longmapsto x^*$ when $x^*(y) \equiv f(x,y) \; \forall \; (x,y) \in M$, called the canonical isomor-

-phism of M onto M*. A normed linear space is a linear space with a function $\| \; \| : M \to \Lambda$ which obeys (i) $\| x \| \geq 0$, $\| x \| = 0$ iff $x = 0$ (ii) $\| \lambda x \| = |\lambda| \, \| x \|$ (iii) $\| \lambda_1 x_1 + \lambda_2 x_2 \| \geq |\lambda_1| \, \| x_1 \| + |\lambda_2| \, \| x_2 \|$ Clearly an inner product space is also normed via $\| x \|^2 = x^*(x)$. The elements of M* are called Λ linear functionals, elements of $C^1(M, \Lambda)$ and functionals. Functionals like f above are called bilinear. If M_1 and M_2 are linear spaces over Λ, the tensor product space of M_1 and M_2 over Λ denoted by:- $M_1 \otimes_\Lambda M_2$ has underlying set isomorphic to $M_1 \otimes M_2$ with addition of elements $x \otimes y$ in $M_1 \otimes_\Lambda M_2$ defined by the rule $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$, $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$, and module operation $\lambda(x \otimes y) = (\lambda x \otimes y) = (x \otimes \lambda y)$. $\forall \; \lambda \in \Lambda$. Given a linear space M over Λ, $M \otimes_\Lambda M$ is isomorphic to the dual of the space of bilinear functionals on M via $x \otimes y : f \mapsto f(x, y) \; \forall \; f \in B^2(M, \Lambda)$ the space of bilinear functionals on M. We will only consider the case $\Lambda = \mathbb{R}$.

Section (7). Categories and Functors.

In this section, we shall briefly discuss the notions of categories and functors from a very elementary standpoint. The concept of categories and functors is due mainly to Eilenberg and Maclane and has a large unifying effect on algebra. We shall use different algebraic constructions due to these authors in the main text of the thesis.

A category $\mathcal{C}$ consists of a pair $(Ob(\mathcal{C}), Ar(\mathcal{C}))$. $Ob(\mathcal{C})$ is a collection of albegraic objects such as sets or group etc. $Ar(\mathcal{C})$ is the set of all morphisms or relations between pairs of elements of $Ob(\mathcal{C})$. We have $Ar(\mathcal{C}) \equiv \dot{\bigcup} \{ Mor(A, B) \mid A, B \in Ob(\mathcal{C}) \}$. For three elements A, B, C $\in$ Ob(C) there is defined a binary relation from Mor(B, C) x Mor(A, B) into Mor(A, C). The relation must satisfy the requirement that $\forall\ A \in Ob(\mathcal{C}) \exists\ 1_A \in Mor(A, A)$ such that $1_A \circ f = f\ \forall\ f \in Mor(A, B)$, $f' \circ 1_A = f'\ \forall\ f' \in Mor(A, B)$, $B \in Ob(\mathcal{C})$ The composition must also be associative when defined. A morphism $f \in Mor(A, B)$ is an isomorphism iff $\exists\ f' \in Mor(B, A) \ni f \circ f' = 1_B$, $f' \circ f = 1_A$. When A = B an isomorphism of Mor(A, A) is called an automorphism. The set Mor(A, A) is called the set of endomorphism of A; End(A). End(A) is a monoid. Let $\mathcal{C}$ be a category: we may endow $Ar(\mathcal{C})$ with the structure of a category as follows. Let $(f_1, f_2) \in Mor(A, B) \times Mor(A', B')$. We define a morphism $f_1 \longmapsto f_2$ to be a pair $(\phi_1, \phi_2) \in Mor(A, A') \times Mor(B, B')$ such that the below diagram is commutative.



One can, in fact endow the category of commutative diagrams between objects of a category $\mathcal{C}$ with the structure of a category by defining translations or morphisms between diagrams. The above operation is a morphism between Mor(A, B) and Mor(A', B') where we define a morphism:-

$Mor(\phi_1,\phi_2)$: $Mor(A, B) \to Mor(A, B')$; $Mor(\phi_1,\phi_2)$: $f_1 \mapsto f_2 = \phi_2 \circ f_2 \circ \phi_1$. We will supply an interpretation of the relation 'Mor' later.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two categories, we call any relation F. $\mathcal{C}_1 \to \mathcal{C}_2$ such that $\forall$ $A \in Ob(\mathcal{C}_1)$, $F(A) \in Ob(\mathcal{C}_2)$; $\forall$ $f \in Mor(A, B)$ $F(f) \in Mor(F(A), F(B))$ and $F(g \circ f) = F(g) \circ F(f) \in Mor(F(A), F(C))\ \forall\ g \in Mor(B, C)$, $f \in Mor(A, B)$; a covariant functor. Similarly a relation from $\mathcal{C}_1$ into $\mathcal{C}_2$ such that $F(A) \in Ob(\mathcal{C}_2)\ \forall\ A \in Ob(\mathcal{C}_1)$, $F(f) \in Mor(F(B), F(A))\ \forall\ f \in Mor(A,B)$, $F(g \circ f) = F(f) \circ F(g) \in Mor(C, A)$ $\forall\ f \in Mor(A, B)$, $g \in Mor(B, C)$; is called a contravariant functor. We will give examples of functors which relate the categories of sets, semi-groups, monoids and groups shortly. Consider first the relations $M_A: \mathcal{C} \to S$, M_A: $X \to Mor(A, X)\ \forall X \in Ob(A)$. If $\phi \in Mor(X,X')$, define $M_A: \phi \mapsto Mor(\mathbb{1}_2, \phi)$: $Mor(A, X) \to Mor(A, X')$, $Mor(\mathbb{1}', \phi)$: $f \mapsto f \circ \phi\ \forall\ f \in Mor(A, X)$. M_A is a covariant functor from $\mathcal{C}$ into $Ar(\mathcal{C})$. Similarly $\forall A \in OB(\mathcal{C})$ we can define a covariant functor M^A: $\mathcal{C} \to Ar(\mathcal{C})$, M^A: $X \mapsto Mor(X, A)$. If $\phi \in Mor(X'\ X)$ $M^A: \phi \mapsto Mor(\phi, \mathbb{1})$ $Mor(\phi, \mathbb{1})$: $f \to f \circ \phi\ \forall\ f \in Mor(X, A)$. We have already defined a functor Mor: $\mathcal{C}^2 \to Ar(\mathcal{C})$, Mor: $(A, B) \mapsto Mor(A, B)\ \forall (A, B) \in \mathcal{C} \times \mathcal{C}$ and Mor: $(\phi_1,\phi_2) \mapsto Mor(\phi_1,\phi_2)\ \forall\ (\phi_1,\phi_2) \in Mor(A', A) \times Mor(B,B')$; $Mor(\phi_1,\phi_2)$: $f \to \phi_1 o f o \phi_2$; $Mor(\phi_1,\phi_2)$: $Mor(A, B) \to Mor(A', B')$, $\forall\ f \in Mor(A, B)$. Mor is both contra and covariant. The category S consists of sets and functions $Mor(S_1, S_2) = C^1(S_1, S_2)$. The category $\mathcal{G}$ consists of groups and homomorphisms $Mor(G_1, G_2) = Hom(G_1, G_2)$.

An important functor is from a category onto the category of sets and mappings. This functor is called the stripping functor. Consider the groups $(S_i, \odot_i) = G_i \in Ob(\mathcal{G})$ ad $f \in Hom(G_1, G_2)$ then F: $G_i \mapsto S_i$ and F: $f \mapsto f \in C^1(S_1, S_2)$, is the covariant stripping functor. Similarly let $(S, \odot_1, \odot_2)$ be a ring; then there exist functors from R into the category of Abelian groups F: $R \mapsto (S, \odot_1)$ and to the category of semi-groups F: $R \mapsto (S, \odot_2)$. If R is a ring with identity then F: $R \mapsto (S, \odot_2)$ to the category of monoids. This ends our summary of the algebraic notions to be used in this thesis. From now on, concepts will be defined as they arise.

CHAPTER (2)

SPACE-TIME, RELATIVITY AND GROUP THEORY

This thesis presents a discussion of various relativity models of physics from an algebraic point of view. In this chapter we shall attempt to introduce the work to follow in later chapters. No apologies are made for the semi-axiomatic approach[1)] to relativity models which we shall use. Whilst some may feel that such an approach obscures the physical content of a theory, it is necessary to present a theory in such a way to make it amenable to the use of the extremely powerful arsenel of modern algebra. The task we attempt to fulfill in this chapter is the translation of physical experience into algebraic language, so that using the logic embedded in that language, as many physical conclusions can be extracted as possible, in the spirit of the quotations in the preface. The algebraic tools used in the analysis of this chapter were presented, in summary, in chapter one. The ideas we obtain from this preliminary analysis are followed up in the next three chapters using the fairly modern algebraic tools developed in the chapter following this.

Theories of relativity are broadly concerned with the structure of a set '$\mathbb{W}$', the 'event-world' or 'space-time'. Elements $x \in \mathbb{W}$ are called 'events' and represent physical phenomena. A theory of relativity assigns a mathematical structure to $\mathbb{W}$ via sets of relations which enable one to formulate relationships between any two events $x_1, x_2 \in \mathbb{W}$, such as "x_1 is 'before x_2" or "x_1 is 'near' x_2" etc. The structures assigned to $\mathbb{W}$ are drawn from experimental observation of the natural world. The assignment of a structure to $\mathbb{W}$ is, loosely, a physical theory. A physical theory is judged in relation to others via its ability to mirror as closely as possible the structure of the natural world as determined by experiment, in its ability to prompt the experimental search for hithertoo unobserved phenomina and in its ability to explain the phenomina hithertoo un-understood. Apart from the philosophical speculations of the ancients, the first theory of relativity (in the modern sense) was proposed by Issac Newton. Newton drew up on the work of immumerable past workers, prominent amongst whom were Copernicus, Kepler and Galileo Galilei. Such was the quality of Newton's relativity, that within its framework physics continued quite happily along to the beginning of the century when the first tremors of the downfall of the relativity were making themselves felt, through the well-known paradoxes, such as the null result of the Michelson-Morley experiment. In order to explain these paradoxes, Albert Einstein, in 1905 proposed a new system of relativity called after him, or the 'Special Theory of Relativity'. In the formulation of his Special Theory, Einstein, like Newton, drew upon the work of his predecessors and contempories such as Lorentz, Poincaré and Minkowski[2)]. After ten years or so, Einstien proposed his General Theory of Relativity as a more perfect theory than his first. The General theory dealt with the 'global' structure of the event-world, reducing, in the 'locality' of an event to the Special Theory, under certain circumstances. Since Einstein's beautiful General Theory was propounded, many new ones, similar in spirit to it have been suggested, in all of them the cosmological structure of space-time is discussed, whilst, in the local case, the Special Theory of relativity remains.

In the 1920's, Einstein's theory of Special Relativity seemed to breakdown when one attempted to explain microscopic phenomena. The explanation of these atomic and sub-atomic phenomena lead to the invention of quantum mechanics by a group of mainly German physicists. Most prominent amongst the inventors of quantum mechanics, were Bohr, Planck, Heisenberg, Born and especially E. Schrodinger. The first attempts at the formulation of a quantum mechanics only used the relativity ideas of Newton. Almost immediately, attempts were made to improve the mechanics by incorporating in in it the Special Theory of Relativity. The first attacks by Schrodinger and Dirac were not particularly successful. It was not until 1939 that Wigner[3)] was able to imbed Special Relativity into quantum theory in a highly successful way. The extreme beauty of Wigner's theory was not, however,matched to its limited practicability. Fairly recently[5,6,7,8)], it was shown that if Newton's relativity was incorporated into Wigner's quantum mechanics, the result was the Schrodinger quantum mechanics, a highly practical theory. The great stumbling block of Wigner's quantum mechanics incorporating the Special Theory of Relativity is its present weakness in describing the interaction between material bodies. Schrodinger's theory was able to do this in a highly successful way, employing the classical description of interactions developed by Newton, Lagrange and Hamilton. This weakness, until comparatively recently, lead to the neglect of Wigner's quantum mechanics in preference to the highly 'slippery' relativistic quantum field theory. The latter was very successful when describing the weaker forms of interaction between sub-atomic particles, but was completely unsuccessful when dealing with the strong interactions between certain of them. The newest of interaction theories, which can deal with these strong interactions is the so called 'S matrix'[8)] theory of strong interactions developed in the last decade. The S matrix theory attempts to side step the descriptions of fields and interactions, dealing instead with the experimentally observable consequences of the interactions. Wigner's theory of quantum mechanics is the natural frame-work for this modern theory.

We cannot discuss the quantum-mechanical theories mentioned here any further due to lack of expertise and to some slight degree, space. They were mentioned however, because most of the material presented in this thesis have quantum mechanical applications[9,10,11)]. It would take another volume to do them justice. The mechanics we discuss will be classical; the extensions of the results we obtain to quantum mechanics is very easy but will not be performed here. Also, although we discuss Hamilton's scheme for the description of interactions, the only results we shall obtain are for free bodies undergoing no interaction with the external world. Neither will the cosmological theories of relativity be discussed, we will concentrate on the Special Theory of relativity, Newtonian relativity and some relativity models derivable from each of them.

In parts (1) and (2) of this chapter, we shall discuss Newton's and Einstien's Special Theories of relativity. In part (3) some degenerate cousins of the first two will be discussed as valid, if unphysical relativity models. The discussion will largely through the use of group theory which is introduced in the first part.

Part (1). Newtonian Relativity.

One can mathematically formulate Newton's notion of the world in modern terms as follows. Our treatment is an extension of a briefer one due to W.Noll in Ref.(1). The Newtonian world $\mathcal{W}(N)$ is a triple $(\mathcal{W}, \tau, \ni)$. Here $\mathcal{W}$ is the event-world or space-time. The object 'τ' is a function $\tau \in C^2(\mathcal{W}, \mathbb{R})$ which assigns to every pair of events $x_1, x_2 \in \mathcal{W}$ a real number $\tau(x_1, x_2)$ called the time-lapse function. According to physical experience, one must assign the following properties to τ. These are (i) $\tau(x_1, x_2) = -\tau(x_2, x_1)$ (ii) $\tau(x_1, x_2) + \tau(x_2, x_3) = \tau(x_1, x_3) \;\forall\; x_1, x_2, x_3 \in \mathcal{W}$; and (iii) $\forall\; (x, t) \in \mathcal{W} \times \mathbb{R} \;\exists\; y \in \mathcal{W} \ni \tau(x, y) = t$. The time-lapse function enables us to assign a relation F in $\mathcal{W}$ via:- $F \equiv \{(x, y) \in \mathcal{W}^2 \ni \tau(x, y) > 0\}$. Condition (ii) then implies that F is a transitive relation, $F \circ F \subset F$. However, F is not a partial order since if $P \equiv F^{-1}$, $P \cap F \equiv S$ then $\Delta \subseteq S$. However, the relation S is an equivalence relation and the relations F, P and S define subsets of $\mathcal{W}$ via:- $F(x) \equiv \{y \in \mathcal{W} \ni (y,x) \in F\}$, $P(x) = F^{-1}(x)$ and $S(x) = F(x) \cap P(x)$ $\forall\; x \in \mathcal{W}$. Clearly from condition (iii) $F(x) \cup P(x) = \mathcal{W}$ and from condition (i) $y \in F(x) \Leftrightarrow x \in P(y) \; \forall \; x, y \in \mathbb{W}$. The equivalence relation S is called simultenaity. If $y \in S(x)$, we say that the events x and y are simultaneous. Given $x \in \mathbb{W}$ we call F(x), P(x) and S(x) the future, past and instant of x. In our definition the map S can be chosen as the canonical map $S: \mathbb{W} \longrightarrow \mathbb{W}/S$, which induces a portion $\mathcal{S}$ of $\mathbb{W}$ into a disjoint family of subsets, i.e.:- $\mathbb{W} = \dot{\bigcup} \{\sigma \mid \sigma \in \mathcal{S}\}$, $S(x_1) \cap S(x_2) = \emptyset$ if $(x_1, x_2) \notin S$ and $S(x_1) \cap S(x_2) = S(x)$ where $x_1, x_2 \in S(x)$. The time lapse function gives rise to a set function $\bar{\tau} \in C^2(\mathcal{S}, \mathbb{R})$ where $\bar{\tau}(\sigma_1, \sigma_2) \equiv \tau(x_1, x_2)$ when $(x_1, x_2) \in \sigma_1 \times \sigma_2$. For each pair of instants $\sigma_1, \sigma_2 \in \mathcal{S}$, the number $\bar{\tau}(\sigma_1, \sigma_2)$ is called the time-lapse between them. Clearly, we have (i) $\bar{\tau}(\sigma_1, \sigma_2) = -\bar{\tau}(\sigma_2, \sigma_1)$ (ii) $\bar{\tau}(\sigma_1, \sigma_2) = 0$ iff $\sigma_1 = \sigma_2$ (iii) $\bar{\tau}(\sigma_1, \sigma_2) + \bar{\tau}(\sigma_2, \sigma_3) = \bar{\tau}(\sigma_1, \sigma_3)$ and (iv) $\forall \; \sigma_1 \in \mathcal{S}$, $t \in \mathbb{R} \; \exists \sigma_2 \in \mathcal{S} \ni \bar{\tau}(\sigma_1, \sigma_2) = t$. In this light, define a relation A in $\mathcal{S}$ via $A \equiv \{(\sigma_1, \sigma_2) \in \mathcal{S} \times \mathcal{S} \ni \bar{\tau}(\sigma_1, \sigma_2) > 0\}$, then we must have $A \cap B = \Delta$ where $B \equiv A^{-1}$ so that A is a partial order. If $\sigma_2 \in A(\sigma_1)$ or $(\sigma_1, \sigma_2) \in A$ we say that the instant σ_2 is after σ_1 or if $(\sigma_1, \sigma_2) \in B$, σ_2 is before σ_1. We must have $A(\sigma) \cap B(\sigma) = \sigma \; \forall \sigma \in \mathcal{S}$ and $A(\sigma) \cup B(\sigma) = \mathcal{S} \; \forall \; \sigma \in \mathcal{S}$ that is A is a total order in $\mathcal{S}$. It is induced in $\mathcal{S}$ via the natural equivalence between $\mathcal{S}$ and $\mathbb{R}$ (to be defined) and the natural total order ($>$) in $\mathbb{R}$. The natural equivalence between $\mathcal{S}$ and $\mathbb{R}$ is obtained as follows. Choose an instant $\sigma_0 \in \mathcal{S}$. Then a function $\mathcal{D}': \mathcal{S} \longrightarrow \mathbb{R}$ is defined by $\mathcal{D}': \sigma \longmapsto \bar{\tau}(\sigma, \sigma_0) \; \forall \; \sigma \in \mathcal{S}$. By condition (iv), $\mathcal{D}'$ is onto, and since $\bar{\tau}(\sigma_1, \sigma_0) = \bar{\tau}(\sigma_2, \sigma_0) \Leftrightarrow \bar{\tau}(\sigma_1, \sigma_2) = 0$ with

$\overline{\tau}(\sigma_1,\sigma_2) = 0 \Leftrightarrow \sigma_1 = \sigma_2$ we see that $\mathcal{D}'(\sigma_1) = \mathcal{D}'(\sigma_2) \Leftrightarrow \sigma_1 = \sigma_2$ whence $\mathcal{D}'$ is one to one and hence bijective. If $\mathcal{D} = \mathcal{D}'^{-1}$ then $\mathcal{D}$ is an equivalence between $\mathbb{R}$ and $\mathcal{D}$ and moreover $(\mathcal{D}(t_1), \mathcal{D}(t_2)) \in A \Leftrightarrow (t_1, t_2) \in (>)$. Also $\mathcal{D}(0) = \sigma_0$. Now let us choose a σ_0' in place of σ_0. A function $\mathcal{D}'^*: \mathcal{S} \longrightarrow \mathbb{R} \ni \mathcal{D}'^*: \sigma \longmapsto \overline{\tau}(\sigma, \sigma_0')$ is obtained which is bijective and whoose inverse naturally preserves the partial order relations; $(\mathcal{D}^*(t_1), \mathcal{D}^*(t_2)) \in A \Leftrightarrow (t_1, t_2) \in (>)$ with $\mathcal{D}^*(0) = \sigma_0'$. We thus see that there exists a family $\mathcal{E} = \langle \mathcal{E}(\sigma) \rangle \sigma \in \mathcal{S}$ of equivalences between $\mathbb{R}$ and $\mathcal{S}$ which preserve the partial orders and satisfy $\mathcal{E}(\sigma): 0 \longmapsto \sigma \ \forall \sigma \in \mathcal{S}$ defined by $\mathcal{E}(\sigma)^{-1}: \sigma' \longmapsto \overline{\tau}(\sigma, '\sigma) \ \forall \sigma' \in \mathcal{S}$. In terms of the axioms for $\overline{\tau}$ we must have (i) $\mathcal{E}(\sigma_1)^{-1}(\sigma_2) = -\mathcal{E}(\sigma_2)^{-1}(\sigma_1)$ (ii) $\mathcal{E}(\sigma_1)^{-1}(\sigma_2) = 0 \Leftrightarrow \sigma_1 = \sigma_2$ (iii) $\mathcal{E}(\sigma_2)^{-1}(\sigma_1) + \mathcal{E}(\sigma_3)^{-1}(\sigma_2) = \mathcal{E}(\sigma_3)^{-1}(\sigma_2) \forall \sigma_1, \sigma_2, \sigma_3 \in \mathcal{S}$. Let us again fix an instant $\sigma_0 \in \mathcal{S}$ with $\mathcal{E}(\sigma_0)^{-1} \equiv \mathcal{D}$. We must then have the consequence that $\mathcal{E}(\mathcal{D}(t_1))^{-1}(\mathcal{D}(t_2)) = \mathcal{E}(\mathcal{D}(0))^{-1}(\mathcal{D}(t_1)) - \mathcal{E}(\mathcal{D}(0))^{-1}(\mathcal{D}(t_2))$ or $\mathcal{E}(\mathcal{D}(t_1))^{-1}(\mathcal{D}(t_2)) = \mathcal{D}^{-1} \circ \mathcal{D}(t_1) - \mathcal{D}^{-1} \circ \mathcal{D}(t_2) = t_1 - t_2$, which is the time-lapse between the instants $\mathcal{D}(t_1)$ and $\mathcal{D}(t_2)$. In particular, if we choose $\mathcal{D}(t_1) = \sigma_0'$, then the time-lapse between σ_0' and σ_0 is t_1 whilst between σ_0' and an arbitrary $\sigma \in \mathcal{S}$ the time lapse is $\mathcal{E}(\sigma_0')(\sigma) = t_1 - \mathcal{D}^{-1}(\sigma)$, the relative time between σ_0' and σ w.r.t. σ_0. If the relative time between σ_0 and σ_0' is T, the 'time' of σ is $t_2 + T$ relative to σ_0. The instant σ_0 chosen to fix the scale of time is called the 'present', if another 'present' is chosen, the total time-lapse is the sum of the time-lapse between the original present a nd the new one and the relative time lapse between the instant and the new present.

As subsets of $\mathbb{W}$, each instant is isomorphic to any other and whence to a given present. Physical experience shows that the present is isomorphic to the set $\mathbb{R}^3$ corresponding to the independent notions of 'lengths', 'breadth' and 'height'. Also, σ_0 must be a real linear space corresponding to the relative nature of these notions. That is, choosing an event $y \in \sigma_0$, we can affix the basis of a real linear space V(y) to y, where V(y) is isomorphic to the real linear space $\mathbb{R}^3$, $y \longmapsto \underline{0}$ and $x \longmapsto \underline{x}\ \forall\, x \in \sigma_0$. $\underline{x}$ is a triple (x_1, x_2, x_3) defining the position of x relative to y in V(y); the vector $\underline{x}$ being called the location of x in V(y) when $x, y \in \sigma_0$. Given two real numbers α_1, α_2 and a pair of events x_1 and x_2, we define the event $\alpha_1 x_1 + \alpha_2 x_2$ in V(y) as the event whose location in $\mathbb{R}^3$ is $\alpha_1 \underline{x}_1 + \alpha_2 \underline{x}_2$. Let us choose an event $y' \in \sigma_0$ whose location in $\mathbb{R}^3$ is $\underline{y}'$ relative to y, and define a vector space V(y') in σ_0 by $y' \longmapsto \underline{0}$ and $x \longmapsto \underline{x}'\ \forall\ x \in \sigma_0$ then there exists a isomorphism $V(y') \longrightarrow V(y)$ defined by $\underline{x}' \longmapsto \underline{x} + \underline{y}'$ when in V(y') $x \longmapsto \underline{x}'$ and in V(y) $x \longmapsto \underline{x}\ \forall\, x \in \sigma_0$. We write V(y') = V(y) + $\underline{y}$ and say that V(y') is a translate of V(y). Clearly $V \equiv \langle V(y) \rangle_{y \in \sigma_0}$ is a family of vector spaces isomorphic to $\mathbb{R}^3$ and whence mutually isomorphic as sets. The family V is called a frame, the members of the family are frames attached to $x \in \sigma_0$ if $\mathcal{U} \in V$ and $\mathcal{U} = V(x)$. Since $\mathbb{R}^3$ is a linear space it is also an Abelian group, each element $V(y) \in V$ can be regarded as a point in the orbit of $\mathbb{R}^3$, the underlying Abelian group of an arbitrary fixed $U \in V$ via $V(y) = \mathcal{U}^{\underline{y}} = \mathcal{U} \circ \rho(\underline{y})$ when ρ is the R.R. of $\mathbb{R}^3$ and σ_0 is taken as the linear space $U. = V(0) = \sigma_0$. That is all frames in V are translates of each other. The underlying Abelian group of $\mathbb{R}^3$ is called the spatial translation group written as $\mathbb{R}^3$, the three fold direct product of the Abelian group $\mathbb{R}_{add}$.

Experience also insists that σ_0 is a Euclidian metric space inheriting the usual topology a nd Euclidian geometry of $\mathbb{R}^3$. The object ∂ in $\mathcal{W}$(N) is a family $\partial = \langle \partial(\sigma) \rangle_{\sigma \in \mathcal{S}}$ of Euclidian metric functions $\partial(\sigma): \sigma \times \sigma \longrightarrow \mathbb{R}$. The topology endowed to σ_0 enables a precise formulation of neighbourhood or nearness and defines relations in σ_0 in the obvious way.

It is fairly clear that the underlying set of $\mathcal{W}$ can be taken as $\mathbb{R}^3 \times \mathbb{R}^1$ or $\sigma_0 \times \mathbb{R}^1$. The subset σ_0 of $\mathcal{W}$ is sometimes called the absolute space of Newton's world. When an event $x \in \mathcal{W}$ is written as $x = (\underline{x}, t)$ one means that $x \in \mathcal{D}(t)$ and that its location in σ_0 (or $\mathcal{U}$) is $\underline{x}$. We can also write $\mathcal{D}(t) = \{(\underline{x}, t) \mid \underline{x} \in \mathbb{R}^3\}$. $\mathcal{D}(t)$ is a linear space with vector addition and module action defined by the rule :-
$\alpha_1 \cdot (\underline{x}_1, t) + \alpha_2 \cdot (\underline{x}_2, t) = (\alpha_1 \underline{x}_1 + \alpha_2 \underline{x}_2, t) \ \forall \ \alpha_1, \alpha_2 \in \mathbb{R}; (\underline{x}_1 t), (\underline{x}_2, t) \in \mathcal{D}(t)$. As in the discussion of frames, all instants in $\mathcal{S}$ can be regarded as translates of σ_0 ,points in the orbit of σ_0 under $\mathbb{R}^1$, the underlying Abelian group of the time axis $\mathbb{R}$ via $\mathcal{D}(t) = \sigma_0{}^t$. The partition $\mathcal{S}$ is endowed with the usual topology on $\mathbb{R}$ which enables one to define the proximity of instants; the distance between two

instants $\sigma_1, \sigma_2 \in \mathcal{S}$ being $d(\sigma_1, \sigma_2) \equiv |\tau(\sigma_1, \sigma_2)|$. The topology on $\mathcal{W}$ is thus the product topology of the usual topologies on $\mathbb{R}^3$ and $\mathbb{R}$.

Let us begin to discuss kinematics. In order to do so, we utilise some rather formal definitions given by Noll. A 'material universe' is a set $\mathcal{U}$, whose elements are called 'bodies'. $\mathcal{U}$ is partially ordered set with order ($\precsim$), one says that if A, B $\in \mathcal{U}$ with $A \precsim B$, A is a 'part of' B. A body B is separate from a body C if $B \cap C = \emptyset$ where $B \cap C \precsim$ B and $B \cap C \precsim$ C. The material universe appropriate to a system of 'particles' is the power set of a finite set S ($\#(S) < \aleph_0$), elements $p \in S$ are particles. A motion of a material universe $\mathcal{U}$ is a function $M: \mathcal{U} \longrightarrow P(\mathcal{W})$ (the power set of $\mathcal{W}$) such that $M(B) \subset M(C)$ if $B \precsim C$ and $M(B) \cap \sigma \neq \emptyset \ \forall \sigma \in \mathcal{S}$ (Call $\mathcal{M}$ the set of motions at $\mathcal{U}$, $M \in \mathcal{M}$). The set $M(B) \subset \mathcal{W}$ is called the set of events experienced by the body B and more commonly the 'world tube' of B. If $\mathcal{U}$ is a system of discrete particles and $p \in S$, then M(p) is called a 'world line'. Now given $p \in S$ $M(p) \cap \sigma$ is a discrete element of $\mathcal{W}$ for let $x_1, x_2 \in M(p) \cap \sigma$, then $x_1, x_2 \in M(p)$ and $x_1, x_2 \in \sigma$ whence x_1 and x_2 are simultaneous events. If we make the assumption that a 'particle' cannot be in two places at the same time then $\underline{x}_1 = \underline{x}_2$ and $x_1 = x_2$. We can thus define a function $f_p(M) : \mathbb{R} \to \mathcal{W} \ni f_p(M): t \longmapsto M(p) \cap \mathcal{S}(t) \ \forall t \in \mathbb{R}$, or $f(M): t \longmapsto (\underline{x}(M)(t), t)$ where $\underline{x}_p(M) \mathbb{R}^1 \longrightarrow \mathbb{R}^3$ is called the trajectory of t he particle p under the motion M. Sometimes we will call $f_p(M)$ the world line since we have $\bigcup\{fp(M)(t) \mid t \in \mathbb{R}\} = \bigcup\{M(p) \cap \mathcal{P}(t) \mid t \in \mathbb{R}\} = M(p) \cap (\bigcup\{\mathcal{P}(t) \mid t \in \mathbb{R}\}) = M(p)$

Having laid our formal basis we turn now to the introduction of group theory into the discussion of relativity models, a technique which we'll use throughout this thesis. Recall that the set $B(\mathbb{W})$ of bijective mappings of $\mathbb{W}$ onto itself forms a group. Consider a subset $\mathcal{Q}(\mathbb{W}) \subset B(\mathbb{W})$ of bijective functions which preserve the structure of the world model $\mathbb{W}(N) = (\mathbb{W}, \tau, \partial)$, called 'world automorphisms'. We place $\alpha \in B(\mathbb{W})$ in $\mathcal{Q}(\mathbb{W})$ iff (i) $\tau \circ (\alpha \times \alpha) = \tau$ or $\tau(\alpha(x_1), \alpha(x_2)) = \tau(x_1, x_2) \; \forall \; x_1, x_2 \in \mathbb{W}$ and (ii) $\partial(\sigma^\alpha)(\alpha(x_1), \alpha(x_2)) = \partial(\sigma)(x_1, x_2) \; \forall \; x_1, x_2 \in \sigma, \sigma \in \mathcal{S}$, where σ^α is the set $x \in \sigma \Rightarrow \alpha(x) \in \sigma^\alpha$. It is immediate that $\mathcal{Q}(\mathbb{W}) < B(\mathbb{W})$ for if $\alpha_1, \alpha_2 \in \mathcal{Q}(\mathbb{W})$ then $\alpha_1 \circ \alpha_2 \in \mathcal{Q}(\mathbb{W})$, also $\mathbb{1} \in \mathcal{Q}(\mathbb{W})$ and if $\alpha \in \mathcal{Q}(\mathbb{W})$ then $\alpha^{-1} \in \mathcal{Q}(\mathbb{W})$ since (i) $\tau \circ (\alpha \times \alpha) = \tau \Rightarrow \tau \circ (\alpha \times \alpha) \circ (\alpha \times \alpha)^{-1} = \tau \circ (\alpha \times \alpha)^{-1}$ or $\tau \circ (\alpha \times \alpha) \circ (\alpha^{-1} \times \alpha^{-1}) = \tau \circ (\alpha \circ \alpha^{-1} \times \alpha \circ \alpha^{-1}) = \tau = \tau \circ (\alpha \times \alpha)^{-1}$ and (ii) we have $\partial(\sigma^\alpha) \circ (\alpha \times \alpha) = \partial(\sigma) \; \forall \; \sigma \in \mathcal{S}$, whence $(\partial(\sigma^\alpha) \circ (\alpha \times \alpha)) \circ (\alpha \times \alpha)^{-1} = \partial(\sigma) \circ (\alpha \times \alpha)^{-1} = \partial(\sigma^\alpha)$ whence $\partial(\sigma^{\alpha^{-1}}) \circ (\alpha \times \alpha)^{-1} = \partial(\sigma^{\alpha \circ \alpha^{-1}}) = \partial(\sigma)$. The important thing to remember about the group $\mathcal{Q}(\mathbb{W})$ is that it is uniquely determined by $\mathbb{W}$ and hence determines $\mathbb{W}$. A discussion of $\mathcal{Q}(\mathbb{W})$ can replace one of $\mathbb{W}$ in many respects.

The first condition defining $\mathcal{Q}(\mathbb{W})$ means that each $\alpha \in \mathcal{Q}(\mathbb{W})$ must preserve the relation F and, in the form $\bar{\tau}(\sigma_1^\alpha, \sigma_2^\alpha) = \bar{\tau}(\sigma_1, \sigma_2) \; \forall \; \sigma_1, \sigma_2 \in \mathcal{S}$ is must preserve the total order on $\mathcal{S}$. Thus the world automorphism must preserve the causal order on Newton's

world which means that an event 'y' of P(x) always satisfies $\alpha(y) \in P(\alpha(x))$:- effects cannot precede causes. We saw above how each world automorphism also preserves the Euclidean distance between simultaneous events. Also, via the definition of the function $f(\alpha): \sigma \longmapsto \sigma^{\alpha}$ we see that $\mathfrak{a}(\mathbb{W})$ is also a group of automorphisms of the partition $\mathcal{S}$. In order to study $\mathfrak{a}(\mathbb{W})$ in more detail, it will be ncessary to write:- $\alpha: (\underline{x}, t) \longmapsto (\beta_1(\alpha)(\underline{x}) + \beta_2(\alpha)(t),\ \beta_3(\alpha)(\underline{x}) + \beta_4(\alpha)(t))$ where $\beta_1 \in C^1(\mathfrak{a}(\mathbb{W}), C^1(\mathbb{R}^3, \mathbb{R}^3))$; $\beta_2 \in C^1(\mathfrak{a}(\mathbb{W}), C^1(\mathbb{R}^1, \mathbb{R}^3))$; $\beta_3 \in C^1(\mathfrak{a}(\mathbb{W}), C^1(\mathbb{R}^3, \mathbb{R}^1))$ and $\beta_4 \in C^1(\mathfrak{a}(\mathbb{W}), C^1(\mathbb{R}^1, \mathbb{R}^1))$. When we have discussed cohomology theory more fully, we shall be able to analyse the mappings $\langle\beta_i\rangle$ $1 \leq i \leq 4$ in greater detail than we can here. At the moment, we will discuss a few more obvious points about this analysis of α, which are more amenable to discussion. From the first condition on α we can determine some facts about β_3 and β_4. Recall that:-
$\tau((\underline{x}_1, t_1), (\underline{x}_2, t_2)) = t_1 - t_2\ \forall\ (\underline{x}_1, t_1), (\underline{x}_2, t_2) \in \mathbb{W}$, so that we must have $\beta_3(\alpha)(\underline{x}_1) + \beta_4(\alpha)(t_1) - \beta_3(\alpha)(\underline{x}_2) \mp \beta_4(\alpha)(t_2) = t_1 - t_2 .\ \forall\ \alpha \in \mathfrak{a}(\mathbb{W})$. Thus when $t_1 = t_2$ we must have $\beta_3(\alpha)(\underline{x}_1) - \beta_3(\alpha)(\underline{x}_2) = 0$ since $\beta_4(\alpha)(t_1) = \beta_4(\alpha)(t_2)$ when $t_1 = t_2$ (since $\beta_4(\alpha)$ is one to one $\forall\ \alpha \in \mathfrak{a}(\mathbb{W})$ as can easily be shown). We surmise that $\beta_3(\alpha)(\underline{x}_1) = \beta_3(\alpha)(\underline{x}_2)$ or that $\beta_3(\alpha): x \longmapsto T'(\alpha)$ where $T' \in C^1(\mathfrak{a}(\mathbb{W}), \mathbb{R}^1)$. Thus, when $t_1 \neq t_2$ we infer that $\beta_4(\alpha)(t_1) - \beta_4(\alpha)(t_2) = t_1 - t_2\ \forall\ \alpha \in \mathfrak{a}(\mathbb{W})$, we have $\beta_4(\alpha)(t_1) - t_1 = \beta_4(\alpha)(t_2) - t_2 \equiv T''(\alpha)$ where $T'' \in C^1(\mathfrak{a}(\mathbb{W}), \mathbb{R}^1)$. Writing $T(\alpha) \equiv T'(\alpha) + T''(\alpha)$, we must have $\alpha: (\underline{x}, t) \longmapsto (\beta_1(\alpha)(\underline{x}) +$

$+ \beta_2(\alpha)(\underline{t})$, $t + T(\alpha)$) $\forall \alpha \in \mathcal{A}(W)$. One can readily show (cf. chapter 3), that $T(\alpha_1 \circ \alpha_2) = T(\alpha_1) + T(\alpha_2)$, so that $T \in \mathrm{Hom}(\mathcal{A}(W), \mathbb{R}^1)$, where $\mathbb{R}^1$ is the group of translations of the time axis which is isomorphic to $\mathbb{R}$ add. Clearly, the automorphism $f(\alpha)$ of $\mathcal{S}$ induced by $\alpha \in \mathcal{A}(W)$ can be written as $f(\alpha)$: $\mathcal{D}(t) \longmapsto \mathcal{D}(t+T(\alpha))$ with $\mathcal{A}(W)$ acting transitively on $\mathcal{S}$:- $f(\alpha) \circ \mathcal{D} = \mathcal{D} \circ \rho(T(\alpha))$. The functions β_1 and β_2 will be discussed after a few more definitions concerning motions have been introduced.

Consider a finite set $S = \{ p_i \mid i \in I \}$ with $\#(I) < \aleph_0$ of particles. A motion M of S defines the individual motions $\{M_i \mid i \in I\}$ of the individual particles where $M(i) \equiv M \mid p_i \; \forall i \in I$, and world lines $f_i: t \longmapsto M(i) \cap \mathcal{D}(t)$. The world lines give rise to trajectories via $f_i(t) \equiv (\underline{x}_i(t), t)$. The trajectory $\underline{x}_i$ is continuous if $\underline{x}_i$ is a continuous map from the usual topology on $\mathbb{R}$ to that on $\mathbb{R}^3$. Given $\underline{x}_i$ is continuous we can define the functions $\dot{\underline{x}}_i$ called the instantaneous velocity and $\ddot{\underline{x}}_i$, called the instantaneous acceleration. The motion M_i is said to be uniform iff $\ddot{\underline{x}}_1 = 0$, and inertial iff $\ddot{\underline{x}}_i = 0$ $\forall i \in I$ when $S = \mathcal{U}(S)$. Select particle 1 say, as the origin of a Frame $\mathcal{V}(f) \equiv \langle V(f(t)) \rangle_{t \in \mathbb{R}}$. When $\ddot{\underline{x}}_1 = 0$ and $\ddot{\underline{x}}_i = 0 \; \forall i \in I$ we can lable the frame $\mathcal{V}(f)$ by $\ddot{\underline{x}}_1 = \underline{V}$ say, the frame $\mathcal{V}(f)$ is then written $\mathcal{I}(\underline{V})$ and called an inertial frame. Corresponding to the set of all inertial motions of S is the family $\langle I(\underline{V}) \rangle_{\underline{V} \in \mathbb{R}^3_T} \equiv I$ of all inertial frames, where $\mathbb{R}^3_T$ is the set of constant tangent vectors in $\mathbb{R}^3$. We also call $\mathbb{R}^3 \wedge$ T the underlying Abelian group of $\mathbb{R}^3$T and note that I can be regarded as the orbit under $\mathbb{R}^3$T I(0) viz. $I(0)^{\underline{v}} = I \circ \beta(\underline{v})$, β being the $\mathbb{R}$.R. of $\mathbb{R}^3$T.

Let us go back to β_1 and β_2. Now the second condition of $\mathfrak{a}(\mathcal{W})$ is that $\alpha \in \mathfrak{a}(\mathcal{W})$ iff $\partial(\mathcal{B}(t + T(\alpha)))(\alpha(x_1), \alpha(x_2)) = \partial(\mathcal{B}(t))(x_1, x_2) \,\forall\, x_1, x_2 \in \mathcal{B}(t)$, $t \in \mathbb{R}^1$. Write $\partial(\mathcal{B}(t)) = d(t)$ then $d(t + T(\alpha))(\alpha(\underline{x}_1), \alpha(x_2)) = d(t)(x_1, x_2)$. We have $d(t)(x_1, x_2)) = \| \underline{x}_1(t) - \underline{x}_2(t) \|$, (the norm on the inner product space $\mathbb{R}^3$). Whence, since $\alpha(\underline{x}_1(t)) - \alpha(\underline{x}_2(t)) = \beta_1(\alpha)(\underline{x}_1(t)) + \beta_2(\alpha)(t) - \beta_1(\alpha)(\underline{x}_2(t)) - \beta_2(\alpha)(t)$, we must have

$$\| \beta_1(\alpha)(\underline{x}_1(t')) - \beta_1(\alpha)(\underline{x}_2(t')) \|^2 = \| \underline{x}_1(t) - \underline{x}_2(t) \|^2$$

which means that the distance apart of two (particles in motion say) must be the same at all times and for stationary particles that:- $\| \beta_1(\alpha)(\underline{x}_1) - \beta_1(\alpha)(\underline{x}_2) \|^2 = \| \underline{x}_1 - \underline{x}_2 \|^2$ or that $\beta_1(\alpha)$ is a linear isometry of $\mathbb{R}^3_q$ with the inner product $\underline{x}_1 \cdot \underline{x}_2 = x_1^{\,i}\, x_2^{\,i}$. These transformations will be discussed further in chapter (3). Note that no restrictions are yet placed on $\beta_2 \in C^1(\mathfrak{a}(\mathcal{W}), C^1(\mathbb{R}^1, \mathbb{R}^3))$! Under a world automorphism with $\beta_1(\alpha) = \mathbb{1}$ we have $(\underline{x}, t) \longmapsto (\underline{x} + \beta_2(\alpha)(t), t)$ or if $\underline{x} : \mathbb{R}^1 \longrightarrow \mathbb{R}^3$ is a trajectory then under α: $\underline{x} \longmapsto \underline{x}^{\alpha}$ where $\underline{x}^{\alpha}(t) = \underline{x}(t) + \beta_2(\alpha)(t)$. We must have $\dot{\underline{x}} \overset{\alpha}{\longmapsto} \dot{\underline{x}}^{\alpha}$, $\dot{\underline{x}}^{\alpha}(t) = \dot{\underline{x}}(t) + (\beta_2(\alpha)(t))^{\bullet}$, $\ddot{\underline{x}} \overset{\alpha}{\longmapsto} \ddot{\underline{x}}^{\alpha}$, $\ddot{\underline{x}}^{\alpha}(t) = \ddot{\underline{x}}(t) + \beta_2(\alpha)^{\cdot\cdot}(t)$, ... Thus an automorphism $\alpha \in \mathfrak{a}(\mathcal{W})$ can map a uniform motion into a non-uniform one. Let $I(\mathcal{W}) \cap \mathfrak{a}(\mathcal{W})^{*}$ be the subgroup of world automorphisms of $\mathcal{W}$ which map uniform motions into uniform

* $I(\mathcal{W}) \cap \mathfrak{a}(\mathcal{W}) \equiv I_{\mathfrak{a}}(\mathcal{W})$

motions. We call $I_a(\mathbb{W})$ the group of inertial world automorphisms of $\mathbb{W}$. Thus, by definition, $\forall \alpha \in I(\mathbb{W})$, we must have $\beta_2(\alpha)(t)^{\cdot\cdot} = 0$ or $\beta_2(\alpha)(t) = \underline{u}(\alpha)\, t$ where $\underline{u} \in C^1(I(\mathbb{W}), \mathbb{R}^3_T)$. We shall show later that in the circumstances that $\beta_1(\alpha_1) = \mathbb{1}$ and $T(\alpha_2) = 0$ we must have $\underline{u}(\alpha_1 \circ \alpha_2) = \underline{u}(\alpha_1) + \underline{u}(\alpha_2)$. Such motions induced by $I(\mathbb{W})$ must always be inertial. In chapter (5) we shall discuss the group theory of non-inertial group motions in $\mathcal{Q}(\mathbb{W})$ and the above ideas in greater detail. Recall that we may choose a subgroup: $i: I'_a(\mathbb{W}) < I_a(\mathbb{W}) \ni$ if $\underline{u}' \equiv \underline{u} \circ i$ then $\underline{u}' \in \mathrm{Hom}(I(\mathbb{W}), \mathbb{R}^3_T)$. In this way $I(\mathbb{W})$ acts transitively on the set $I = \langle I(\underline{v}) \rangle\ \underline{v} \in \mathbb{R}^3_T$ of inertial frames via $\alpha: I(\underline{v}) \longmapsto I(\underline{v} + \underline{u}(\alpha))$. Note that we can choose $\underline{u}$ such that $\underline{u}: \alpha \longmapsto 0\ \forall \alpha \in I(\mathbb{W})$, we will discuss this point later.

The subgroup $I_a(\mathbb{W})$ is called the Galilei group. In chapter (3), we shall discuss it in great detail. However we note that each Galilei transformation can be expressed as a quadruplet $(\underline{a}, b, \underline{v}, R) \in \mathbb{R}^3 \times \mathbb{R}^1 \times \mathbb{R}^3_T \times O(3, \mathbb{R})$ which operates on an event $(\underline{x}, t) \in \mathbb{W}$ like $(a, b, \underline{v}, R): (\underline{x}, t) \longmapsto (R\,\underline{x} + \underline{v}\,t + a, t + b)$.

In his celebrated laws of motion, Newton laid the basis for the laws of motion in his world. He postulated that two particles would influence each other in some way such that this influence changed the state of each others motion in a reciprocal fashion. This influence he called a force, which could be propagated instantaneously between two events. His laws of motion were

(i) 'Every particle remains in a state of uniform motion unless acted upon by external forces'.

(ii)'The instantaneous acceleration produced by a force is directly proportional to the force'.

(iii) 'Given two particles interacting with each other, the force exerted on one particle by the other is equal and opposite to the force exerted by the second particle on the first'.

All these laws were experimental, the second law provided a means of defining a force via the acceleration it produced on test bodies. The constant of proportionality Newton called the 'intertial mass'. Of course, knowing the force, one can compute the trajectory of the particle in principle by using Newton's law to provide a second order differential equation for the trajectory. Then the trajectory and its associated world line provides a complete description of the past and future events encountered by the particle. One is led to seek a more general prescription than Newton's law of Motion to describe dynamical situations. Such a prescription was provided by Hamilton via his enunciation of his Principle of Least Action, a prescription which can be applied in any world model to yield the relevant dynamics. The principle will be discussed algebraically in chapter (4). Here we shall see how the relativity group of inertial world automorphisms of a world model completely describes the 'free' motions via Hamilton's principle. These calculations are made by combining Galileo's Principle of Relativity with Hamilton's Principle. The former principle is the one which is the backbone of all group-

-theoretical analyses in physics, perhaps the most important principle of physics. Its statement is very brief:- '<u>The laws of physics must be of the same form in all inertial frames</u>'. Einstein's Principle of Relativity includes Galileo's and is the basis of the Special Theory of Relativity which we shall soom discuss.

Combining Galileo's Principle with Hamilton's will lead us to see that such notions as kinetic energy, inertial mass etc. are group theoretical in origin.

Part (1). Einstein's Special Theory of Relativity.

Recall how in Newton's relativity, signals cones be propagated between simultaneous events. This implies that there is no finite upper limit to signal velocities. To see this more clearly[11)], combine Galileo's relativity principle with the velocity addition law of Newtonian relativity. Consider two particles in an inertial frame moving with velocity $\underline{V}$ relative to the present. If a signal is propagated between the two particles with velocity $\underline{U}$, then relative to the present it propagates between them with velocity $\underline{V} + \underline{U}$. Thus there can be no finite maximum signal velocity.

Einstein combined Galileo's Principle of Relativity with the additional postulate that there is a finite upper limit to signal velocities, the velocity of light 'C', a finite quantity. This combined principle is called 'Einstein's Principle of Relativity'. If the speed of light is to be maximal, it should be invariant under all inertial transformations. Consider the form Q defined on $\mathbb{W} \times \mathbb{W}$ $:- Q(x_1, x_2) \equiv (C^2(x_1^o - x_2^o)^2 - \|\underline{x}_1 - \underline{x}_2\|^2) \forall x_1, x_2 \in \mathbb{W}$. If $Q(x_1, x_2) \geqslant 0$ for the pair of events x_1 and x_2, then signals may propagate between them with constant speeds less or equal to that of light. This follows from:- $\|\underline{x}_1 - \underline{x}_2\|^2 / (x_1^o - x_2^o)^2 \leqslant C^2$ when $Q(x_1, x_2) \geqslant 0$. Thus, fixing an event $x_o \in \mathbb{W}$, the event can only communicate with events in $\mathbb{W}$ such that $x \in \{x \in \mathbb{W} \ni Q(x_o, x) \geqslant 0\}$. Also, the quantity Q must be invariant under all inertial automorphisms to accord with Einstein's principle of relativity.

These two new facts suggest the following formalisations of Einstein's world model, which we shall denote by $\mathbb{W}(E)'$. We must have $\mathbb{W}(E)' = (\mathbb{W}, \tau, \partial, Q)$, where $(\mathbb{W}, \tau, \partial)$ is Newton's world and $Q \in C^2(\mathbb{W}, \mathbb{R})$ was defined above. Given the group $B(\mathbb{W})$ of bijective functions from $\mathbb{W}$ onto its self, we define the subgroup $\mathfrak{a}(\mathbb{W}(E)) \leq B(\mathbb{W})$ of functions in $B(\mathbb{W})$ such that the pair $(\mathbb{W}, Q)$ is invariant via $Q \circ (\alpha \times \alpha) = Q \ \forall \alpha \in \mathfrak{a}(\mathbb{W}(E))$. Note that we do not require the time lapse or the instantaneous distance to be invariant. Let us discuss Q a little more.

Recall $Q(x_1, x_2) = ((x_1^o - x_2^o)^2 C^2 - \|\underline{x}_1 - \underline{x}_2\|^2) \forall x_1, x_2 \in \mathbb{W}$. If the events x_1 and x_2 are simultaneous, the Poincaré invariant $Q(x_1, x_2) = -\|\underline{x}_1 - \underline{x}_2\|^2 \leqslant 0$ is obviously related to their Newtonian distance apart. When two events are coincident, then the Poincaré invariant $Q(x_1', x_2')$ is related to their Newtonian time-lapse by

$Q(x_1, x_2) = C^2\tau(x_1, x_2)^2 \gg 0$. Note that simultaneous events are not necessarily simultaneous in all inertial frames connected by Poincaré transformations, although they are always the same distance apart and that the time-lapse between coincident events is always the same. For two coincident events the quantity $\sqrt{(Q(x_1, x_2)/c^2)}$ is called their proper time lapse $\tau_o(x_1, x_2)$. Given two events which can communicate then their proper time-lapse between them is $\sqrt{(Q(x_1, x_2)/C^2)} = \sqrt{(\tau(x_1, x_2)^2 - \|\underline{x}_1 - \underline{x}_2\|^2/C^2)}$, a Poincaré invariant. If the event x_1 is regarded as the origin of the present, then $\tau(x_1, x) = t(x_1)(x)$ and $\|\underline{x}_1-\underline{x}\|^2 = d(x_1)(x_2)$, then the proper time lapse between x_1 and x_2 is $(t_o(x_1))(x) = \sqrt{(t(x_1)(x)^2 - d(x_1)(x)^2/C^2)}$ or $t_o = \sqrt{(t^2 - \underline{x}^2/C^2)}$ for two events which can communicate. We must have $dto/dt = \sqrt{(1 - \dot{\underline{x}}^2/c^2)} = \gamma(\dot{x})^{-1}$. Thus if the two events lie on a world line with trajectory $\underline{x}$, the time lapse between them is $\int_{\underline{x}} (1-\dot{\underline{x}}^2/C^2)^{\frac{1}{2}}dt$. The quantity $Q(x_1, x_2)$ is called the interval between the events x_1 and x_2.

We will replace $\mathbb{W}(E)'$ with $\mathbb{W}(E) = (Q, \mathcal{W})$, $\mathfrak{a}(\mathbb{W}(E)) = (\mathfrak{a}(\mathbb{W}(E))')$. Let us examine $\mathbb{W}(E)$ a little more closely. In Newton's relativity, we were able to define the relations F, P and S and the corresponding subsets F(x), P(x) and S(x) for an event $x \in \mathcal{W}$; called the future, past and instant of x. The function Q enables us to define new relations on $\mathcal{W}$. Call $T = \{(x, y) \in \mathcal{W}^2 \ni Q(x, y) > 0\}$, $\mathcal{S} = \{(x, y) \in \mathcal{W}^2 \ni Q(x, y) < 0\}$ and $L = \{(x, y) \in \mathcal{W}^2 \ni Q(x, y) = 0\}$. The relations T, $\mathcal{S}$ and L are all symmetric, L is also reflexive

$\Delta \subset L$. It is sometimes useful to define the relations $\mathcal{E}' \equiv \mathcal{E} \cup \Delta$, $T' = T \cup \Delta$ and $L' = L - \Delta$. We must have $T \cap \mathcal{E} = \emptyset$, $T \cap L = \emptyset$ and $\mathcal{E} \cap L = \emptyset$ which means that $\mathcal{E}' \cap T' = \Delta$, $\mathcal{E}' \cap L = \Delta$, $T' \cap L = \Delta$. The relations $\mathcal{E}'$ and T' are called augmented and L' subtracted. Via the relations we have defined, the subsets $T'(x)$, $\mathcal{E}'(x)$ and $L(x)$ naturally arise. The set $T'(x)$ is called the interior of the time-tube of x, it is the subset of $\mathcal{W}$ with which the event x can communicate via signals with speed less than light. Also, the set $\mathcal{E}'(x)$ is the subset of $\mathcal{W}$ forever seperated from x, consisting of the events in $\mathcal{W}$ with which x cannot communicate. The set $\mathcal{E}(x)$ is called the 'elsewhere' or the interior of the space tube of x. The set $L(x)$ is called the light cone through x and is the boundary between the time tube and the space tube of x. We call the subsets $T(x) \cup L(x) \equiv \tilde{T}(x)$ and $\mathcal{E}(x) \cup L(x) \equiv \tilde{\mathcal{E}}(x)$ the time and space tubes through x. $\tilde{T}(x) \cap \tilde{\mathcal{E}}(x) = L(x)$ is their boundary. $\tilde{T}(x)$ is the set of all events in $\mathcal{W}$ with which x can communicate.

The relation F in $\mathcal{W}$ allows the definition of two extremely important relations in $\mathcal{W}$. Call $T' \cap F$ and $T' \cap P$, V_+ and V_- respectively. The subsets $V_+(x)$ and $V_-(x)$ are called the interiors of the past and future cones at x, where the future cone is $L(x) \cap F(x)$ and the past cone $L(x) \cap P(x) = L_+(x)$ and $L_-(x)$ respectively. We must have $L_+(x) \cup L_-(x) = L(x)$ and $L_+(x) \cap L_-(x) = \{x\}$ since $L(x) \cap (F(x) \cap P(x)) = L(x) \cap S(x) = \{x\}$ $(\because \| \underline{x}_1 - \underline{x}_2 \| = 0$

$\Leftrightarrow \underline{x}_1 = \underline{x}_2$). One readilly sees that $\mathcal{V}_- = \mathcal{V}_+^{-1}$ and that $\mathcal{V}_+ \cap \mathcal{V}_- = \Delta$. The latter follows from the fact that $\mathcal{V}_+(x) \cap \mathcal{V}_-(x) = T'(x) \cap (F(x) \cap P(x)) = T'(x) \cap S(x)$. But $S(x) \subseteq \mathcal{E}(x)\ \forall\ x \in \mathcal{W}$ since let $x_1 \in S(x)$ then $Q(x_1, x_2) = -\|\underline{x}_1 - \underline{x}\|^2 \leq 0$. Thus $T'(x) \cap S(x) \subset T'(x) \cap \mathcal{E}'(x) = \{x\}$, clearly $x \in T'(x) \cap S(x)$ so that $\mathcal{V}_+(x) \cap \mathcal{V}_-(x) = \{x\}$ or $\mathcal{V}_+ \cap \mathcal{V}_- = \Delta$. Thus $\mathcal{V}_+$ is 'antisymmetric'. But $\mathcal{V}_+$ is also transitive, since let $x_1 \in \mathcal{V}_+(x_2)$, $x_2 \in \mathcal{V}_+(x_3)$. Then $x_1^0 > x_2^0 > x_3^0$ which implies $x_1^0 > x_3^0$; and since $Q(x_1, x_2) \geq 0$, $Q(x_2, x_3) \geq 0$, and $= 0 \Leftrightarrow x_1 = x_2$ or $x_2 = x_3$; by Schwartz's inequality $Q(x_1, x_3) \geq 0$, $= 0$ iff $x_1 = x_3$. Thus $x_1 \in \mathcal{V}_+(x_3)$. So we see that $\mathcal{V}_+$ is an assymetric transitive relation which is a partial order on $\mathcal{W}$. The order is called the Zeeman[13)] causal order and we will discuss it in more detail later. It is interesting to see its connection with the total order A on the partition $\mathcal{S}$ of $\mathcal{W}$ into instants.

Let $\langle \mathcal{P}(t) \rangle$ $t \in \mathbb{R}$ be the partition $\mathcal{S}$ of $\mathcal{W}$ into disjoint instants relative to the present $\mathcal{P}(0)$. Consider an event $x \in \mathcal{P}(0)$, we can define its future tube $T'(x)$ and hence a partition $\underline{\mathcal{S}}$ of it into disjoint regions $\underline{\mathcal{P}}(x)(t) \equiv T'(x) \cap \mathcal{P}(t)\ \forall\ t \in \mathbb{R}$. The natural order ($>$) on $\mathbb{R}$ induces a partial order on $\underline{\mathcal{S}}$ given by $\mathcal{P}(x)(t_1) > \mathcal{P}(x)(t_2)$ iff $t_1 > t_2$, which is the partial order on $\mathcal{S}$ inherited from $\mathcal{V}_+$ in $\mathcal{W}$. Given that $\mathcal{W}$ is equivalent to $\mathbb{R}^4$ we define it as a real linear space in the natural way. It is interesting to note that the module structure of $\mathbb{R}^4$ is an additional structure to that which we have so far used. Define the natural isomorphism $x \longmapsto x^*$ on to the linear space $\mathcal{M}^*$ which arises, by $x_1^*(x_2) \equiv \sum_{\mu=0}^{3} x_\mu^1 x_\mu^2$. Now $\mathcal{M}$ is a left $M(4,\mathbb{R})$ module and its dual $\mathcal{M}^*$ a right $M(4,\mathbb{R})$ module via:- $(x_1^* \cdot M^T)(x_2) \equiv x_1^*(M \cdot x_2)\ \forall\ M \in M(4,\mathbb{R})$, where $M \longmapsto M^T$ is the involute anti-endomorphism 'transpose' of $M(4,\mathbb{R})$. Consider the matrix $G \in M(4,\mathbb{R})$ where $(G_{ij}) \equiv -\delta_{ij}\ \forall\ i,j \in \{1, 2, 3\}$ and $G_{io} \equiv G_{io} \equiv 0$ with $G_{oo} \equiv 1$. Then $G = G^T = G^{-1}$. Let us consider next the bilinear functional $g: (x_1, x_2) \longmapsto (x_1^* \cdot G)(x_2) = x_1^*(G\, x_2)\ \forall\ x_1, x_2 \in \mathcal{W}$. One can define an 'inner produce space' $\mathcal{M}(\mathcal{W})$ via $\mathcal{M}(\mathcal{W}) = (\mathcal{M}, g)$ where g is regarded as an inner product. The map $x \overset{\wedge}{\longmapsto} x^* \cdot G$ is regarded as the canonical isomorphism of $\mathcal{M}(\mathcal{W})$ onto its 'dual' $\mathcal{M}(\mathcal{W})^\wedge$ where then $x_1^\wedge(x_2) \equiv x_1^*(G \cdot x_2) \equiv x_1^* \cdot G(x_2) \equiv g(x_1, x_2) = (x_1^o x_2^o - \sum_{i=1}^{3} x_1^i x_2^i$. Then the functional $Q(x_1, x_2) = g(x_1 - x_2, x_1 - x_2)$ is the derived pseudo norm of $\mathcal{M}(\mathcal{W})$ which will enable us to write $Q(x_1, x_2) = \| x_1 - x_2 \|^2.\ \forall\ x_1, x_2 \in M(\mathcal{W})$; Q is a pseudo-metric on $\mathcal{M}(\mathcal{W})$. However $M(\mathcal{W})$ is not to be regarded as a linear topological space with the 'pseudo-metric' topology. Zeeman[14)] has introduced a topology in $\mathcal{W}$ which reduces as it must, to the topology on $\mathbb{R}$ or $\mathbb{R}^3$ when restricted to time or space axes. The neighbourhoods are defined in terms of the E(4) open balls by removing the deleted light cone:- I.e.:- if $B(\delta)$ is an E(4) open ball then the set $(B(\delta) - L'(x)) \equiv B'(\delta)$ is defined as an open ball centered on x when $B(\delta)$ is centered on x. This topology on $M(\delta)$ which is finer than that implicit in E(3) is called by Zeeman the 'fine topology'. We shall not elaborate these topological notions further, except to note that Zeeman's fine topology is the finest which reduces to the usual topologies on time or space axes, i.e.:- preserves the Newtonian notions of 'distance' on these axes.

Clearly, if we regard $\mathcal{M}(\mathbb{W})$ which is called Minkowski space as a metric space, the subgroup of linear 'isometries' of $\mathcal{M}(\mathbb{W})$ is a subgroup of $\mathcal{A}(\mathbb{W})$. We shall call this group the Poincaré group. It is immediate that it must have the structure $\mathbb{R}^4 \boxtimes_n O(1, 3; \mathbb{R})$. Here $\mathbb{R}^4$ is the underlying Abelian group of $\mathcal{M}(\mathbb{W})$ which operates on $\mathcal{M}(\mathbb{W})$ via the regular representation. $O(1, 3; \mathbb{R})$ is the subgroup of pseudo-orthogonal matrices in $M(4, \mathbb{R})$ such that $\Lambda \in O(1, 3; \mathbb{R})$ iff $\Lambda^{-1T} = G \cdot \Lambda \cdot G^{-1}$. It is well known as the Lorentz group. The semi-direct product of two groups was discussed in chapter one, we shall use the notation $P(\mathbb{R}) \equiv \mathbb{R}^4 \boxtimes_n O(1, 3; \mathbb{R})$. Its action on $\mathbb{W}$ is defined via the monomorphism:- $p: P(\mathbb{R}) < \mathcal{A}(\mathbb{W})$, $p(x', \Lambda) \equiv (p \circ i_1(x')) \circ (p \circ i_2)(\Lambda) = p_1(x') \circ p_2(\Lambda): x \longmapsto \Lambda x + x'$ $\forall (x', \Lambda) \in P(\mathbb{R})$; with the monomorphisms $i_1: \mathbb{R}^4 \lhd P(\mathbb{R})$, $i_2: L(\mathbb{R}) < P(\mathbb{R})$; (where we introduce the notation $L(\mathbb{R})$ for the Lorentz group). $P(\mathbb{R})$ is then the group of world automorphisms of $\mathbb{W}(E)$ with its linear space structure.

Now the Poincaré transformation (a, Λ) transforms the event $(\underline{x}, x^0)$ into the event $(x_i', x'^0) = (\Lambda_{ij} x_j + \Lambda_{io} x_o + a_i, \Lambda_{oi} x_i + \Lambda_{oo} x_o + a_o)$ so that we clearly see that if two events x and y are simultaneous, the events x' and y' have a time lapse $\Lambda_{oi}(x_i - y_i)$ and are no longer simultaneous. So whilst this concept was invariant in Newtonian relativity, it is no-longer true in Einstein's world. The inertial nature of the Poincaré transformations is also immediate. If $d^2\underline{x}/dt^2 = 0$ for an event $(\underline{x}, t)$ then we must have $d^2x'/dt^2 = 0$ for the event (x', t'), since $d\underline{x}'/dt' = (d\underline{x}'/dt)(dt/dt') = (d\underline{x}'/dt)/(dt'/dt)$. Now $(dt'/dt) = d/dt(\Lambda_{oi}x_i + \Lambda_{oo}t_o) = (\Lambda_{oi}\dot{x}_i + \Lambda_{oo})$ also $(dx'/dt) = (\Lambda_{ij}\dot{x}_j + \Lambda_{io})\,x$. Thus $dx'/dt' = (\Lambda_{ij}\dot{x}_j + \Lambda_{io}) / (\Lambda_{oi}\dot{x}_i + \Lambda_{oo})$. So $d^2x'/dt^2 = d/dt\,(dx'/dt') / (\Lambda_{oi}\dot{x}2 + \Lambda_{oo}) = ((\Lambda_{ij}\ddot{x}_i)/(\Lambda_{oi}\dot{x}j + \Lambda_{oo})^2 - {}_{oi}\ddot{x}_i/(\Lambda_{oj}\dot{x}_j + \Lambda_{oo})^2) = 0$.

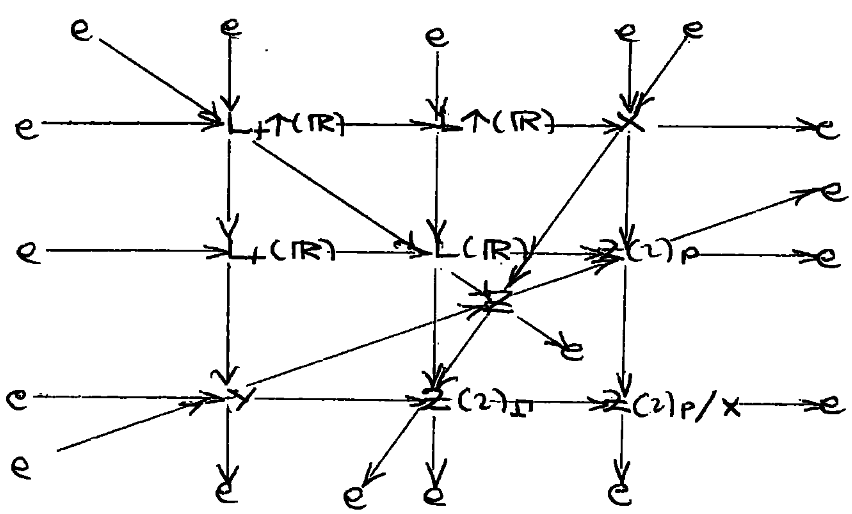
The replacement by invariant proper time of time leads to simpler calculation. Given the world line $x_p(M)$ of a particle p under a motion M, one calls $d(x_p(M)(t))/dt_o$ the 4 velocity, $d^2(x_p(M)(t))/dt_o^2$ the 4 acceleration etc. The relationship between these Minkowskian notions and the Newtonian ones is immediate from $dx(t)/dt_o = (dx(t)/dt)(dt/dt_o) = \gamma(\dot{\underline{x}})\,dx(t)/dt$.

Let us consider now some of the subgroup structure of the Poincaré and Lorentz groups. Recall that the Lorentz group $L(\mathbb{R}) = O(1, 3;\mathbb{R})$ was the subgroup of matrices of $GL(4,\mathbb{R})$ such that $\Lambda^{-1T} = G.\Lambda.G^{-1} = In(G)(\Lambda)$. The function $det: GL(4,\mathbb{R}) \to \mathbb{R}_m$ is an epimorphism of $GL(4,\mathbb{R})$ onto the multiplicative group $\mathbb{R}_m$. Its kernel is written $SL(4,\mathbb{R}) \triangleleft GL(4,\mathbb{R})$. Let us write $L(\mathbb{R}) \cap SL(4,\mathbb{R}) = L_+(\mathbb{R})$ or $SO(1, 3;\mathbb{R})$. If $det' \equiv det\ o\ i$ where $i: L(\mathbb{R}) < GL(4,\mathbb{R})$, then we must have $\mathrm{Ker}(\det') = L_+(\mathbb{R})$ and $\mathrm{Im}(\det') = Z(2)$ since from $\Lambda^{-1T} = G.\Lambda.G^{-1}$, $\det(\Lambda)^{-1} = \det(\Lambda)$ or $(\det(\Lambda))^2 = +1 \; \forall \Lambda \in L(\mathbb{R})$. Define an injective function in $C^1(Z(2), L(\mathbb{R}))$ via $\det' \circ d = \mathbb{1}$ and $d : (-1) \mapsto G$ and $d : (+1) \mapsto e$. Then d is a monomorphism since $G^2 = e$. Whence, via the definitions of chapter (1), $L(\mathbb{R})$ must have the semi-direct product structure $L_+(\mathbb{R}) \boxtimes_K Z(2)_p$, where $\mathrm{Im}(d) = Z(2)_p$, where $p = G$ is called the parity operator or space reflection. The homomorphism $K \in \mathrm{Hom}(Z(2)_p, \mathrm{Aut}(L_+(\mathbb{R})))$ is defined by $K(p) : \Lambda \mapsto \Lambda^{-1T} = \mathrm{Im}(G)(\Lambda) \; \forall \Lambda \in L_+(\mathbb{R})$. Note the set structure $L(\mathbb{R}) = L_+(\mathbb{R}) \mathbin{\dot{\cup}} L_-(\mathbb{R})$ where $L_-(\mathbb{R}) \equiv P \cdot L_+(\mathbb{R})$.

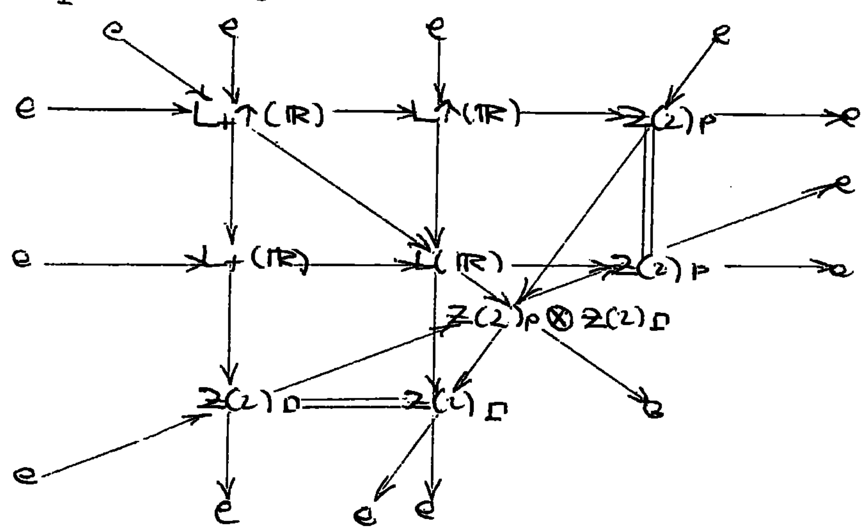
Let us now define a function $\alpha \in C^1(L(\mathbb{R}), \mathbb{R})$ via $\alpha : \Lambda \mapsto \Lambda_{00} \; \forall \Lambda \in L(\mathbb{R})$. The identity $\Lambda^T.G.\Lambda = G$ or $\Lambda^T_{0\alpha} g_{\alpha\beta} \Lambda_{\beta 0} = g_{00} = 1$, enables us to make the following construction. Define a map $\lambda \in C^1(L(\mathbb{R}), \mathbb{R}^4)$ via $\lambda(\Lambda)_\mu = \Lambda_{\mu 0} \; \forall \; \Lambda \in L(\mathbb{R})$, then we must have $\|\lambda(\Lambda)\|^2 = 1 \; \forall \Lambda$ or $\Lambda_{00}^{\;2} = (1 + \|\lambda(\Lambda)\|^2) \geq 1$. We surmise that α is surjective from $L(\mathbb{R})$ onto the subset $(\geq +1) \cup (\leq -1)$ of $\mathbb{R}$. If we define a section $\beta \in C^1(\mathbb{R}, L(\mathbb{R}))$, $\alpha \circ \beta = \mathbb{1}$, by $\beta(+1) \equiv e$ and $\beta(-1) \equiv -e$ and $\beta[\geq +1] \equiv L\!\uparrow\!(\mathbb{R})$ and $\beta[\leq -1] \equiv L\!\downarrow\!(\mathbb{R})$ then via $i : Z(2) \subset (\geq +1) \cup (\leq -1)$, $\beta \circ i$ is a monomorphism from $Z(2)$ into $L(\mathbb{R})$ such that as a function $\alpha' \circ (\beta \circ i) = \mathbb{1}$, the function α' being $\alpha'(\Lambda) \equiv \Lambda_{00}/|\Lambda_{00}| = \alpha(\Lambda)/|\alpha(\Lambda)|$. One can show that $\alpha'(\Lambda_1\Lambda_2) = \alpha'(\Lambda_1)\alpha'(\Lambda)$ which means that since α' is trivially surjective in $C^1(L(\mathbb{R}), Z(2))$, it is also an epimorphism, and $\mathrm{Ker}(\alpha') = L\!\uparrow\!(\mathbb{R})$. That $\alpha' \circ (\beta \circ i) = \mathbb{1}$ implies that

$L(\mathbb{R})$ is a direct product $L^{\uparrow}(\mathbb{R}) \otimes Z(2)_{\Gamma}$ where $\mathrm{Im}(\beta \circ i) = Z(2)_{\Gamma}$, $\Gamma = -e$. The central character of the extension arises from the fact that $Z(2)_{\Gamma}$ is isomorphic to the central $Z(2)$ of $L(\mathbb{R})$. The operator Γ is called the space-time reflection operator. As sets $L(\mathbb{R}) = L^{\uparrow}(\mathbb{R}) \dot\cup L^{\downarrow}(\mathbb{R})$.

We have thus obtained the two isomorphic structures $L_+(\mathbb{R}) \boxtimes_k Z(2)p$ and $L^{\uparrow}(\mathbb{R}) \otimes Z(2)_{\Gamma}$ of $L(\mathbb{R})$. The two distinct invariant subgroups $L^{\uparrow}(\mathbb{R})$ and $L_+(\mathbb{R}) \triangleleft L(\mathbb{R})$ are usually called the orthochronous or the 'proper' Lorentz groups. Given the four subsets $L_+(\mathbb{R})$, $L_-(\mathbb{R})$, $L^{\uparrow}(\mathbb{R})$, $L^{\downarrow}(\mathbb{R}) \subset L(\mathbb{R})$, there arise the subsets obtained by intersection:- $L_+^{\uparrow}(\mathbb{R}) \equiv L_+(\mathbb{R}) \cap L^{\uparrow}(\mathbb{R})$, $L_+^{\downarrow}(\mathbb{R}) \equiv L_+(\mathbb{R}) \cap L^{\downarrow}(\mathbb{R})$ etc. $L_-^{\uparrow}(\mathbb{R})$ and $L_-^{\downarrow}(\mathbb{R})$. All these subsets are mutually disjoint and define a partition of $L(\mathbb{R})$ into the four subsets. Recall that $L_+(\mathbb{R}), L^{\uparrow}(\mathbb{R}) \triangleleft L(\mathbb{R}) \Rightarrow L_+^{\uparrow}(\mathbb{R}) = L_+(\mathbb{R}) \cap L^{\uparrow}(\mathbb{R}) \triangleleft L(\mathbb{R})$. One can draw the commutative diagram:-



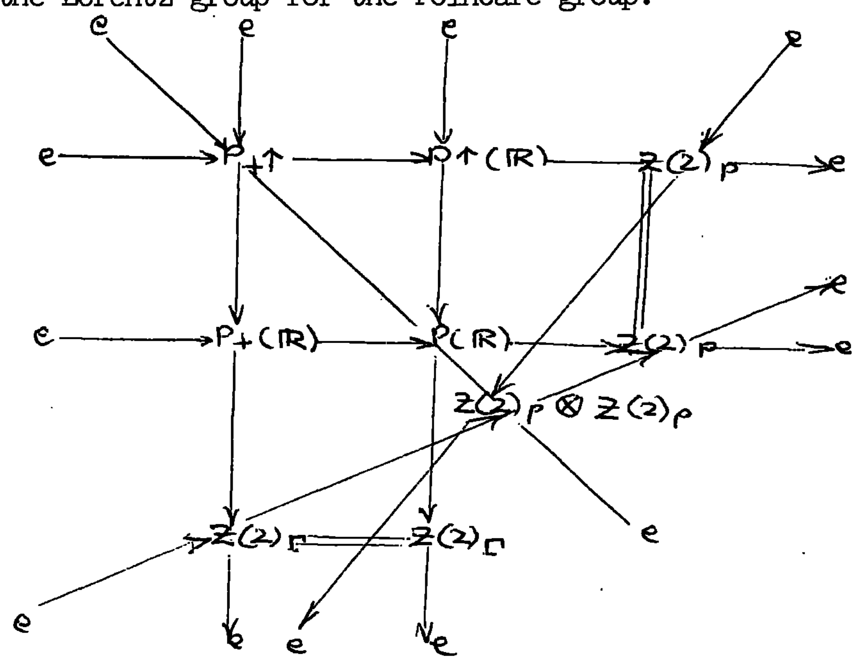
Here the groups X, Y and Z are defined by $X = L_+(\mathbb{R})/L_+^\uparrow(\mathbb{R})$; $Y = L_+(\mathbb{R})/L_+^\uparrow(\mathbb{R})$ and $Z = L(\mathbb{R})/L_+^\uparrow(\mathbb{R})$. Clearly $L^\uparrow(\mathbb{R})/L_+^\uparrow(\mathbb{R}) = Z(2)_P$, $L_+(\mathbb{R})/L_+^\uparrow(\mathbb{R}) = Z(2)_\Gamma$. Then, we must have $Z(2)_P/X = e$. Now Z is an extension of $Z(2)_\Gamma$ by $Z(2)_P$ from our diagram, as well as an extension of $Z(2)_P$ by $Z(2)_\Gamma$ and hence must be $Z(2)_P \otimes Z(2)_\Gamma$. The diagram reduces to:-



which conveniently summarises the relations between the groups $L(\mathbb{R})$, $L^\uparrow(\mathbb{R})$, $L_+(\mathbb{R})$, $L_+^\uparrow(\mathbb{R})$, $Z(2)_P$ and $Z(2)_\Gamma$. Let us consider the subgroup $Z(2)_P \otimes Z(2)_\Gamma < L(\mathbb{R})$. Its underlying set is $\{e, \Gamma, P$ and $P\Gamma\}$. It is more convenient to consider instead the group $Z(2)_P \otimes Z(2)_T < L(\mathbb{R})$ whose underlying set is $\{e, P, T, PT\}$. The operator $T = P\Gamma \equiv -G$ is called the time reversal operator, $PT = \Gamma$. Using T instead of Γ introduces the complication that T doesn't operate trivially on $L^\uparrow(\mathbb{R})$ as does Γ. We still have $Z(2)_T < L(\mathbb{R})$ with $L(\mathbb{R})/L^\uparrow(\mathbb{R}) \doteq Z(2)_T$, that is $L(\mathbb{R}) = L^\uparrow(\mathbb{R}) \boxtimes_{K'}$

$Z(2)_{\mathbb{T}}$, where $K' \in \mathrm{Hom}(Z(2)_{\mathbb{T}}, \mathrm{Aut}(L^{\uparrow}(\mathbb{R})))$ is defined by $K'(\mathbb{T}): \Lambda \longmapsto \Lambda^{-1T} \; \forall \Lambda \in L^{\uparrow}(\mathbb{R})$. Thus we have $L^{\uparrow}(\mathbb{R}) \otimes Z(2)_{\mathbb{T}} \rightleftharpoons L^{\uparrow}(\mathbb{R}) \rtimes_{k'} Z(2)_{\mathbb{T}}$, we shall explain how this is possible in chapter (3).

Combining our definitions of $P(\mathbb{R})$ with the above decompositions for $L(\mathbb{R})$ leads to a diagram similar to the one we obtained from the Lorentz group for the Poincaré group.



The groups $P_+^{\uparrow}(\mathbb{R})$, $P^{\uparrow}(\mathbb{R})$ and $P_+(\mathbb{R})$ are defined by the diagram.

The action of the operators in $Z(2)_P \otimes Z(2)_{\mathbb{T}}$ on Minkowski space are defined as follows:- $P: (\underline{x}, x^0) \longmapsto (-\underline{x}, x^0)$, $\mathbb{T}: (\underline{x}, x^0) \longmapsto (\underline{x}_1 - x^0)$ and $P\mathbb{T} \equiv \Gamma : (\underline{x}, x^0) \longmapsto -(\underline{x}, x^0) \; \forall \; (\underline{x}, x^0) \in \mathbb{R}^4$. From these definitions, it is clear that:- $\mathbb{T}(V_+(x)) = V_-(x)$ $\forall \; x \in \mathbb{W}$. Thus, if $\mathbb{T}$ is regarded as a valid inertial transformation, the Principle of Causality will be violated by each Lorentz transformation in $L^{\downarrow}(\mathbb{R}) \subseteq L(\mathbb{R})$. Clearly, each element of the orthochronous

Poincaré group $P^{\uparrow}(\mathbb{R}) = \mathbb{R}^4 \boxtimes_n L^{\uparrow}(\mathbb{R})$ will preserve causality. It is of great interest to compute the group of automorphisms of $\mathbb{W}(E)$ which preserve the causal order V_+. Let $C^{\uparrow}(\mathbb{R}) < B(\mathbb{W})$ be the subgroup of causal automorphisms of $\mathbb{W}$, note that causal automorphisms are not necessarily linear. Evidently $P^{\uparrow}(\mathbb{R}) < C^{\uparrow}(\mathbb{R})$ Consider also the fact by M in an $\mathbb{R}$ module, $\mathbb{R}_m$ acting on $\mathcal{M}$ as a group of automorphisms, via $\alpha \in Hom(\mathbb{R}_m, Aut(\mathbb{R}^{+})) \ni \alpha(\lambda): x \mapsto \lambda x \ \forall \lambda \in \mathbb{R}_m,\ x \in \mathbb{R}^4$. We see that the subgroup $\mathbb{R}^+_m < \mathbb{R}_m$ of positive reals preserves the causal order also, so that $\mathbb{R}^+_m < C^{\uparrow}(\mathbb{R})$ Moreover as subgroups of $GL(4,\mathbb{R})$, $\mathbb{R}_m \cong \mathcal{C}(GL(4)\mathbb{R}))$, so that $\alpha \cdot \Lambda = \Lambda \cdot \alpha \ \forall (\alpha, \Lambda) \in \mathbb{R} \times L(\mathbb{R})$. We can easily show that $\mathbb{R}^+_m < \mathcal{N}(C^{\uparrow}(\mathbb{R}))(P^{\uparrow}(\mathbb{R}))$ the normaliser of $P^{\uparrow}(\mathbb{R})$ in $C^{\uparrow}(\mathbb{R})$ with $In(\lambda)(x,\Lambda) \mapsto (\lambda x, \Lambda) \ \forall \lambda \in \mathbb{R}^+_m,\ (x,\Lambda) \in P^{\uparrow}(\mathbb{R})$, viz $\exists \ g \in Hom(\mathbb{R}^+_m, Aut(P^{\uparrow}(\mathbb{R}))) \ni g(\lambda):(x,\Lambda) \mapsto (\lambda x, \Lambda)$. Now $P^{\uparrow}(\mathbb{R}) \cap \mathbb{R}^+_m = \{e\}$ so that we can define a subgroup $D^{\uparrow}(\mathbb{R}) = \mathbb{R}^+_m \cdot P^{\uparrow}(\mathbb{R}) \cong P^{\uparrow}(\mathbb{R}) \boxtimes_g \mathbb{R}^+_m < C^{\uparrow}(\mathbb{R})$. An automorphism $x \mapsto \lambda x$ is called a 'dilatation'. E.C. Zeeman has ~~shown that~~ proved the result:- $D^{\uparrow}(\mathbb{R}) = C^{\uparrow}(\mathbb{R})$, all causal automorphisms of M are thus linear! L. Michel has shown that $D^{\uparrow}(\mathbb{R}) \cong Aut(P^{\uparrow}(\mathbb{R}))$. It would be extremely interesting to prove directly that $C^{\uparrow}(\mathbb{R}) = Aut(P^{\uparrow}(\mathbb{R}))$. Michel has also shown that $Aut(P^{\uparrow}(\mathbb{R})) = Aut(P(\mathbb{R}))$ and thus that all algebraic automorphisms of $P(\mathbb{R})$ are continuous in the topology of $P(\mathbb{R})$ which is the product of the <u>Euclidean one</u> on $\mathbb{R}^4$ and the locally homeomorphic to E(6) topology on the Lie-group $L(\mathbb{R})$.

It does not follow that all automorphisms of $P^{\uparrow}(\mathbb{R})$ are continuous when $\mathbb{R}^4$ is endowed with Zeeman's fine topology and hence $P(\mathbb{R})$ with the appropriate product topology.

By our definition of $C^{\uparrow}(\mathbb{R})$, as the group of world automorphisms which preserves the causal order V_+, a similarity between the Galilei group which preserves the temporal order on the partition $\mathcal{S}$ of $\mathcal{W}$, emerges. But physics is only required to be Poincaré invariant, the dilations introduced above also act on velocities and hence the dilatations will violate Einstein's principle of relativity:- that C is maximal if included as valid inertial transformations. In chapter (4), we shall see that if the group $C^{\uparrow}(\mathbb{R})$ is regarded as a valid relativity group no motions at all can occur. To conclude our discussion of Special Relativity for the moment let us write down the velocity addition rule for pure inertial boosts along an axis. If a Lorentz transformation along a direction parallel to a space axis is parametrised by a velocity $\underline{V}$ then we have

$\Lambda(\underline{V}_1) \circ \Lambda(\underline{V}_2) = \Lambda((\underline{V}_1 + \underline{V}_2)/(1 + \frac{V_1 \cdot V_2}{C^2}))$, clearly $\Lambda(\underline{C}) \circ \Lambda(\underline{V}) = \Lambda(C) = \Lambda(\underline{V}) \circ \Lambda(\underline{C})$. Each Lorentz transformation in $L_+^{\uparrow}(\mathbb{R})$ can be parameterised by a rotation R in $SO(3,\mathbb{R})$ and a velocity $\underline{V}$. The action of $\Lambda(R, \underline{V})$ on an event $(\underline{x}, x^0)$ is given by:- $\Lambda(R, \underline{V})$:

$$(\underline{x}, t) \longmapsto (R\underline{x} + \gamma^2/(\gamma+1)(\underline{\beta}\cdot R\underline{x})\_ + \gamma\underline{\beta}\,\underline{x}^0,\ \gamma(x^0 + \underline{\beta}\cdot R\ \underline{x}))$$

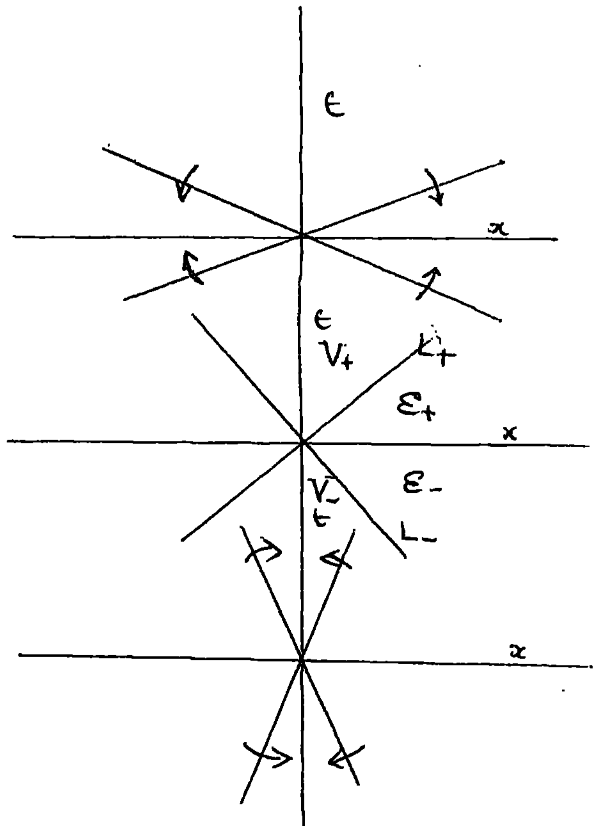
where $\gamma \equiv \gamma(\underline{V})$ and $\underline{\beta} \equiv \underline{V}/C$.

In part (3) we shall discuss two relativity models derivable from the Newtonian and Einstein's Special Relativity World.

Part (3). Other Related Relativity Models.

Both relativity models discussed here are due to J.M. Lévy-Leblond, the first[15)] we shall discuss was introduced by him individually and the second in a joint paper with H. Bacry.[16)]. Lévy-Leblond was lead to the first relativity model through his investigations of the so called 'non-relativistic' limits of the Poincaré group. The criterion for the deformation of the Einstein Special Relativity World into the Newtonian World is that 'the first deforms into the latter as one allows the speed of light to approach infinity' is quoted by most authors (including H. Minkowski). However, as Lévy-Leblond observed, this criterion is not quite correct. He illustrates this with an example. Consider a two-dimensional Minkowski space and two events separated by the vector $(\Delta x, \Delta t)$. Then under a pure Lorentz boost along the space axis, the separation becomes:- $(\gamma(u)(\Delta x + u \Delta t), \gamma(u)(\Delta t + u \Delta x))$, where u is the boost to a frame moving with velocity u with respect to the first, and $C \equiv 1$. The Galilean approximation is then taken as $u \ll 1$ and writing:- $(\Delta x, \Delta t) \overset{u}{\longmapsto} (\Delta x + u \Delta t, \Delta t)$. However, one can easily see that the validity of the latter approximation is ensured iff $u \ll 1$ <u>and</u> $\Delta x/\Delta t \ll 1$ when $u \Delta x \ll \Delta t$ without $u \Delta t \ll \Delta x$. Thus iff $\Delta x / \Delta t \ll 1$, $u \ll 1$ then $\Delta x'/\Delta t' \ll 1$ when $(\Delta x, \Delta t) \overset{u}{\longmapsto} (\Delta x', \Delta t')$. Thus the Galilean approximation is valid when considering large time-like intervals with small velocities:- $\| (\Delta x, \Delta t) \|^2 = (\Delta t^2 - \Delta x^2) \gg 0$

This situation immediately raises the interesting question of the limit $u \ll 1$ and $\Delta x/\Delta t \gg 1$ with small velocities and large space-like intervals. In the fashion of the above approximation we obtain $u\,\Delta t \ll \Delta x$, but $u\,\Delta x \not\ll \Delta t$, corresponding to $(\Delta x, \Delta t) \overset{u}{\longmapsto} (\Delta x, \Delta t + u\,\Delta x)$! Diagrammatically the situations are summarised as follows:-



(1)

(2)

(3)

The Galilean approximation is summarised by (1) $\mapsto$ (2) when the light cone falls back on the space axis corresponding to $V_+(x) \longrightarrow F(x)$ and $V_-(x) \longrightarrow P(x)$, The new approximation corresponds to (1) $\longmapsto$ (3) when the light cone is deformed onto the time axis corresponding to $\mathcal{E}_+(x) \longrightarrow F(x)$ and $\mathcal{E}_-(x) \longrightarrow P(x)$, where $\mathcal{E}_+(x) \equiv \mathcal{E}(x) \cap F(x)$, $\mathcal{E}_-(x) \equiv \mathcal{E}(x) \cap P(x)$.

Recall the action of the Lorentz transformation given at the end of part (2). Under writing $t = x/C$; $\underline{V} = \beta C$ and making a similar definition $t' = a_0/C$ for a translation a, the former approximation defined by Lévy-Leblond leads to:- $(\underline{x}, t^0) \longmapsto (R\,\underline{x} + \underline{v}\,t + \underline{a}', t + t')$ a Galilei transformation:- $(\underline{x}', t', \underline{v}, R)$. If we write $t \equiv C\,x\,o$, $\underline{V} \equiv C\beta$ and $t' = C\,a_0$ and choose $C \to \infty$, we obtain the transformation $(\underline{x}, t) \longmapsto (R\,\underline{x} + a', t + \underline{v} \cdot R\,\underline{x} + t')$. Such transformations form a group (which we shall discuss futher in chapter (3)), which Lévy-Leblond called the Carroll group after Charles Dodgeson the author and mathematician whose pen-name Lewis Carroll labels him as the author of Alice in Wonderland. The name was chosen for the correspondence between the lact of causality in a world where the Carroll group is the relativity group and the lack of causality in the adventures of Alice in Wonderland! The name is thus very apt! The world model corresponding to the Carroll group must be of the form $(\mathbb{W}, \tau, \partial)$ where ∂ is the family of Euclidean metrics and one does not require $\tau \circ (\alpha \times \alpha) = \tau$ for α to be a world automorphism.

In another direction, recall how in our preliminary discussions of the Galilei group we introduced the function $\beta_2 \in C^1(\mathcal{A}(\mathbb{W}), C^1(\mathbb{R}^1, \mathbb{R}^3$ required $\beta_2(\alpha)(t) = \underline{U}(\alpha)t \;\forall\, \alpha \in \mathcal{A}(\mathbb{W}),\ t \in \mathbb{R}^1$. This gave rise to the notion of velocity boosts, where $\underline{U} \in \mathrm{Hom}(\mathcal{A}(\mathbb{W}), \mathbb{R}^3_T)$. We stated that we are perfectly free to choose $\underline{U} \equiv 0$. In this case, each automorphism of $\mathbb{W}$ (N) takes the form:- $(\underline{x}, t) \longmapsto (\beta_1(\alpha)(\underline{x}),$

$t + t(\alpha))$ where $\beta_1(\alpha)$ is a linear isometry of three dimensional Euclidian space and $t(\alpha)$ is a time translation. Under such inertial world automorphisms, which form a group, (called by Lévy-Leblond the Static group via a different unrelated context), world automorphisms cannot permute inertial frames, hence the name of the group. The Static group is thus the world group of Newtonian world where the velocity of a moving particle appears the same from all inertial frames moving relative to it. Thus, if a body is static in one frame, it must appear static in all frames!

We shall not pursue the physical interpretations of the Carroll and Static groups as they obviously describe worlds very widely separated from reality. The groups will be discussed in Chapter (4) however for their large algebraic and slight physical interest. Let us close this chapter by noting some characteristics of the Galilei and Carroll groups. Firstly we note that under pure boosts $\underline{v}:(\underline{x}, t) \longmapsto (\underline{x} + \underline{v}\, t, t)$ and $\underline{v}:(\underline{x}, t) \longmapsto (x, t + \underline{v}\cdot\underline{x})$ respectively. Under these conditions one says that space is 'absolute', or in the second case time is absolute in that no 'mixing' of space with time or time with space occurs in the respective cases. We shall pursue this point later when the event $(\underline{x}, t)$ is replaced by a spatio-temporal translation.

CHAPTER (3)

COHOMOLOGY THEORY OF GROUPS AND GROUP EXTENSIONS.

In this chapter, we provide a survey of the cohomology theory of abstract groups as formulated by S. Eilenberg and S. Maclane. The theory makes use of the algebraic methods developed in algebraic topology when dealing with homology and cohomology properties of topological spaces. The subject matter falls under the general heading of homological algebra. The algebraic method is exceptionally powerful when discussing group extensions in particular, and has numerous other group theoretical applications.

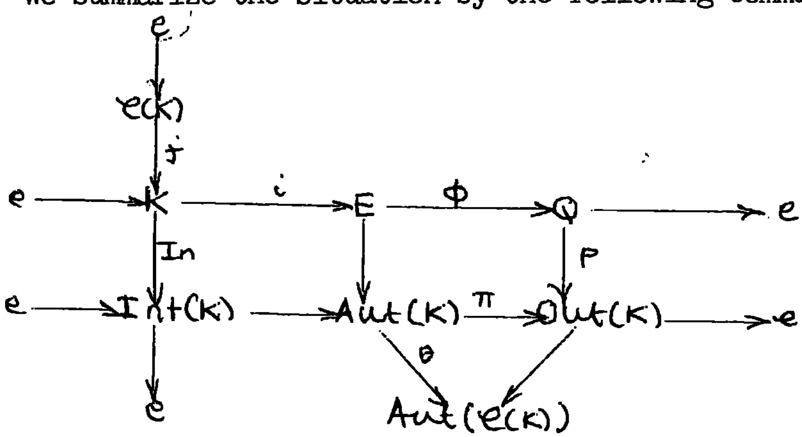
We may formulate the problem of group extensions as follows. Given two groups K and Q, find all groups E such that:-(i) $K \lhd E$ and (ii) $E/K \cong Q$. This problem occurs time and again when one applies group theory to theoretical physics. For instance, when we computed the group of world automorphisms of Minkowski space:- $\mathfrak{a}(\mathcal{W}) = P(\mathbb{R})$, we were able to compute the structure of $P(\mathbb{R})$ knowing that $L(\mathbb{R}) < P(\mathbb{R})$, $\mathbb{R}^4 \lhd P(\mathbb{R})$ and $P(\mathbb{R})/\mathbb{R}^4 \cong L(\mathbb{R})$. We wrote $P(\mathbb{R}) \cong \mathbb{R}^4 \boxtimes_n L(\mathbb{R})$ which embodies the latter properties. Again, when computing $C\uparrow(\mathbb{R})$ we used the property that $\mathbb{R}^+_m < C\uparrow(\mathbb{R})$, $P\uparrow(\mathbb{R}) \lhd C\uparrow(\mathbb{R})$ and $C\uparrow(\mathbb{R})/P\uparrow(\mathbb{R}) \cong \mathbb{R}^+_m$; whence we wrote $C\uparrow(\mathbb{R}) = P\uparrow(\mathbb{R}) \boxtimes_g \mathbb{R}^+_m$ where the homomorphism $g \in \mathrm{Hom}(\mathbb{R}^+_m, \mathrm{Aut}(P\uparrow))$ specified $\mathbb{R}^+_m$ as a group of automorphisms of $P\uparrow(\mathbb{R})$. Extensions of the above type, semi-direct products, are called trivial since $Q < E$ specifies Q as a group of automorphisms of K in an unambiguous manner. The situation is not, in general, so easy.

Let us return to the general problem, given K, Q find all E such that $K \lhd E$, $E/K \cong Q$. We can re-write this as meaning that $\exists$ $i \in Mon(K,E)$, $\phi \in Ep(E, Q)$ $\ni$ $\phi \circ i = 0$ i.e.:- $Im(i) = Ker(\phi)$, the equality sign arises from the isomorphism of $E/i(K)$ with Q. This means we can draw the exact sequence

$$e \longrightarrow K \longrightarrow E \longrightarrow Q \longrightarrow e$$

Conversely given a pair (E,ϕ) $\phi \in Ep(E, Q)$ we say that E is an extension of Q by K iff $K \cong Ker(\phi)$. Let us take i as the identity monomorphism $i: K \lhd E$. Now since $K \lhd E$, $\exists$ an $f \in Mon(E, Aut(K))$ defined in the natural way:- $f(g)(k) = In(g)(k)$ $\forall$ $(g, k) \in E \times K$.

We summarize the situation by the following commutative diagram:-



Here π is the canonical map of $Aut(K)$ onto $Out(K)$. Consider now two elements $g_1, g_2 \in E$ $\ni$ $g_2^{-1} g_1 \in K$, $g_1 = g_2 k$, $k \in K$, then, $f(g_1) = f(g_2) \circ In(k)$. That is iff $g_1 \approx g_2$ mod(k) then $f(g_1) \approx f(g_2)$ mod $Int(K)$, so there arises a canonical map $p: E/K \longrightarrow Aut(K) / Int(K)$ i.e.:- $p: Q \longrightarrow Out(K)$. Thus $Ker(\phi)$, $\phi \in Ep(E, Q)$ involves two items i.e.:- $K \lhd E$, $K \cong Ker(\phi)$, and the way in which Q operates on K as a group of outer automorphisms, specified by $p \in Hom(Q, Out(K))$. We say that the pair (E,ϕ) is a group extension of Q

by the 'kernel' (K, p). Conversely, given a pair (K, p) where $p \in Hom(Q, Out(K))$ we call the pair (K, p) a 'Q kernel'. Let G be the centre $\mathcal{C}(K)$ of K. Since $\mathcal{C}(K) \triangleleft K$, there is a canonical mapping θ of Aut(K) into Aut(G) where if $j: G \triangleleft K$, $\theta(f) \equiv f \circ j \; \forall \; f \in Aut(K)$. Also, by definition $In(k)(g) = g \; \forall \; (k, g) \in K \times G$ thus $Ker(\theta) = Int(K)$. Whence we can naturally define a homomorphism from Out(K) onto Aut(G) and whence from Q into Aut(G) via $p' \in Mon(Q, Aut(G))$ $p' q \mapsto (\theta \circ p)(q) \; \forall \; q \in Q$. Thus Q operates on G as a group of automorphisms, and on K only as a group of outer automorphisms. An extension (E, ϕ) of Q by the Q kernel (K, p) is called <u>central</u> iff $p = 0$ in Hom(Q, Out(K). Recall the situation for a semi-direct product when Q operates on K as a group of automorphisms and simplified matters rather. We see a similar situation occurs when K is Abelian, $Out(K) = Aut(K)$, $Int(K) = 0$. By definition Aut(K) is an extension of Out(K) by Int(K), Out(K) being not necessarily homomorphic to Aut(K), we cannot always specify Q as a group of automorphisms of K in an unambiguous manner.

In this chapter, we will proceed in three stages. The first stage will consist of a general discussion of cohomology theory, and group extensions when the kernel K is Abelian. The second stage will then involve us in a discussion of the general case when K is not necessarily Abelian, whilst the third stage will be concerned with the discussion of the theory of G enlargements invented by Eilenberg. Having introduced cohomology theory in this chapter, we will be free to introduce cohomological theorems and definitions as the need arises.

Before embarking on a study of the special case when K is Abelian, we introduce a few concepts which are basic to the rest of the chapter.

(i) Multiplication of Kernels.

We define here the notion of the 'G product of two Q kernels relative to a notion of equivalence. Two Q kernels $(K_1,p_1),(K_2,p_2)$ are 'G equivalent' written $((K_1, p_1),(K_2,p_2)) \in (\triangleq)$ iff $\exists\ \sigma: K_1 \rightleftharpoons K_2 \ni \sigma(g) = g \ \forall\ g \in G$ and $\forall\ \alpha(q) \in p_1(q) \subset Aut(K_1)$ $\sigma \circ \alpha(q) \circ \sigma^{-1} \in p_2(q) \subset Aut(K_2)\ \forall\ q \in Q$. With this notion of equivalence the classes $\mathcal{Q}$ of Q kernels with center (G, p_0) is a monoid under the G product where $\mathcal{Q} \equiv \mathcal{Q}/(\triangleq)$, $\mathcal{Q}$ being the set of Q kernels with centre G.

Recall that by definition, $Int(K \otimes K) \rightleftharpoons Int(K_1) \otimes Int(K_2)$ which means $\mathcal{C}(K_1 \otimes K_2) \rightleftharpoons G \otimes G$. The set $\nabla(G \times G) \equiv \{(g_1, g_2) \in G \times G \mid g_2 = g_1^{-1}\}$ is a subgroup $\nabla(G \otimes G)$ of $G \otimes G$ moreover, it is the kernel of the epimorphism $h: G \otimes G \rightarrow G$; $h: (g_1, g_2) \mapsto g_1 g_2 \ \forall\ (g_1, g_2) \in G$. Define a group K_3 as a homomorph[illegible] of $K_1 \otimes K_2$ via $H: K_1 \otimes K_2 \longrightarrow K_3$; $H: (k_1,k_2) \longmapsto k_1 k_2$ $\forall\ (k_1,k_2) \in K_1 \otimes K_2$. Then $Ker(H) = \nabla(G \otimes G)$ and $K_1 \otimes K_2/\nabla(G \otimes G) \rightleftharpoons K_3$. Moreover $H|G \otimes G \equiv h$ is onto $\mathcal{C}(K_3)$, thus $\mathcal{C}(K_3) \rightleftharpoons G \otimes G / \nabla(G \otimes G) \rightleftharpoons G$, where $\mathcal{C}(K_3) \rightleftharpoons G$. The group K_3 with center G can be endowed with the structure of a Q kernel by defining $p_3 \in Hom((Q, Out(K_3))$ as follows. Let $(\alpha_1,\alpha_2) \in p_1(q) \times p_2(q) \subset Aut(K_1) \times$

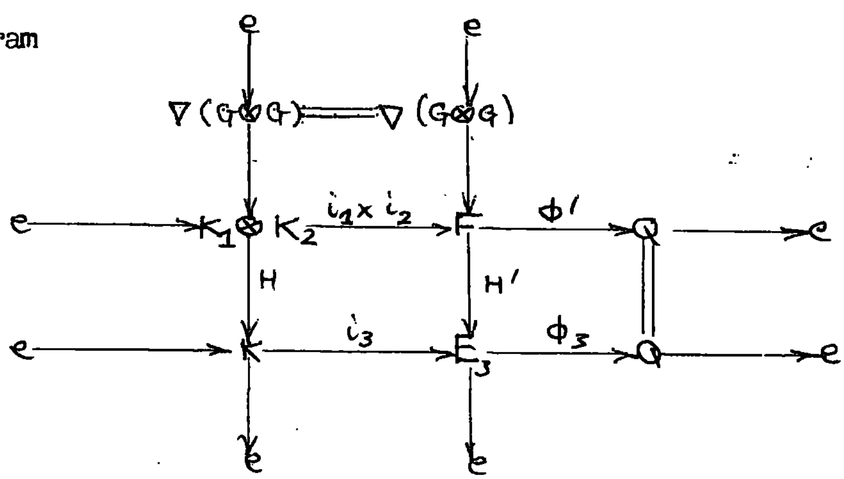
$Aut(K_2)$. Then $\alpha_1 \times \alpha_2 : (k_1, k_2) \longmapsto (\alpha_1(k_2), \alpha_2(k_2))$ is an automorphism of $K_1 \otimes K_2 \; \forall \; q \in Q$ moreover, $\alpha_1 \times \alpha_2$ only depends on $q \in Q$. Also $\alpha_1 \times \alpha_2 : \nabla(G \otimes G) = \nabla(G \otimes G) \; \forall \; q \in Q$, since $\alpha_1 \times \alpha_2 : (g, g^{-1}) \longmapsto (\alpha_1(g), \alpha_2(g)^{-1}) = (p_o(q)\, g, p_o(q)(g)^{-1} \; \forall \; g \in G$. So that β defined by $\beta \circ H = H \circ (\alpha_1 \times \alpha_2)$ is an automorphism of $K_3 \; \forall \; q \in Q$, only depending on $q \in Q$ up to an inner automorphism of K_3. Thus there is a homomorphism $p_3 \in Hom(Q, Out(K_3))$, the pair (K_3, p_3) is a Q kernel with centre G and is called the 'G product' of the Q kernels (K_1, p_1) and (K_2, p_2), written $(K_3, p_3) = (K_1, p_1) \wedge (K_2, p_2)$. The centre (G, p_o) satisfies

$(K, p) \wedge (G, p_o) \cong (K, p) \; \forall \; (K, p) \in \mathcal{Q}$, and acts as the identity for the monoid $\mathcal{Q} = \mathcal{Q} / (\cong)$.

(ii) Multiplication of Extensions.

Let (E_1, ϕ_1) and (E_2, ϕ_2) be extensions of Q by Q kernels (K_1, p_1) and (K_2, p_2). Consider the group $E_1 \otimes E_2$, there is a ntural epimorphism $\phi_1 \times \phi_2 : E_1 \otimes E_2 \longrightarrow Q \times Q$; $\phi_1 \times \phi_2 : (e_1, e_2) \longmapsto (\phi_1(e_1), \phi_2(e_2)) \; \forall \; (e_1, e_2) \in E_1 \otimes E_2$. The diagonal subset $\Delta(Q \otimes Q)$ of $Q \otimes Q$ is a subgroup of $Q \otimes Q$ isomorphic to Q under $(q, q) \longmapsto q$. Let $F < E_1 \otimes E_2$ be the group $(\phi_1 \times \phi_2)^{-1}(\Delta(Q \otimes Q))$. Then F is a natural epimorph of Q under $\phi' \in Ep(F, Q)$, $\phi' : (e_1, e_2) \longmapsto \phi_1(e_1) = \phi_2(e_2) \; \forall \; (e_1, e_2) \in F$. Evidently, $Ker(\phi') = K_1 \otimes K_2$ since if $(e_1, e_2) \in Ker(\phi')$, $\phi_1(e_1) = \phi_2(e_2) = \phi'(e_1, e_2) = e$ which means $(e_1, e_2) \in Ker(\phi_1) \times Ker(\phi_2)$, whence $K_1 \otimes K_2 > Ker(\phi')$. Moreover if $(e_1, e_2) \in K_1 \otimes K_2$, $(e_1, e_2) \in Ker(\phi')$ whence $Ker(\phi') = K_1 \otimes K_2$.

Thus F is an extension of Q by $K_1 \otimes K_2$. Define a homomorph E_3 of F under H': $F \longrightarrow E_3$, $H':(e_1, e_2) \mapsto e_1 e_2$. Evidently Ker(H) = $\nabla(G \otimes G)$. One now defines a homomorphism $\phi_3 : E_3 \longrightarrow Q$ by $\phi_3 \circ H = \phi'$. By a classical lemma $\mathrm{Ker}(\phi_3) \rightleftharpoons \mathrm{Ker}(\phi')/\mathrm{Ker}(H)$, that is $\mathrm{Ker}(\phi') \rightleftharpoons K_1 \otimes K_2/\nabla(G \otimes G) \rightleftharpoons K_3$. Thus we see that E_3 is an extension of Q by K_3. We write $(E_3,\phi_3) = (E_1,\phi_1) \wedge (E_2,\phi_2)$ and call (E_3,ϕ_3) the extension product of the extensions (E_1,ϕ_1) and (E_2,ϕ_2). The constructions used above may be summarised by the diagram



We now proceed with the study of Abeliam extensions, embedding the theory in the cohomology theory of groups.

<u>Cohomology Theory of Groups</u> (1) Abelian Kernels

Consider a sequence of pairs $\mathcal{C} = \langle (C^n, \delta^n) \rangle$ $n \in Z$ where $\forall n \in Z$, C^n is an additive Abelian group and $\delta^n \in \mathrm{Hom}(C^n, C^{n+1})$. Using $Z = Z^+ \cup Z^-$ we write $C = C^+ \cup C^-$ where $C^+ = \langle (C^n, \delta^n) \rangle$ $n \in Z^+$ and $C^- = \langle (C^n, \delta^n) \rangle$ $n \in Z^-$. With $Z^+ \cap Z^- = \{0\}$, we write $(C^0, \delta^0) = (0, 0)$. Now $C^- = \langle (C^{-n}, \delta^{-n}) \rangle$ $n \in Z^+$, so we relabel the sequence as $C^- \equiv \langle(C_n, \partial_n)\rangle$ $n \in Z^+$ where $(C_n, \partial_n) \equiv (C^{-n}, \delta^{-n}) \forall$ $n \in Z^+$. Here $\partial n \in \mathrm{Hom}(C_n, C_{n-1}) \forall n \in Z^+$. One calls the sequence $\langle(C^n, \delta^n)\rangle$ $n \in Z^+$ an upper sequence and $\langle(C_n, \partial_n)\rangle$ $n \in Z^+$ a lower sequence.

Now if $\delta^n \circ \delta^{n-1} = 0$ $\forall n \in Z^+$ one calls $\langle(C^n, \delta^n)\rangle$ $n \in Z^+$ a semi-exact sequence similarly, for $\langle(C_n, \partial_n)\rangle$ $n \in Z^+$ such a lower sequence is semi-exact iff $\partial_{n-1} \circ \partial_n = 0 \forall n \in Z^+$. The conditions are just that in the former case $\mathrm{Im}(\delta^{n-1}) < \mathrm{Ker}(\delta^n)$ and in the latter $\mathrm{Im}(\partial_n) < \mathrm{Ker}(\partial_{n-1})$. Let $\langle(Cn, \partial n)\rangle n \in Z_+$ be a lower semi-exact sequence, one calls C_n the group of n-dimensional chains of the complex C; ∂n is an n-dimensional boundary operator; $\mathrm{Im}(\partial_{n+1}) \equiv B_n < C_n$ is called the group of n - dimensional boundaries of C and $\mathrm{Ker}(\partial n) \equiv Z_n < C_n$ is the group of n-dimensional cycles of the complex C. That $\partial_n \circ \partial_{n+1} = 0 \forall n \in Z_+$ means that $B_n \lhd Z_n \lhd C_n \forall n \in Z_+$. The group $H_n \equiv Z_n/B_n$ is called the 'n-dimensional homology group of the complex C'. If $f_1, f_2 \in Z_n$ and $f_1 - f_2 \in B_n$ one says that the two n-cycles f_1 and f_2 are 'homologous'.

Consider now the case of an upper semi-exact sequence C = $\langle(C^n, \delta^n)\rangle$ $n \in Z^+$. Here, we call C^n the group of n-dimensional cochains of the complex, $\mathrm{Im}(\delta^{n-1}) \equiv B^n < C^n$ is the group of n-dimensional coboundaries of the complex; δ^{n-1} is an n-dimensional coboundary operator and $Z^n \equiv \mathrm{Ker}(\delta^n) < C^n$ is the group of n-dimensional co-cycles of the complex. Finally $Z^n/B^n \equiv H^n$ is the n-dimensional cohomology group of the complex C. Two n-cocycles $f_1, f_2 \in Z^n$ are 'cohomologous' iff

$f_1 - f_2 \in B^n$.

This completes our preliminary definitions, note that every exact-sequence is semi-exact but not necessarily vice-versa.

Consider now two groups Q and K where K is an Abelian group and Q is an operator group on K via $p \in \mathrm{Hom}(Q, \mathrm{Aut}(K))$. Let $X_{i=1}^{n} \{ Q \}$ denote the n-fold Cartesian product set of isomorphic images of Q Call $C_p^n(Q, K)$ the set of all functions: $f: X_{i=1}^{n} Q \longrightarrow K$ $C_p^n(Q, K)$ is an Abelian group $\forall n \in Z^+$. We identify $C_p^0(Q, K)$ as K itself, the set of constant functions from Q into K. Define now a homomorphism $\delta^n \in \mathrm{Hom}(C_p^n(Q, K), C_p^{n+1}(Q, K)) \forall n \in Z_+$ by

$$\delta^n(f)(q_1,\ldots,q_{n+1}) = (p(q_1) \circ f)(q_2,\ldots,q_{n+1})$$

$$+ \sum_{i=1}^{n}(-1)^i f(q_1,\ldots,q_i q_{i+1},\ldots q_{n+1}) + (-)^{n+1} f(q_1,\ldots q_n)$$

One can show(1), by rather tedious algebra, that the complex:- $C_p(Q,K) \equiv \langle C_p^n(Q,K), \delta^n \rangle \; n \in Z^+$, is a semi-exact sequence, $\delta^n \circ \delta^{n-1} = 0$ $\forall n \in Z^+$. Also, one can always choose normalised cochains, where a normalised cochain satisfied $f(q, q_i, q_n) = 0$ if $q_i = e$ for some $1 \leq i \leq n$.(1) Recall that $C_p^0(Q, K) \equiv K$ and $\delta^{-1} \equiv 0$. The important point to note is that $C_p(Q, K)$ depends on $p \in \mathrm{Hom}(Q, \mathrm{Aut}(K))$. This latter set is never empty, it always contains at least the trivial homomorphism $T \ni \mathrm{Ker}(T) = Q$. Let us adopt the convention that we drop indices on the coboundary operators $\forall \delta^n, n \in Z^+$ and attempt to interpret the groups of cochains.

<u>(0)</u>. We consider here the group of zero dimensional cochains. $C_p^0(Q,K)$

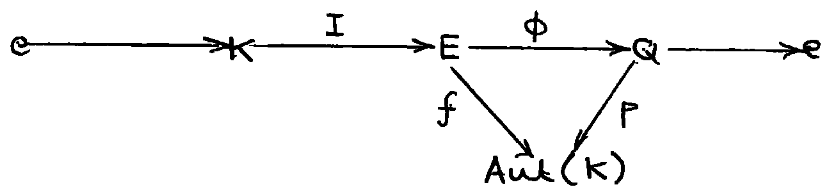
These are the constant functions of Q into K, identified with K. Let $k \in C^0_p(Q, K)$ then $\delta(k) \in B^1_p(Q, K)$ is a 1 co-boundary, $\delta(k)(q) = p(q)(k) - k$. Thus $k \in Z^0_p(Q, K)$ iff $p(q,k) = k \; \forall q \in Q$ Whence $Z^0_p(Q, K)$ is the set of $k \in K$ on which Q operates simply. By definition $\delta^{-1} \equiv 0$ whence $Z^0_p(Q, K) = H^0_p(Q, K)$, and then $B^0_p(Q, K) = 0$.

(1). A one cochain $f \in C^1_p(Q, K)$ is a function $f: Q \longrightarrow K$. $f \in B^1_p(Q, K)$ iff $\exists \; k \in K \ni f(q) = p(q)(k) - k$, and $f \in Z^1_p(Q, K)$ if $\delta(f) = 0$ or:- $\delta(f)(q_1, q_2) = p(q_1)(f(q_2)) + f(q_1) = 0$, where $\delta(f) \in B^2_p(Q, K)$. The 1 cocycles of the complex $C_p(Q, K)$ are called crossed homomorphisms, the 1 co-boundaries principal homomorphisms, thus the group $H^1_p(Q, K)$ is the group of crossed homomorphisms modulus the principal homomorphisms. If $p = T$ one sees that a principle homomorphism vanishes automatically whilst $Z^1_T(Q, K) = H^1_T(Q,K) = \mathrm{Hom}(Q, K)$. We will need the properties of 1 cocycles in the sequel at least twice.

(2). Next, we consider the two cochains of $C^2_p(Q, K)$. These are functions $f: Q \times Q \longrightarrow K$. A two cochain is a two coboundary and a-priori a two cocycle iff $\exists \; f' \in C^1_p(Q, K) \ni f(q_1, q_2) = \delta(f')(q_1, q_2) = p(q_1)(f'(q_2)) + f'(q_1)$. f is a two cocycle iff $\delta(f)(q_1, q_2, q_3) = 0$ i.e.:- $p(q_1)(f(q_2, q_3)) - f(q_1 \cdot q_2, q_3) + f(q_1, q_2, q_3) - f(q_1, q_2) = 0$. The second cohomology group is interpreted via the theory of group extensions of Q by K when the natural action of Q on K as a group of automorphisms induced by the extension coincides with p.

That is, in the Q kernel of the extension (E, ϕ):- (K, p'), where

$p' \in Hom(Q, Out(K)) = Hom(Q, Aut\ K)$ (since $Int(K) = 0$), $p'(q) \equiv p(q)$ $\forall q \in Q$. Conversely we can construct group extensions (E,ϕ) of Q by any Q kernel (K, p) $p \in Hom(Q, Aut(K))$ when K is Abelian. Recall (E,ϕ) is a group extension of Q by the Abelian Q kernel (K, p) when the following diagram is commutative:-



Here I is the identity monomorphism $I: K \triangleleft E$. Define a section $j: Q \longrightarrow E$ such that $\phi \circ j = \mathbb{1}$. Now since ϕ is a homomorphism $j(q_1)\cdot j(q_2)$ and $j(q_1,q_2)$ are in the same E/K coset, since $\phi(j(q_1) j(q_2)) = \phi(j(q_1))\phi(j(q_2)) = \phi(j(q_1\ q_2)) = q_1\ q_2 \forall\ (q_1,q_2) \in Q \times Q$. Thus $f: Q \times Q \longrightarrow K$ defined by $f: (q_1, q_2) \longmapsto j(q_1)\ j(q_2) (j(q_1,q_2))^{-1}$ enables us to use the two cochain f to write $j(q_1) j(q_2) = f(q_1,q_2)\ j(q_1,q_2) \forall\ (q_1,q_2) \in Q$. Since ϕ is a homomorphism, $\phi(j(e)) = e \Rightarrow j(e) = e$. On E, composition must be associative which means that $j(q_1)\cdot(j(q_2)j(q_3)) = (j(q_1)j(q_2))j(q_3) \forall\ q_1,q_2,q_3 \in Q$. This is just that:-

$$j(q_1)f(q_2,q_3)j(q_2 \cdot q_3) = f(q_1,q_2)j(q_1 \cdot q_2)j(q_3)$$

or

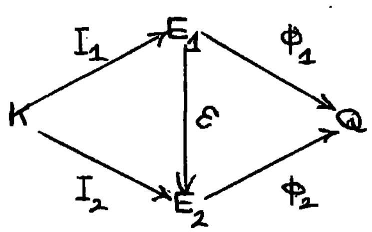
$$j(q_1f(q_2,q_3)j(q_1)^{-1}\ j(q_1)j(q_2 \cdot q_3) = f(q_1, q_2)\ f(q_1 \cdot q_2,q_3)j(q_1 \cdot q_2 \cdot q_3).$$

By definition $In(j(q_1))\ (f(q_2,q_3)) = p(q_1)(f(q_2,q_3))$ so we have:-

$p(q_1)(f(q_2,q_3))f(q_1,q_2 . q_3)\ j(q_1 . q_2 . q_3) =$
$f(q_1 . q_2,q_3)f(q_1,q_2)j(q_1,q_2,q_3)$. Which, passing to the additive notation, just means that $p(q_1)(f(q_2,q_3))-f(q_1 . q_2,q_3)+f(q_1,q_2 . q_3) - f(q_1,q_2) = 0$ i.e.:- $\delta(f)(q_1,q_2,q_3) = 0$. Thus $f \in Z^2_p(Q; K)$. Let us choose another section $j_1: Q \longrightarrow E$ such that $\phi \circ j_1 = 1$. One can then define a 1 cochain β via $\beta(q) \equiv j_1(q)j(q)^{-1} \forall q \in Q$, $\beta \in C^1_p(Q,K)$ since we have $\phi(\beta(q)) = \phi(j_1(q))\phi(j(q))^{-1} = qq^{-1} = e$. Write $j_1(q) = \beta(q)j(q)$ then we must have $j_1(q_1)j_1(q_2) = \beta(q_1)j(q_1)\beta(q_2)$ $j(q_2)$ or $j_1(q_1)j_1(q_2) = \beta(q_1)j(q_1)\beta(q_2)j(q_1)^{-1})j(q_1)j(q_2)$ which is $j_1(q_1)j_2(q_2) = \beta(q_1)p(q_1)(\beta(q_2))f(q_1,q_2)\ j(q_1,q_2)$ or $j_1(q_1)j_2(q_2)$ $= \beta(q_1)p(q_1)(\beta(q_2))f(q_1,q_2)\beta(q_1,q_2)^{-1}j_1(q_1,q_2)$. This means that $j_1(q_1)j_1(q_2) = f(q_1,q_2)\delta(\beta)(q_1,q_2)j_1(q_1,q_2) = f'(q_1,q_2)j_1(q_1,q_2)$. Where, passing to the additive notation we write $f'(q_1,q_2) = f(q_1,q_2) + \delta(\beta)(q_1,q_2)$, or $f' = f + \delta(\beta)$.

Choose a section $j: Q \longrightarrow E$ and, $\forall k \in K$, $q \in Q$ write $k \cdot j(q) \equiv (k, q)$. The composition of these pairs is then $(k_1,q_1)(k_2,q_2) = (k_1 + p(q_1)(k_2) + f(q_1,q_2), q_1 q_2)$. The cocycle $f \in Z^2_p(Q, K)$ specifies (E,ϕ) up to an isomorphism since if $f' - f \in B^2_p(Q, K)$ then this cocycle f' determines an isomorphic group. That is if $k \cdot j_1(q) = [k, q]$ then $[k_1, q_2][k_2, q_2] = [k_1+p(q_1)(k_2)+f'(q_1,q_2),q_1 . q_2]$ the isomorphism is $[k, q] \longmapsto (k, q)$ where $[k, q] = (k + \beta(q),q)$

$B^2_p(Q, K)$. Given $f \in Z^2_p(Q, K)$ we define the extension corresponding to f as $K \boxtimes_f Q$. If f' is cohomologous to f, then $K \boxtimes_{f'} Q \cong K \boxtimes_f Q$. We call two extensions of Q by the Q kernel (K, p) equivalent iff the following diagram is commutative:-



If the diagram is commutative we write $(E_1, E_2) \in (\sim)$, which is an equivalence relation. One can show (using the five-lemma) that $(\sim) \subseteq (\cong)$. Given two cohomologous cocycles $f_1, f_2 \in Z^2{}_p(Q,K)$ then $(K \boxtimes_{f_1} Q, K \boxtimes_{f_2} Q) \in (\sim)$, clearly this is true iff $f_1 - f_2 \in B^2{}_p(Q, K)$. Let $\mathcal{E}xt(Q, K)p$ be the set of all extensions of Q by K characterised by $p \in Hom(Q, Aut(K))$. Then the sets $\mathcal{E}xt(Q, K)\,p$ and $Z^2{}_p(Q, K)$ are isomorphic. Recall that $\mathcal{E}xt\ p(Q, K)$ was a semi-group under the multiplication of extensions defined in (ii). From our earlier definitions we surmise that (j_1, j_2) are sections from Q to (E_1, E_2) such that $\Phi_1 \circ j_1 = \Phi_2 \circ j_2 = \mathbb{1}$, then the product of the extensions (E_1, Φ_1) and (E_2, Φ_2) viz $(E_3, \Phi_3) = (E_1, \Phi_1) \wedge (E_2, \Phi_2)$ is isomorphic to the group whose underlying set is $\{(k_1 + k_2, (j_1(q), j_2(q)) \mid k_1, k_2 \in K, q \in Q\}$, where $\Phi_3((k_1+k_2, (j_1(q), j_2(q)) \longmapsto \Phi_1(j_1(q)) = \Phi_2(j_2(q)) = q$. The composition in E_3 is defined by $(0, (j_1(q), j_2(q)))\cdot(0, (j_1(q'), j_2(q')) \longmapsto (0, (j_1(q)j_1(q'), j_2(q)j_2(q')) = (f_1(q,q') + f_2(q, q'), (j_1(q\,q') . j_2(q\,q')))$. Thus we have $(E_1, \Phi_1) \wedge (E_2, \Phi_2) = (K \boxtimes f_1 Q) \wedge (K \boxtimes f_2 Q) =$

$K \boxtimes_{f_1 * f_2} Q$. If $[(E_i, \Phi_i)] = [K \boxtimes_{f_i} Q]$ is the equivalence class of (E_i, Φ_i) we have $[K \boxtimes_f Q]$ depends only on the cohomology class of f in $Z^2_p(Q,K)$ and that $[(E_1,\Phi_1) \wedge (E_2,\Phi_2)]$ depends only on the equivalence class of $f_1 + f_2$ which is the product of equivalance classes. This the group $Ext(Q, K)p = \mathcal{E}xt(Q, K)p/(\sim)$ is isomorphic to $H^2_p(Q, K)$.

The identity of $H^2_o(Q, K) :- B^2_p(Q, K)$ corresponds to the set of extensions of Q by K in which Q is a subgroup, each extension of this class being equivalent, and hence isomorphic to the trivial extension $K \boxtimes_o Q = K \boxtimes_p Q$, $p \in Hom(Q, Aut(K))$. Let j_1 and j_2 be monomorphisms from Q into $K \boxtimes_p Q$ such that $\phi \circ j_1 = \phi \circ j_2 = \mathbb{1}$. Then $j_1(q)\, j_2(q)^{-1} \in K$ or $j\ (q) = \gamma(q) j_2(q)$ where $\gamma \in C^1_p(Q, K)$. Since j_1 and j_2 are monomorphisms we must have $j_1(q)j_1(q') =$ $\gamma(q)j_2(q) \cdot \gamma(q')j_2(q') = \gamma(q)(j_2(q)\gamma(q')j_2(q)^{-1})j_2(q)j_2(q') =$ $\gamma(q)\, p(q)(\gamma(q'))\, j_2(q\, q') = j_1(q\, q') = \gamma(q\, q')j_2(q\, q')$. Thus we must have $\gamma(q\, q') = \gamma(q) + p(q)(\gamma(q'))$ or $\delta(\gamma)(q,q') = 0$, $\gamma \in Z^1_o(Q, K)$. The group $H^1_p(Q, K)$ corresponds then to the different ways of placing Q onto $K \boxtimes_p Q$. $B^1_p(Q, K) \cong K$ is just the ways in which Q is injected into E in conjugate manners. Z^1/B^1 different ways up to a conjugation:-

Consider $j : Q \longrightarrow E$. Then $j'(q) \equiv (In(k^{-1}) \circ j')(q)$ $\forall q \in Q$, $k \in C^0_p(Q, K)$; $\equiv k^{-1} j(q)\, k = k^{-1}\, q \cdot k\, j(q) = \delta(k)(q) j(q)$. Thus $\dot{j} = j + \delta(k)$.

When the extension is central with K Abelian, we must have p = 0 in the Q kernel (K, P) since Out(K) = Aut(K). The group of inequivalent central extensions is just $H_0^2(Q, K)$, the extension of Q by K labelled by the two cocycle $f \in Z_0^2(Q, K)$ is denoted by $K \otimes_f Q$. The elements of $B_0^2(Q, K)$ correspond to extensions equivalent to the direct product $K \otimes Q \equiv K \otimes_0 Q$. In this case, as in a similar way above, the group $Z_0^1(Q, K) = H_0^1(Q, K) = \mathrm{Hom}(Q, K)$ corresponding to the possible ways of embedding Q as a subgroup of $K \otimes Q$.

We now return to the general discussion of group extensions having introduced the cohomological apparatus in the preceding discussion, and briefly discuss the group theoretic interpretation of the group $H_0^3(Q, K)$. The group theoretic interpretations of the cohomology groups for $n > 3$. Eilenberg has conjectured that their application might come in the theory of 'loops' or not ~~not~~ necessarily associative groups, the higher cocycles providing the measure of the degree of associativity of the loop. Recall how the 2 cocycles in the above construction arose from the requirement of associativity. We shall use Eilenberg's theory of loops and prolongations of groups in Chapter (5).

Extensions with non-Abelian Kernels.

Recall that when we defined the nature of an extension (E, Φ) of Q by K a homomorphism $\rho \in \mathrm{Hom}(Q, \mathrm{Out}(K))$ arose in a natural way. When K is non-Abelian this fact causes a major complication in one's efforts to acertain the natures of such extensions, since, Out(K) only operates on K modules an inner automorphism, in order to specify (E,ϕ) as a group we have to specify the action of Q on K unambiguously.

Consider a Q kernel (K, p). In each automorphism class p(q), $q \in Q$, select an automorphism $\alpha(q)$ of Q. If π is the canonical mapping of Aut(K) onto Out(K) then $p \in \mathrm{Hom}(Q, \mathrm{Out}(K))$ determines a map α from Q into Aut(K) via $\alpha(q) \equiv (\beta o\, p)(q)$ where β is a section of Out(K) in Aut(K). Given $q_1, q_2 \in Q$, we have $\alpha(q_1)\alpha(q_2)$ o $(\alpha(q_1\, q_2)^{-1}) \in \mathrm{Int}(K)$ since $\pi(\alpha(q_1) o\, \alpha(q_2)(\alpha(q_1\, q_2)^{-1})) = \pi(\alpha(q_1))\pi(\alpha(q_2))(\pi(\alpha(q_1, q_2))^{-1}) = p(q_1)\cdot p(q_2).\ p(q_1,q_2)^{-1} = \mathbb{1}$. Thus we can define a function $\gamma : Q \times Q \longrightarrow K$ via $\alpha(q_1)$ o $\alpha(q_2) = \mathrm{In}(\gamma(q_1, q_2))\alpha(q_1,q_2) \forall q_1,q_2 \in Q$. Using the associativity requirement on Aut(K) we find that $\alpha o(q_1) o(\alpha(q_2) o\, \alpha(q_3)) \equiv (\alpha(q_1)$ o $\alpha(q_2))$ o $\alpha(q_3) \implies \alpha(q_1)$ o $\mathrm{In}(\gamma(q_2, q_3))$ o $\alpha(q_1\, q_2) = \mathrm{In}(\gamma(q_1, q_2))$ o $\alpha(q_1 \cdot q_2)$ o $\alpha(q_3)$. Or $\alpha(q_1)$ o $\mathrm{In}(\gamma(q_2,q_3))$ o $\alpha(q_1)^{-1}$ o $(\alpha(q_1)$ o $\alpha(q_2 \cdot q_3)) = \mathrm{In}(\gamma(q_1,q_2))$ o $\mathrm{In}(\gamma(q_1 \cdot q_2,q_3))$ o $\alpha(q_1 \cdot q_2 \cdot q_3)$. Which is $\mathrm{In}(\alpha(q_1)(\gamma(q_2,q_3))$ o $\mathrm{In}(\gamma(q_1, q_2 \cdot q_3))$ o $\alpha(q_1 \cdot q_2 \cdot q_3) = \mathrm{In}(\gamma(q_1,q_2))$ o $\mathrm{In}(\gamma(q_1 \cdot q_2,q_3))$ o $\alpha(q_1 \cdot q_2 \cdot q_3)$. Finally $\mathrm{In}(\alpha(q_1)(\gamma(q_2,q_3))\ \gamma(q_1,q_2 \cdot q_3)) = \mathrm{In}(\gamma(q_1,q_2)\gamma(q_1 \cdot q_2,q_3))$. This means that we must have $\alpha(q_1)(\gamma(q_2,q_3))\gamma(q_1,q_2 \cdot q_3)\gamma(q_1 \cdot q_2,q_3)^{-1}\ \gamma(q_1,q_2)^{-1} \in G$, the centre of K. Thus we can define a^3 cochain of Q in G i.e.:- and $f \in C^3_{p_o}(Q, G)$ where (G, p_o) is the centre of (K, p) via:-

$$f(q_1, q_2, q_3) \equiv \alpha(q_1)(\gamma(q_2, q_3))\gamma(q_1,q_2 \cdot q_3)\gamma(q_1 \cdot q_2,q_3)^{-1}$$

$\gamma(q_1,q_2)^{-1}$. One can show by tedious algebra that in fact $f \in Z^3_{po}$ (Q, G) or that $\delta(f)(q_1,q_2,q_3,q_4) = p(q_1)(f(q_2,q_3,q_4)) - f(q_1 \cdot q_2,q_3,q_4)$

$f(q_1,q_2.q_3,q_4) - f(q_1,q_2,q_3.q_4) + f(q_1,q_2,q_3) = 0$. The expression $\alpha(q_1) \circ \alpha(q_2) = In(\gamma(q_1,q_2)) \circ \alpha(q_1,q_2)$ is to a degree arbitary since we see that if $\gamma'(q_1,q_2) \equiv \gamma(q_1,q_2)\phi(q_1,q_2)$ where $\phi \in C^2_{po}$ (Q, G) then it is unchanged. Under this mapping $\gamma \longmapsto \phi'$, we find $f \longmapsto f' = f + \delta(\phi)$. Let us now choose a new section β': $Out(K) \longrightarrow Aut(K)$ then we must have $\alpha'(q) \equiv (\beta' \circ p)(q) = In(\sigma(q)) \circ \alpha(q)$ where $\sigma: Q \longrightarrow K$. Then $\alpha'(q_1) \circ \alpha'(q_2) = In(\sigma(q_1)) \circ \alpha(q_1) \circ In(\sigma(q_2)) \circ \alpha(q_2) = In(\sigma(q_1)) \circ In(\alpha(q_1)(\sigma(q_2))) \circ (\alpha(q_1) \circ \alpha(q_2)) = In(\sigma(q_1)\alpha(q_1)(\sigma(q_2))\gamma(q_1,q_2)) \circ \alpha(q_1,q_2)$. Or $\alpha'(q_1) \circ \alpha'(q_2) = In(\gamma'(q_1,q_2)) \circ \alpha'(q_1.q_2)$ where $\gamma'(q_1,q_2) \equiv \sigma(q_1)\, \alpha(q_1)(\sigma(q_2))\, \gamma(q_1,q_2)\, \sigma(q_1,q_2)^{-1}$. Now $\alpha'(q_1)(\gamma'(q_2, q_3))\, \gamma'(q_1,q_2.q_3)\, \gamma'(q_1,q_2,q_3)^{-1}\, \gamma'(q_1,q_2)^{-1} = f(q_1,q_2,q_3)$ after some manipulation. Thus the cocycle $f \in Z^3_{\ p}(Q, G)$ is unchanged by the map $\alpha \longmapsto \alpha'$ and, changing γ by a two cochain of $C^2_{\ p}(Q, G)$ changes f by a 3 coboundary, the latter mapping corresponding to mapping the kernel (K, p) onto a G equivalent kernel. Each cohomology class in $H^3_{\ p_o}(Q, G)$ corresponds one to one with an equivalence class of G equivalent kernels.

We now show how of a kernel (K, p) is extendible when the three cocycle associated with it vanishes. Let (E, ϕ) be an extension of Q by the Q kernel (K, p). Then, as before we can define a section j: $Q \longrightarrow E$ such that $\phi \circ j = 1$, which means that we must have $j(q_1) j(q_2) = f(q_1,q_2)j(q_1q_2)$ where $f : Q \times Q \longrightarrow K$. The automorphism $f'(j(q))$ of K defined by $i \circ f'(j(q)) = In(j(q)) \circ i$ is in the class $p(q)\ \forall\ q \in Q$, we may thus choose the map ' defined above via

$\alpha(q) \equiv f(j(q)) \; \forall \, q \in Q$. Then we have $f(j(q_1) \circ f(j(q_2)) = f(j(q_1)j(q_2)) = f(\rho(q_1,q_2)j(q_1,q_2)) = In(\rho(q_1,q_2)) \circ f(j(q_1,q_2)$ since f is a homomorphism and $f(k) = In(k) \forall k \in K \lhd E$. Thus the corresponding 3 cocycle 'F' is given by:-

$$F(q_1,q_2,q_3) = f \circ j(q_1)(\rho(q_2,q_3)) \, \rho(q_1,q_2 \cdot q_3)$$
$$\rho(q_1 \cdot q_2,q_3)^{-1} \rho(q_1,q_2)^{-1}. \text{ Th is just}$$
$$F(q_1,q_2,q_3) = f(q_1) \, j(q_2) \, j(q_3) \, j(q_2 \cdot q_3)^{-1} j(q_1)^{-1}$$
$$j(q_1)j(q_2,q_3) \, j(q_1 \cdot q_2 \cdot q_3)^{-1} \, j(q_1 \cdot q_2 \cdot q_3)j(q_3)j(q_1 \cdot q_2)^{-1} \, j(q_1,q_2)$$
$$j(q_2)^{-1}j(q_1)^{-1} = e.$$

Thus F = 0 if (K, p) is extendible. Conversely let F = 0. Define the group E as the set of pairs $(k, q) \in K \times Q$ with the composition $(k_1,q_1)(k_2,q_2) = (k_1\alpha(q_1)(k_2) \, \rho(q_1,q_2), \, q_1 \cdot q_2)$ where α is the map from Q into Aut(K) defined above and $\alpha(q_1) \circ \alpha(q_2) = In(\rho(q_1,q_2)) \circ \alpha(q_1 \cdot q_2) \, \forall \, q_1,q_2 \in Q$. E is easily seen to be a group. The map $\phi: (k, q) \longmapsto q$ is a homomorphism of E onto Q. The kernel of is the subgroup $\{(k, e) \mid k \in K\}$, isomorphic to K. Choosing the section $j: Q \longrightarrow E; \; j: q \longmapsto (e, q)$ then In(e, q) is the automorphism class of p(q). Whence F = 0 for a kernel (K, p) implies that (K, p) is extendible. An incidental result of the above construction is that we have explicitly constructed one extension of an extendible kernel (K, p) where Q operates on K via the section ,
β: Out(K) $\longrightarrow$ Aut(K), $\alpha(q) \equiv \beta \circ p(q) \forall \; q \in Q$. The construction of this extension will enable us to generate all extensions of Q by the extendible Q kernel (K, p) with centre (G, p_0). In certain favourable circumstances, we shall be able to choose α as a homomorphism,

depending of course on α being a homomorphism.

Recall our earlier definition of extension products and the notions of equivalence of extensions and G equivalence of Q kernels. Now if $\sigma_i: (E_i, \phi_i) \longrightarrow (E_i', \phi_i')$ $i = 1, 2$ are Q - K_i equivalences then one can easily see that $\sigma_1 \times \sigma_2: (E_1, \phi_1) \wedge (E_2, \phi_2) \rightarrow (E_1', \phi_1') \wedge (E_2', \phi_2')$ is a Q - $(K_1, p) \wedge (K_2, p)$ equivalance, where $\sigma_1 \times \sigma_2: (e_1, e_2) \longmapsto (\sigma_1(e_1), \sigma_2(e_2)) \forall (e_1, e_2) \in E_1 \times E_2$

Now let (K, p) be an extendible Q kernel with centre (G, p_o). We exhibited above extensions of Q by (K, p), let this be (E, ϕ). Consider next an extension (F, ψ) of Q by the central Q kernel (G, p_o), then the product $(E, \phi) \wedge (F, \psi)$ is an extension of Q by the Q kernel $(K, p) \wedge (G, p_o)$ which is G equivalent to (K, p), i.e.:- $(E, \phi) \wedge (F, \psi)$ is an extension of Q by (K, p). The map $(F, \phi) \longmapsto (F, \psi) \wedge (E, \phi)$ is a map from the classes of extensions of Q by (G, p_o) to the classes of extensions of Q by (K, p). One can show that the map is onto and that (Q, G) equivalent extensions of Q by (G, p_o) map onto (Q, K) equivalent extensions. Because of the importance of the construction, we indicate the proofs of the above assertions.

Firstly, $\forall\ q \in Q$, select an $\alpha(q) \in p(q)$ and a $j(q) \in E$ with $\phi \circ j(q) = q$ and $\pi(\alpha(q)) = p(q)$, subject to the requirement that $In(j(q)) \circ i = i \circ \alpha(q)\ \forall\ q \in Q$ where $i: K \lhd E$. We must have $j(q_1)\ j(q_2) = f(q_1, q_2)\ j(q_1 . q_2)$ where $f: Q \times Q \rightarrow K$. The requirements $j(q)\ k\ j(q)^{-1} = \alpha(q)(k) \forall k \in K$ with $j(q_1)\ j(q_2) =$

$= f(q_1,q_2)\ j\ (q_1 \cdot q_2)$ fix the composition on E whose underlying set may be taken as the set K x Q in a unique manner, $k \cdot j\ (q) \longmapsto (k, q)$ where $(e, q)\ (k, e)\ (e, q)^{-1} = (\alpha(q)(k), e)$ and $(e, q_1)(e, q_2) = (f(q_1,q_2), q_1 \cdot q_2)$. Associativity requires $f(q_1,q_2)\ f(q_1 \cdot q_2, q_3) = \alpha(q_1)(f(q_2,q_3))\ f\ (q_1, q_2 \cdot q_3)$ and $\alpha(q_1) \circ \alpha(q_2) = \text{In}\ (f(q_1,q_2)) \circ \alpha(q_1 \cdot q_2)$. Similarly, the extensions (F, Ψ) are constructed in the manner outlined in our preliminary discussions on Abelian kernels. We may write each (F, Ψ) as $G \boxtimes g\ Q$ with $g \in H^2{p_o}\ (Q, G)$. In the group $G \boxtimes g\ Q$ we use the notation where the underlying set is $G \times Q$. Thus in $(E, \Phi) \wedge (F, \Psi)$, which we take as $\{(k_1, k_2, (j_1(q), j_1(q)) \mid (k, q) \in K \times Q\}$, we must have the composition

$$(e_1, (j_1(q_1), j_2(q_1))(e_2, (j_1(q_2), j_2(q_2)) =$$
$$(h(q_1,q_2)\ f\ (q_1,q_2), (j_1(q_1 \cdot q_2), j_2(q_1 \cdot q_2))) \quad \text{and also that}$$

$$\text{In}(e, (j_1(q), j_2(q))((k_1(e, e))) = (\alpha(q)(k), (e,e))$$

Thus the factor set associated with $(E, \Phi) \wedge (F, \Psi)$ is then the 'product' of the factor sets $(h\ \circledast f)$. Now let (E', Φ') be any extension of Q by K with kernel (K, p) Define a section $j': Q \rightarrow E'$, $\Phi' o j' = \mathbb{1}$ and $\text{In}(j(q)) \circ i = i \circ \alpha(q)$. This determines a factor set f' with $f'(q_1,q_2) f'(q_1 \cdot q_2, q_3) = \alpha(q_1)(f'(q_2,q_3))\ f'(q_1, q_2 \cdot q_3)$ and $\alpha(q_1) \circ \alpha(q_2) = \text{In}(f'(q_1,q_2)) \circ \alpha(q_1 \cdot q_2)$. This means that we must have $\text{In}(f'(q_1,q_2) f(q_1,q_2)^{-1}) = \mathbb{1}$ or that $f'(q_1,q_2) = d(q_1,q_2) f(q_1,q_2)$ where $d' \in C^2{p_o}\ (Q, G)$. Using the factor set properties of f and f' then we must have $d' \in Z^2 p_o\ (Q, K)$. We must also have that

$(E,\Phi)\wedge(F,\Psi)$ is Q-K equivalent to (E',Φ') since they have the same composition and isomorphic underlying sets. Thus the map $(F,\Psi)\longmapsto(F,\Psi)\wedge(E,\Phi)$ is onto the set of extensions of Q by (K, p), modulus the Q-K equivalence.

Next let (F_1,Ψ_1) and (F_2,Ψ_2)be two extensions of Q by G labelled by cocycles h_1, $h_2 \in Z^2{p_o}(Q, G)$ respectively. Then form the extensions $(E,\Phi)\wedge(F_1,\Psi_1)\equiv(E_1,\Phi_1)$ and $(E,\Phi)\wedge(F_2,\Psi_2)$ $\equiv (E_2,\Phi_2)$ of Q by (K, p). Assume that (E_1,Φ_1) and (E_2,Φ_2) are Q-K equivalent, then since $\Phi_2 \circ \sigma = \Phi_1$ where $\sigma : E_1 \sim E_2$, $\sigma(j_1(q))$ must be of the from $\Phi(q)j_2(q)$ where j_1 and j_2 are sections from Q to E_1 and E_2 respectively and $\Phi \in C^1_p(Q, K)$. Since we have $j_1(q)\, k\, j_1(q)^{-1} = \alpha(q)(k)$ and $j_2(q) k\, j_2(q)^{-1} = \alpha(q)(k)\,\forall(k,q)$ $K \times Q$ and $\sigma = 1$ on K then, $\sigma(j_1(q)\, k\, j_1(q)^{-1}) = \sigma(j_1(q))\, k$ $\sigma(j_1(q))^{-1} = \alpha(q)(k) = \Phi(q)\, j_2(q)\, k\, j_2(q)^{-1}\, \Phi(q)^{-1} = j_2(q)\, k\, j_2$ $(q)^{-1}\ \forall\, (k, q) \in K \times Q$. Thence $In(\Phi(q)) = 1 \Rightarrow \Phi \in C^1{p_o}(Q,G)$. We also have $\sigma(j_1(q_1))\ \sigma(j_1(q_2)) = f_2(q_1,q_2)\delta(\Phi)(q_1,q_2)$ $\sigma(j_1(q_1,q_2))$. Thus $d_1 = d_2 + \delta(\Phi)$ which means that (F_1,Ψ_1) $\sim (F_2,\Psi_2)$ under Q-G equivalence.

This completes the proof of our assertion that the map α: $H^2{p_o}(Q,G) \longrightarrow Ext(Q,(K,p))/(\sim), \alpha : f \longrightarrow (G \boxtimes_f Q)\wedge(E,\Phi)\, 0$ where $(E,\Phi)\, 0$ is a fixed extension of Q by K is a set isomorphism. We were able to prove the existence of a fixed extension of Q by (K, p), by constructing a multiplication table defined by a factor set. In some cases however we can choose the fixed extension in a particularly simple way. We discuss these in the next few paragraphs.

(a) Central Extensions.

Let us consider the case of the central extensions of Q by the Q kernel (K, 0). The extension $K \otimes Q$ is a central extension of Q by K and we choose it as the fixed one in our discussion above. All other extensions of Q by (K,0) are then of the form $(K \otimes Q) \wedge (G \otimes f\, Q)$, $f \in H^2_0(Q, K)$ since $p = 0$, $p_0 = 0$.

(b) Semi-direct products.

The example above $K \otimes Q$ is the trivial example of a semi-direct product. We discuss now more general examples. The necessary and sufficient condition that a semi-direct product is one solution of the problem of finding extensions of Q by the Q kernel (K, p) $\rho \in$ Hom (Q, Out(K)) is that there exists a $g \in$ Hom(Q, Aut(K)) such that $\pi \circ g = p$ where π is the canonical epimorph π : Aut(K) $\rightarrow$ Out(K). Several situations when this is so are immediate. One is when K is Abelian and then Aut(K) = Out(K); another when Aut(K) is a semi-direct product, when there exists a monomorphism j: Out(K) < Aut(K). In the cases when such a $g \in$ Hom(Q, Aut(K)) exists, we can obtain a solution to the problem of finding all solutions to the problem of extending Q by (K, p) = (K, $\pi \circ g$). These extensions are just:- $(E, \phi) = (K \boxtimes_g Q) \wedge (G \boxtimes_f Q)$ $f \in H^2_{g_0}(Q, G)$.

Let us just note in conclusion of our discussion of extensions that a semi-direct product may be equivalent to a central extension - the direct product. Consider the case of the central extension $K \boxtimes_r Q$ when $r \in$ Hom(Q, Int(K)). Then we have $\pi \circ r = 0$ since $\pi \circ r(q) = 0 \ \forall \ q \in Q$. The necessary and sufficient condition for $K \boxtimes_r Q \rightleftharpoons K \otimes Q$ is that $\exists \ t \in$ Hom(Q, K) with $r = I\,n \circ t$. The necessity is obvious in the light of chapter '1''s diagram language. The sufficiency follows by noting that if j: $Q < K \boxtimes_r Q$ then $p(q)(k) = j(q)k\,j(q)^{-1}$ where $p = \mathbf{1} \circ r$; I: Int(K) $\lhd$ Aut(K). Let $t \in$ Hom(Q, K) with $r = I\,n \circ t$. Then $\mathbf{1} \circ$ In(t(q)) In(j(q)) $\circ\, i = i \circ p(q)$. Then we have:-

$i(t(q))\, i(k) i(t(q))^{-1} = j(q)\, i\,(k)\, j(q)^{-1} \forall\, (k, q) \in E \times Q$ which is $i(t(q))j(q)^{-1} \in \mathcal{C}_E(i(K))$ the centraliser in E of i(K). Define a map:- $j' : Q \rightarrow E$ via $j'(q) = i(t(q))j(q)^{-1}$ then $j'(q)j'(q') = i(t(q))j(q')^{-1} i(t(q'))j(q')^{-1} = j'(qq')$ our above definitions. Moreover $\phi \circ j' = 0$ since $\phi \circ j = 0$, and $Im(j) \lhd E$, so $K \boxtimes_r Q$ is equivalent to $K \otimes Q$. Recall how we write $L\uparrow(\mathbb{R}) \otimes Z(z)_{pT} \rightleftharpoons L\uparrow(\mathbb{R}) \boxtimes_{\gamma'} (Z(z))_T$. Here, the action of T on $L\uparrow(\mathbb{R})$ is defined by $\gamma'(T) = In(G) = In(-G) = In(PT.G)$. Thus $\exists\, t \in Hom(Z(z)_T, L\uparrow(\mathbb{R}))$ defined by $T \mapsto In(G)(T) = T^{-1T} = -T = G$. The injection $j' : Z(z)_T \rightleftharpoons Z(z)_{pT} < L(\mathbb{R})$ is $j(T) = t(T)j(T)^{-1} = G(-G) = -e = PT$.

In the final section of this chapter which follows, we will discuss a concept related to extensions and which involves cohomology theory. This is the theory of 'G enlargements' of a group Q by an Abelian group K.

Theory of G Enlargements

Let G and Q by arbitary groups and K an Abelian group. Moreover let G operate on Q and on K via the homomorphisms $(p_1, p_2) \in Hom(G, Aut(K)) \times Hom(G, Aut(Q))$. A group (E, ϕ) is said to be a G enlargement of Q by K iff (i) (E, ϕ) is a trivial group extension of Q by K. (ii) the epimorphism $\phi : E \rightarrow Q$ is an operator epimorphism $\phi \in OpHom(E, Q)$. (iii) The injection $i: K \lhd E$ with $\phi \circ i = 0$ is an operator monomorphism $i \in Op\, Hom(K,E)$. (iv) G is a group of automorphisms for the group E via $P \in Hom(G, Aut(E))$.

Two G enlargements of Q by K are called equivalent if the corres-

-ponding group extensions are Q - K equivalent; with the isomorphism establishing the equivalence an operator homomorphism. A G enlargement of Q by K is said to be 'inessential' if $j: Q \prec E$ is an operator monomorphism. Let (E,ϕ) be a G enlargement whose underlying group is $K \boxtimes_F Q$ where $F \in Hom(Q, Aut(K))$. Then (E,ϕ) inessential implies that $P(g)(k, q) = (P_1(g)(k), p_2(g)(q)) \forall (g, k, q) \in G \times K \times Q$. Since $p(g)(k, q) = p(g)(k,e)p(g)(e, q) = (p(g) \circ i)(k)(p(g) \circ j)(e,q)$ where $(i,j): (K, Q) \prec E$. Since $(i, j) \in Op\ Hom(K, E) \times Op\ Hom(Q, E)$, $p(g) \circ i = i \circ p_1(g)$, $p(g) \circ j = j \circ p_2(g)\ \forall\ g \in G$; so that $p(g)(k, q) = (p_1(g)(k), p_2(g)(k))$ or $p(g) = p_1(g) \times p_2(g) \forall g \in G$. The inessential G enlargements thus form an equivalence class. Let $\mathfrak{E}nl(G,(Q, K))$ be the set of all G enlargements of Q by K. We can endow the set $Enl(G, (Q,K))$ defined by $Enl \equiv \mathfrak{E}nl/(\simeq)$, (where $(\simeq)$ is the equivalence of G enlargements), with the structure of a group. Let $(E_1,\phi_1) \vee (E_2,\phi_2)$ be the G object whose underlying group is $(E_1,\phi_1) \wedge (E_2,\phi_2)$ and where the action of G on $(E_1,\phi_1) \wedge (E_2,\phi_2)$ is just $p(g) = p'(g) \times p''(g) \forall g \in G$ where (p',p'') are the actions of G on (E, E') respectively. Then $(E_1,\Phi_1) \vee (E_2,\phi_2)$ is again a G enlargements of Q by K.

We proove here that $Enl(G, (Q, K))$ is isomorphic to the group $H^1_{p_3}(G, H^1_F(Q, K))$ where the action p_3 of G on $H^1_F(Q, K)$ is defined by $p_3(g): f \mapsto p_1(g) \circ f \circ p_2(g)^{-1} \forall\ f \in H^1_F(Q, K)$. Let (E,ϕ) be a G enlargement of Q by K with $E = K \boxtimes_F Q$. Then there is a monomorphism $j: Q \prec E \ni \phi \circ j = \mathbb{1}$. j is an operator monomorphism iff (E,ϕ) is inessential. Define $\rho(g)(\Psi(p_2(g)^{-1}(q))\Psi(q^{-1})$ as $J(g)(q)$ where $J \in C^1_{p_3}(G, C^1_F(Q, K))$ since we must have $\Phi(J(g)(q)) = \Phi(p(g)(\Psi(p_2(g)^{-1}(q))\Phi(\Psi(q^{-1})) = p_2(g) \circ \Phi(\Psi(p_2(g)^{-1}(q))q^{-1} = q\, q^{-1} = e$. Moreover $J(g)(q_1 \cdot q_2) = p(g)(\Psi(p_2(g)^{-1}(q_1 \cdot q_2))\Psi(q_1 \cdot q_2)^{-1} = p(g)(\Psi(p_2(g)^{-1}(q_1)))p(g)(\Psi(p_2(g)^{-1}(q_2))\Psi(q_2)^{-1}\Psi(q_1) = p(g)\Psi(p_2(g)^{-1}(q_1))\, J(g)(q_2)\Psi(q_1) = p(g)\Psi(p_2)(g)^{-1}(q_1))\Psi(q_1)^{-1}$ $\Psi(q_1)J(g)(q_2)\Psi(q_1)^{-1} = J(g)(q_1)F(q_1)(J(g)(q_2))$. Thus we have $J(g)(q_1, q_2) = J(g)(q_1)F(q_1)(J(g)(q_2))$, or $J(g)(q_1, q_2) = J(g)(q_1) + F(q_1)(J(g)(q_2))$ using the additive notation. So $J(g) \in Z^1_F(Q,K)$ $\forall g \in G$ or :- $J \in C^1_{p_3}(G, Z^1_F(Q,K))$. Similarly $J \in Z^1_{p_3}(G, Z^1_F(Q,K))$ since $J(g_1 \cdot g_2)(q) = p(g_1 g_2)(\Psi(p_2(g_1, g_2)^{-1}(q))\Psi(q)^{-1}$, or $J(g_1 \cdot g_2)(q) = p(g_1)(p(g_2)(\Psi(p(g_2)^{-1}(p(g_1)^{-1}(q)))$. $\Psi(g_1^{-1}q)^{-1}$ $p(g_1)^{-1}\Psi(p(g_1^{-1})(q))\Psi(q)^{-1}$. Or $J(g_1, g_2)(q) = p_3(g_1)(J(g_2))(q))$ $J(g_1)(q)$, which is :- $J(q_1, q_2) = p_3(g_1)(J(g_2)) + J(g_1)$. Choose a new monomorphism $\Psi' : Q < E \ni \Phi \circ \Psi' = 1$; we must then have $\Psi'(q) = \Pi(q)\Psi(q) \forall q \in Q; \Pi \in Z^1_F(Q,K)$, as we saw before. In terms of the deviation of Ψ' from being an operator monomorphism define:- $J'(g)(q) = p(g)(\Psi'(p_2(g)^{-1}(q))\Psi'(q)^{-1}$. Then $J'(g)(q) = p(g)$ $(\Pi(q)(g)^{-1}(q))$. $p(g)\Psi(p_2(g)^{-1}(q))\Psi(q)^{-1}\Pi(q)^{-1}$, $=$ $p_1(g)(\Pi(p_2)(g)^{-1}(q))\, J(g)(q)\Pi(q)^{-1} = p_3(g)(\Pi)(q))\Pi(q)^{-1}$ $J(g)(q)$, since $J(g)(q) \in K$ and $i: K \lhd E$ is an operator monomorphism. Thus:-$J(g)' = J(g) + \delta(\Pi)$, $\delta(\Pi) \in B^1_{p_3}(G, Z^1_F(Q,K))$, $\Pi \in C^0_{p_3}(G, Z^1_F(Q, K)) = Z^1_{p_3}(Q, K)$. The G enlargements corresponding to Ψ and Ψ' are equivalent, implying that the appropriate cocycles are cohomologous. The elements of $B^1_F(Q, K)$ corresponding to the conjugate images of Q in E; $(e, q) \overset{k}{\longmapsto} (k, e)^{-1} (e, q)(k, e) = (k^{-1} q \cdot k, q) = (\sigma(k)(q), q)$. Thus we must have $\mathrm{Ehl}\ (G,(Q,K))/(\simeq) \cong H^1_{p_3}(G, H^1_F(Q, K))$, as sets. The isomorphism also extending to a group isomorphism in a way which does not concern us.

The above discussion of G enlargements will be quite frequently referred to in the next chapter, and it ends our summary of the cohomology theory of groups.

CHAPTER (4)

COHOMOLOGY THEORY OF THE KINEMATICAL GROUPS

In this chapter we will discuss the algebraic structures of relativity groups using the tools developed in Chapter (3). The study was initiated in Chapter (2) where we were able to derive the structure of the Poincaré group of world automorphisms of Minkowski-spacé, the Einsteinian world. The structure of the Galilei group of inertial automorphisms of Newtonian relativity was only really hinted at. We shall see that the structure of the Galilei group is considerably more complex than that of the Poincaré group. Moreover we shall see that it has several isomorphic algebraic structures as a group extension, corresponding to permutations of its underlying set, a four-fold Cartesian product.

The last part of Chapter (2) saw the introduction of the Carroll and Static groups and their interpretations as kinematical groups. Although the Carroll group has no connection with physical reality at all, the Static group is somewaht more plausible, and it is of interest to discuss its algebraic structure along with the static group, as generalisations of the Galilei group. After this discussion we attempt to solve the problem of listing all possible classical kinematical groups where either space, time or both are 'absolute' in a mathematical sense to be elaborated in that calculation. This excercise, apart from being of mathematical interest, lends a rather deep insight into the various algebraic structures of the Galilei, Carroll and Static groups, enabling one to relate them as members of more exotic families of groups. No physical interpretations of the groups, apart from the obvious three, will be attempted.

The chapter is divided into three parts. In the first section we will discuss the structure of the Galilei group in some detail. In the second, much shorter, section we will discuss the structures of the Carroll and Static groups and in the third, the generalised relativity groups will be calculated (in principle only, since these turn out to be $\aleph_1$!)

Part (1). The Galilei Group

Recall that the Galilei group could be expressed as $\mathcal{A}(\mathbb{W}) \cap I(\mathbb{W})$, where $\mathbb{W} = \mathbb{W}(N)$ and $\mathcal{A}(\mathbb{W})$ was the group of world automorphisms of $\mathbb{W}(N)$ i.e.:- $f \in \mathcal{A}(\mathbb{W}) < B(\mathbb{W})$ iff (i) $\tau \circ (f \times f) = \tau$ or $\tau(f(x_1), f(x_2)) = \tau(x_1, x_2)\ \forall\, x_1, x_2 \in W$ and (ii) $\partial(x)(x_1, x_2) = \partial(f(x))(f(x_1), f(x_2))\ \forall\, x_1, x_2 \in S(x),\ x \in \mathbb{W}$. $I(\mathbb{W})$ was the subgroup of inertial functions of $B(\mathbb{W})$ or $I(\mathbb{W}) \equiv \{ f \in B(\mathbb{W}) \mid \ddot{\underline{x}} = 0 \Rightarrow f(\underline{x})^{\cdot\cdot} = 0 \}$. Given an $\alpha \in \mathcal{A}(\mathbb{W}) \cap I(\mathbb{W})$, let us write $\alpha: (\underline{x}, t) \mapsto (\beta_1(\alpha)(\underline{x}) + \beta_2(\alpha)(t), \beta_3(\alpha)(\underline{x}) + \beta_4(\alpha)(t))$, where $\beta_1 \in C^1(\mathcal{A}(\mathbb{W}) \cap I(\mathbb{W}), \mathrm{Sym}(\mathbb{R}^3))$; $\beta_2 \in C^1(\mathcal{A}(\mathbb{W}) \cap I(\mathbb{W}), C^1(\mathbb{R}^1, \mathbb{R}^3))$; $\beta_3 \in C^1(\mathcal{A}(\mathbb{W}) \cap I(\mathbb{W}), C^1(\mathbb{R}^3, \mathbb{R}^1))$ and $\beta_4 \in C^1(\mathcal{A}(\mathbb{W}) \cap I(\mathbb{W}), \mathrm{Sym}(\mathbb{R}^1))$. Let us now consider the restrictions placed on α by $\alpha \in \mathcal{A}(\mathbb{W}) \cap I(\mathbb{W})$. Firstly we must have $\tau(\alpha(x_1), \alpha(x_2)) = \tau(x_1, x_2)\ \forall\, x_1, x_2 \in \mathbb{W}$. Write $(x_1, x_2) = ((\underline{x}_1, t_1), (\underline{x}_2, t_2))$. Then the condition is that $\beta_3(\alpha)(\underline{x}_1) + \beta_4(\alpha)(t_1) - \beta_3(\alpha)(\underline{x}_2) + \beta_4(\alpha)(t_2) = t_1 - t_2$. Let $t_1 = t_2$, then we must have:- $\beta_4(\alpha)(t_1) = \beta_4(\alpha)(t_2)$ since α is injective, which means that $\beta_3(\alpha)(\underline{x}1) -$

$\beta_3(\alpha)(\underline{x}_2) = 0.\ \forall\ \underline{x}_1,\ \underline{x}_2 \in \mathbb{R}^3$ or $\beta_3(\alpha)$ is a constant function from $\mathbb{R}^3$ into $\mathbb{R}^1$, an element of $C^0(\mathbb{R}^3, \mathbb{R}^1)$. We write $\beta_3(\alpha)(x) = T'(\alpha)$, $T' \in C^1(\mathfrak{A}(W) \cap I(W), \mathbb{R}^1)$. Thus we must have $\beta_4(\alpha)(t_1) - \beta_4(\alpha)(t_2) = t_1 - t_2$ when $t_1 \neq t_2$, or $\beta_4(\alpha)(t_1) - t_1 = \beta_4(\alpha)(t_2) - t_2 \equiv \tau(\alpha)$ i.e.:- $\beta_4(\alpha)(t) = t + \tau(\alpha)$, where $\tau \in C^1(\mathfrak{A}(W) \cap I(W), \mathbb{R}^1)$. We write:- $T(\alpha) \equiv T'(\alpha) + \tau(\alpha)$ or $T \equiv T' + \tau$. Thus we have to have $\beta_4(\alpha) = \mathbb{1}$ modulo an element of $\mathbb{R}^1$ and $\beta_3(\alpha)(x) = T(\alpha)\ \forall\ \alpha \in \mathfrak{A}(W)$. The second condition on α is that $\partial(\alpha(x))(\alpha(x_1), \alpha(x_2)) = \partial(x)(x_1, x_2)\ \forall\ x_1, x_2 \in S(x), x \in W$, or $d(t + T(\alpha))(\alpha(x_1), \alpha(x_2)) = d(t)(x_1, x_2)$. Let us take x_1 and x_2 as fixed events, '$\underline{x}_i \neq \underline{x}_i(t)$', then we must have $\| \underline{x}_1 - \underline{x}_2 \|^2 = \| \beta_1(\alpha)(\underline{x}_1) + \beta_2(\alpha)(t) - \beta_1(\alpha)(\underline{x}_2) - \beta_2(\alpha)(t) \|^2$ or $\| \beta_1(\alpha)(\underline{x}_1) - \beta_1(\alpha)(\underline{x}_2) \|^2 = \| \underline{x}_1 - \underline{x}_2 \|^2$. Thus $\beta_1(\alpha)$ is a linear isometry of three dimensional Euclidean space i.e.:- $\beta_1(\alpha) \in E(3, \mathbb{R})$ the three dimensional Euclidean group, whose structure is $\mathbb{R}^3 \rtimes_n O(3, \mathbb{R})$. Here $\mathbb{R}^3$ is the group of translations in the vector space $\mathbb{R}^3 \int \mathbb{R}$ and $O(3, \mathbb{R})$ is the three-dimensional orthogonal group or rotation group and $n \in \mathrm{Hom}(O(3, \mathbb{R}), \mathrm{Aut}(\mathbb{R}^3))$ is the natural action of $M(3, \mathbb{R})$ on its module $\mathbb{R}^3$, restricted to $O(3, \mathbb{R}) \subset M(3, \mathbb{R})$. We thus have $\beta_1 \in C^1(\mathfrak{A}(W) \cap I(W), E(3, \mathbb{R}))$, we shall write $\beta_1(\alpha) = \underline{X}(\alpha) \circ R(\alpha)$ where $\underline{X} \in C^1(\mathfrak{A}(W) \cap I(W), \mathbb{R}^3)$ and $R \in C^1(\mathfrak{A}(W) \cap I(W), O(3, \mathbb{R}))$. No restrictions have yet appeared on $\beta_2 \in C^1(\mathfrak{A}(W) \cap I(W), C^1(\mathbb{R}^1, \mathbb{R}^3))$, since we have not yet imposed the condition that $\alpha \in I(W)$. Before doing so, let

us examine the structure of $\mathcal{A}(W)$, the not-necessarily inertial world automorphisms. We write $\alpha = ((\underline{x}(\alpha), R(\alpha)), \beta_2(\alpha), T(\alpha))$ where $((\underline{x}(\alpha), R(\alpha)), \beta_2(\alpha), T(\alpha)) : (\underline{x}, t) \longmapsto (R(\alpha)(\underline{x}) + \underline{x}(\alpha) + \beta_2(\alpha)(t), t + T(\alpha))$. Thus we must have $\alpha_1 \circ \alpha_2$:
$(\underline{x}, t) \longmapsto (R(\alpha_1)(R(\alpha_2)(\underline{x}) + \underline{x}(\alpha_2) + \beta_2(\alpha_2)(t)) + \underline{x}(\alpha_1) + \beta_2(\alpha_1)(t + T(\alpha_2)), t + T(\alpha_1) + T(\alpha_2)) =$
$((\underline{x}(\alpha_1 \circ \alpha_2), R(\alpha_1 \circ \alpha_2)), \beta_2(\alpha_1 \circ \alpha_2), t + T(\alpha_1 \circ \alpha_2))(\underline{x}, t)$
We thus surmise that $\underline{x}(\alpha_1 \circ \alpha_2) = \underline{x}(\alpha_1) + R(\alpha_1)(\underline{x}(\alpha_2))$,
$R(\alpha_1 \circ \alpha_2) = R(\alpha_1) \circ R(\alpha_2)$, $T(\alpha_1 \circ \alpha_2) = T(\alpha_1) + T(\alpha_2)$ and
$\beta_2(\alpha_1 \circ \alpha_2) = \quad R(\alpha_1)(\beta_2(\alpha_2)(t)) + \beta_2(\alpha_1)(t + T(\alpha_2))$. We thus see that (i) $\underline{x} \in Z^1_{noR}(\mathcal{A}(W), \mathbb{R}^3)$; $R \in \text{Hom}(\mathcal{A}(W), O(3, \mathbb{R}))$ which means that $\beta_1 \in \text{Hom}(\mathcal{A}(W), E(3, \mathbb{R}))$, since $\underline{x}(\alpha_1) \circ R(\alpha_1) \circ \underline{x}(\alpha_2) \circ R(\alpha_2) = \underline{x}(\alpha_1) \circ R(\alpha_1)(\underline{x}(\alpha_2)) \circ R(\alpha_1 \circ \alpha_2) = x(\alpha_1 \circ \alpha_2) \circ R(\alpha_1 \circ \alpha_2)$. Also $T \in \text{Hom}(\mathcal{A}(W), \mathbb{R}^1_+)$ the additive group of the real line. The most interesting feature of the Galilei-group has been left till the end. This is β_2. Firstly, if we neglect the $T(\alpha)$, we have $\beta_2(\alpha_1 \circ \alpha_2)(t) = R(\alpha_1)(\beta_2(\alpha_2)(t)) + \beta_2(\alpha_1)$. So that $\beta_2 \in Z^1_{noR}(\mathcal{A}(W), C^1(\mathbb{R}^1, \mathbb{R}^3))$ where $(NoR)(\alpha)(f) \equiv R(\alpha) \circ f \; \forall f \in C^1(\mathbb{R}^1, \mathbb{R}^3)$.

Also, if we neglect the R, we have:-
$\beta_2(\alpha_1 \circ \alpha_2)(t) = \beta_2(\alpha_2)(t) + \beta_2(\alpha_1)(T(\alpha_2) + t)$ or $\beta_2(\alpha_1 \circ \alpha_2) = \beta_2(\alpha_2) + p(\alpha_2)(\beta_2(\alpha_1))$ where $p(\alpha_2)(\beta) \equiv \beta \circ T(\alpha_2) \; \forall \beta \in C^1(\mathbb{R}^1, \mathbb{R}^3$; so that if $\beta_2(\alpha) \equiv \gamma(\alpha^{-1}) \; \forall \alpha \in \mathcal{A}(W)$, we must have

$\gamma(\alpha_1 \circ \alpha_2) = \gamma(\alpha_1) + P'(\alpha_1)(\gamma(\alpha_2))$. Where $P'(\alpha) \equiv P(\alpha^{-1}) = - P(\alpha)$. So that we see $\gamma \in Z^1_{p'}(\mathfrak{A}(\mathbb{W}), C(\mathbb{R}^1, \mathbb{R}^3))$. We cannot elicit much information yet about β_2 apart from its cocycle properties, so let us impose the property that $\alpha \in I(\mathbb{W})$. Consider a trajectory $\underline{x}: \mathbb{R}^1 \longrightarrow \mathbb{R}^3$ such that $\ddot{\underline{x}}(t) = 0$, then $\alpha \in I(\mathbb{W})$ iff $\ddot{\underline{x}}^{\alpha}(t) = 0$, this is just $(\underline{x}(t) + \beta_2(\alpha)(t))^{\cdot\cdot} = 0$ or $(\beta_2(\alpha)(t))^{\cdot\cdot} = 0$. We must then have $\beta_2(\alpha)(t) = \underline{U}(\alpha)t + \underline{C}(\alpha)$ where $\underline{U}(\alpha) \equiv (\beta_2(\alpha)(t))^{\cdot}$ and $C(\alpha) = \beta_2(\alpha)(0) = 0$, so that $\beta_2(\alpha)(t) = \underline{U}(\alpha)t$, where $\underline{U} \in C^1(\mathfrak{A}(\mathbb{W}) \cap I(\mathbb{W}), \mathbb{R}^3_T)$, $\mathbb{R}^3_T$ being the Abelian group of tangent vectors in $\mathbb{R}^3 \int \mathbb{R}$. We derived

$\beta_2(\alpha_1 \circ \alpha_2)(t) = R(\alpha_1)(\beta_2(\alpha_2)(t)) + \beta_2(\alpha_1)(t + T(\alpha_2))$

The above shows that $\beta_2 \in C^1(\mathfrak{A}(\mathbb{W}) \cap I(\mathbb{W}), Z^1_0(\mathbb{R}^1, \mathbb{R}^3))$ where of course $Z^1_0 = H^1_0 = \text{Hom}$, thus we have:- $\beta_2(\alpha_1 \circ \alpha_2)(t) = R(\alpha_1)(\beta_2(\alpha_2)(t)) + \beta_2(\alpha_1)(t) + \beta_2(\alpha_1)(T(\alpha_2))$, or that:-

$\underline{U}(\alpha_1 \circ \alpha_2)\, t = R(\alpha_1)(\underline{U}(\alpha_2))\, t + \underline{U}(\alpha_1)(t) + \underline{U}(\alpha_1)(T(\alpha_2))$

Recall that we wrote $\alpha_1 \circ \alpha_2 : (\underline{x}, t) \longmapsto (R(\alpha_1) \circ R(\alpha_2)(\underline{x}) + R(\alpha_1)(\underline{x}(\alpha_2)) + R(\alpha_1)(\beta_2(\alpha_2)(t)) + \beta_2(\alpha_1)(T(\alpha_2) + t), t + T(\alpha_1) + T(\alpha_2))$. If we group $\underline{U}(\alpha_1)(T(\alpha_2))$ with the $\mathbb{R}^3$ translations i.e.:- $\underline{X}(\alpha_1 \circ \alpha_2) = R(\alpha_1)\underline{X}(\alpha_2)) + \underline{X}(\alpha_1) + \underline{U}(\alpha_1)T(\alpha_2)$ we can write $\underline{u} \in Z^1_{noR}(\mathfrak{A}(\mathbb{W}) \cap I(\mathbb{W}), \mathbb{R}^3_T)$ i.e.:- $\underline{U}(\alpha_1 \circ \alpha_2) = \underline{U}(\alpha_1) + R(\alpha_1)(\underline{U}(\alpha_2))$. Thus, we can take the underlying set of $\mathfrak{A}(\mathbb{W}) \cap I(\mathbb{W})$ as $\mathbb{R}^3 \times \mathbb{R}^1 \times \mathbb{R}^3_T \times O(3, \mathbb{R})$ where the group law is $(\underline{x}_1, T_1, \underline{U}_1, R_1)(\underline{x}_2, T_2, \underline{U}_2, R_2) = (R_1 \cdot \underline{X}_2 + \underline{X}_1 + \underline{U}_1 T_2, T_1 + T_2, \underline{U}_1 + R_1 \underline{U}_2, R_1 R_2)$, and where $(\underline{X}, T, \underline{U}, R) : (\underline{x}, t) \longmapsto (R\, \underline{x} +$

$+ \underline{x} + \underline{U}\, t,\; t + T)$.

Having established this group law, we will return to it later after discussing some cohomological complications. If we neglect rotations, we showed above that we must have $\underline{x}(\alpha_1 \circ \alpha_2) = \underline{X}(\alpha_1) + \underline{X}(\alpha_2) + \underline{U}(\alpha_1)T(\alpha_2)$, which we write as $\underline{X}(\alpha_1 \circ \alpha_2) = \underline{X}(\alpha_1) + \underline{X}(\alpha_2) + \xi(\alpha_1, \alpha_2)$ where $\xi(\alpha_1,\alpha_2) \equiv \underline{U}(\alpha_1)\, T(\alpha_2)$ with $\underline{U} \in \mathrm{Hom}(\mathcal{A}(W), \mathbb{R}^3_T)$ and $T \in \mathrm{Hom}(\mathcal{A}(W), \mathbb{R}^1)$. Then we have $\xi \in C^2(\mathcal{A}(W), \mathbb{R}^3)$ and moreover that $\xi(\alpha_1,\alpha_3) - \xi(\alpha_1 \circ \alpha_2, \alpha_3) + \xi(\alpha_1, \alpha_2 \circ \alpha_3) - \xi(\alpha_1,\alpha_2) \equiv \delta(\xi)(\alpha_1,\alpha_2,\alpha_3) = \underline{U}(\alpha_2)T(\alpha_3) - (\underline{U}(\alpha_1) + U(\alpha_2))\, T(\alpha_3) + \underline{U}(\alpha_1)(T(\alpha_1) + T(\alpha_2)) - \underline{U}(\alpha_1)T(\alpha_2) = 0$. Thus $\xi \in Z^2_0(\mathcal{A}(W), \mathbb{R}^3)$. Choosing a cohomologous cocycle $\xi'(\alpha_1,\alpha_2) \equiv \xi(\alpha_1,\alpha_2) + \delta(\phi)(\alpha_1,\alpha_2) = U(\alpha_1)\, T(\alpha_2) + \phi(\alpha_1) + \phi(\alpha_2) - \phi(\alpha_1 \circ \alpha_2)$. Corresponds to choosing an $\underline{X}' = \underline{X} + \underline{\phi} \in Z'_{noR}(\mathcal{A}(W), \mathbb{R}^3)$, so up to an arbitrary translation:-

$$(\underline{X}', T, \underline{U}, R) : (\underline{x}, t) \longmapsto (R\underline{x} + \underline{X} + \underline{\phi} + \underline{u}\, t,\; t + T)$$

the problem is unchanged, and hence depends only on the cohomology class of $\xi \in Z^2_0(\mathcal{A}(W), \mathbb{R}^3)$

Also, in the case where rotations are not neglected we have

$X(\alpha_1 \circ \alpha_2) = X(\alpha_1) + R(\alpha_1)(X(\alpha_2)) + \xi''(\alpha_1,\alpha_2)$

Where $\xi'' = \underline{U}(\alpha_1)\, T(\alpha_2)$ and $\xi'' \in C^2_{noR}(\mathcal{A}(W), \mathbb{R}^3)$. It is fairly easy to show that we must have $\xi'' \in Z^2_{noR}(\mathcal{A}(W), \mathbb{R}^3)$ since we now have:-

$\underline{U}(\alpha_1 \circ \alpha_2) = \underline{U}(\alpha_1) + R(\alpha_1) \bullet \underline{U}(\alpha_2)$ i.e.:- $\underline{U} \in Z^1_{noR}\mathcal{A}(W)$ $\mathbb{R}^3$). Whence:- $\delta(\xi'')(\alpha_1,\alpha_2,\alpha_3) = \alpha_1 \cdot \xi''(\alpha_2,\alpha_3) - \xi''(\alpha_1 \circ \alpha_2,$

$\alpha_3) + \xi''(\alpha_1, \alpha_2 \circ \alpha_3) - \xi''(\alpha_1, \alpha_2) = R(\alpha_1)\cdot\underline{U}(\alpha_2)\, T(\alpha_3) - \underline{U}(\alpha_1 \circ \alpha_2)\, T(\alpha_3) + \underline{U}(\alpha_1)T(\alpha_2 \circ \alpha_3) - \underline{U}(\alpha_1)T(\alpha_2)$. Which is $\delta(\xi'')\,(\alpha_1, \alpha_2, \alpha_3) = R(\alpha_1)\cdot\underline{U}\,(\alpha_2)T(\alpha_3) - \underline{U}(\alpha_1)T(\alpha_3) - R(\alpha_1)\cdot\underline{U}\,(\alpha_2)\, T(\alpha_3) + \underline{U}(\alpha_1)(T(\alpha_2) + T(\alpha_3)) - \underline{U}(\alpha_1)T(\alpha_2) = 0$. So that, explicitly $\xi'' \in Z^2_{noR}(\mathfrak{a}(W)'', \mathbb{R}^3)$. The cocycle $\xi \in Z^2_0(\mathfrak{a}(W)', \mathbb{R}^3)$ determines a central extension $\mathbb{R}^3 \otimes_\xi \mathfrak{a}(W)'$ where $\mathfrak{a}(W)' <$ $\mathfrak{a}\,(W)$ omits the rotational subgroup of $E(3,\mathbb{R})$, whilst the cocycle ξ'' defined a non-central non-trivial Abelian extension $\mathbb{R}^3 \boxtimes_{\xi''}$ $\mathfrak{a}(W)''$ when $\mathfrak{a}''(W) = \mathfrak{a}\,(W)/\mathbb{R}^3$. More will be said about these and their related cochains when we discuss the use of Eilenberg Maclane's 'cup products' of cocycles. Thus, bearing in mind that a non-trivial factor systemwill be involved, we shall analyse the structure of $\mathfrak{a}\,(W) \cap I(W)$, which we will write as $G(3,\mathbb{R})$, using the simple method of knowing the group law. The method will closely follow that used by the author in a preprint (Ref.),the first to analyse fully the global structure of the Galilei group. We shall see that due to the peculiar nature of the 2 cocycles ξ, permutations of the underlying sets of $G(3,\mathbb{R})$ enable one to see a semi-direct product group as well as non-trivial-non-central Abelian extensions. The author feels that the analysis of the structure of the Galilei group repays the small effort due to the number of examples of different kinds of group extensions one finds, apart from the physical interest! Precisely why one can 'swap' semi-direct product for non-trivial extensions will emergy in the third section of this paper when we will use some powerful cohomological theorems due to Eilenberg-Maclane and one generalised from a theorem due to Mackey by the author.

Now given the group law on $G(3,\mathbb{R})$:- $(\underline{x}_1, t_1, \underline{v}_1, R_1), (\underline{x}_2, t_2, \underline{v}_2, R_2) \longmapsto (R_1 \cdot \underline{x}_1 + \underline{x}_2 + \underline{v}_1 t_2, R_1 \underline{v}_2 + \underline{v}_1, t_1 + t_2, R_1 R_2)$, it is easy to see that the injections $\alpha_1 : \underline{x} \longmapsto (\underline{x}, 0, 0, e)$; $\alpha_2 : t \longmapsto (0, t, 0, e)$;

$\alpha_3 : \underline{v} \longmapsto (0, 0, \underline{v}, e)$; $\alpha_4 : R \longmapsto (0, 0, 0, R)$ of $\mathbb{R}^3, \mathbb{R}^1, \mathbb{R}^3_T$ and $O(3,\mathbb{R})$ into $G(3,\mathbb{R})$ are, in fact, monomorphisms. To express $G(3,\mathbb{R})$ as an extension must involve a pair of groups and moreover the set theoretic images of these groups must satisfy $G(3,\mathbb{R})/K \rightleftharpoons Q$ and $K \triangleleft G(3,\mathbb{R})$ for a pair (K, Q) which extend to $G(3,\mathbb{R})$. Given the four injections $\langle \alpha_0 \rangle$ $1 \leq i \leq 4$ we can, of course, define combinations e.g.:- $\alpha_1 \times \alpha_2 : \mathbb{R}^3 \times \mathbb{R}^1 \longrightarrow G(3, \quad)$; $\alpha_1 \times \alpha_2 : (\underline{x}, t) \longmapsto (\underline{x}, t, 0, e)$; $\alpha_1 \times \alpha_2 \times \alpha_3 : \mathbb{R}^3 \times \mathbb{R}^1 \times \mathbb{R}^3_T \longrightarrow G(3,\mathbb{R})$; $\alpha_1 \times \alpha_2 \times \alpha_3 : (\underline{x}, t, \underline{v}) \longmapsto (\underline{x}, t, \underline{v}, e)$ etc. Consider those combinations where $\text{Im}(\alpha'_1) \cdot \text{Im}(\alpha'_2) \rightleftharpoons G(3,\mathbb{R})$(has sets) and $\text{Im}(\alpha'_1) \cap \text{Im}(\alpha'_2) = e$. We can form ${}^4C_2 = 6$ injections of the required form:- $\alpha_i \times \alpha_j$; ${}^4C_3 = 4$ injections of the form $\alpha_i \times \alpha_j \times \alpha_k$. That is, we can attempt to extend $\text{Im}(\alpha_i \times \alpha_j \times \alpha_k)$. That is we can attempt to extent $\text{Im}(\alpha_1 \times \alpha_2)$ by $\text{Im}(\alpha_3 \times \alpha_4)$; $\text{Im}(\alpha_1 \times \alpha_3)$ by $\text{Im}(\alpha_2 \times \alpha_4)$; $\text{Im}(\alpha_1 \times \alpha_4)$ by $\text{Im}(\alpha_2 \times \alpha_3)$ or vice versa; and in a similar manner attempt to extend $\text{Im}(\alpha_1)$ by $\text{Im}(\alpha_2 \times \alpha_3 \times \alpha_4)$, $\text{Im}(\alpha_2)$ by $\text{Im}(\alpha_1 \times \alpha_3 \times \alpha_4)$; $\text{Im}(\alpha_3)$ by $\text{Im}(\alpha_1 \times \alpha_2 \times \alpha_4)$ and $\text{Im}(\alpha_4)$ by $\text{Im}(\alpha_1 \times \alpha_2 \times \alpha_3)$ or vice-versa.

There are seven possibilities, where we take the injections and isomorphisms of the sets onto their images equipped with the composition on $G(3,\mathbb{R})$ which the original sets inherit. (e.g.:- α_i are monomorphisms). Let us call the pairs which we defined above $\langle Pi \rangle$ $1 \leq i \leq 7$ in their order of definition. We discuss the pairs, testing elements for normality or as subgroups of $G(3,\mathbb{R})$ and whether or not $G(3,\mathbb{R})$ can be written as an extension of elements of the pair.

P1 This is the pair $(\mathrm{Im}(\alpha_1 \times \alpha_2),\ \mathrm{Im}(\alpha_2 \times \alpha_3)) \cong ((\mathbb{R}^3 \times \mathbb{R}^1), (\mathbb{R}^3_T \times O(3,\mathbb{R}))$. Now consider the group law on $\mathrm{Im}(\alpha_1 \times \alpha_2)$: $(\underline{x}_1, t_1, o, e)(\underline{x}_2, t_2, o, e) = (\underline{x}_1 + \underline{x}_2, t_1 + t_2, o, e)$. Thus we have $\mathbb{R}^3 \otimes \mathbb{R}^1 < G(3,\mathbb{R})$. Similarly $(0, 0, \underline{v}_1\ R_1)(0, 0, \underline{v}_2\ R_2) = (0, 0, R_1\ \underline{v}_2 + \underline{v}_1, R_1R_2)$. So that $\mathbb{R}^3_T \boxtimes {}_n O(3,\mathbb{R}) = E(3,\mathbb{R}) < G(3,\mathbb{R})$. Also $\mathbb{R}^3 \otimes \mathbb{R}^1 \lhd G(3,\mathbb{R})$ and $G(3,\mathbb{R}) / \mathbb{R}^3 \otimes \mathbb{R}^1 \cong E(3,\mathbb{R})$ since $(o, 0, \underline{v}, R)(\underline{x}, t_1, 0, e)(0,0,\underline{v}, R)^{-1} = \mathrm{In}(0,0,\underline{v}, R)\ (\underline{x}, t, 0,0)) = ((\underline{v}, R) : (\underline{x}, t),0,0) = (R\,\underline{x} + \underline{v}\, t, 0, 0)$. Thus $\mathbb{R}^3 \otimes \mathbb{R}_1 \lhd G(3,\mathbb{R})$ and $E(3,\mathbb{R})_T < G(3,\mathbb{R})$, $G(3,\mathbb{R}) / \mathbb{R}^3 \otimes \mathbb{R}^1 \cong E(3,\mathbb{R})_T$ which means that $G(3,\mathbb{R})$ can be written as a semi-direct product:-

$G(3,\mathbb{R}) \cong (\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes g_1\ E(3,\mathbb{R})_T$, where $g \in \mathrm{Hom}(E(3,\mathbb{R}), \mathrm{Aut}(\mathbb{R}^3 \otimes \mathbb{R}^1))$ is defined via $g: (\underline{v}, R): (\underline{x}, t) \longmapsto (R\,\underline{x} + \underline{v}\,t, t)$.

This structure is the one with the structure which bears the easiest physical interpreations. $E(3,\mathbb{R})_T$ plays the role of the Lorentz group in the Poincaré group as an operator group on $\mathbb{R}^3 \otimes \mathbb{R}^1$, whose trivial homogeneous space $\mathbb{R}^3 \otimes \mathbb{R}^1 / \{(0,0)\}$ is isomorphic to W on which $\mathbb{R}^3 \otimes \mathbb{R}^1$ acts via the regular representation.

(P2). We next consider the pair $(\mathrm{Im}(\alpha_1 \times \alpha_3), \mathrm{Im}(\alpha_2 \times \alpha_4))$ or $((\mathbb{R}^3 \times \mathbb{R}^3_T), (\mathbb{R}^1 \times O(3,\mathbb{R}))$. Now $(\underline{x}_1, \underline{v}_1, 0, E)\ (\underline{x}_2, v_2, 0, e)$ $G(3,\mathbb{R})$. Similarly $(0, t_1, 0, R_1)(0, t_2, 0, R_2) = (0, t_1 + t_2, 0, R_1 \circ R_2)$ so that also $\mathbb{R}^1 \otimes O(3,\mathbb{R}) < G(3,\mathbb{R})$. Also $\mathbb{R}^3 \otimes \mathbb{R}^3_T \triangleleft G(3,\mathbb{R})$, $\mathbb{R}^1 \otimes O(3,\mathbb{R}) < G(3,\mathbb{R})$ and $G(3,\mathbb{R})/\mathbb{R}^3 \otimes \mathbb{R}^3_T \cong \mathbb{R}^1 \otimes O(3,\mathbb{R})$, where $\mathrm{In}(0, 0, t, R)((\underline{x}, \underline{v}, 0, e)) \equiv (g_2(t, R)(\underline{x}_1, \underline{v}, \underline{v}), 0, e) = (R(\underline{x} - \underline{v}\ t), R\ \underline{v}, 0, e)$, $g_2 \in \mathrm{Hom}(\mathbb{R}^1 \otimes O(3,\mathbb{R}), \ \mathrm{Aut}(\mathbb{R}^3 \otimes \mathbb{R}^3_T)$.
So again $G(3,\mathbb{R})$ may be regarded as a semi-direct product:-

$$(\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes_{g_2} (\mathbb{R}^1 \otimes O(3,\mathbb{R}))$$

(P3) This is the pair $(\mathrm{Im}(\alpha_1 \times \alpha_4), \mathrm{Im}(\alpha_2 \times \alpha_3))$. Now $(\underline{x}_1, 0, 0, R_1)(\underline{x}_2, 0, 0, R_2) = (\underline{x}_1 + R_1\ x_2, 0, 0, R_1, R_2)$ so that $E(3,\mathbb{R}) < G(3,\mathbb{R})$. Also $(0, t_1, \underline{v}, e)\ (0, t_2, \underline{v}_2, e) = (\underline{v}_1, t_2, t_1+t_2, \underline{v}_1 + \underline{v}_2, e)$ so that $\mathbb{R}^1 \otimes \mathbb{R}^3_T \ntriangleleft G(3,\mathbb{R})$, moreover $E(3,\mathbb{R}) \ntriangleleft G(3,\mathbb{R})$ since $\mathrm{In}(0, t, \underline{v}, e) : (\underline{x},0,0,R) \longmapsto (\underline{x} + R\ \underline{v}\ t - \underline{v}\ t, 0, \underline{v} - R\ \underline{v})\ R)$ so that $G(3,\mathbb{R})$ cannot be expressed as an extension involving P .
(P4) P4 is the pair $(\mathbb{R}^3, \mathbb{R}^1 \times \mathbb{R}^3_T \times O(3,\mathbb{R}))$. We have $\mathbb{R}^3 < G(3,\mathbb{R})$ and $\mathbb{R}^3 \triangleleft G(3,\mathbb{R})$ since $\mathrm{In}(0, t_1\ \underline{v}, R)$: $(\underline{x}_1, 0, 0, e) \longmapsto (R\underline{x}, 0, 0, e)$. Now $(0, t_1, \underline{v}_1\ R_1)(0, t_2, \underline{v}_2, R_2) = (\underline{v}_2, t_2, t_1+t_2, \underline{v}_1 + R_1\ \underline{v}_2, R_1\ R_2)$ so that $G(3,\mathbb{R})/\mathbb{R}^3 \cong E(3,\mathbb{R}) \otimes \mathbb{R}^1$, associated with the injection is a cochain $\xi((t_1, \underline{v}_1, R_1),(t_2,\underline{v}_2, R_2)) \equiv \underline{v}_1\ t_2$ where $\xi \in C^2_N(E(3,\mathbb{R})_T \otimes \mathbb{R}_1, \mathbb{R}^3)$ where $N \in \mathrm{Hom}(E(3,\mathbb{R})_T \otimes \mathbb{R}^1, \mathrm{Aut}(\mathbb{R}^3))$ is defined by $N((\underline{v}, R),t) : \underline{x} \longmapsto R\ \underline{x}\ \forall((\underline{v}, R),t) \in E(3,\ \ ) \times \mathbb{R}^1$,

$\underline{x} \in \mathbb{R}^3$. Moreover $\xi \in Z^2{}_N(E(3,\mathbb{R})_T \otimes \mathbb{R}^1, \mathbb{R}^3)$ since we have:-

$$\delta(\xi)(((\underline{v}_1, R_1),t_1),((\underline{v}_2,R_2),t_2),((\underline{v}_3,R_3),t_3)) =$$

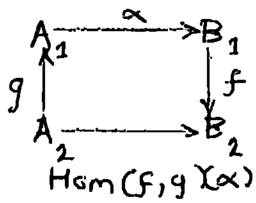
$$N((\underline{v}_1,R_1),t_1))(\ \xi(((\underline{v}_2, k_2),t_2),((\underline{v}_3,R_3),t_3))-$$
$$\xi((\underline{v}_1 + R_1\underline{v}_2, R_1 R_2),t_1+t_2),((\underline{v}_3,R_3),t_3)) +$$
$$\xi(((\underline{v}_1, R_1),t_1), ((\underline{v}_2 + R_2\, \underline{v}_3, R_1 R_3),t_2 + t_3)) -$$
$$\xi(((\underline{v}_1, R_1),t_1),((\underline{v}_2,R_2),t_2)) =$$

$$R_1\underline{v}_2\ t_3 - (\underline{v}_1 + R_1\ \underline{v}_2)\ t_3 + \underline{v}_1(t_2+t_3) - \underline{v}_1\ t_2 = 0.$$

Thus we can write $G(3,\mathbb{R}) \cong \mathbb{R}^3 \boxtimes_{\xi} (E(3,\mathbb{R})_T \otimes \mathbb{R}^1), \xi \in Z^2{}_N(E(3,\mathbb{R})_T \otimes \mathbb{R}_1, \mathbb{R}^3)$.

For the same reason that $G(3,\mathbb{R})$ can have several isomorphic algebraic structures so can $E(3,\mathbb{R})_T \otimes \mathbb{R}^1$. In fact, there are isomorphisms $(\beta_1, \beta_2) \in \mathrm{Hom}(E(3,\mathbb{R})_T \otimes \mathbb{R}^1, (\mathbb{R}^1 \times \mathbb{R}^3{}_T) \boxtimes_{q_1} O(3,\mathbb{R})) \times \mathrm{Hom}(E(3,\mathbb{R})_T \otimes \mathbb{R}^1, \mathbb{R}^3{}_T \boxtimes_{q_2} (O(3,\mathbb{R}) \otimes \mathbb{R}^1))$ where $\beta_1 : ((\underline{v}, R),t) \longmapsto ((t, \underline{v}),R)$ and $\beta_2 : ((\underline{v}, R), t) \longmapsto (\underline{v}, (k, t))\ \forall\ (t, \underline{v}, R) \in \mathbb{R}^1 \times \mathbb{R}^3{}_T \times O(3,\mathbb{R})$. The homomorphisms: $q_1 \in \mathrm{Hom}(O(3,\mathbb{R}), \mathrm{Aut}(\mathbb{R}^1 \otimes \mathbb{R}^3{}_T))$; $q_1(R):(t, \underline{x}) \longmapsto (t, R\underline{v})$ and $q_2(R,t): (\underline{v}) \longmapsto R\underline{v}$, $q_2 \in \mathrm{Hom}(\mathbb{R}^1 \otimes O(3,\mathbb{R}), \mathrm{Aut}(\mathbb{R}^3{}_T))$.

Recall the existence of the functor 'Hom' from $\mathcal{G} \times \mathcal{G}$, where $\mathcal{G}$ is the category of groups, into the category $Ar(\mathcal{G})$ i.e.:- $\forall\ (A,B) \in \mathcal{G} \times \mathcal{G}\ \exists\ \mathrm{Hom}(A, B) \in Ar(\mathcal{G})$ and when $(g, f) \in \mathrm{Hom}(A_2,A_1) \times \mathrm{Hom}(B_1,B_2)$, $\mathrm{Hom}(g, f): \mathrm{Hom}(A_1, B_1) \longrightarrow \mathrm{Hom}(A_2,B_2)$, $\mathrm{Hom}(g, f): \alpha \longmapsto f \circ \alpha \circ g$, i.e.:-



Given N Hom$(E(3,\mathbb{R})_T \otimes \mathbb{R}^1$, Aut$(\mathbb{R}^3))$ then one can define $N_1 \equiv$ Hom$(\beta_1^{-1}, 1\!\!1)(N)$, $N_2 \equiv$ Hom$(\beta_2^{-1}, 1\!\!1)(N)$ in Hom$((\mathbb{R}^1 \otimes \mathbb{R}^3) \boxtimes_{q_1} O(3,\mathbb{R})$, Aut $(\mathbb{R}^3))$ and Hom$(\mathbb{R}^3_T \boxtimes_{q_2}(\mathbb{R}^1 \otimes O(3))$, Aut$(\mathbb{R}^3))$, respectively. Similarly given $\xi \in C^2_N(E(3,\mathbb{R})_T \otimes \mathbb{R}^1, \mathbb{R}^3)$ then there are cochains ξ_1 and ξ_2 in the group $C^2_{N_1}((\mathbb{R}^1 \otimes \mathbb{R}^3_T) \boxtimes_{q_1} O(3,\mathbb{R}), \mathbb{R}^3)$ and $C^2_{N_2}(\mathbb{R}^3_T \boxtimes_{q_2} \mathbb{R}^1 \otimes O(3,\mathbb{R}), \mathbb{R}^3)$ respectively where $\xi_1 \equiv \xi\, O(\beta_1 \times \beta_1)$; $\xi_2 \equiv \xi\, O(\beta_2 \times \beta_2)$, since β_1 and β_2 are isomorphisms, $\xi \in Z^2_N(E(3,\mathbb{R}) \otimes \mathbb{R}^1, \mathbb{R}^3)$ implies that $\xi_1 \in Z^2_{N_1}((\mathbb{R}^1 \otimes \mathbb{R}^3_T) \boxtimes_{q_1} O(3,\mathbb{R}), \mathbb{R}^3)$ and $\xi_2 \in Z^2_{N_2}(\mathbb{R}^3_T \boxtimes_{q_2}(\mathbb{R}^1 \otimes O(3,\mathbb{R})), \mathbb{R}^3)$. So that given the extension $\mathbb{R}^3 \boxtimes_\xi (E(3,\mathbb{R})_T \otimes \mathbb{R}^1)$ is isomorphic to $G(3,\mathbb{R})$, the $E(3,\mathbb{R})_T \times \mathbb{R}^1 - \mathbb{R}^3$ equivalent extensions $\mathbb{R}^3 \boxtimes_{\xi_1} ((\mathbb{R}^1 \otimes \mathbb{R}^3_T) \boxtimes_{q_1} O(3,\mathbb{R}))$ $\xi_1 \in Z^2_{N_1}((\mathbb{R}^1 \otimes \mathbb{R}^3_T) \boxtimes_{q_1} O(3,\mathbb{R}), \mathbb{R}^3)$ and $\mathbb{R}^3 \boxtimes_{\xi_2}(\mathbb{R}^3_T \boxtimes_{q_2} (\mathbb{R}^1 \otimes O(3,\mathbb{R}))$, $\xi_2 \in Z^2_{N_2}(\mathbb{R}^3_T \boxtimes_{q_1}(\mathbb{R}^1 \otimes O(3,\mathbb{R})), \mathbb{R}^3)$ are also isomorphic to $G(3,\mathbb{R})$. Thus $G(3,\mathbb{R})$ cal also be expressed in three equivalent ways as an extension by $\mathbb{R}^3$:-

$$\mathbb{R}^3 \boxtimes_\xi (E(3,\mathbb{R})_T \otimes \mathbb{R}^1),\ \xi \in Z^2_N(E(3,\mathbb{R}) \otimes \mathbb{R}^1, \mathbb{R}^3)$$
$$\mathbb{R}^3 \boxtimes_{\xi_1}((\mathbb{R}^1 \otimes \mathbb{R}^3_T) \boxtimes_{q_1} O(3,\mathbb{R})), \xi_1 \in Z^2_{N_1}((\mathbb{R}^1 \otimes \mathbb{R}^3_T) \boxtimes_{q_1} O(3), \mathbb{R}^3)$$
$$\mathbb{R}^3 \boxtimes_{\xi_2}(\mathbb{R}^3_T \boxtimes_{q_2}(\mathbb{R}^1 \otimes O(3,\mathbb{R}));\ \xi_2 \in Z^2_{N_2}(\mathbb{R}^3 \boxtimes_{q_2}(\mathbb{R}^1 \otimes O(3), \mathbb{R}^3)$$

(P5). We now consider the pair $(\mathbb{R}^1, \mathbb{R}^3 \times \mathbb{R}^3_T \times O(3,\mathbb{R}))$. Now that:- $(t_1, 0, 0, e)(t_2, 0, 0, e) = (t_1 + t_2, 0, 0, e) \Longrightarrow \mathbb{R}^1 < G(3,\mathbb{R})$. However $\mathbb{R}^1 \ntriangleleft G(3,\mathbb{R})$ since In$(\underline{x}, 0, \underline{v}, R)$: $(t, 0, 0, e)$ —— $(\underline{v}\, t, t, 0, e) \notin \mathbb{R}^1$. Also, we have:-

$\underline{x}_1, 0, \underline{v}_1, R_1)(\underline{x}_2, 0, \underline{v}_2, R_2) = (\underline{x}_1 + R_1 \underline{x}_2, 0, \underline{v}_1 + R_2 \underline{v}_2, R_1 R_2)$ so

$(\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes_{k_1} O(3,\mathbb{R})$; (where $k_1 \in \mathrm{Hom}(O(3,\mathbb{R}), \mathrm{Aut}(\mathbb{R}^3 \otimes \mathbb{R}^3_T))$ and $k_1(R): (\underline{v}, \underline{x}) \longmapsto (R\,\underline{x}_1\; R\underline{v})$); $< G(3,\mathbb{R})$. Moreover $(\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes k_1 O(3,\mathbb{R}) \lhd G(3,\mathbb{R})$, since $\mathrm{In}(0, t, 0, e):(\underline{x}, \underline{v}, 0, R) \longmapsto (\underline{x} + \underline{v}\, t, 0, \underline{v}, R)$. With $\mathbb{R}^1 \rightleftharpoons G(3,\mathbb{R})/(\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes_{k_1} O(3,\mathbb{R})$, and $P \in \mathrm{Hom}(\mathbb{R}^1, \mathrm{Aut}((\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes k_1\, O(3,\mathbb{R}))$ defined by $p(t):((\underline{x}, \underline{v}),R) \longrightarrow ((\underline{x} + \underline{v}\, t, \underline{v}), k)$; we can obviously write:-

$$G(3,\mathbb{R}) \rightleftharpoons ((\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes k_1\; O(3,\mathbb{R})) \boxtimes_p \mathbb{R}^1$$

Now the group $(\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes_{k_1} O(3,\mathbb{R})$ has the isomorphic structures $\mathbb{R}^3 \boxtimes_{k_2} E(3,\mathbb{R})_T$ and $\mathbb{R}^3_T \boxtimes k_3\, E(3,\mathbb{R})$. Here, k_2 and k_3 respectively elements of $\mathrm{Hom}(E(3,\mathbb{R})_T, \mathrm{Aut}(\mathbb{R}^3))$ and $\mathrm{Hom}(E(3,\mathbb{R}), \mathrm{Aut}(\mathbb{R}^3_T))$ are defined by $k_2\, (\underline{v}, R): \underline{x} \longmapsto R\,\underline{x}$ and $k_3(\underline{x}, R): \underline{v} \longmapsto R\underline{v}$, $\forall\, (\underline{x}, \underline{v}, R) \in \mathbb{R}^3 \times \mathbb{R}^3_T \times O(3,\mathbb{R})$. Let γ_1 and γ_2 be the isomorphisms $\gamma_1 : ((\underline{x}, \underline{v}), R) \longmapsto (\underline{x}, (\underline{v}, R))$ and $\gamma_2:((\underline{x}, \underline{v}),R) \longmapsto (\underline{v}, (\underline{x}, R))$ of $(\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes k_1\; O\; (3,\mathbb{R})$ onto $\mathbb{R}^3 \boxtimes\; k_2 E(3,\mathbb{R})_T$ and $\mathbb{R}^3_T \boxtimes k_3\, E(3,\mathbb{R})$, respectively. Then, given the homomorphism $p \in \mathrm{Hom}(\mathbb{R}^1, \mathrm{Aut}((\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes k_1\; O(3,\mathbb{R}))$ there exist homomorphisms (p_1, p_2) in the sets $\mathrm{Hom}\,(\mathbb{R}^1, \mathrm{Aut}(\mathbb{R}^3 \boxtimes k_2 E(3,\mathbb{R})_T))$ and $\mathrm{Hom}(\mathbb{R}^1, \mathrm{Aut}(\mathbb{R}^3_T \boxtimes k_2 E(3,\mathbb{R}))$ respectively, where $p_1(t) = \mathrm{Hom}(\gamma_1^{-1}, \gamma_1)(p(t))$ and $p_2(t) = \mathrm{Hom}(\gamma_2^{-1}, \gamma_2)(p(t))\ \forall\ t \in \mathbb{R}^1$. The isomorphisms $\gamma_1 \times \mathbb{1}$ and $\gamma_2 \times \mathbb{1}$ set up equivalences between the extensions defined by p_1 and p_2 of $\mathbb{R}^1$ by $\mathbb{R}^3 \boxtimes_{k_3} E(3,\mathbb{R})_T$ and $\mathbb{R}^3_T \boxtimes k_3\; E(3,\mathbb{R})$ and whence we can assign to $G(3,\mathbb{R})$ the structure of the three equivalent extensions

$((\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes k_1 \; O(3,\mathbb{R})) \boxtimes p \; \mathbb{R}^1$

$(\mathbb{R}^3 \boxtimes k_2 \; E(3,\mathbb{R})_T) \boxtimes p_1 \mathbb{R}^1$

$(\mathbb{R}^3_T \boxtimes k_3 \; E(3,\mathbb{R})) \boxtimes p_2 \mathbb{R}^1$

<u>(P6)</u>. Here we consider the pair $(\mathbb{R}^3_T, \mathbb{R}^3 \times \mathbb{R}^1 \times O(3,\mathbb{R}))$. Now, $(0,0, \underline{v}_1, e)(0,0, \underline{v}_2\, e) = (0,0, \underline{v}_1, +\underline{v}_2, e)$ means that $\mathbb{R}^3_T < G(3,\mathbb{R})$. However $\mathbb{R}^3_T \not\triangleleft G(3,\mathbb{R})$ since $In(\underline{x}, t, 0, R) : (0, 0, \underline{v}, e) \longmapsto (-R\, \underline{v}\, t, 0, R\, \underline{v}, e) \notin \mathbb{R}^3_T$. Also $(\underline{x}_1, t_1, 0, R)(\underline{x}_2, t_2, 0, R_2) = (\underline{x}_1 + R_1\, \underline{x}_2', 0, R_1, R_2) \Longrightarrow \mathbb{R}_1 \otimes E(3,\mathbb{R}) < G(3,\mathbb{R})$. Now $In(0,0,\underline{v},e) : (\underline{x}, t, 0, R) \longmapsto (\underline{x} + \underline{v}\, t, t, \underline{v} - R\, \underline{v}, R) \notin \mathbb{R}^1 \otimes E(3,\mathbb{R})$ so that $\mathbb{R}^1 \otimes E(3,\mathbb{R}) \not\triangleleft G(3,\mathbb{R})$. Thus $G(3,\mathbb{R})$ cannot be expressed as an extension involving this pair. Since $\mathbb{R}^1 \otimes E(3,\mathbb{R}) < G(3,\mathbb{R})$, the groups $(\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes \alpha_1 O(3,\mathbb{R})$ and $\mathbb{R}^3 \boxtimes \alpha_2 (\mathbb{R}^1 \otimes O(3,\mathbb{R}))$, where $\alpha_1(R) : (\underline{x}, t) \longmapsto (R\, \underline{x}, t)$ and $\alpha_2(t, R) : \underline{x} \longmapsto R\underline{x}$, are isomorphic to $\mathbb{R}^1 \otimes E(3,\mathbb{R})$ and are also subgroups.

<u>(P7)</u>. The final pair which we have to consider is the pair $(\mathbb{R}^3 \times \mathbb{R}^1 \times \mathbb{R}^3_T, O(3,\mathbb{R}))$. Now $(0,0,0, R_1)(0,0,0,R_2) = (0,0,0, R_1\, R_2)$ means that $O(3,\mathbb{R}) < G(3,\mathbb{R})$. Let K be the subgroup of elements in $\mathbb{R}^3 \times \mathbb{R}^1 \times \mathbb{R}^3_T$. $K < G(3,\mathbb{R})$ since $(\underline{x}_1, t_1, \underline{v}_1, e)(\underline{x}_2, t_2, \underline{v}_2, e) = (\underline{x}_1 + \underline{x}_2 + \underline{v}_1 t_2, t_1 + t_2, \underline{v}_1 + \underline{v}_2, e)$, and $K \triangleleft G(3,\mathbb{R})$ since $In(0,0,0,R) : (\underline{x}, t, \underline{v}, e) \longmapsto (R\, \underline{x}, t, R\underline{v}, e)$. Let $\beta \in Hom(O(3,\mathbb{R}), Aut(K))$ be defined by $\beta(R) : (\underline{x}, t, \underline{v}) \longmapsto (R\, \underline{x}, t, R\, \underline{v})$. With $G(3,\mathbb{R})/K \cong O(3,\mathbb{R})$, $G(3,\mathbb{R})$ is then an extension of $O(3,\mathbb{R})$ by K.

Let us consider, in some detail, the structure of the kernel K which is rather interesting. Firstly, we note that $\mathbb{R}^3 \triangleleft K$ with $In(0, \underline{v}, t):(\underline{x}, 0, 0) \longmapsto (\underline{x}, 0, 0)$ also $K/\mathbb{R}^3 \doteqdot \mathbb{R}^3_T \otimes \mathbb{R}^1$ with section $j: \mathbb{R}^3_T \otimes \mathbb{R}^1 \longrightarrow K$ defined by $j\_(\underline{v}, t) \longmapsto (0,(\underline{v}, t))$. With $(0, (\underline{v}_1, t_1))(0,(\underline{v}_2, t_2)) \longrightarrow (\underline{v}_1 . t_2,((\underline{v}_1 + \underline{v}_2),t_1 + t_2))$ we can define a two cocycle $\xi \in Z^2{}_0(\mathbb{R}^3_T \otimes \mathbb{R}^1, \mathbb{R}^3)$; $\xi((\underline{v}_1, t_1), (\underline{v}_2, t_2)) = \underline{v}_2\, t_1$. Thus K can be written as a central extension:- $\mathbb{R}^3 \otimes_\xi (\mathbb{R}^3_T \otimes \mathbb{R}^1)$. Next, we see that $\mathbb{R}^3 \otimes \mathbb{R}^1 \triangleleft K$ since $In(0, 0, \underline{v})$: $(\underline{x}, t, 0) \longmapsto (\underline{x} + \underline{v}\, t)$, also $K/\mathbb{R}^3 \otimes \mathbb{R}^1 \doteqdot \mathbb{R}^3_T$ and since $(0, 0, \underline{v}_1)(0,0, \underline{v}_2) = (0,0, \underline{v}_1 + \underline{v}_2), \mathbb{R}^3_T < K$. Thus K can be expressed as a semi-direct product $K \doteqdot (\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes_{\gamma_1} \mathbb{R}^3_T$, where $\gamma_1 \in Hom(\mathbb{R}^3_T, Aut(\mathbb{R}^3 \times \mathbb{R}^1)$ is defined by $\gamma_1(\underline{v}):(\underline{x}, t) \longmapsto (\underline{x} + \underline{v}\, t)\ \forall\, (\underline{v}, \underline{x}, t) \in \mathbb{R}^3_T \times \mathbb{R}^3 \times \mathbb{R}^1$. Lastly, we have $\mathbb{R}^3 \otimes \mathbb{R}^3_T \triangleleft K$ since $(\underline{x}_1, \underline{v}_1, 0)(\underline{x}_2, \underline{v}_2, 0) = (\underline{x}_1 + \underline{x}_2, \underline{v}_1 + \underline{v}_2, 0)$ and $In(0,0,t)$: $(\underline{x}, \underline{v}, 0) \longmapsto (\underline{x} + \underline{v}\, t, \underline{v}, 0)$. Also $K/\mathbb{R}^3 \otimes \mathbb{R}^3_T \doteqdot \mathbb{R}^1$ and $(0, 0, t_1)(0, 0, t_2) = (0, 0, t_1 + t_2)$ implies $\mathbb{R}^1 < K$. So K is once again a semi-direct product:- $(\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes_{\gamma_2} \mathbb{R}_1$ with $\gamma_2 \in Hom(\mathbb{R}^1, Aut(\mathbb{R}^3 \otimes \mathbb{R}^3_T))$ defined by $\gamma_2(t):(\underline{x}, \underline{v}) \longmapsto (\underline{x} + \underline{v}\, t, \underline{v})\ \forall\, (t, \underline{v}, \underline{x}) \in \mathbb{R}^1 \times \mathbb{R}^3_T \times \mathbb{R}^3$. Let σ_1, σ_2 be the isomorphisms $\sigma_1: \mathbb{R}^3 \otimes_\xi (\mathbb{R}^3_T \otimes \mathbb{R}_1) \longrightarrow (\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes \gamma_1 \mathbb{R}^3_T, \sigma_1$: $(\underline{x},(\underline{v}, t)) \longmapsto ((\underline{x}, t),\underline{v})$ and $\sigma_2:(\underline{x}, (\underline{v}, t)) \longmapsto ((\underline{x}, \underline{v}), t)$. Then with $p \in Hom(O,(3,\mathbb{R})$, $Aut(\mathbb{R}^3 \otimes_\xi (\mathbb{R}^3_T \otimes \mathbb{R}_1))$ defined by $p(R)$: $(\underline{x}, (\underline{v}, t)) \longmapsto (R\, \underline{x}, (R\, \underline{v}, t))$, we can define the homomorphisms $p_1 \in Hom(O(3,\mathbb{R}), Aut(\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes \gamma_1 \mathbb{R}^3_T))$; $p_1(R) = Hom(\sigma_1^{-1}, \sigma_1)$

$(p(R))\ \forall\ R \in O(3,\mathbb{R})$; $p_2 \in \text{Hom}(O(3,\mathbb{R}), \text{Aut}(\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes \gamma_2 \mathbb{R}^1))$.
$p_2(R) = \text{Hom}(\sigma_2^{-1}, \sigma_2)(p(R))\ \forall\ R \in O(3,\mathbb{R})$; we can express $G(3,\mathbb{R})$ as the three equivalent extensions:-

$$(\mathbb{R}^3 \otimes_{\xi} (\mathbb{R}^3_T \otimes \mathbb{R}_1)) \boxtimes p\ O(3,\mathbb{R})$$
$$((\mathbb{R}^3 \otimes \mathbb{R}_1) \boxtimes \gamma_1 \mathbb{R}^3_T) \boxtimes p_1\ O(3,\mathbb{R})$$
$$(\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes \gamma_2 \mathbb{R}^1) \boxtimes p_2\ O(3,\mathbb{R})$$

This completes our analysis of the algebraic structure of the Gailiei group for the present. We list the structures we have obtained:-

(1) $(\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes p_1\ E(3,\mathbb{R})_T$. Semi-direct product with $p_1(\underline{v}, R)$: $(\underline{x}, t) \longmapsto (R\,\underline{x} + \underline{v}\,t, t)$

(2) $(\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes p_2(\mathbb{R}^1 \otimes O(3,\mathbb{R}))$. Semi-direct product with $p_2(t, R)$: $(\underline{x}, \underline{v}) \longmapsto (R(\underline{x} - \underline{v}\,t), R\,\underline{v})$

(3) Three equivalent extensions of a non-trivial type with Abelian kernel. Representative is

$\mathbb{R}^3 \otimes_{\xi} (\mathbb{R}^1 \otimes E(3,\mathbb{R})_T)$ where ξ is a 2 cocycle of $Z^2_N(\mathbb{R}^1 \otimes E((3,\mathbb{R}), \mathbb{R}^3)$ where the automorphism N is defined by $N((t),(\underline{v},R))$: $\underline{x} \longmapsto R\,\underline{x}$ and where $\xi((t_1,(\underline{v}_1, R_1)), (t_2,(\underline{v}_2, R_2))) \equiv \underline{v}_1\, t_2$.

(4) A semi-direct product structure, with three equivalent extensions. The representative of the equivalence class is:-

$$(\mathbb{R}^3 \boxtimes_k E(3,\mathbb{R})_T) \boxtimes_{p_3} \mathbb{R}^1$$

where $k(\underline{v}, R): \underline{x} \longmapsto R\,\underline{x}$, $p_3(t): (\underline{x}, (\underline{v}, R)) \longmapsto (\underline{x} + \underline{v}\, t, (\underline{v}, R))$

(5) A semi-direct product structure again. The kernel can be expressed as a central extension or a semi-direct product:-

$$(\mathbb{R}^3 \otimes_{\xi} (\mathbb{R}^3_T \otimes \mathbb{R}_1)) \boxtimes_{p_4} O(3,\mathbb{R})$$

$$((\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes_{\gamma_1} \mathbb{R}^3_T) \boxtimes_{p_4'} O(3,\mathbb{R}).$$

where $\xi \in Z^2_0(\mathbb{R}^3_T \otimes \mathbb{R}^1, \mathbb{R}^3)$; $\xi((\underline{v}, t_1),(\underline{v}_2,t_2)) \equiv \underline{v}_2\, t$

$p_4(R): (\underline{x}, (\underline{v}, t)) \longmapsto (R\,\underline{x}, (R\,\underline{v}, t))$; $\gamma_1(\underline{v}):(\underline{x}, t) \longmapsto (\underline{x}+\underline{v}\, t, t)$;

$p_4'(R)((\underline{x}, t),\underline{v}) \longmapsto ((k\,\underline{x}, t),R\,\underline{v})$.

The structure (1) was introduced in chapter (2). The group-theoretic interest lies in the two representations of the equivalence class in (5). Here a generalisation is suggested, emphasised even more in the next section where we discuss the Carroll group. The interesting point is how a premutation   the cocycle 'ξ' into an automorphism.

Part (ii) The Carroll and Static Groups.

The underlying sets of the Carroll and Static group, which we denote by $C(3,\mathbb{R})$ and $S(3,\mathbb{R})$ respectively, are the same as that of the Galilei group, viz:- $\mathbb{R}^3 \times \mathbb{R}^1 \times \mathbb{R}^3_T \times O(3,\mathbb{R})$. The composition on $C(3,\mathbb{R})$ is:-

$(\underline{x}_1, t_1, \underline{v}_1, R_1)(\underline{x}_2, t_2, \underline{v}_2, R_2) = (\underline{x}_1 + R_1\,\underline{x}_2, t_1+t_2+\underline{v}_1 \cdot R_1\,\underline{x}_2, \underline{v}_1 + R_1\,\underline{v}_2, R_1\,R_2)$ whilst the composition on $S(3,\mathbb{R})$ is:-

$(\underline{x}_1, t_1, \underline{v}_1, R_1)(\underline{x}_2, t_2, \underline{v}_2, R_2) = (\underline{x}_1 + R_1\,\underline{x}_2, t_1 + t_2, \underline{v}_1 + R, \underline{v}_2,$
$R_1\,R_2)$.

The structure of the latter is somewhat more simple than that of the former. In fact, we may write down the structure of the group $S(3,\mathbb{R})$ immediately, following the system of the first section. We write the isomorphic structures as extensions as:-

(1) $(\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes h_1\ E(3,\mathbb{R})_T$

(2) $(\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes h_2\ (\mathbb{R}^1 \otimes O(3,\mathbb{R}))$

(3) $(\mathbb{R}^3 \otimes h_3(\mathbb{R}^1 \otimes E(3,\mathbb{R}))$

(4) $((\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes h_4\ O(3,\mathbb{R})) \otimes \mathbb{R}^1$

(5) $((\mathbb{R}^3 \otimes \mathbb{R}^1) \otimes \mathbb{R}^3_T) \boxtimes h_5\ O(3,\mathbb{R})$.

The decoupling of the Galilean boosts $\mathbb{R}^3_T$ from space-time considerably simplifies the algebraic structure. The homomorphisms $h_1 - h_5$ are defined by

$h_1(\underline{v}, R):(\underline{x}, t) \longmapsto (R\underline{x}, t)$

$h_2(t,R):\ (\underline{x}, \underline{v}) \longmapsto (R\underline{x}, R\underline{v})$

$h_3(t,(\underline{v}, R)):\ \underline{x} \longmapsto R\underline{x}$

$h_4(R):\ (\underline{x}, \underline{v}) \longmapsto (R\underline{x}, R\underline{v})$

$h_5(R):\ ((\underline{x},\underline{v}),t) \longmapsto (h_4(R)(\underline{x}, \underline{v}), t)$

$\forall\ (\underline{x}, t, \underline{v}, R) \in \mathbb{R}^3 \times \mathbb{R}^1 \times \mathbb{R}^3_T \times O(3,\mathbb{R})$.

Let us now turn to an analysis of the structure of the Carroll group, using the techniques of section(1), i.e.:-considering the pairs $\langle P_i \rangle$ $1 \leq i \leq 7$ and attempting to express $C(3,\mathbb{R})$ as an extension involving these pairs.

(P1). The pair P_1 is $(\mathbb{R}^3 \times \mathbb{R}^1, \mathbb{R}^3_T \times O(3,\mathbb{R}))$. Now $(\underline{x}_1, t_1, 0, e)(\underline{x}_2, t_2, 0, e) = (\underline{x}_1 + \underline{x}_2, t_1 + t_2, 0, e)$ implies that $\mathbb{R}^3 \otimes \mathbb{R}^1 < C(3,\mathbb{R})$. Moreover $In(0, 0, \underline{v}, R): (\underline{x}, t, 0, e) \longmapsto (R\,\underline{x}, t+\underline{v}\cdot R\underline{x}, 0, e) \in \mathbb{R}^3 \otimes \mathbb{R}^1$ implies $\mathbb{R}^3 \otimes \mathbb{R}^1 \lhd C(3,\mathbb{R})$. That $C(3,\mathbb{R})/\mathbb{R}^3 \otimes \mathbb{R}^1 \rightleftharpoons E(3,\mathbb{R})_T$ means that $C(3,\mathbb{R})$ is an extension of $E(3,\mathbb{R})_T$ by $(\mathbb{R}^3 \otimes \mathbb{R}^1)$ and that $(0, 0, \underline{v}_1, R_1)(0, 0, \underline{v}_2, R_2) = (0, 0, \underline{v}_1 + R_1\,\underline{v}_2, R_1 R_2)$ implies that $C(3,\mathbb{R})$ is a semi-direct product of $E(3,\mathbb{R})_T$ by $\mathbb{R}^3 \otimes \mathbb{R}^1$, with a $p_1' \in Hom(E(3,\mathbb{R})_T, Aut(\mathbb{R}^3 \otimes \mathbb{R}^1))$ defined by $p_1'(\underline{v}, R): (\underline{x}, t) \longmapsto (R\,\underline{x}, t + \underline{v}\cdot R\underline{x})$. Thus we have $C(3,\mathbb{R}) \rightleftharpoons (\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes_{p_1'} E(3,\mathbb{R})$.

(P2). This is the pair $(\mathbb{R}^3 \times \mathbb{R}^3_T, \mathbb{R}^1 \times O(3,\mathbb{R}))$. Now $(\underline{x}_1, 0, \underline{v}_1, e)(\underline{x}_2, 0, \underline{v}_2, e) = (\underline{x}_1 + \underline{x}_2, \underline{v}_1\cdot \underline{x}_2, \underline{v}_1 + \underline{v}_2, e)$, thus $\kappa_1 \times \alpha_3$ is an injection of $\mathbb{R}^3 \otimes \mathbb{R}^3_T$ into $C(3,\mathbb{R})$ which involves a factor system $\xi \in C^2(\mathbb{R}^3 \otimes \mathbb{R}^3_T, \mathbb{R}^1 \otimes O(3,\mathbb{R}))$ defined by $\xi((\underline{x}_1, \underline{v}_1),(\underline{x}_2,\underline{v}_2)) = (\xi'((\underline{x}_1, \underline{v}_1),(\underline{x}_2, \underline{v}_2)),e))$ where $\xi' \in C^2_0(\mathbb{R}^3 \otimes \mathbb{R}^3_T, \mathbb{R}^1)$. $\mathbb{R}^1 < \mathscr{C}(\mathbb{R}^1 \otimes O(3,\mathbb{R}))$. The three cocycle associated with ξ is a rule which implies that $\delta(\xi') = 0$ or $\xi' \in Z^2_0(\mathbb{R}^3 \otimes \mathbb{R}^3_T, \mathbb{R}^1)$. We can thus see the existence of the central extension $\mathbb{R}^1 \otimes_{\xi} (\mathbb{R}^3 \otimes \mathbb{R}^3_T)$ of $\mathbb{R}^3 \otimes \mathbb{R}^3_T$ by the centre $\mathbb{R}^1$ of $\mathbb{R}^1 \otimes O(3,\mathbb{R})$. However, although $\mathbb{R}^1 \otimes O(3,\mathbb{R}) < C(3,\mathbb{R})$, $\mathbb{R}^1 \otimes O(3,\mathbb{R}) \ntriangleleft C(3,\mathbb{R})$ since $In(\underline{x}, 0, \underline{v}, e)$: $(0, t, 0, R) \longmapsto (\underline{x} - R\,\underline{x}, t + \underline{x}\cdot\underline{v} - \underline{v}\cdot R\,\underline{x}, \underline{v} - R\,\underline{v}, R) \notin \mathbb{R}^1 \otimes O(3)$. Thus $C(3,\mathbb{R})$ cannot be expressed as an extension involving the pair P_2

(P3). We now come to the pair $(\mathbb{R}^3 \times O(3,\mathbb{R}),\ \mathbb{R}^1 \times \mathbb{R}^3_T)$. The fact that:- $(\underline{x}_1, 0, 0, R_1)(\underline{x}_2, 0, 0, R_2) = (\underline{x}_1 + R_1\,\underline{x}_2, 0, 0, R_1\,R_2)$ implies that $E(3,\mathbb{R}) < C(3,\mathbb{R})$. Also $(0, t_1, \underline{v}_1\ e)\ (0, t_2, \underline{v}_2, e) = (0, t_1 + t_2, \underline{v}_1 + \underline{v}_2, e)$ means that $\mathbb{R}^1 \otimes \mathbb{R}^3_T < C(3,\mathbb{R})$. Now $In(\underline{x}, 0, 0, R):(0, t, \underline{v}, e) \longmapsto (0, t - R\,\underline{v} \cdot \underline{x}, R\,\underline{v}, e)$ implies that $\mathbb{R}^1 \otimes \mathbb{R}^3_T \lhd C(3,\mathbb{R})$. Write $g(\underline{x}, R):(t, \underline{v}) \longmapsto (t - R\underline{v}\cdot\underline{x}, R\,\underline{v})$, then $g \in Hom(E(3,\mathbb{R}), Aut(\mathbb{R}^1 \otimes \mathbb{R}^3_T))$. Since $C(3,\mathbb{R})/\,\mathbb{R}^1 \otimes \mathbb{R}^3 \rightleftharpoons E(3,\mathbb{R}) < C(3,\mathbb{R})$, ~~the latter has the structure E(3,) C(3,~~), the latter has the structure of a semi-direct product, $(\mathbb{R}^1 \otimes \mathbb{R}^3_T) \boxtimes_g E(3,\mathbb{R})$.

(P4). P_4 is the pair of sets:- $(\mathbb{R}^3,\ \mathbb{R}^1 \times \mathbb{R}^3_T \times O(3,\mathbb{R}))$. Here $\mathbb{R}^3,\ \mathbb{R}^1 \otimes E(3,\mathbb{R})_T < C(3,\mathbb{R})$ since we have $(\underline{x}_1, 0, 0, e)\ (\underline{x}_2, 0, 0, e) = (\underline{x}_1 + \underline{x}_2, 0, 0, e)$ and $(0, t_1, \underline{v}_1, R_1)(0, t_2, \underline{v}_2, R_2) = (0, t_1 + t_2, \underline{v}_1 + R_1\,\underline{v}_2, R_1\,R_2)$.
Moreover, $\mathbb{R}^1 \otimes E(3,\mathbb{R})_T \ntriangleleft C(3,\mathbb{R})$ since $In(\underline{x}, 0, 0, e):(0, t, \underline{x}, R) \longmapsto (\underline{x} - R\,\underline{x}, t - \underline{v} \cdot R\,\underline{x}, \underline{v}, R) \notin \mathbb{R}^1 \otimes E(3,\mathbb{R})_T$. Also $\mathbb{R}^3 \ntriangleleft C(3,\mathbb{R})$ since $In(0, t, \underline{v}, R):(\underline{x}, 0, 0, e) \longmapsto (k\,\underline{x}, \underline{v} \cdot R\,\underline{x}, 0, e) \notin \mathbb{R}^3$. Whence $C(3,\mathbb{R})$ cannot be expressed as an extension involving the pair P_4.

(P5). Next, we deal with the pair $(\mathbb{R}^1, \mathbb{R}^3 \times \mathbb{R}^3_T \times O(3,\mathbb{R}))$. Now $\mathbb{R}^1 < C(3,\mathbb{R})$, but $\alpha_1 \times \alpha_3 \times \alpha_4$ is an injection only of $(\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes_q O(3,\mathbb{R})$ into $C(3,\mathbb{R})$ where:- $(\underline{x}_1, 0, \underline{v}_1, R_1)(\underline{x}_2, 0, \underline{v}_2, R_2) = (\underline{x}_1 + R_1\,\underline{x}_2, \underline{v}_1 \cdot R_1\,\underline{x}_2, \underline{v}_1 + R_1\,\underline{v}_2, R_1\,R_2$ associated with the injection is the factor system $\xi \in C^2((\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes_q O(3,\mathbb{R}), \mathbb{R}^1)$, where

$\xi(((\underline{x}_1, \underline{x}_1), R_1), ((\underline{x}_2, \underline{v}_2), R_2)) = \underline{v}_1 \cdot R_1 \underline{x}_2$. Now $\mathbb{R}^1 \lhd C(3, \mathbb{R})$ since $In(\underline{x}, 0, \underline{v}_1 R) : (0, t, 0, e) \longmapsto (0, t, 0, e)$, so that $\mathbb{R}^1 < \mathcal{C}(C(3, \mathbb{R}))$. We see that $C(3, \mathbb{R})$ is a central extension of $(\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes q\ O(3, \mathbb{R})$ by $\mathbb{R}_1$ and that we must have $\xi \in Z^2{}_o((\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes q\ O(3, \mathbb{R}), \mathbb{R}^1)$ where $q \in Hom((O(3, \mathbb{R})), Aut(\mathbb{R}^3 \otimes \mathbb{R}^3{}_T))$ is defined by $q(R) : (\underline{x}, \underline{v}) \longmapsto (R\,\underline{x}, R\,\underline{v}) \forall (R_1\ \underline{v}, \underline{x}) \in O(3, \mathbb{R}) \times \mathbb{R}^3_T \times \mathbb{R}^3$. That $\delta(\xi) = 0$, we show explicitly. $\delta(\xi)((((\underline{x}_1, \underline{v}_1), R_1), ((\underline{x}_2, \underline{v}_2), (R_2)), ((\underline{x}_3, \underline{v}_3), R_3)) = \xi(((\underline{x}_2, \underline{v}_2), R_2), ((\underline{x}_3, \underline{v}_3), R_3)) - \xi((\underline{x}_1 + R_1\ x_2, \underline{v}_1 + R_1\ \underline{v}_2), R_1 R_2), ((\underline{x}_3, \underline{v}_3), R_3)) + \xi(((\underline{x}_1, \underline{v}_1), R_1), ((\underline{x}_2 + R_2\ \underline{x}_3, \underline{v}_2 + R_2\ \underline{v}_3, R_2\ R_3))) - \xi((\underline{x}_1, \underline{v}_1), R_1), ((\underline{x}_2, \underline{v}_2), R_2)) = \underline{v}_2 \cdot R_2\ \underline{x}_3 - (\underline{v}_1 + R_1\ \underline{v}_2) \cdot R_1\ R_2\ \underline{x}_3 + \underline{v}_1 \cdot R_1\ (\underline{x}_2 + R_2\ \underline{x}_3) - \underline{v}_1 \cdot R_1\ \underline{x}_2 = \underline{v}_2 \cdot R_2\ \underline{x}_3 - v_1 \cdot R_1\ R_2\ \underline{x}_3 - \underline{v}_2 \cdot R_2\ \underline{x}_3 + \underline{v}_1 \cdot R_1\ \underline{x}_2 + \underline{v}_1 \cdot R_1\ R_2\ \underline{x}_3 - \underline{v}_1 \cdot R_1\ \underline{x}_2 = 0$. So we have explicitly, $\xi \in Z^2{}_o((\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes_q O(3, \mathbb{R}), \mathbb{R}^1)$. The structure of $C(3, \mathbb{R})$ can thus be expressed as a central extension $\mathbb{R}^1 \otimes_\xi ((\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes {}_q O(3, \mathbb{R}))$.

In the same way as in section (1), the structure of $(\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes O(3, \mathbb{R})$ can be expressed in the isomorphic ways. $\mathbb{R}^3 \boxtimes {}_{q1}\ E(3, \mathbb{R})_T$ and $\mathbb{R}^3_T \boxtimes {}_{q2}\ E(3, \mathbb{R})$, which are isomorphic to the group $(\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes {}_q\ O(3, \mathbb{R})$ under β_1, β_2 onto the former. The cochaincs $\xi_1 = \xi 0 (\beta_1{}^{-1} \times \beta_2{}^{-1})$ and $\xi 0 (\beta_2{}^{-1} \times \beta_2{}^{-1})$ are 2 cocycles of $Z^2{}_o(\mathbb{R}^3 \boxtimes {}_{q1} E(3, \mathbb{R})_T, \mathbb{R}^1)$ and of $Z^2_o(\mathbb{R}^3_T \boxtimes q_2\ E(3, \mathbb{R}), \mathbb{R}^1)$ respectively. Thus the isomorphisms $I \times \beta_1$ and $I \times \beta_2$ set up equivalences between the extensions:- $\mathbb{R}^1 \otimes \xi_1\ (\mathbb{R}^3 \boxtimes q_1\ E(3, \mathbb{R})_T)$, $\mathbb{R}^1 \otimes \xi_2 (\mathbb{R}^3_T \boxtimes q_2\ E(3, \mathbb{R}))$ and the extension $\mathbb{R}^1 \otimes_\xi ((\mathbb{R}^3 \otimes \mathbb{R}^3) \boxtimes {}_q\ O(3, \mathbb{R}))$ of

$(\mathbb{R}^3 \otimes \mathbb{R}^3_T) \otimes q\ O(3,\mathbb{R})$ by $\mathbb{R}^1$ to $C(3,\mathbb{R})$.

<u>(P6)</u>. The penultimate pair is $(\mathbb{R}^3_T, \mathbb{R}^3 \times \mathbb{R}^1 \times O(3,\mathbb{R}))$. $C(3,\mathbb{R})$ cannot be expressed as an extension involving this pair since neither $\mathbb{R}^3_T$ or $\mathbb{R}^1 \otimes E(3,\mathbb{R})$ which are subgroups, is invariant. We have:- $In(\underline{x}, t, 0, R):(\underline{v}, 0, \underline{v}, e) \longmapsto (0, R\,\underline{v} \cdot \underline{x}, R\,\underline{v}, e) \notin \mathbb{R}^3_T$ and $In(0, 0, \underline{v}, e) : (\underline{x}, t, 0, R) \longmapsto (\underline{x}, t + \underline{v} \cdot \underline{x}, \underline{v} - R\,\underline{v}, R) \notin \mathbb{R}^1 \otimes E(3,\mathbb{R})$. All we can say is that $\mathbb{R}^3_T, \mathbb{R}^1 \otimes E(3,\mathbb{R}) < C(3,\mathbb{R})$.

<u>(P7)</u>. The final pair is $(O(3,\mathbb{R}), \mathbb{R}^3 \times \mathbb{R}^1 \times \mathbb{R}^3_T)$. Now $O(3,\mathbb{R}) < C(3,\mathbb{R})$ since $(0,0,0,R_1)(0,0,0,R_2) = (0,0,0,R_1 R_2)$, also $O(3,\mathbb{R}) \ntriangleleft C(3,\mathbb{R})$. Let K be the group whose underlying set is $\mathbb{R}^3 \times \mathbb{R}^1 \times \mathbb{R}^3_T$ and where the composition is $(\underline{x}_1, t_1, \underline{v}_1, e)(\underline{x}_2, t_2, \underline{v}_2, e) = (\underline{x}_1 + \underline{x}_2, t_1 + t_2 + \underline{v} \cdot \underline{x}_2, \underline{v}_1 + \underline{v}_2)$. Clearly $K \triangleleft C(3,\mathbb{R})$:- $In(0,0,0,R):(\underline{x}, t, \underline{v}, e) \longmapsto (R\,\underline{x}, t, R\,\underline{v}, e)$ is an automorphism of K and defines $q \in Hom(O((3,\mathbb{R}), Aut(K))$; $q(R): (\underline{x}, t, \underline{v},) \longmapsto (R\,\underline{x}, t, R\,\underline{v},)\ \forall R \in O(3,\mathbb{R})$, $(x, t, \underline{v}) \in \mathbb{R}^3 \times \mathbb{R}^1 \times \mathbb{R}^3_T$. Let us note that:- $C(3,\mathbb{R})/K \cong O(3,\mathbb{R})$ and $O(3,\mathbb{R}) < C(3,\mathbb{R})$ implies $C(3,\mathbb{R})$ is a semi-direct product of $O(3,\mathbb{R})$ by K, and study the algebraic structure of K. First we note that $\mathbb{R}^1 \triangleleft K$ and moreover $\mathbb{R}^1$ being Abelian, is also in the centre of K:- $In(\underline{x}, 0, \underline{v}):(0, t, \underline{v}) \longmapsto (0, t, \underline{v})$. Now $(\underline{x}, \underline{v}) \longmapsto (\underline{x}, 0, \underline{v})$ is a section from $\mathbb{R}^3 \otimes \mathbb{R}^3_T$ into K with a factor set $\xi \in C^2_0(\mathbb{R}^3 \otimes \mathbb{R}^3_T, \mathbb{R}^1)$ defined by $\xi((\underline{x}_1, \underline{v}_1),(\underline{x}_2, \underline{v}_2)) \equiv \underline{v}_1 \cdot \underline{x}_2$. Moreover $\xi \in Z^2_0(\mathbb{R}^3 \otimes \mathbb{R}^3_T, \mathbb{R}^1)$. So that K can be written as a central extension of $\mathbb{R}^3 \otimes \mathbb{R}^3_T$ by $\mathbb{R}^1$, $K \cong \mathbb{R}^1 \otimes_\xi (\mathbb{R}^3 \otimes \mathbb{R}^3_T)$. Again, $\mathbb{R}^3 \otimes \mathbb{R}^1 \triangleleft K$ with

$K/\mathbb{R}^3 \otimes \mathbb{R}^1 \doteq \mathbb{R}^3_T < K$, i.e.:- K is a semi-direct product of $\mathbb{R}^3_T$ by $\mathbb{R}^3 \otimes \mathbb{R}^1$ defined by $q_1 \in \mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Aut}(K))$ where with $\mathrm{In}(0, 0, \underline{v})$: $(\underline{x}, t, 0) \longmapsto (\underline{x}, t + \underline{v} \cdot \underline{x}\ 0)$; $q_1(\underline{v}):(\underline{x}, t) \longmapsto (\underline{x}, t + \underline{v}\cdot\underline{x})$ $\forall\ \underline{v} \in \mathbb{R}^3_T$, $(\underline{x}, t) \in \mathbb{R}^3 \otimes \mathbb{R}^1$. Thus we write $K \doteq (\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes q_1 \mathbb{R}^3_T$. Finally $\mathbb{R}^1 \otimes \mathbb{R}^3_T \lhd K$ and $K/\mathbb{R}^1 \otimes \mathbb{R}^3_T \doteq \mathbb{R}^3 < K$, that $\mathrm{In}(\underline{x}, 0, 0)$: $(0, t, \underline{v}) \longmapsto (0, t - \underline{v}\cdot\underline{x}, \underline{v})$ means that $\exists\ q_2 \in \mathrm{Hom}(\mathbb{R}^3, \mathrm{Aut}(\mathbb{R}^1 \otimes \mathbb{R}^3_T)$ where $q_2(\underline{x})$: $(t, \underline{v}) \longmapsto (t - \underline{v}\cdot\underline{x}, \underline{v}) \forall\ \underline{x} \in \mathbb{R}^3$, $(t, \underline{v}) \in \mathbb{R}^1 \otimes \mathbb{R}^3_T$.

So we have found that $K \doteq \mathbb{R}^1 \otimes_\xi (\mathbb{R}^3 \otimes \mathbb{R}^3_T) \overset{\gamma_1}{\doteq} (\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes q_1 \mathbb{R}^3_T$; $\mathbb{R}^1 \otimes_\xi (\mathbb{R}^3 \otimes \mathbb{R}^3_T) \overset{\gamma_2}{\doteq} (\mathbb{R}^3_T \otimes \mathbb{R}^1) \boxtimes_{q_1} \mathbb{R}^3$ where $\gamma_1 : (t, (\underline{x}, \underline{v})) \longmapsto ((\underline{x}, t), \underline{v})$; γ_2: $(t, (\underline{x}, \underline{v})) \longmapsto ((t, \underline{v}), \underline{x})$ $\forall$ $(t, \underline{x}, \underline{v}) \in \mathbb{R}^1 \times \mathbb{R}^3 \times \mathbb{R}^3_T$. Given $g \in \mathrm{Hom}(O(3,\mathbb{R}), \mathrm{Aut}(K)) \exists (g_1, g_2) \in \mathrm{Hom}(O(3,\mathbb{R}), \mathrm{Aut}((\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes q_1 \mathbb{R}^3_T)), \mathrm{Hom}(O(3,\mathbb{R}), \mathrm{Aut}((\mathbb{R}^1 \otimes \mathbb{R}^3_T) \boxtimes q_2 \mathbb{R}^3)))$ where $\forall\ R \in O(3,\mathbb{R})$, $q_1(k) = \mathrm{Hom}(\gamma_1^{-1}, \gamma_,)(q(k))$; $q_2(k) = \mathrm{Hom}(\gamma_2^{-1}, \gamma_1)(q(k)$. So we have the three equivalent extensions of $O(3,\mathbb{R})$ by K to $C(3,\mathbb{R})$ viz:

$$(\mathbb{R}^1 \otimes_\xi (\mathbb{R}^3 \otimes \mathbb{R}^3_T)) \boxtimes q\ O(3,\mathbb{R})$$
$$((\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes q_1 \mathbb{R}^3_T) \boxtimes_{q_1} O(3,\mathbb{R})$$
$$((\mathbb{R}^1 \otimes \mathbb{R}^3_T) \boxtimes_{q_2} \mathbb{R}^3) \boxtimes q_2\ O(3,\mathbb{R}).$$

Let us list the various structures that we've obtained for the Carroll group. These are:-

(1) A semi-direct product $(\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes g'\ E(3,\mathbb{R})_T$ where $g'(\underline{v}, R)$: $(\underline{x}, t) \longmapsto (R\,\underline{x}, t + \underline{v} \cdot R\,\underline{x})$

(2) A semi-direct product $(\mathbb{R}^1 \otimes \mathbb{R}^3_T) \boxtimes g'' E(3,\mathbb{R})$ where $g''(\underline{x}, R)$: $(\underline{V}, t) \longmapsto (R\underline{v}, t - R\underline{v} \cdot \underline{x})$.

(3) An Abelian central extension $\mathbb{R}^1 \otimes_{\xi} ((\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes g\, O(3,\mathbb{R}))$ where $\xi \in Z^2_0((\mathbb{R}^3 \otimes \mathbb{R}^3)_T \boxtimes g\, O(3,\mathbb{R}), \mathbb{R}^1)$ and $\xi(((\underline{x}_1, \underline{v}_1), R_1), ((\underline{x}_2, \underline{v}_2), R_2)) \equiv \underline{v}_1 \cdot R_1 \underline{x}_2$, and $\forall\ R \in O(3,\mathbb{R})$, $g(R):(\underline{x}, \underline{v}) \longmapsto (R\underline{x}, R\underline{v})$

(4) A semi-direct product of $O(3,\mathbb{R})$ by a kernel which can be expressed either as a central extension by an Abelian kernel or as a semi-direct product :- $C(3,\mathbb{R}) \doteqdot (\mathbb{R}^1 \otimes_{\xi} (\mathbb{R}^3 \otimes \mathbb{R}^3_T)) \boxtimes g\, O(3,\mathbb{R})$ where $\xi \in Z^2{}_0((\mathbb{R}^3 \otimes \mathbb{R}^3_T), \mathbb{R}^1)$ and $\forall\ R \in O(3,\mathbb{R})$, $(t,(\underline{x}, \underline{v})) \in \mathbb{R}^1 \otimes_{\xi} (\mathbb{R}^3 \otimes \mathbb{R}^3_T)$, $g(R):(t,(\underline{x}, \underline{v})) \longmapsto (t, (R\underline{x}, R\underline{v}))$.

We have now completed this section. In the final section of this chapter, we will discuss, in general terms extension of the type we have found for $G(3,\mathbb{R})$, $C(3,\mathbb{R})$ and $S(3,\mathbb{R})$.

PART (iii). Algebraic Theory of 'Kinematical Groups'.

Consider the first members of each list of structures we obtain for the groups $G(3,\mathbb{R})$, $S(3,\mathbb{R})$ and $C(3,\mathbb{R})$. These were of the form $(\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes g\, E(3,\mathbb{R})_T$, where for the group $G(3,\mathbb{R})$ the form of the homomorphism $g \in \mathrm{Hom}(E,(3,\mathbb{R})_{T_1}\ \mathrm{Aut}(\mathbb{R}^3 \otimes \mathbb{R}^1))$ was $g_1(\underline{v}, R):(\underline{x}, t) \longmapsto (R\underline{x} + \underline{v}t, t)$; for $S(3,\mathbb{R})$, $g_3(\underline{v}, R):(\underline{x}, t) \longmapsto (R\underline{x}, t)$ and for $C(3,\mathbb{R})$; $g_3(\underline{v}, R):(\underline{x}, t) \longmapsto (R\underline{x}, t + \underline{v} \cdot R\underline{x})$. The three groups differ in which the group $E(3,\mathbb{R})_T$, sometimes called the 'homogeneous' Galilei Group', acts on the group $\mathbb{R}^3 \otimes \mathbb{R}^1$ of spatio-temporal translations. The underlying set $\mathbb{R}^3 \otimes \mathbb{R}^1$ of the spatio-

- temporal translation group, is just that of the event-world $\mathbb{W}$. The group $E(3,\mathbb{R})_T$ can be looked upon as an operator group on the set W of $\mathbb{W}$, which also acts, in the case of the Galilei groups, as a group of inertial world automorphisms, the static group also filling this role in a trivial sense. In other way we can regard W as the trivial homogeneous space $\mathbb{R}^3 \otimes \mathbb{R}^1 / \{0,0\}$ of the spatio-temporal group on which the latter acts via the regular representation. There is again the clear analogy between the Poincaré group $P(\mathbb{R})$ and these groups $G(3,\mathbb{R})$, $S(3,\mathbb{R})$ and $C(3,\mathbb{R})$, where the role of the homogeneous Galilei group is taken by the Lorentz group or vice-versa. If we consider the causality group:- $C\uparrow(\mathbb{R}) = P\uparrow(\mathbb{R}) \boxtimes_q \mathbb{R}^+m$, or $(\mathbb{R}^4 \boxtimes_{q_1} L\uparrow(\mathbb{R})) \boxtimes_q \mathbb{R}^+m \rightleftharpoons \mathbb{R}^4 \boxtimes_{q_2} (L\uparrow(\mathbb{R}) \otimes \mathbb{R}^+m)$ where $(\Lambda,\alpha) \in L\uparrow(\mathbb{R}) \otimes \mathbb{R}^+m$; $q_2(\Lambda,\alpha): \underline{x} \mapsto \Lambda\alpha x$, then the subgroup $(\mathbb{R}^4 \boxtimes q' \mathbb{R}^+m)$; $q'(\alpha) \equiv q(e,\alpha)\ \forall \alpha \in \mathbb{R}^+m$ states explicitly that $\mathbb{R}^4$ is a vector space over $\mathbb{R}$ which is a highly non-trivial statement, endowing the spatio-temporal translation group as a <u>vector space</u>, isomorphic to Minkowski space. $C\uparrow(\mathbb{R})/L\uparrow(\mathbb{R})$ can be looked at as a vector, in this case. It seems that $C\uparrow(\mathbb{R})$ builds in a lot more physics than does $P(\mathbb{R})$!

In the cases of $G(3,\mathbb{R})$, $S(3,\mathbb{R})$ and $C(3,\mathbb{R})$ the rotation subgroup $O(3,\mathbb{R}) \prec E(3,\mathbb{R})_T$ acts in the natural way as a group of automorphisms of the spatio-temporal group, we factor out this action, and by studying the kernels of the extensions in the seven cases we considered discovered exactly how the kinematical boosts to a moving frame act on

the spatio-temporal group and hence on a space-time. For the Galilei group, the kernel K could be expressed in three ways:-
$(\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes p_1 \mathbb{R}^3_T$, $\mathbb{R}^3 \otimes_\xi (\mathbb{R}^3_T \otimes \mathbb{R}^1)$ $\xi \in Z^2_o(\mathbb{R}^3_T \otimes \mathbb{R}^1, \mathbb{R}^3)$, or $(\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes p_2 \mathbb{R}^1$, where we showed that $p \in \mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Aut}(\mathbb{R}^3 \otimes \mathbb{R}^1))$ was just $p_1(\underline{v}):(\underline{x}, t) \longmapsto (\underline{x} + \underline{v}\, t, t)\ \underline{v} \in \mathbb{R}^3_T$; $(\underline{x}, t) \in \mathbb{R}^3 \otimes \mathbb{R}^1$. Here p_1 specifies $\mathbb{R}^3_T$ as a group of automorphisms of the spatio-temporal group and hence as an operator group on the world-set $\mathbb{W}$.
For the static group, $S(3,\mathbb{R})$, the corresponding kernels were:-
$(\mathbb{R}^3 \otimes \mathbb{R}^1) \otimes \mathbb{R}^3_T$, $\mathbb{R}^3 \otimes (\mathbb{R}^3_T \otimes \mathbb{R}_1)$ and $(\mathbb{R}^3 \otimes \mathbb{R}^3_T) \otimes \mathbb{R}^1$. Thus we see explicitly that $\mathbb{R}^3_T$ acts trivially in this case, greatly simplifing the algebra, but not allowing inertial motion to occur. In the case of the Carroll group, we wrote $\mathbb{R}^1 \otimes_{\xi^1} (\mathbb{R}^3 \otimes \mathbb{R}^3_T)$, $\xi^1 \in Z^2_o(\mathbb{R}^3 \otimes \mathbb{R}^3_T, \mathbb{R}_1)$; $(\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes q_1 \mathbb{R}^3_T$ and $(\mathbb{R}^3_T \otimes \mathbb{R}^1) \boxtimes q_2 \mathbb{R}^3$ where we defined $q_1(\underline{v}):(\underline{x}, t) \longmapsto (\underline{x}, t + \underline{v} \cdot \underline{x})\ \forall\ \underline{v} \in \mathbb{R}^3_T$ and $(\underline{x}, t) \in \mathbb{R}^3 \otimes \mathbb{R}^1$.
In the Carroll, Static and Galilei groups then, the essential differences arise in the way in which the group of pure boosts operates on the spatio-temporal group and hence on space-time. The actions of the former group allows us to form a semi-direct product group in a natural way, and then we find that the semi-direct product group is expressible as a central extension of a permutation of the three underlying sets, or again as a semi-direct product.

Recall how in Newtonian relativity one could picture time as an absolute independent parameter labelling instants. We note that in the Galilei group, the pure boosts act 'as if' $\mathbb{R}^3$, the subgroup of spatial translations of the spatio-temporal group was characteristic:-

$\mathbb{R}^3 \lhd \mathbb{R}^3 \otimes \mathbb{R}^1$, the rotations naturally preserving this characteristic nature:- $\alpha(\mathbb{R}^3) = \mathbb{R}^3 \ \forall \ \alpha \in \mathrm{Aut}(\mathbb{R}^3 \otimes \mathbb{R}^1)$. In the Carroll group, the roles of space and time are swapped, each automorphism of $\mathbb{R}^3 \otimes \mathbb{R}^1$ induced by $\mathbb{R}^3_T$ leaving $\mathbb{R}^1$ intact i.e.:- $\mathbb{R}^1 \lhd \mathbb{R}^3 \otimes \mathbb{R}^1$, the rotations also preserve this intactness as they act trivially on $\mathbb{R}^1$. In the static group where $\mathbb{R}^3_T$ acts trivially we have $\mathbb{R}^3$, $\mathbb{R}^1 \lhd \mathbb{R}^3 \otimes \mathbb{R}^1$. In all cases, the pure boosts operate trivially on the separate subgroups. In this section we attempt to explain and generalise the results of the first two sections using the notion of $\mathbb{R}^3_T$ 'characteristic' subgroups. After discussing the algebra in a simple-minded way, we will corrobarate our results using the notions of G enlargement theory.

Pure Galilei boosts as an operator group on Space-Time

In this part of section (3) we will compute all 'physically acceptible' automorphisms of the spatio-temporal group induced by the Abelian group of pure Galilei boosts. We will call such an automorphism 'acceptible' iff the automorphisms of 'space'and the automorphisms of 'time' it induces are trivial. That is we will require the Galilian group to operate simply on space axes and time axes.

One can formally picture three-types of classical space-time; a world with absolute time, a world with absolute space or a world where both are absolute. We shall take the meaning of 'absoluteness' as follows. That time is absolute we take to mean that $\mathbb{R}^3 \lhd \mathbb{R}^3 \otimes \mathbb{R}^1$, space is absolute $\mathbb{R}^1 \lhd \mathbb{R}^3 \otimes \mathbb{R}^1$ and that 'both are absolute' that

$\mathbb{R}^1 \lhd \mathbb{R}^3 \otimes \mathbb{R}^1$ and $\mathbb{R}^3 \lhd \mathbb{R}^3 \otimes \mathbb{R}^1$. The symbol '$\lhd$' can be called '$\mathbb{R}^3_T$ absoluteness'. Assume then, that $\mathbb{R}^3_T$ is a group of automorphisms of $\mathbb{R}^3 \otimes \mathbb{R}^1$, and that space, time, or both, are absolute.

(a) <u>'Absolute Time':- $\mathbb{R}^3 \lhd \mathbb{R}^3 \otimes \mathbb{R}^1$</u>

Select an $F \in \mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Aut}(\mathbb{R}^3 \otimes \mathbb{R}^1))$. We want to compute the $F(\underline{v})(\underline{x}, t) \forall \underline{v} \in \mathbb{R}^3_T$, $(\underline{x}, t) \in \mathbb{R}^3 \otimes \mathbb{R}^1$ when $\mathbb{R}^3 \lhd \mathbb{R}^3 \otimes \mathbb{R}^1$. That $\mathbb{R}^3 \lhd \mathbb{R}^3 \otimes \mathbb{R}^1$ implies $\forall \Phi \in \mathrm{Aut}(\mathbb{R}^3 \otimes \mathbb{R}^1), \exists \phi \in \mathrm{Aut}(\mathbb{R}^3)$ where, if $i: \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \otimes \mathbb{R}^1 \,\}\, i : \underline{x} \mapsto (\underline{x}, 0) \forall \underline{x} \in \mathbb{R}^3$, $\Phi \circ i \equiv i \circ \phi$. Whence, given $F \in \mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Aut}(\mathbb{R}^3 \otimes \mathbb{R}^1)) \exists f \in \mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Aut}(\mathbb{R}^3)) \ni F(\underline{v}) \circ i = i \circ f(\underline{v}) \forall \underline{v} \in \mathbb{R}^3_T$; i.e.:- $F(\underline{v})(\underline{x}, 0) \equiv (f(\underline{v})(\underline{x}), 0) \forall \underline{x} \in \mathbb{R}^3$. Define $F(\underline{v})(0, t) \equiv (\phi(\underline{v})(t), \Phi(\underline{v})(t))$ where $\phi \in C^1(\mathbb{R}^3_T, C^1_0(\mathbb{R}^1, \mathbb{R}^3_T))$ and $\Phi \in C^1(\mathbb{R}^3_T, \mathrm{Sym}(\mathbb{R}^1))$, $\mathrm{Sym}(\mathbb{R}^1)$ being the set of permutations of $\mathbb{R}^1$. Since $F(\underline{v})$ is an automorphism, we must have $F(\underline{v})(\underline{x}, t) = F(\underline{v})(\underline{x}, 0)F(\underline{v})(0, t)$, whence $F(\underline{v})(\underline{x}, t) = (f(\underline{v})(\underline{x}), 0) \ (\phi(\underline{v})(t), \Phi(\underline{v})(t)) = (f(\underline{v})(\underline{x}) + \phi(\underline{v})(t), \Phi(\underline{v})(t))$. Also $F(\underline{v})(0, t_1) F(\underline{v})(0,t_2) = F(\underline{v})(0,t+t_2)$ means that $(\phi(\underline{v})(t_1+t_2), \Phi(\underline{v})(t_1+t_2)) = (\phi(\underline{v})(t_1), \Phi(\underline{v})(t_1))(\phi(\underline{v})(t_2), \Phi(\underline{v})(t_2)) = (\phi(\underline{v})(t_1) + \phi(\underline{v})(t_2), \Phi(\underline{v})(t_1) + \Phi(\underline{v})(t_2))$. Thus we surmise that (i) $\phi(\underline{v})(t_1 + t_2) = \phi(\underline{v})(t_1)+\phi(\underline{v})(t_2)$ and (ii) $\Phi(\underline{v})(t_1+t_2) = \Phi(\underline{v})(t_1) + \Phi(\underline{v})(t_2)$, so that $\phi \in C^1(\mathbb{R}^3_T, Z^1_0(\mathbb{R}^1, \mathbb{R}^3)) = C^1(\mathbb{R}^3, \mathrm{Hom}(\mathbb{R}^1, \mathbb{R}^3))$ and $\Phi \in C^1(\mathbb{R}^3_T, \mathrm{End}(\mathbb{R}^1))$. Now we also have:- $F(\underline{v}_1 + \underline{v}_2) = F(\underline{v}_1) \circ F(\underline{v}_2)$. Thus $F(\underline{v}_1 + \underline{v}_2)(0, t) = (\phi(\underline{v}_1 + \underline{v}_2)(t), \Phi(\underline{v}_1 + \underline{v}_2)(t)) = F(\underline{v}_1)(\phi(\underline{v}_2)(t), \Phi(\underline{v}_2)(t)) = ((f(\underline{v}_1) \circ \phi(\underline{v}_2))(t) + \phi(\underline{v}_1)(\Phi(\underline{v}_2)(t)), \Phi(\underline{v}_1) \circ \Phi(\underline{v}_2)(t))$. So we have:- $\phi(\underline{v}_1+\underline{v}_2)(t)$

$f(\underline{v}_1)(\phi(\underline{v}_2)(t)) + \phi(\underline{v}_1)(\mathbb{D}(\underline{v}_2)(t))$ and $\mathbb{D}(\underline{v}_1 + \underline{v}_2)(t) = [\mathbb{D}(\underline{v}_1) \circ \mathbb{D}(\underline{v}_2)]$ (t). The latter shows that $\mathbb{D} \in Z^1_0(\mathbb{R}^3_T, \mathrm{End}(\mathbb{R}^1)) = \mathrm{Hom}(\mathbb{R}^3_T, \mathrm{End}(\mathbb{R}^1))$ and since $\mathbb{D}(\underline{v})^{-1} = \mathbb{D}(-\underline{v})$ exists and is injective, $\mathbb{D} \in \mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Aut}(\mathbb{R}^1))$. We now inject the first of our physics by requiring that $\mathbb{D} = 0$, which implies for ϕ that $\phi(\underline{v}_1 + \underline{v}_2)(t) = \phi(\underline{v}_1)(t) + f(\underline{v}_1)(\phi(\underline{v}_2)(t))$ or $\phi(\underline{v}_1 + \underline{v}_2) = \phi(\underline{v}_1) + f(\underline{v}_1) \circ \phi(\underline{v}_2)$ Defining the action of $\mathbb{R}^3_T$ on $\mathrm{Hom}(\mathbb{R}^1, \mathbb{R}^3)$ by $f'(\underline{v})(\beta)(t) \equiv (f(\underline{v}) \circ \beta)(t)\ \forall\ \underline{v} \in \mathbb{R}^3_T$, we must have $\phi \in Z^1_{f'}(\mathbb{R}^3_T, \mathrm{Hom}(\mathbb{R}^1, \mathbb{R}^3))$. Whence if we make the physical assumption that f = 0, we have $\phi \in \mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Hom}(\mathbb{R}^1, \mathbb{R}^3))$. Thus, with $\mathbb{R}^3_T$ operating simply on $\mathbb{R}^3$ and on $\mathbb{R}^1$ with $\mathbb{R}^3 \lhd \mathbb{R}^3 \otimes \mathbb{R}^1$ we must have $F(\underline{v})(\underline{x}, t) \longmapsto (\underline{x} + \phi(\underline{v})(t), t)$ where $\phi \in Z^1{}_0(\mathbb{R}^3_T, \mathrm{Hom}(\mathbb{R}^1, \mathbb{R}^3))$. This correspondance sets up a map from the set T of all such acceptible homomorphisms of $\mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Aut}(\mathbb{R}^3 \otimes \mathbb{R}^1))$ onto $\mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Hom}(\mathbb{R}^1, \mathbb{R}^3))$. Call this $\phi : \mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Aut}(\mathbb{R}^3 \otimes \mathbb{R}^1)) \longrightarrow \mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Hom}(\mathbb{R}^1, \mathbb{R}^3))$ then $\phi: F \longmapsto \phi(F)$ where $\forall\ \underline{v} \in \mathbb{R}^3_T\ F(\underline{v}):(\underline{x}, t) \longmapsto (\underline{x} + \phi(F)(\underline{v})(t), t)\ \forall\ F \in T$.

(b) <u>'Absolute Space' $\mathbb{R}^1 \lhd \mathbb{R}^3 \otimes \mathbb{R}^1$.</u>

Again choose an $F \in \mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Aut}(\mathbb{R}^3 \otimes \mathbb{R}^1))$ and require $\mathbb{R}^1 \lhd \mathbb{R}^3 \otimes \mathbb{R}^1$. We see that $\exists\ f \in \mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Aut}(\mathbb{R}^1)) \ni F(\underline{v}) \circ j \equiv j \circ f(\underline{v})\ \forall\ \underline{v} \in \mathbb{R}^3_T$, where $j: \mathbb{R}^1 \longrightarrow \mathbb{R}^3 \otimes \mathbb{R}^1$, $j: t \longmapsto (0, t)\ \forall\ t \in \mathbb{R}^1$. Whence we write $F(\underline{v})(0, t) = (0, f(\underline{v})(t))$. The argument now preceeds in an exactly similar way to before. If $F(\underline{v}):(\underline{x}, 0) \longmapsto (A(\underline{v})(\underline{x}), \alpha(\underline{v})(\underline{x}))$, then we must have $A \in \mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Aut}(\mathbb{R}^3))$ and

$\alpha \in Z^1{}_{f'}(\mathbb{R}^3_T, \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^1))$. Thus each acceptible $F \in \mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Aut}(\mathbb{R}^3 \otimes \mathbb{R}^1))$ which leaves space absolute, gives rise to a 1 cocycle $\alpha(F) \in \mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^1))$ i.e.:- α is a map from $\mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Aut}(\mathbb{R}^3 \otimes \mathbb{R}^1))$ onto $\mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^1))$ which is a set isomorphism of the subset $S \subset \mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Aut}(\mathbb{R}^3 \otimes \mathbb{R}^1))$ of acceptible homomorphisms which leave space absolute.

$F(\underline{v}):(\underline{x}, t) \longmapsto (\underline{x}, t + [\alpha(F)(\underline{v})](\underline{x}))$.

(c) Absolute space and Absolute time.

Evidently $S \cap T = \{0\}$, whence the Gailiei group only acts simply on the spatio-temporal group if it is required to leave both absolute. I.e.:- $F \in S \cap T \Rightarrow F(\underline{v}):(\underline{x}, t) \longmapsto (\underline{x}, t)\ \forall\ \underline{v} \in \mathbb{R}^3_T$.

Now $\forall\ F \in \mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Aut}(\mathbb{R}^3 \otimes \mathbb{R}^1))$ we can define the trivial group extensions $(\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes_F \mathbb{R}^3_T$. That is there is a map $\mathcal{E}$ of $\mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Aut}(\mathbb{R}^3 \otimes \mathbb{R}^1))$ onto the set of all semi-direct products of $\mathbb{R}^3_T$ by $\mathbb{R}^3 \otimes \mathbb{R}^1$; $\mathcal{E} : F \longmapsto (\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes_F \mathbb{R}^3_T$. The set $\mathcal{E}(T)$ of semi-direct products are groups in which $\mathbb{R}^3_T$ acts on $\mathbb{R}^3 \otimes \mathbb{R}^1$ preserving the 'absoluteness' of time. Also the set $\mathcal{E}(S)$ of semi-direct products is the set where the absoluteness of space is preserved. Clearly $\mathcal{E}(S \cap T) = \mathcal{E}(S) \cap \mathcal{E}(T) = \{(\mathbb{R}^3 \otimes \mathbb{R}^1) \otimes \mathbb{R}^3_T\}$. In section (C) we shall relate the groups in $\mathcal{E}(T)$ to a particular family of central extensions $\mathbb{R}^3 \otimes_{\xi} (\mathbb{R}^3_T \otimes \mathbb{R}^1)$, the groups in $\mathcal{E}(S)$ to a family of central extensions of the form $\mathbb{R}^1 \otimes_{\xi'} (\mathbb{R}^3_T \otimes \mathbb{R}^3)$. I.e.:- estalish maps $\mathcal{C}_1$ and $\mathcal{C}_2$ from T and S into $H^2_0(\mathbb{R}^3_T \otimes \mathbb{R}^1, \mathbb{R}^3)$ and into $H^2_0(\mathbb{R}^3_T \otimes \mathbb{R}^3, \mathbb{R}^1)$ respectively.

(c) To establish the injections $\mathcal{C}_i$ note that, $\forall F \in T \exists$ a one cochain $\gamma(F) \in C_0^1(\mathbb{R}^3_T \otimes \mathbb{R}^1, \mathbb{R}^3)$ where $\gamma(F)(\underline{v}, t) \equiv [\phi\ (F)(\underline{v})]$ (t). $\forall$ $(\underline{v}, t) \in \mathbb{R}^3_T \otimes \mathbb{R}^1$. Thus there is a map $\gamma : T \longrightarrow C_0^1$ $(\mathbb{R}^3_T \otimes \mathbb{R}^1, \mathbb{R}^3)$. Similarly note that $\exists$ $\gamma' : S \rightarrow C_0^1(\mathbb{R}^3_T \otimes \mathbb{R}^3, \mathbb{R}^1)$ where $\gamma'(F)(\underline{v}, \underline{x}) \equiv [\alpha(F)(\underline{v})]\ (\underline{x})\ \forall F \in S$, $(\underline{v}, \underline{x}) \in \mathbb{R}^3_T \otimes \mathbb{R}^3$.

Let us state Eilenberg and Maclane's '1st Reduction Theorem'.

Eilenberg-Maclane's Theorem

"Given Abelian groups Π and G, Π acting simply on G, the correspondence:-

$$\sigma_n : H_0^n(\Pi,\ C_0^1(\Pi, G)) \longrightarrow H_0^{n+1}(\Pi; G)$$

σ_n (f) $(\pi_1, \ldots\ldots, \pi_{n+1}) \equiv f(\pi_2, \ldots, \pi_{n+1})\ (\pi_1) \forall f \in H_0^n(\Pi, C_0^1$ $(\Pi, G))$; is a group isomorphism $\forall\ n \in Z^+$."

Recall how we obtained 1 cochains $(\gamma(F), \gamma'(F')) \in C_0^1(\mathbb{R}^3_T \otimes \mathbb{R}^1,$ $\mathbb{R}^3) \times C_0^1\ (\mathbb{R}^3_T \otimes \mathbb{R}^3, \mathbb{R}^1)$ where $(F, F') \in T \times S$. Define a 1 cochain of $C_0^1(\mathbb{R}^3_T \otimes \mathbb{R}^1, C_0^1(\mathbb{R}^3_T \otimes \mathbb{R}^1, \mathbb{R}^3)$ by the rule $[\lambda(F)(\underline{v}_1, t_1)]\ (\underline{v}_2, t_2) \equiv$ $\gamma(F)(\underline{v}_1, t_2)\ \forall\ F \in T$ and $(\underline{v}_1, t_1), (\underline{v}_2, t_2) \in \mathbb{R}^3_T \otimes \mathbb{R}^1$. Then we must have $\lambda(F) \in Z_0^1(\mathbb{R}^3_T \otimes \mathbb{R}^1,\ C_0^1(\mathbb{R}^3_T \otimes \mathbb{R}^1, \mathbb{R}^3))$, since we see that

$[\lambda(F)(\underline{v}_1, t_1)(\underline{v}_2, t_2))]\ (\underline{v}_3, t_3) - [\lambda(F)(\underline{v}_1, t_1)]\ (\underline{v}_3, t_3)$
$- [\lambda(F)(\underline{v}_2, t_2)]\ (\underline{v}_3, t_3) = 0\ \forall\ (\underline{v}_1, t_1), (\underline{v}_2, t_2), (\underline{v}_3, t_3) \in \mathbb{R}^3_T \otimes \mathbb{R}^1$,

i.e.:-

$\lambda(F)(\underline{v}_1 + \underline{v}_2, t_1 + t_2) = \lambda(F)(\underline{v}_1, t_1) + \lambda(F)(\underline{v}_2, t_2)$. This follows from $\gamma(F)((\underline{v}_1 + \underline{v}_2), t_3) - \gamma(F)(\underline{v}_1, t_3) - \gamma(F)(\underline{v}_2, t_3) = \phi(\underline{v}_1 + \underline{v}_2)$

$(t_3) - \Phi(\underline{v}_1)(t_3)$ ... $- \Phi(\underline{v}_2)(t_3) = 0$. In a similar manner, we define a 1 cocycle $\lambda'(F) \in Z_0^1(\mathbb{R}^3_T \otimes \mathbb{R}^3, C_0^1(\mathbb{R}^3_T \otimes \mathbb{R}^3, \mathbb{R}^1))\ \forall\ F \in S$, via $(\lambda'(F)(\underline{v}_1, \underline{x}_1))(\underline{v}_2, \underline{x}_2) \equiv \gamma'(F)(\underline{v}_1, \underline{x}_2) = (\alpha(F)(\underline{v}_1))(\underline{x}_2)$. Using the theorem, we see that the 2 cochains $\xi(F\ )$ and $\xi'(F')$ defined by $\xi(F)((\underline{v}_1, t_1), (\underline{v}_2, t_2)) = (\Phi(F)(\underline{v}_1))(t_2)\ \forall\ F \in T$ and $\xi'(F')((\underline{x}_1, \underline{v}_1), (\underline{x}_2, \underline{v}_2)) = (\lambda'(F)(\underline{x}_1, \underline{v}_1))(\underline{x}_2, \underline{v}_2) = \gamma'(F)(\underline{x}_2, \underline{v}_1) = (\alpha(F)(\underline{v}_1))(\underline{x}_2)\ \forall\ F \in S$ are also two cocycles of $Z_0^2((\mathbb{R}^3_T \otimes \mathbb{R}^1), \mathbb{R}^3)$ and $Z^2_0((\mathbb{R}^3_T \otimes \mathbb{R}^3, \mathbb{R}^1)$ respectively. Thus we have established an injective map $\xi : T \longrightarrow Z_0^2(\mathbb{R}^3_T \otimes \mathbb{R}^1, \mathbb{R}^3)$ and a similar one $\xi' : S \longrightarrow Z_0^2(\mathbb{R}^3_T \otimes \mathbb{R}^3, \mathbb{R}^1)$. So, in effect a pair of maps from T and S into the sets of central extensions of $\mathbb{R}^3_T \otimes \mathbb{R}^1$ by $\mathbb{R}^3$ and of $\mathbb{R}^3_T \otimes \mathbb{R}^3$ by $\mathbb{R}^1$ Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be these maps. Then $\mathcal{C}_1(F) = \mathbb{R}^3 \otimes \xi(F)(\mathbb{R}^3_T \otimes \mathbb{R}^1)$ and $\mathcal{C}_2(F') = \mathbb{R}^1 \otimes \xi'(F')\ (\mathbb{R}^3_T \otimes \mathbb{R}^3)\ \forall\ (F, F') \in T \times S$. Clearly $F'' \in T \cap S$ means that $\xi(F'') = 0$ whence $\mathcal{C}_1(0) = \mathcal{C}_2(0) = \mathbb{R}^3 \otimes (\mathbb{R}^3_T \otimes \mathbb{R}^1)) = \mathbb{R}^1 \otimes\ (\mathbb{R}^3_T \otimes \mathbb{R}^3)$.

We shall discuss the central extensions of this particular type in a little more detail in section (d). To do so, we develope a theorem which is a generalisation, by the author, of one due to Mackey. Thus we postpone our discussion to the end of section (d) We proove our theorem in full generality, as it will be used extensivly in later chapters.

(d). Generalised Mackey[(1.9)] theorem.

"Given the trivial group extension $K \boxtimes p\, Q$ of Q by a group K, specified by $p \in \mathrm{Hom}(Q, \mathrm{Aut}(K))$, each $\nu \in H_0^2(K \boxtimes p\, Q\,, A)$, ($A$ being an Abelian group), can be written down as:-

$$\nu((k_1, q_1),(k_2,q_2)) = \xi_1(k_1, q_1 \cdot k_2) + \xi_2(q_1,q_2) + \gamma(k_2, q_1)$$

where:-

$q_1 \cdot k \equiv p(q)(k)\ \forall\ (q, k) \in Q \times K$ and $(\xi_1, \xi_2) \in (H_0^2(K, A)$ $H_0^2(Q, A))$ and where also the pair (ξ_1, γ) with $\gamma \in C_0^1(K \boxtimes p\, Q, A)$ satisfy:-

(1) $\gamma(k, e) = 0\ \forall\ k \in K$

(2) $\gamma(k_1 \cdot k_2, q) - \gamma(k_1, q) - \gamma(k_2, q) = \xi_1(q \cdot k_1, q \cdot k_2) - \xi_1(k_1, k_2)$

(3) $\gamma(k, q_1 q_2) = \gamma(q_2 \cdot k, q_1) + \gamma(k, q_2)$ ".

Proof Let $\nu \in Z_0^2(K \times p\, Q, A)$. Then there exists a group extension $A \otimes_\nu (K \boxtimes p\, Q)$, where $A < (A \otimes_\nu (K \boxtimes p\, Q))$ and $K \boxtimes p\, Q$ is embedded in the extension via the section $j:(k, q) \longmapsto (0, (k, q))$ and $(0, (k_1, q_1))(0, (k_2, q_2)) = (\nu((k_1, q_1),(k_2,q_2)), (k_1,q_1)(k_2,q_2)) = (\nu((k_1, q_1),(k_2,q_2)),(k_1\, q_1 \cdot k_2, q_1\, q_2))$. We must then have $k \longmapsto (0, (k, e))$ is a section with associated 2 cocycle $\xi_1 \equiv \nu | K \times K$, and $q \longmapsto (0, (e, q))$ a factor system with 2 cocycle $\xi_2 \equiv \nu | Q \times Q$. Also, we must have the result that $(0, (k, e))(0, (e, q)) = (\nu((k, e), (e, q))(k, q)) = (\phi(k, q),(k,q))$ where $\phi \in C_0^1(K \boxtimes p\, Q, A)$. Whence, up to a two coboundary $\delta(\phi) \in B_0^2(K \boxtimes p\, Q, A)$, we must have $(0, (k_1,e))(0, (e,q_1))(0, (k_2,e))(0, (e, q_2)) = (\nu((k_1,q_1),(k_2,q_2)),(k_1 q_1 \cdot k_2, q_1\, q_2))$. The last line can be written as:-

$(\nu((k_1, q_1), (k_2, q_2)), (e, e))(0, (k_1\, q_1 \cdot k_2, q_1\, q_2))$ and then as:-
$(\nu((k_1\, q_1), (k_2, q_2)) - \xi_1(k_1, q_1 \cdot k_2) - \xi_2(q_1, q_2), (e, e)) \cdot (0, (k_1, e))(0, (q_1 \cdot k_2, e))(0, (e, q_1))\,(0, (e, q_2))$.

This is just the identity:-

$$(0, (k_2, e))(0, (e, q_1))\ [(0, (q_1 \cdot k_2, e))]^{-1}\ [(0, (e, q_1))]^{-1} =$$

$(\nu((k_1, q_1), (k_2, q_2)) - \xi_1(k_1, q_1 \cdot k_2) - \xi_2(q_1, q_2), (e, e))$. Now the first expression depends only on k_2 and q_1 and defines a one cocycle $\gamma \in C_0^1(K \boxtimes p\, Q, Q)$, via $\gamma(k_2, q) \equiv \nu((k_1, q_1), (k_2, q_2)) - \xi_1(k_1, q_1 \cdot k_2) - \xi_2(q_1, q_2)$.

We must have:-

$(0, (e, q))\,(0, (k, e))\,(0, (e, q))^{-1} = (\gamma(k, q), (q \cdot k, e))$.

$\forall (k, q) \in K \times Q$. Which leads to (i) $\gamma(k, e) = 0\ \forall\ k \in K$ (ii) $\gamma(k_1\, k_2, q) - \gamma(k_1, q) - \gamma(k_2, q) = \xi_1((q \cdot k_1 . q \cdot k_2) - \xi_1(k_1, k_2)$ and (iii) $\gamma(k, q_1 q_2) = \gamma(q_2 \cdot k, q_1) + \gamma(k, q_2)$. Condition (iii) can be re-written as follows. Define a cochain $\Phi \in C_{p'}^1(Q, C_0^1(K, A))$ (where $\forall\ f \in C_0^1(K, A)$, we define $p'(q^{-1}(f) \equiv f \circ p(q)\ \forall\ q \in Q)$, by $\Phi(q)(k) \equiv \gamma(k, q)$. Condition (iii) is that $\Phi(q_1\, q_2) = q_2^{-1} \cdot \Phi(q_1) + \Phi(q_2)$. Define $\varpi(q) \equiv \Phi(q^{-1})$, then we must have:-

$\varpi(q_1 \cdot q_2) = q_1 \cdot \varpi(q_2) + \varpi(q_1)$ i.e.:- $\varpi \in Z_{p'}^1(Q, C_0^1(K, a))$.

In terms of ϖ condition (i) is that $\varpi(e) = 0$ i.e.: is a normalised cocycle and (ii) reads:-

$\xi_1(q^{-1} \cdot k_1, q^{-1} \cdot k_2) - \xi_1(k_1, k_2) = \delta(\varpi(q))(k_1\, k_2)$. So conversely, we consider the two cochain $\mu \in C_0^2(K \boxtimes p\, Q, A)$ defined by :-

$$\mu((k_1, q_1), (k_2, q_2)) \equiv \xi_1(k_1, q_1 \cdot k_2) + \xi_2(q_1, q_2) + \varpi(q_1^{-1})(k_2)$$

Where $\xi_1 \in Z_0^2(K, A), \xi_2 \in Z_0^2(Q, A)$ and $\Phi \in Z_p^1, (Q, C_0^1(K, A))$ with two coboundary $\delta(\Phi(q))$ defined, $\forall\, q \in Q$ by, $\delta(\Phi(q))(k_1,k_2) = \xi_1(q^{-1}\cdot k_1, q^{-1}\cdot k_2) - \xi_1(k_1,k_2)$ To prove our theorem we have to show that $\mu \in Z_0^2(K \boxtimes p\, Q, A)$ or $\delta(\mu) = 0$, which is:-

$$\delta(\mu)((k_1,q_1),(k_2,q_2),(k_3,q_3)) = \mu((k_2,q_2),(k_3,k_3))$$
$$- \mu((k_1\, q_1\cdot k_2, q_1\, q_2),(k_3, q_3)) + \mu((k_1,q_1),(k_2\, q_2\cdot k_3,q_2\, q_3))$$
$$- \mu((k_1,q_1),(k_2,q_2)) = 0.$$ We must have:-

$$\delta(\mu)((k_1,q_1),(k_2,q_2),(k_3,q_3)) = \xi_1(k_2,q_2\cdot k_3) + \xi_2(q_2,q_3) + \Phi(q_2^{-1})(k_3) - \xi_1(k_1\, q_1\cdot k_2, q_1\, q_2\cdot k_3) - \xi_2(q_1\, q_2,q_3) - \Phi(q_2^{-1}q_1^{-1})(k_3) + \xi_1(k_1, q_1\cdot(k_2\, q_2\cdot k_3)) + \xi_2(q_1,q_2,q_3) + \Phi(q_1^{-1})(k_2\, q_2\cdot k_3) - \xi_1(k_1,q_1\cdot k_2) - \xi_2(q_1,q_2) - \Phi(q_1^{-1})(k_2).$$

Gathering terms in ξ_1, ξ_2 and Φ, we obtain:-

$$\delta(\mu)((k_1, q_1),(k_2,q_2),(k_3,q_3)) = [\xi_1(k_2,q_2\cdot k_3) - \xi_1(k_1\, q_1\cdot k_2,q_1\, q_2\cdot k_3) + \xi_1(k_1, q_1\cdot k_2\, q_1\, q_2\cdot k_3) - \xi_1(k_1,q_1\cdot k_2)] + [\xi_2(q_2,q_3) - \xi_2(q_1\, q_2,q_3) + \xi_2(q_1,q_2\, q_3) - \xi_2(q_1,q_2)] + [\Phi(q_2^{-1})(k_3) - \Phi(q_2^{-1}q_1^{-1})(k_3) - \Phi(q_1^{-1})(k_2) + \Phi(q_1^{-1})(k_2\, q_2\cdot k_2)$$

The square bracketed term in ξ_2 is $\delta(\xi_2)(q_1,q_2,q_3) = 0$ since $\xi_2 \in Z_0^2(Q,A)$. Consider the term $\Phi(q_1^{-1})(k_2\, q_2 \cdot k_3)$ in the third square bracket. We have :-

$$\Phi(q_1^{-1})k_2\, q_2\cdot k_3) = \xi_1(q_1\cdot k_2,q_1\, q_2\cdot k_3) - \xi_1(k_2,q_2\cdot k_3) + \Phi(q_1^{-1})(k_2) + \Phi(q_1^{-1})(q_2\cdot k_3).$$ Inserting this in our three coboundary we have $\delta(\mu)((k_1,q_1),(k_2,q_2),(k_3,q_3)) =$

$$[\xi_1(q_1\cdot k_2,q_1\, q_2\cdot k_3) - \xi_1(k_1\, q_1\cdot k_2,q_1\, q_2\cdot k_3) + \xi_1(k_1,q_1\cdot k_1\, q_1\, q_2\cdot k_3)$$

$- \xi_1(k_1, q_1 \cdot k_2)] + [\Phi(q_2^{-1})(k_3) + \Phi(q_1^{-1})(q_2 \cdot k_3) - \Phi(q_2^{-1} q_1^{-1})$ $(k_3)] =$

$\delta(\xi_1)(k_1, q_1 \cdot k_2, q_1 q_2 \cdot k_3) + [\delta(\Phi)(q_2^{-1} \cdot q_1^{-1})](k_3) \equiv 0$

Since $\xi_1 \in Z_0^2(K, A)$ and $\Phi \in Z_{p'}^1(Q, C_0^1(K, A))$. Thus $\delta(\mu)((k_1, q_1),$ $(k_2, q_2), (k_3, q_3)) = 0 \; \forall \; k_1, k_2, k_3 \in K; \; q_1, q_2 \; q_3 \in Q$. Thus our theorem can be re-written as:-

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Each $\nu \in H_0^2(K \otimes p\, Q, A)$ where A is Abelian, has the form

$\nu((k_1, q_1), (k_2, q_2)) = \xi_1(k_1, q \cdot k_2) + \xi_2(q_1, q_2) + \Phi(q_1^{-1})(k_2)$

where $\xi_1 \in H_0^2(K, A)$, $\xi_2 \in H_0^2(Q, A)$ and $\Phi \in Z'_{p'}(Q, C_0^1(K, A))$ with associated 2 coboundary of $B_0^2(K, A)$ given by $\delta(\Phi(q))(k_1, k_2) =$ $\xi_1(q^{-1}k_1, q^{-1}k_2), \; - \xi_1(k_1, k_2). \; \forall \; q \in Q$ or $\delta(\Phi(q)) = \xi_1 \circ p(q^{-1}) \times$ $p(q^{-1}) - \xi_1 \forall q \in Q'$.

Let us consider then, the case of a direct product $K \otimes Q$. We must then have $\delta(\Phi(q)) = 0 \; \forall q \in Q$ whence $\Phi(q) \in Z_0^1(K, A)$. Thus each 2 cocycles of $Z_0^2(K \otimes Q, A)$, when A is Abelian, may be expressed as:-

$\nu((k_1, q_1), (k_2, q_2)) = \xi_1(k_1, k_2) + \xi_2(q_1, q_2) + \Phi(q_1^{-1})(k_2)$ where $\xi_1 \in Z_0^2(K, A)$, $\xi_2 \in Z_0^2(Q, A)$ and $\Phi \in Z_0^1(Q, Z_0^1(K, A)) =$ Hom(Q, Hom (K, A)); up to a 2 coboundary of $B^2{}_0(K \otimes Q, A)$. Let $\mathcal{Z}_0^2(K \otimes Q, A)$ be the image of the map ν:Hom(Q, Hom(K, A)) $\longrightarrow Z_0^2(K \otimes Q, A)$;

$\nu(\Phi)((k_1 \; q_1), (k_2, q_2)) \equiv \Phi(q_1^{-1})(k_2) \; \forall \; \Phi \in \mathrm{Hom}(Q, \mathrm{Hom}(K, A))$

Returning to our discussion we see that each element $F \in T$ gives rise to an element $\xi(F) \in \mathcal{Z}_0^2(\mathbb{R}_T^3 \otimes \mathbb{R}^1, \mathbb{R}^3)$ and each $F' \in S$ to an element $\xi'(F') \in \mathcal{Z}_0^2(\mathbb{R}_T^3 \otimes \mathbb{R}^3, \mathbb{R}^1)$ where $\xi'(F') \equiv \nu(\alpha(F'))$, $\xi(F) =$

$\vartheta(\alpha(F))$. $\forall F \in T$, $F' \in S$. Thus there is seen to be a one to one correspondence :-

$$\langle(\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes_F \mathbb{R}^3_T\rangle \; F \in T \longleftrightarrow \langle \mathbb{R}^3 \otimes_\xi (\mathbb{R}^3_T \otimes \mathbb{R}^1)\rangle \; \xi \in H(\mathbb{R}^3_T \otimes \mathbb{R}^1, \mathbb{R}^3) \text{ and}$$

$$\langle(\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes_{F'} \mathbb{R}^3_T\rangle \; F' \in S \longleftrightarrow \langle \mathbb{R}^1 \otimes_{\xi'} (\mathbb{R}^3_T \otimes \mathbb{R}^3)\rangle \xi' \in H^2_o(\mathbb{R}^3_T \otimes \mathbb{R}^3, \mathbb{R}^1), \text{ where } H^2_o \equiv Z^2_o/B^2_o$$

The two coboundaries are interesting in that they in fact describe the arbitraries of the origins of time and velocity. To pursue this extremely interesting point we reserve the final comments of this chapter. Next however, we go onto discuss on the correspondence between the groups Hom($\mathbb{R}^3_T$, Hom($\mathbb{R}^1$, $\mathbb{R}^3$)), and T and Hom($\mathbb{R}^3_T$, Hom($\mathbb{R}^3$, $\mathbb{R}^1$)) and S; in terms of G enlargements in section (e). In this section, we shall also discuss the O(3, $\mathbb{R}$) enlargements of the families of semi-direct products: $\langle(\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes_F \mathbb{R}^3_T\rangle \; F \in S \cup T$. We will only discuss the inessential enlargements in this case.

(e) Connection of G enlargements with (a) and (b).

In sections (a) and (b), we computed the acceptible actions of $\mathbb{R}^3_T$ on the spatio-temporal group, assuming that either space or time or both were absolute in the sense that the adjoint pairs were $\mathbb{R}^3_T$ characteristic. Let us couch the problem in the language of G enlargment theory. We want to compute the $\mathbb{R}^3_T$ enlargements of the spatio-temporal group. The latter is a group extension of $\mathbb{R}^3$ by $\mathbb{R}^1$ or of $\mathbb{R}^1$ by $\mathbb{R}^3$, since it is a direct product. If we regard it as an extension of $\mathbb{R}^3$ by $\mathbb{R}^1$, then the set Enl($\mathbb{R}^3_T$, ($\mathbb{R}^1$, $\mathbb{R}^3$)) =

$H^1_0(\mathbb{R}^3_T, (H^1_0(\mathbb{R}^1, \mathbb{R}^3)) = \mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Hom}(\mathbb{R}^1, \mathbb{R}^3))$ as we deduced earlier. This corresponds to choosing automorphisms $F \in T$. If we regard the spatio-temporal group as an extension of $\mathbb{R}^1$ by $\mathbb{R}^3$ and compute its $\mathbb{R}^3_T$ enlargements when $\mathbb{R}^3_T$ acts trivially on $\mathbb{R}^1 \triangleleft \mathbb{R}^3 \otimes \mathbb{R}^1$ and on $\mathbb{R}^3 \cong \mathbb{R}^3 \otimes \mathbb{R}^1/\mathbb{R}^1$, then $\mathrm{Enl}(\mathbb{R}^3_T, (\mathbb{R}^3, \mathbb{R}^1)) = \mathrm{Hom}(\mathbb{R}^3_T, (\mathbb{R}^3, \mathbb{R}^1))$ as we also deduced earlier. Our earlier results fall immediately out of G enlargement theory.

(f) <u>$O(3,\mathbb{R})$ enlargements of $\mathbb{R}^3_T$ by $\mathbb{R}^3 \otimes \mathbb{R}^1$.</u>

We have constructed the family of groups $\langle(\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes_F \mathbb{R}^3_T\rangle$ $F \in SUT$, corresponding to all possible $\mathbb{R}^3_T$ enlargements of the spatio-temporal group. An isomorphism between the above family of group extensions and the family:-

$\langle \mathbb{R}^3 \otimes_\xi (\mathbb{R}^3_T \otimes \mathbb{R}^1)\rangle$ $\xi \in Z^2_0(\mathbb{R}^3_T \otimes \mathbb{R}^1, \mathbb{R}^3) \cup \langle \mathbb{R}^1 \otimes_{\xi'} (\mathbb{R}^3_T \otimes \mathbb{R}^3)\rangle$ $\xi' \in Z^2_0(\mathbb{R}^3_T \otimes \mathbb{R}^3, \mathbb{R}^1)$, was also shown to exist. The subgroup $Z^2_0(\mathbb{R}^3_T \otimes \mathbb{R}^1, \mathbb{R}^3) < Z^2_0(\mathbb{R}^3_T \otimes \mathbb{R}^1, \mathbb{R}^3)$ consists of all cocycles whose restriction to the invariant subgroups vanish.

The calculations of the $O(3,\mathbb{R})$ enlargements of the semi-direct products $(\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes_F \mathbb{R}^3_T$ will enable us to define semi-direct products $((\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes F\, \mathbb{R}^3_T) \boxtimes N\, O(3,\mathbb{R})$ for those $F \in SUT$ which allow inessential $O(3,\mathbb{R})$ enlargements. We assume the existence of homomorphisms N_1 and N_2:- $N_1 \in \mathrm{Hom}(O(3,\mathbb{R}), \mathrm{Aut}(\mathbb{R}^3 \otimes \mathbb{R}^1))$, $N_2 \in \mathrm{Hom}(O(3,\mathbb{R}), \mathrm{Aut}(\mathbb{R}^3_T)))$ given by $N_1(R): (\underline{x}, t) \longmapsto (R\,\underline{x}, t)$ $N_2(R): \underline{V} \mapsto R\underline{v}$. The inessential enlargements then must have $N(R) = N_1(R) \times N_2(R) \forall R \in O(3,\mathbb{R})$ where $N_1(R) \times N_2(R)$:

$((\underline{x},t),\underline{v}) \longmapsto (N_1(R)(\underline{x}, t), N_2(R)(\underline{v}))\ \forall\ ((\underline{x},t),\underline{v}) \in (\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes_F \mathbb{R}^3{}_T$; $F \in S \cup T$. Let us divide our discussion into two stages case (i) with $F \in T$ and case (ii) with $F \in S$.

Case (i) Here, since N(R) is to be an automorphism of $(\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes_F \mathbb{R}^3{}_T$, we must have:-

$((R\,\underline{x}_1, t_1), R\,\underline{v}_1)((R\,\underline{x}_2,t_2), R\,\underline{v}_2) = (R(\underline{x}_1 + y_2 + \phi(F)(\underline{v}_1)(t_2)$ $t_1 + t_2), R(\underline{v}_1+\underline{v}_2))$. Or $\phi(F)(R\underline{v}_1)(t_2) = R\phi(F)(\underline{v}_1)(t_2). \forall R \in O(3,\mathbb{R})$, $(\underline{v}_1, t_2) \in \mathbb{R}^3_T \times \mathbb{R}^1$, which is just $\phi(F)(R\,\underline{v}) = R \circ \phi(F)(\underline{v})$, $F \in T$

Case (ii) In a similar way to the above, we must have $\alpha(F)(R\underline{v}_1)(R\underline{x}_2) = \alpha(F)(\underline{v}_1)(x_2)$ or $\alpha(F)(R\underline{v})(R\underline{x}) = \alpha(F)(\underline{v})(\underline{x})$, $F \in S$.

Evidently, conditions of a very restrictive nature are placed on the type of automorphism $F \in S \cup T$ allowing of an inessential $O(3,\mathbb{R})$ enlargement. Before we exploit these restrictions, we shall consider these automrophisms in $S \cup T$ which have a physical interpretation in that they can be regarded as peculiar 'inertial world automorphisms'. We shall in fact, only consider those automorphs on T. That $F \in T$ gives rise to inertial automorphisms means that $\ddot{\underline{x}}(t) = 0 \Longrightarrow (\ddot{\underline{x}}^F(t)) = 0$. We have $\underline{x}^F(t) = \underline{x}(t) + \phi(F)(\underline{v})(t) \forall\ F \in T$. Thus $(\underline{x}^F(t))^{\cdot\cdot} = 0$ means that $(\phi(F)(\underline{v})(t))^{\cdot\cdot} = 0$. Or $\phi(F)(\underline{v})(t) = \pi(F)(\underline{v})\,t$, $\pi(E) \in \mathrm{End}(\mathbb{R}^3)$, since $\pi(F)(\underline{v}_1 + \underline{v}_2) = \pi(F)(\underline{v}_1) + \pi(F)(\underline{v}_2)$(since $\phi(F) \in \mathrm{Hom}(\mathbb{R}^3{}_T, \mathrm{Hom}(\mathbb{R}^1, \mathbb{R}^3)))$; $\pi(F)$ must be a matrix of $GL(3,\mathbb{R})$ if we require it to be $\mathbb{R}$ linear which a reasonable requirement of continuity would show. For such elements F which give rise to inertial world automorphisms, we must then require that $\pi(F) \circ R = R \circ \pi(F)$ if an $O(3,\mathbb{R})$ enlargement is to exist. That means that $\pi(F) \in \mathcal{C}(GL(3,\mathbb{R}))(O(3,\mathbb{R}))$

the centraliser in $GL(3,\mathbb{R})$ of $O(3,\mathbb{R})$. Now $O(3,\mathbb{R})$ is irreducible so that $\mathcal{C}(GL(3,\mathbb{R}))(O(3,\mathbb{R})) \cong \mathbb{R}_m$. Whence, the only automorphisms of T which are inertial and allow inessential $O(3,\mathbb{R})$ enlargements are those where $\phi(F)(\underline{v})(t) = \beta(F)\,\underline{v}\,t$, $\beta(F) \in \mathbb{R}$. Also, if we require $\mathbb{R}$ linearity then $\forall F \in S$, $\alpha(F) \in B_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R})$, the space of bilinear functionals on the vector space $\mathbb{R}^3$. The vector space $\mathbb{R}^3 \otimes_{\mathbb{R}} \mathbb{R}^3_T$, where $\mathbb{R}^3 \otimes_{\mathbb{R}} \mathbb{R}^3$ denotes the tensor product of (field) modules, is the dual o f $B_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R})$. Whence, via the conjugate isomorphism $\forall f \in B_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R})$, $\exists\ \underline{A}, \underline{B} \in \mathbb{R}^3$ such that $(\underline{x} \otimes \underline{y})(f) = (\underline{A} \otimes \underline{B})(\underline{x} \otimes y) = (\underline{A} \cdot \underline{x})(\underline{B} \cdot \underline{y})$. This is the most general element, which is $\mathbb{R}$ linear, of the group $Hom(\mathbb{R}^3_T, Hom(\mathbb{R}^3, \mathbb{R}_1)) \supseteq Hom(\mathbb{R}^3_T, (\mathbb{R}^3\ \ *)$. For an $O(3,\mathbb{R})$ enlargement to exist $\alpha(F) \in B_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R})$ must satisfy $\alpha(F)\ o(R \times R) = \alpha(F)\ \forall R \in O(3,\mathbb{R})$. Thus one must have $\alpha(F)(\underline{v}_1\ \underline{x}) \equiv \beta(F)(\underline{v} \cdot \underline{x})$ $\forall\ (\underline{v}, \underline{x})\ \forall \in \mathbb{R}^3_T \times \mathbb{R}^3$, where $\beta\ (F) \in \mathbb{R}$.

We have seen then, that the only acceptible, inertial automorphisms of $\mathbb{R}^3 \otimes \mathbb{R}^1$ induced by $\mathbb{R}^3_T$ which admit $O(3,\mathbb{R})$ enlargement are of the form:-

$F(\underline{v})$: $(\underline{x}, t) \longmapsto (\underline{x} + \beta(F)\underline{v}\ t,\ t)\ \forall\ (\underline{v}, \underline{x}, t) \in \mathbb{R}^3_T \times \mathbb{R}^3 \times \mathbb{R}^1$. Defining the corresponding semi-direct product $(\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes_F \mathbb{R}^3_T$ where the composition is:-

$((\underline{x}_1, t_1), \underline{v}_1)((\underline{x}_2, t_2), \underline{v}_2) = ((\underline{x}_1 + \beta(F)\underline{v}_1\ t_2, t_1 + t_2), \underline{v}_1 + \underline{v}_2)$ we found that there exist $O(3,\mathbb{R})$ enlargements with a natural definition of a semi-direct product using the inessential enlargements:- $((\mathbb{R}^3 \otimes \mathbb{R}^1)) \boxtimes_F \mathbb{R}^3_T) \boxtimes_{N_1 x N_2} O(3,\mathbb{R})$, whose group law is:-

$(((\underline{x}, t_1), \underline{v}_1), R_1)(((\underline{x}_2, \underline{t}_2), \underline{v}_2), R_2) = (((\underline{x}_1 + R_1 \underline{x}_2 + (F)\underline{v}_1 t_2)$ $\underline{v}_1 + R_1 \underline{v}_2), R_1 R_2))$. Similarly there are $\mathbb{R}^3_T$ enlargements of $\mathbb{R}^1$ by $\mathbb{R}^3$ defining semi-direct products $(\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes_F \mathbb{R}^3_T$ when $F \in S$, which admit inessential $O(3, \mathbb{R})$ enlargements. On the group $(\mathbb{R}^3 \otimes \mathbb{R}^1)$ $\boxtimes_F \mathbb{R}^3_T$ the composition is:-

$((\underline{x}_1, t_1), \underline{v}_1)((\underline{x}_2, t_2), \underline{v}_2) = ((\underline{x}_1 + \underline{x}_2, t_1 + \underline{v}_1 \cdot \underline{x}_2), \underline{v}_1 + \underline{v}_2)$

The group law on the semi-direct product $((\mathbb{R}^3 \otimes \mathbb{R}^1)) \boxtimes_F \mathbb{R}^3_T) \boxtimes_{N1 x N_2}$ $O(3, \mathbb{R})$ being:- $(((\underline{x}_1, t_1), \underline{x}_1), R_1)(((\underline{x}_2, t_2), \underline{v}_2), R_2) = ((\underline{x}_1 + R_1 \underline{x}_2,$ $t_1 + t_2 + \beta(F)\, \underline{v}_1 \cdot R_1 \underline{x}_2), \underline{v}_1 + R_1 \underline{v}_2), R_1 R_2$

We establish a lemma in order to proceed with our discussion. This is the so called 'freshman theorem'. Let $K, H \triangleleft G$ with $K < H$ and . The theorem states that $G/H \rightleftharpoons$ $(G/H)/(H/K)$. Evidently G is an extension of G/H by H which is an extension of K by H/K. The theorem thus insists that G is an extension of K by G/K which is an extension of G/H by H/K.

Let us label our $O(3, \mathbb{R})$ enlargements defined above by $\alpha \in \mathbb{R}$. Clearly, in the light of the freshman theorem, we must see that $G(\alpha) \equiv$ $((\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes_{F(\alpha)} \mathbb{R}^3_T) \boxtimes_{N_1 x N_2} O(3, \mathbb{R})$ can be written as the extension $((\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes_{(F(\alpha), N_1)} E(3, \mathbb{R})_T$ where $(F(\alpha), N_1)(\underline{v}, R) \equiv F(\alpha)(\underline{v}) \circ$ $N_1(R)$ i.e. for the class 'T' automorphism $(F(\alpha)(\underline{v}) \circ N_1(R)) : (\underline{x}, t) \longmapsto$ $(R\underline{x} + \alpha \underline{v}\, t, t)$ and for the class S automorphisms,

$F(\alpha)(\underline{v}) \circ N_1(R) : (\underline{x}, t) \longmapsto (R\, \underline{x}, t + \alpha \underline{v} \cdot R\, \underline{x})$

<u>(g)</u> Let us now discuss the same problem as in section f, using however, an Abelian-non-central extension point of view. We saw that, how given an acceptible F T we could write,

$(\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes_F \mathbb{R}^3{}_T \rightleftharpoons \mathbb{R}^3 \otimes_{\xi(F)} (\mathbb{R}^3{}_T \otimes \mathbb{R}^1)$ where $\xi(F) \in \mathcal{Z}^2{}_0(\mathbb{R}^3_T \otimes \mathbb{R}^1, \mathbb{R}^3) < Z^2{}_0(\mathbb{R}^3_T \otimes \mathbb{R}^1, \mathbb{R}^3)$. Also with $F \in S$, then $(\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes_F \mathbb{R}^3{}_T \rightleftharpoons \mathbb{R}^1 \otimes_{\xi'}(F)(\mathbb{R}^3{}_T \otimes \mathbb{R}^3)$ where $\xi'(F) \in \mathcal{Z}^2{}_0(\mathbb{R}^3{}_T \otimes \mathbb{R}^3, \mathbb{R}^1)$. So we can transfer to an Abelian extension problem.

Now there exists a natural map $N' \in \mathrm{Hom}(O(3,\mathbb{R}), \mathrm{Aut}(\mathbb{R}^3 \otimes_{\xi(F)} (\mathbb{R}^3_T \otimes \mathbb{R}^1))$ i.e.:- $N(R): (\underline{x}, (\underline{v}, t)) \longmapsto (R\,\underline{x}, (R\,\underline{v}, t))$. We can regard an extension of $O(3,\mathbb{R})$ by an extension of $\mathbb{R}^3$ by $\mathbb{R}^3_T \otimes \mathbb{R}^1$ as an extension by $\mathbb{R}^3$ of an extension of $O(3,\mathbb{R})$ by $\mathbb{R}^3_T \otimes \mathbb{R}^1$. Also $\forall\, R \in O(3,\mathbb{R})$, $N'(R)$ restricted to $\mathbb{R}^3{}_T \otimes \mathbb{R}^1$ i.e.:- $N_2(R)$ is an automorphism of $\mathbb{R}^3 \otimes \mathbb{R}^1$ which shows that the latter extension is a semi-direct product isomorphic to $\mathbb{R}^1 \otimes E(3,\mathbb{R})_T$. Associated with the injection $\mathbb{R}^1 \otimes E(3,\mathbb{R})_T$ is a factor system $\xi \in C^2_{N_1}(\mathbb{R}^1 \otimes E(3,\mathbb{R}), \mathbb{R}^3)$ where:- $\xi((t_1(\underline{v}_1,R_1),(t_2,(\underline{v}_2,R_2)) = (\xi(F)((\underline{v}_1,t_1), (\underline{v}_2,t_2))$ Where the action of $\mathbb{R}^1 \otimes E(3,\mathbb{R})$ on $\mathbb{R}^3$ is the same as N_1, the kernel of our original semi-direct product being a central extension. Since there is an associativity requirement on the original group extension, the three cocycle of $Z^3{}_{N_1}(\mathbb{R}^1 \otimes E(3,\mathbb{R}), \mathbb{R}^3)$, i.e.:- $\delta(\xi)$,

associated with the two cochain ξ must vanish i.e.:- $\delta(\xi) = 0$. We must then have:- $N(t_1,(\underline{v}_1,k_1))(\quad)$

$$\xi((t_2,(\underline{v}_2,R_2),(t_3,(\underline{v}_3,R_3)))- \xi((t_1+t_2,(\underline{v}_1+R_1\,\underline{v}_2, R_1\,R_2)), (t_3,(\underline{v}_3, R_3)) + \xi((t_1,(\underline{v}_1,R_1)), (t_2+t_3(\underline{v}_2 + R_2\,\underline{v}_3, R_2\,R_3))) - \xi((t_1, (\underline{v}_1,R_1)),((t_2,(\underline{v}_2,R_2))) = 0.$$

Which is :-

$$R_1 \cdot \phi(F)(\underline{v}_2)(t_3) - \phi(F)(\underline{v}_1 + R_1\underline{v}_2)(t_3) + \phi(F)(\underline{v}_1)(t_3\; t_2) - \phi(F)(\underline{v}_1)(t_2) = 0 \qquad R_1(\phi(F)(\underline{v}_2)(t_3)) - \phi(F)(R_1\,\underline{v}_2)\,(t_3) = 0 \text{ or } R_1 \circ \phi(F)$$

$= \Phi(F) \circ R$. Which is the same requirement as before. The same construction applied to the existence of a semi-direct product of $O(3, \mathbb{R})$ by $\mathbb{R}^1 \otimes_{\xi'(F')} (\mathbb{R}^3_T \otimes \mathbb{R}^3)$, $F' \in S$. As before, the condition that the canonical cochain ξ be a cocycle is that $\alpha(F)(R\underline{v})(R\underline{x}) = \alpha(F)(\underline{v})(\underline{x})$ $\forall$ $R \in O(3, \mathbb{R}), (\underline{v}, \underline{x}) \in \mathbb{R}^3_T \otimes \mathbb{R}^3$. Thus, given the groups $\mathbb{R}^3 \otimes_{\xi} (\mathbb{R}^3_T \otimes \mathbb{R}^1)$ and $\mathbb{R}^1 \otimes_{\xi'} (\mathbb{R}^3_T \otimes \mathbb{R}^3)$, we can try to form the semi-direct products by $O(3, \mathbb{R})$. These would be non-central extensions of $\mathbb{R}^3$, $\mathbb{R}^1$ by $\mathbb{R}^1 \otimes E(3, \mathbb{R})_T$, $(\mathbb{R}^3_T \otimes \mathbb{R}^3) \boxtimes_p O(3, \mathbb{R})$ respectively and exist only if the above conditions were true. The latter extension would be central. So we can form the groups $\mathbb{R}^3 \otimes_{\xi(F)} (\mathbb{R}^1 \otimes E(3, \mathbb{R})_T)$ and $\mathbb{R}^1 \otimes_{\xi(F)} ((\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes O(3, \mathbb{R}))$ when the implied $O(3, \mathbb{R})$ enlargements of $\mathbb{R}^3_T$ by $\mathbb{R}^3 \otimes \mathbb{R}^1$ are inessential. They exist only for these F where $\Phi(F)(R\underline{v}) = R \circ \Phi(F)(\underline{v})$ or $\alpha(F)(R\underline{v})(R\underline{x}) = \alpha(F)(\underline{v})(\underline{x})$ $\forall$ $(R, \underline{v}, \underline{x}, t) \in O(3, \mathbb{R}) \times \mathbb{R}^3_T \times \mathbb{R}^3 \times \mathbb{R}^1$.

Using our Mackey theorem, we see that the $\xi(F) \in Z^2_0(\mathbb{R}^1 \otimes E(3, \mathbb{R})_T, \mathbb{R}^3)$ are those whose restriction $O(3, \mathbb{R})$ vanishes, and whose restriction to $\mathbb{R}^3_T \otimes \mathbb{R}_1$ is $\xi(F)$ and whose associated $\gamma \in Z^1_p(O(3, \mathbb{R}), C^1_0(K, A))$ is a two coboundary.

This almost concludes the discussion of chapter (4) from the algebraic point of view. However, a few physical remarks would be pertinent. We see from our discussion that the Galilei group is a member of the family $\langle\langle (\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes_F \mathbb{R}^3_T) \boxtimes_N O(3, \mathbb{R}) \rangle\rangle$ $F \in T'$ where T' is the set of inertial actions of $\mathbb{R}^3_T$ which are physically acceptible and allow inessential $O(3, \mathbb{R})$ enlargements of $\mathbb{R}^3_T$ by $(\mathbb{R}^3 \otimes \mathbb{R}^1)$, parameterizing this family in terms of the real line:-

$F \longmapsto F(\alpha); F(\alpha)(\underline{v}):(\underline{x}, t) \longmapsto (\underline{x} + \alpha\underline{v}\, t, t)$, we see that the Galilei group corresponds to $\alpha = 1$. Again, parameterizing the family of groups $\langle((\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes_F \mathbb{R}^3_T) \boxtimes_N O(3,\mathbb{R})\rangle F \in S$ by $\mathbb{R}$ via $F \longmapsto F(\alpha)$; $F(\alpha)\ (\underline{v})(\underline{x}, t) \longmapsto (\underline{x}, t + \alpha\underline{v}\cdot\underline{x})\ \forall \alpha \in \mathbb{R}$ It seems that the physical interpretation of these groups involves only a re-scaling of the velocity parameter. I.e. if $G(\alpha_1)$ and $G(\alpha_2)$ are world groups of Newtonian relativities, where to transform from one system to another involves $\underline{v} \longmapsto \alpha_2/\alpha_1\ \underline{v}\ \forall\ \underline{v} \in \mathbb{R}^3_T$. This only involves a re-definition of inertial veclocity, for in a world where $G(\alpha)$ is the world group $F(\underline{v}):(0, t) \longmapsto (\alpha\underline{v}\, t, t)$ is a boost to a frame where $\dot{\underline{x}} = \alpha\underline{v}$. Note however that we were able to define inertial boosts parameterized by $\mathbb{R}^3_T$ where $F(\underline{v}):(\underline{x}, t) \longmapsto (\underline{x} + \pi(F)(\underline{v})\cdot t, t)$ Here such a boost transforms $(0, t)$ into the inertial frame where $\dot{\underline{x}} = \pi(\underline{u})$ where $\pi \in GL(3,\mathbb{R})$. One can conserve then, (algebraically at least) of worlds where isotropy ($O(3,\mathbb{R})$ invariance), is violated enabling us to discharge the requirements of incorporating $O(3,\mathbb{R})$ as a subgroup of world automorphisms, where boosts like i.e.:- $\underline{v} \longmapsto \underline{v} + \underline{u} \wedge \underline{A}$, ($\underline{A}$ a constant vector) are feasible, for example, on $\underline{u}$: $\underline{v}\ i \longmapsto v_i + g_{ij} u_j$

Let us give two more 'phenomenological' plausible interpretations of our algebra. Firstly recall that $Hom(B, A)$ can be regarded as listing the no. of ways a group B can be i njected into $A \otimes B$. Replace A by $\mathbb{R}^3$ and B by $\mathbb{R}^1$ and recall that $F \in Hom(\mathbb{R}^3 T, Aut(\mathbb{R}^3 \otimes \mathbb{R}^1))$ is uniquely specified by a $\phi(F) \in Hom(\mathbb{R}^3_T\ Hom(\mathbb{R}^1, \mathbb{R}^3))$, each $\underline{v} \in \mathbb{R}^3_T$

then corresponds to choosing a new injection $\phi(F)(\underline{v})$ of $\mathbb{R}_1$ into $\mathbb{R}^3 \otimes \mathbb{R}^1$ that is:- $\underline{v}: t \longmapsto (\phi(F)(\underline{v})(t), t) \equiv (\pi(\underline{v})t, t)$. Or on the world set W, of a new section of the time axis into W of the family $\langle \sigma(t) \rangle_{t \in \mathbb{R}^1}$ of instants into W. Compare this with the notion of path $\underline{x}: \mathbb{R}^1 \to \mathbb{R}^3$, $x(t) \in \sigma(t) \forall t \in \mathbb{R}^1$; $\underline{x} \in C^1(\mathbb{R}^1, \mathbb{R}^3)$ is called the path of $\underline{x}$ in W. $\underline{x} \in \mathrm{Hom}(\mathbb{R}^1, \mathbb{R}^3)$ when $\underline{x}(t_1 + t_2) = x(t_1) + \underline{x}(t_2)$ i.e. when $\ddot{\underline{x}}(t) = 0$ $\underline{x}(t) = \underline{u}\, t$; $\underline{u} \equiv$ 'velocity'. Let us now attempt to glean some information from the fact that $(\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes_F \mathbb{R}^3_T \rightleftharpoons \mathbb{R}^3 \otimes_\xi (\mathbb{R}^3_T \otimes \mathbb{R}^1)$, where $\xi \in Z^2_0(\mathbb{R}^3_T \otimes \mathbb{R}^1, \mathbb{R}^3)$. Let $\Phi \in C^1_0(\mathbb{R}^3_T \otimes \mathbb{R}^1, \mathbb{R}^3)$, then the two groups:- $\mathbb{R}^3 \otimes_\xi (\mathbb{R}^3_T \otimes \mathbb{R}^1)$ and $\mathbb{R}^3 \otimes_{\xi + \delta(\Phi)} (\mathbb{R}^3 \otimes \mathbb{R}^1)$ are equivalent extensions and hence isomorphic. Thus the sections $(\underline{v}, t) \longmapsto (0, (\underline{v}, t))$ and $(\underline{v}, t) \longmapsto (\Phi(\underline{v}, t), (\underline{v}, t))$ induce the same physics!

The final section of this chapter is concerned with a discussion of the notion of cup-product for cocycles, and their relevence to the work of this and ensueing chapters. Let us first formulate the notion of cup product, which like almost all the other algebraic techniques of cohomology theory in groups, due to Eilenberg and Maclane.

<u>(h). 'Cup Products' of Cochains</u>.

Let Π_1, Π_2 and Π be three additive Abelian groups each with a group G as a group of left operators, defined as such via $p, p_1, p_2 \in \mathrm{Hom}(G, \mathrm{Aut}(\Pi_i))$ $i = o, 1, 2$. Π_1 and Π_2 and said to be <u>'paired to Π'</u> if $\forall (\pi_1, \pi_2) \in \Pi_1 \times \Pi_2$, $\exists$ an element $\pi_1 \cup \pi_2 \in \Pi$ such that:-

(i) $\pi_1 \cup (\pi_2 + \pi_2') = \pi_1 \cup \pi_2 + \pi_1 \cup \pi_2'$

(ii) $(\pi_1 + \pi_1') \cup \pi_2 = \pi_1 \cup \pi_2 + \pi_1' \cup \pi_2$

(iii) $p(g)(\pi_1 \cup \pi_2) = p_1(g)(\pi_1) \cup p_2(g)(\pi_2) \; \forall g \in G$.

Let us now consider two cochains $f_1^n \in C^n_{p1}(G,\pi_1)$ $f_2^m \in C^m_{p2}(G,\pi_2)$. Define a pairing via:-

$f_1^n \cup f_2^m(g_1,\ldots g_{n+m}) \equiv f_1^n(g_1,\ldots g_n) \cup p_2(g_1,\ldots g_n) f_2^m(g_1,\ldots g_m)$ which is a map $(f_1^n, f_2^m) \longmapsto f_1^n \cup f_2^m$ from $C^n_{p_1}(G,\pi_1) \times$ $\times \; C^n_{p2}(G,\pi_2)$ into $C^{n+m}_p(G,\pi)$. Moreover, one can show that (cocycle) $\cup$ (cocycle) = cocycle, with other permutations yielding coboundaries Thus we can define a map:-

$$H^n_{p_1}(G,\pi_1) \times H^m_{p_2}(G,\pi_2) \longrightarrow H^{n+m}_p(G,\pi).$$ We are interested in the map :-

$$H^1_{p_1}(G,\pi_1) \times H^1_{p_2}(G,\pi_2) \longrightarrow H^2_p(G,\pi).$$

Consider the case where $p = p_1 = p_2 = 0$. We then have a map:- $\mathrm{Hom}(G,\pi_1) \times \mathrm{Hom}(G,\pi_2) \longrightarrow H^2_0(G,\pi)$. Let $G = \mathbb{R}^3_T \otimes \mathbb{R}^1$, then $\exists$ epimorphisms $(p_1, p_2) \in \mathrm{Hom}(G_1, \mathbb{R}^3_T) \times \mathrm{Hom}(G, \mathbb{R}_1)$ given by:- $p_1 : (\underline{v}, t) \longmapsto \underline{v}$ and $p_2 : (\underline{v}, t) \longmapsto t \; \forall (\underline{v}, t) \in \mathbb{R}^3_T \times \mathbb{R}^1$. So take $\Pi_1 \equiv \mathbb{R}^3_T$, $\Pi_2 \equiv \mathbb{R}_1$. Now $\mathbb{R}^3_T$ and $\mathbb{R}_1$ can be paired to $\mathbb{R}^3$ via the cup product:- $\underline{v} \cup t \equiv \underline{v}\, t \; \forall (\underline{v}, t) \in \mathbb{R}^3 \times \mathbb{R}^1$. Thus there exists a two cocycle of $Z^2_0(\mathbb{R}^3_T \otimes \mathbb{R}^1, \mathbb{R}^3)$ defined by $p_1 \cup p_2((\underline{v}_1, t_1),(\underline{v}_2, t_2)) \equiv p_1(\underline{v}_1, t_1) \cup p_2(\underline{v}_2, t_2)$, viz $p_1 \cup p_2((\underline{v}_1, t_1),(\underline{v}_2, t_2)) = \underline{v}_1 t_2$.

Thus in our theorems concerning G enlargements the cocycles which vanish off $\mathbb{R}^3_T$ and on $\mathbb{R}^1$ are the cup products of cocycles. Similarly, let us consider the case when $G = \mathbb{R}^3_T \otimes \mathbb{R}^3$. There exists a pairing $\mathbb{R}^3_T \otimes \mathbb{R}^3 \longrightarrow \mathbb{R}^1$, via $(\underline{v}, \underline{x}) \longmapsto \underline{v} \cdot \underline{x}$. Via the projections $p_1 : (\underline{v}, \underline{x}) \longmapsto \underline{v}$; $p_2 : (\underline{v}, \underline{x}) \longmapsto \underline{x}$, we have a two cocycle $p_1 \cup p_2 \in Z^2_o(\mathbb{R}^3_T \otimes \mathbb{R}^3, \mathbb{R}_1)$ defined by:-

$p_1 \cup p_2 : ((\underline{v}_1, \underline{x}_1), (\underline{v}_2, \underline{x}_2)) \longmapsto p_1(\underline{v}_1, \underline{x}_1) \cup p_2(\underline{v}_2, \underline{x}_2) \equiv \underline{v}_1 \cdot \underline{x}_2$.

Let us now take $G = \mathbb{R}^1 \otimes E(3, \mathbb{R})_T$, $\Pi_1 = \mathbb{R}^3_T$ and $\Pi_2 = \mathbb{R}^1$, paired to $\Pi = \mathbb{R}^3$ as before. However, the group G operates on Π_1 and Π in this case via $p_1(t, \underline{v}, R) : \underline{v}' \longmapsto R\underline{v}'$, and $p(t, \underline{v}, R) : x \longmapsto R\underline{x}$ $\forall x \in \mathbb{R}^3$. Choose the following 1 cocycles of $Z^1_p(\mathbb{R}^1 \otimes E(3, \mathbb{R})_T, \mathbb{R}^3_T)$ and $Z^1_o(\mathbb{R}^1 \otimes E(3, \mathbb{R}), \mathbb{R}^1)$:- f_1 and f_2 respectively, where $f_1 : (t, \underline{v}, R)) \longmapsto \underline{v}$ and $f_2 : (t, (\underline{v}, \ \ )) \longmapsto t$ $\forall (t, (\underline{v}, R)) \in \mathbb{R}^1 \otimes E(3, \mathbb{R})_T$. In the case of f_1, the one cocycle property follows immediately:-

$f_1((t_1(\underline{v}_1, R_1))(t_2, (\underline{v}_2, R_2)) - p_1(t_1, (\underline{v}_1, R_1)).f_1(t_2, (\underline{v}_2, R_2)) - f_1(t_1, (\underline{v}_1, R_1)) = \underline{v}_1 + R_1 \underline{v}_2 - R_1 \underline{v}_2 - \underline{v}_1 = 0$. The cup product of cocycles in this case becomes:- $f_1 \cup f_2 ((t_1, (\underline{v}_1, R_1)1(t_2, \underline{v}_2, R_2)) \equiv f_1(t_1, (\underline{v}_1, R_1)) \cup f_2(t_2, (\underline{v}_2, R_2)) = \underline{v}_1 \cup t_2 = \underline{v}_1 t_2$. Then we have:-

$\delta(f_1 \cup f_2)((t_1, (\underline{v}_1, R_1)), (t_2, (\underline{v}_2, (\underline{v}_2, R_2)), (t_3, (\underline{v}_3, R_3)) =$
$p((t_1, (\underline{v}, R_1)) f_1 \cup f_2 (t_2, (\underline{v}_2, R_2)), (t_3, (\underline{v}_3, R_3)) - f_1 \cup f_2((t_1 + t_2, (\underline{v}_1 + R_1 \underline{v}_2, R_1 R_2)), (t_3, (\underline{v}_3, R_3)) + f_1 \cup f_2((t_1, (\underline{v}_1 R_1)), (t_2 + t_3, (\underline{v}_2 + R_2 \underline{v}_3, R_2 R_3)) - f_1 \cup f_2 ((t_1, (\underline{v}, R_1)), (t_2, (\underline{v}_2, R_2))$; which is just $R_1 \cdot (\underline{v}_2 \cup t_3) - (\underline{v}_1 + R_1 \underline{v}_2) \cup t_3 + \underline{v}_1 \cup (t_2 + t_3) - \underline{v}_1 \cup t_2 = R_1 \cdot \underline{v}_2 \cup t_3 - \underline{v}_1 \cup t_3$
$R_1 \cdot \underline{v}_2 \cup t_3 + \underline{v}_1 \cup t_2 + \underline{v}_1 \ t_3 - \underline{v}_1 \cup t_2 = 0$.

Again, take $G = (\mathbb{R}^3 \otimes \mathbb{R}^3_T) \boxtimes_p O(3, \mathbb{R})$ and $\Pi_1 = \mathbb{R}^3$, $\Pi_2 = \mathbb{R}^3_T$ and $\Pi = \mathbb{R}^1$ with the inner product as the cup product which pairs Π_1 and Π_2 to Π. In this case G operates on Π_1 and Π_2 and trivially on Π:- viz:- $\exists \ p_i \in \mathrm{Hom}(G, \mathrm{Aut}(\Pi_i))$ $i = 1, 2$, where $p_1((\underline{v}, \underline{x}), R))$: $\underline{x}' \longmapsto R\underline{x}'$, $p_2((\underline{v}, \underline{x}), R) : \underline{v}' \longmapsto R\underline{v}'$, $\forall ((\underline{v}, \underline{x}), R) \in (\mathbb{R}^3 \otimes \mathbb{R}^3_T)$

$\rho O(3,\mathbb{R})$ and $(\underline{v}', \underline{x}') \in \mathbb{R}^3_T \otimes \mathbb{R}^3$. With $\underline{x} \cup \underline{v} \equiv \underline{x} \cdot \underline{v}$, we then have $R\cdot\underline{x} \cup R\cdot\underline{v} = \underline{x}\cdot\underline{v} = R\cdot(\underline{x} \cup \underline{v})$. Define one cocycles $f_i \in Z^1_{p_i}(G, \Pi i)$ via $f_2: ((\underline{x}, \underline{v}), R) \longmapsto \underline{x}$ and $f_1((\underline{x}, \underline{v}) R) \longmapsto \underline{v}$. Whence via the definition :- $f_1 \cup f_2((\underline{x}_1, \underline{v}_1), R_1), ((\underline{x}_2, \underline{v}_2), R_2)) = f_1((\underline{x}_1, \underline{v}_1), R)) \cup R_1 \cdot f_2((\underline{x}_2, \underline{v}_2), R_2)) = \underline{v}_1 \cdot R_1\ \underline{x}$. Thus we see that $\delta(f_1 \cup f_2) = 0$, from $\delta(f_1 \cup f_2)(((\underline{x}, \underline{v}_1), R_1), ((\underline{x}_2, \underline{v}_2), R_2), ((\underline{x}_3, \underline{v}_3), R_3)) = \underline{v}_2 \cdot R_2\ x_3 - (\underline{v}_1 + R_1\ \underline{v}_2) \cdot R_1\ R_2\ x_3 + \underline{v} \cdot R_1(\underline{x}_2 + R_2\ x_3) - \underline{v}_1 \cdot R_1\ x_2 = \underline{v}_2 \cdot R_2\ x_3 - (\underline{v}_1 + R_1\ \underline{v}_2) \cdot R_1\ R_2\ x_3 + \underline{v}_1 \cdot R_1(\underline{x}_2 + R_2\ x_3) - \underline{v}_1 \cdot R_1\ x_2 = \underline{v}_2 \cdot R_2\ x_3 - v_1 \cdot R_1\ R_2\ x_3 - \underline{v}_2 \cdot R_2\ x_3 + \underline{v}_1 \cdot R_1\ x_2 + \underline{v}_1 \cdot R_1\ R_2\ x_3 - v_2 \cdot R_1\ x_2 = 0$.

This concludes this discussion for the moment. Cup products of cochains will be discussed again in chapter (6) in our discussion of non-inertial motions.

CHAPTER (5)

C O H O M O L O G Y   T H E O R Y   I N

C L A S S I C A L   M E C H A N I C S .

COHOMOLOGY THEORY AND CLASSICAL MECHANICS

The work presented in this chapter is an extension by the author of recent work of J.M. Lévy-Leblond[1] on a group theoretical approach to the Lagrangian theory of classical mechanics. It attempts to relate the free motion of a classical particle in space-time to the structure of the relativity model $\mathcal{W}$, through the use of Hamilton's principle, the principle of relativity and the group $\mathcal{A}(\mathcal{W})$ of inertial automorphisms of $\mathcal{W}$.

The chapter is divided into three parts. In the first and shortest part, we discuss the use of Hamilton's principle in classical mechanics at an elementary level, exposing the almost ad-hoc usage of the method. In the second part,we attempt a more rigorous approach leading to the introduction of the notions of homological algebra into Lagrangian mechanics. Having set up the formalism, in the latter part of the chapter, we discuss the intersection between Hamilton's principle and the principle of relativity and how one can almost completely define the free motions of a particle via the relativity group of a world model. We also see how, in this approach, the concept of inertial mass arises in a group-theoretical way, as does the kinetic energy functional and momentum.

Part (1) Intuitive Classical Mechanics

We shall first discuss classical mechanics in the frame work of Newtonian relativity. The underlying principle of classical Mechanics is Hamilton's principle.

Recall how in the Newtonian world model, we were able to partition the world into instants $\langle \mathcal{D}(t) \rangle$ $t \in \mathbb{R}^1$, where each instant $\mathcal{D}(t)$ was a Euclidean three dimensional vector space. We called a 'world-line' a map $x: \mathbb{R}^1 \longrightarrow W$, $x(t) \in \mathcal{D}(t)$ $\forall$ $t \in \mathbb{R}^1$ which lead to the definition of a trajectory as a map $\underline{x}: \mathbb{R}^1 \longrightarrow \mathbb{R}^3$, $x(t) \equiv (\underline{x}(t), t)$, $\underline{x}$ specifying the location, within an instant of an event. Consider two events $(x_1, x_2) \in \mathcal{D}(t_1) \times \mathcal{D}(t_2)$. Let $P(x_1, x_2)$ be the set of all world lines from x_1 to x_2. Let $x \in P(x_1, x_2)$ then $x(0) = x_1$ and $x(1) = x_2$, we can write $x(s) = (\underline{x}(s), st_1 + (1-s)t_2)$ where $\underline{x}$ is a path from $\underline{x}_1$ to $\underline{x}_2$ in $\mathbb{R}^3$. Roughly stated, Hamilton's principle asserts that there exists a function 'S' on $P(x_1, x_2)$ called the "Action function' such that the actual trajectory x followed by a particle from x_1 to x_2 is the solution of $\delta(S(\underline{x})) = 0$. The operation δ corresponding to the variation of S(x) as 'adjacent' paths in $P(x_1, x_2)$ are substituted, i.e.:- $\delta(S(\underline{x})) = 0$ means that S(x) is an extremal value of S(x), either or maximum or a minimum. We discuss this variation in more detail in part (2). One usually writes:-

$$S(\underline{x}) = \int_{[0,1]} ds \; L(\underline{x}(s), \dot{\underline{x}}(s)) =$$

The functional L on $\mathbb{R}^3$ is called a Lagrangian function. That the variation of S(x) is zero is the same as:-

$$(S(x)) = \int_{[0,1]} ds \left(\delta \underline{x} \cdot \frac{\partial}{\partial \underline{x}} + \delta \dot{\underline{x}} \cdot \frac{\partial}{\partial \dot{\underline{x}}} \right) L(\underline{x}(s), \dot{\underline{x}}(s)) = 0$$

where $\delta \underline{x}$ is the variation in path and $\delta \dot{\underline{x}} = (\delta x)^{\cdot}$. We must have

$\delta\underline{x} = \delta\underline{\dot{x}} = 0$ at the end points, whence, using an integration by parts:-

$$\delta(S(x)) = \int_{[0,1]} ds \quad \delta\underline{x} \cdot (\nabla - \frac{d}{ds} \circ \dot{\nabla}) L(\underline{x}(s), \underline{\dot{x}}(s)) = 0$$

the vector operator $\dot{\nabla}$ is defined via $\dot{\nabla}_i \equiv \partial/\partial \dot{x}_i$. We must then have:-

$$\delta\underline{x} \cdot (\nabla - \frac{d}{ds} \circ \dot{\nabla}) L(\underline{x}(s), \dot{x}(s)) = 0$$

Choosing the variations $\delta\underline{x}$ to be linearly independent we must have $1 \leq i \leq 3$:-

$$\left[\frac{\partial}{\partial x_i} - \frac{d}{ds}\frac{\partial}{\partial \dot{x}_i}\right] L(\underline{x}(s), \underline{\dot{x}}(s)) = 0$$

This is a second order differential equation whose solution is $\underline{x}$, the system being called Lagrange's equations. They are equivalent to Newton's equation of motion in the case under discussion. In most books on classical mechanics, one usually writes the 'identity' $L_o(\underline{x}(s), \underline{\dot{x}}(s)) = \frac{1}{2}m\,\underline{\dot{x}}(s)^2$ for a free particle, L_o being called a free-Lagrangian with $L_o(\underline{x}(s), \underline{\dot{x}}(s)) = T(\underline{\dot{x}})$, T being called the 'kinetic energy' functional and m the 'inertial mass' of the body.

The historic (and very inadequate) approach to interactions between particles and the external world is as follows. One assumes that a particle interacts with the external world via the coupling of the particle to a field $\Phi(x)$. A Lagrangian function of the form:-

$L(\underline{x}(s), \underline{\dot{x}}(s)) = L_o(\underline{x}(s), \underline{\dot{x}}(s)) - k\,\Phi(x) = T(\underline{\dot{x}}) - \Phi(x)$ is postulated to hold. The parameter 'k' is called the coupling constant of the particle with the field or its 'charge'. $k\,\Phi(\underline{x}) \equiv V(\underline{x})$ is called the 'potential energy' of the particle in the field $\Phi(\underline{x})$. For such a function L, Lagrange's equations imply equations of motion of the form $m\ddot{\underline{x}} = -k\nabla\Phi(x)$. The term $-k\nabla\Phi(\underline{x})$ is called the force exerted by the field on the particle, which is coupled to it with strength k. Clearly, when $k = 0$, $L = L_o$ and the particle is free, following the trajectory $\underline{x}(t) = \underline{v}\,t + \underline{x}(o)$ where $\underline{v} = \dot{\underline{x}}(t)$.

Let us attempt to apply similar ideas to the theory of special relativity. Given two events $x_1, x_2 \in W$, we consider all world-lines connecting x_1 and x_2 such that if $x_2 \in V_+(x_1)$ then $x(s) \in V_+(x_1)$ $\forall\, s \in [0, 1]$. (Note the interesting lack of a topology on W!) Here we have no Newton's laws of motion to fall back upon, only Einstein's principle of relativity allied with Hamilton's principle guides the mechanics, Einstein's principle guiding us in our selection of a Lagrangian for free motion. Without going into details yet, the choice of Lagrangian implied by Einstein's principle is $L_o(x, \dot{x}) = -\alpha$, when the proper time or 'arc-length' is chosen as the evolution parameter. Making a transformation to a variable 's' $\in [0, 1]$, a free Lagrangian $L_o(x, \dot{x}) = -\alpha\,(d\tau/ds) = -\alpha\sqrt{(\dot{x}(s))^2}$, is obtained. Inserting this formula into Lagrange's equation, we obtain the free motions $\ddot{x} = 0$, similar to Newton's equations. The relativistic description of particle interactions allows notions from Newtonian dynamics. A particle interacts with the world via a vector field $\Phi(x)$, the free Lagrangian being modified by an amount $-k\,\Phi(x)\cdot\dot{x}$ with the field.

Lagrange's equations imply equations of motion of the form

$$\ddot{x}_{\mu} = - k f_{\mu\nu} \dot{x}^{\nu}$$

where the tensor $(f_{\mu\nu})$ is just $(\underrightarrow{\Box} \otimes \Phi - \Phi \otimes \underleftarrow{\Box})$, $\Box$ being D'Alembert's operator. Due to the similarity of this equation of motion with Newton's second law, the Poincaré invariant α is called inertial mass. The use of classical time instead of proper time enables one to specify that the corresponding Newtonian mass is $m = \alpha (d\tau/dt)^{-1}$, which is not Poincaré invariant.

Let us next discuss the use of the principle of relativity in classical mechanics. Einstein's statement of the principle includes Galileo's postulating that the laws of physics must be of the same form in all inertial frames. In Newtonian relativity, inertial frames are connected by the inertial world automorphisms comprising the Galilei group, whilst in Einstein's relativity, they are connected by the Poincaré group. Incorporating this principle into Hamilton's Action principle, the implication is that if $\delta(S(x)) = 0$ for some path (either in Euclidean or Minkowski space, then $\delta(S(x^g)) = 0 \ \forall g \in \mathcal{Q}(W) \cap I(W)$. Which roughly means that if $x \in P(x_1, x_2)$ is the path of 'least action' in an inertial frame F, $x_1, x_2 \in F$ then $x^g \equiv g \circ x$ is the path of lest action in the inertial frame, F^g, $g x_1$, $g x_2 \in F$.

In the case of special relativity, we were able to choose a Poincaré invariant free Lagrangian i.e.:- $L_0(x^g, (x^g)^{\cdot}) \equiv L_0(x, \dot{x})$ $\forall g \in P(\mathbb{R})$. Consider however the case of the Newtonian free Lagrangian $L_0(\underline{x}, \underline{\dot{x}}) = T(\underline{\dot{x}})$. $T(\underline{\dot{x}})$ is obviously invariant under

$\mathbb{R}^3 \otimes \mathbb{R}^1 \triangleleft G(3,\mathbb{R})$ and, since $T(\underline{\dot{x}}) = \frac{1}{2} m \underline{\dot{x}}^2$, it is invariant under $O(3,\mathbb{R}) < G(3,\mathbb{R})$. However, under a pure Galilei boost, we find that $T((\underline{x}\ \underline{v})\ ^{\cdot}) = T(\underline{\dot{x}} + \underline{v}) = T(\underline{\dot{x}}) + \frac{1}{2} m\ (\underline{v}^2 + 2\underline{v}\cdot\underline{\dot{x}}))$. So that $L_0(\underline{x},\underline{\dot{x}})$ is not Galilei invariant , it is called 'variant'! It will be useful to note here that we have $T_{\cdot}(\underline{x}^{\underline{v}}) = T(\underline{x}) + \frac{d}{dt}\ (\frac{1}{2}m\ \underline{v}\ t + 2\underline{v}\cdot\underline{x})T(\underline{x}^{\underline{v}}) = T(\underline{x}) + \dot{\phi}(\underline{v})(\underline{x})$. Since the group of Galilei boosts operates transitively on the tangent space $\mathbb{R}^3_T$ (its trivial homogeneous space), it can be seen that all free Lagrangians can be generated by pure Galilei boosts via its canonical action on T. The point of this chapter, is in fact to investigate the group theoretical properties of functions where if g is an inertial transformation

$$L(x\ g,\ x\ g) = L(x,\ \dot{x}) + \dot{\phi}(g)(x)$$

Such functions 'ϕ' are called guage functions for the group of inertial world automorphisms in question. We are lead to discuss more fully the solution to the question. 'Given two Lagrangian L_1 and L_2 what is the condition that they determine the same laws of motion?' or 'what are the conditions that two action functionals S_1 and S_2 have co-incident minima'? Let us write $S_1 = \mathcal{A}(L_1)$ and $S_2 = \mathcal{A}(L_2)$, where $\mathcal{A}(L)(\underline{x},\ \underline{\dot{x}}) = \int_{[0,1]} ds\ L(\underline{x}(s),\ \underline{\dot{x}}(s))$. Here, $\mathcal{A}$ is a linear map from the set of Lagrange functions into the set of action functions. Clearly, two action functionals have coincident minima iff $\delta(S_1) = 0 \Leftrightarrow \delta(S_2) = 0$. So L_1 and L_2 determine the same equation of motion if $\delta(\mathcal{A}(L_1)) = 0 \Leftrightarrow \delta(\mathcal{A}(L_2)) = 0$. Obviously $\delta(\mathcal{A}(L_1)) = \delta(\mathcal{A}(L_2)) = 0$ means that $\delta(\mathcal{A}(L_1)) - \delta(\mathcal{A}(L_1)) = 0$. Since δ and $\mathcal{A}$ are both

linear functions, we must then have $\mathcal{S}(\mathcal{D}(L_1 - L_2)) \equiv 0$. When $\mathcal{S}(\mathcal{D}(L)) = 0$ then, in our problem, if Δ is the Lagrange operator $(\mathcal{D}(\Delta L) = 0$.
So that $\mathcal{D}(\Delta(L_1)) = 0 \Longleftrightarrow \mathcal{D}(\Delta(L_2)) = 0 \Rightarrow \mathcal{D}(\Delta(L_1 - L_2)) \equiv 0$
Write $L_1(\underline{x}(s),\dot{\underline{x}}(s)) - L_2(\underline{x}(s),\dot{\underline{x}}(s)) \equiv (L_1 - L_2)(\underline{x}(s),\dot{\underline{x}}(s)) \equiv F(\underline{x}(s), \dot{\underline{x}}(s))$. We must then have $\Delta(F(\underline{x}(s),\dot{\underline{x}}(s)) \equiv 0$. Thus we are lead to seek the set of functionals F which identically satisfy the Euler-Lagrange equations. One can show that the necessary and sufficient condition that a function $F(\underline{x}, \dot{\underline{x}})$ satisfies the Euler-Lagrange equations is that $F(\underline{x}, \dot{\underline{x}}) = \frac{d}{ds}(\Phi(x(s))$, Φ being an arbitrary differentiable functional, $F(\underline{x}, \dot{\underline{x}}) = \dot{\underline{x}} \cdot \nabla\Phi(x(s))$. Thus two Lagrange functions specify the same equations of motion iff $\exists\ \Phi$ such that $L_1(x(s), \dot{x}(s)) = L_2(\underline{x}(s),\dot{\underline{x}}(s)) + \dot{\Phi}(\underline{x}(s))$. The set of such Lagrangians is obviously an equivalence class. Using this notion of equivalence, we see that the two Newtonian Lagrangians $L_0^{\underline{v}}$ and L_0 are equivalent:-
$L_0^{\underline{v}}(\underline{x}, \dot{\underline{x}}) = L_0(x^{\underline{v}}, \dot{x}^{\underline{v}}) = L(\underline{x}, \underline{x}) + \frac{m}{2}(\underline{v}^2 t + \underline{v} \cdot \underline{x})^{\bullet}$

It seemed natural to the author that homological algebra could be applied to some purely algebraic aspects of Lagrangian theory. Lévy-Leblond had, of course introduced (in a simple fashion) the cohomology theory of groups into the Lagrangian theory of free classical mechanics, via the group theory of the guage variance functionals for a relativity group. The author has included the homological algebra expounded in section (2) into a expanded re-treatment of Lévy-Leblond's work (in section (3)) to produce a more mathematically coherent approach to the applications of the principle of relativity to Lagrangian theory for <u>free</u> particles. The homological algebra enters Lagrangian theory through the observation that there are many almost blatant prods from at least the notion of equivalence. First we note that the set of real valued functions on a linear space x its tangent space form a vector space in a natural way. Also, regarding S_T and S_{TT} as distinct then so is $C^1(S \times S_T \times S_{TT}, \mathbb{R})$. Given an $f \in C^1(S \times S_T, \mathbb{R})$ and applying the Lagrange operator, we find that $\Delta(f(\underline{x}, \underline{\dot{x}}))$ depends on $\underline{\ddot{x}}$ viz:- $\Delta(f(\underline{x}, \underline{\dot{x}})) = (\nabla - {}^d/ds \circ \dot{\nabla})\, f(\underline{x}, \underline{\dot{x}})$, or, with ${}^d/ds = \underline{\dot{x}} \cdot \nabla + \underline{\ddot{x}} \cdot \dot{\nabla}$, $\Delta(f(\underline{x}_1, \dot{x})) = \nabla(f(\underline{x}, \underline{\dot{x}})) - (\underline{\dot{x}} \cdot \nabla)\dot{\nabla} f(\underline{x}, \underline{\dot{x}})) + \underline{\ddot{x}} \cdot \dot{\nabla}\dot{\nabla}(f(\underline{x}, \underline{\dot{x}})$. So that since Δ is linear, we can define a group homomorphism δ^3: $C^1(S \times S_T, \mathbb{R}) \longrightarrow C^1(S \times S_T \times S_{TT}, \mathbb{R})$ $\delta^3(f)(\underline{x}, \underline{\dot{x}}, \underline{\ddot{x}}) \equiv \Delta(f(\underline{x}, \underline{\dot{x}}))$. The vanishing of $\delta^3(f)$ provides a second order differential equation for $\underline{x}$. The kernel of δ^3 is the set of Lagrangians. The map δ^2: $C^1(S, \mathbb{R}) \longrightarrow C^1(S \times S_T, \mathbb{R})$, defined by $\delta^2(\mathcal{f})(\underline{x}, \underline{\dot{x}}) = \underline{\dot{x}} \cdot \nabla \mathcal{f}(\underline{x}) \equiv \frac{d}{ds} \mathcal{f}(\underline{x})$, is also a homomorphism of $C^1(S, \mathbb{R})$ into $C^1(S \times S_T, \mathbb{R})$. The fact we saw above that $\mathrm{Im}(\delta^2) < \mathrm{Ker}(\delta^3)$ leads us to see that the sequence

$$0 \longrightarrow C^1(S, \mathbb{R}) \xrightarrow{\delta^2} C^1(S \times S_T, \mathbb{R}) \xrightarrow{\delta^3} C^1(S \times S_T \times S_{TT}, \mathbb{R})$$

is semi-exact. The group $\mathrm{Ker}(\delta^3)/\mathrm{Im}(\delta^2)$ is the set o f inequivalent Lagrange functions, it is isomorphic to the set of all inequivalent dynamical situations. We shall discuss the above notions in much greater generality in the next section, extending it considerably.

<u>Part (2). Homological Algebra and Hamilton's Principle.</u>

We shall first discuss the variation process used in part (1) in greater detail than is to be found there, emphasizing the algebra rather than analysis. Let S be a topological space. The paths between the points x_1 and x_2 in S are the continuous maps $\underline{x} \in C^1([0, 1], S)$ such that $\underline{x}(0) = \underline{x}_1$ and $\underline{x}(1) = \underline{x}_2$. Given two paths x_1 and x_2 in $P(x_1, x_2)$ the set of all such paths, then x_1 is said to be homotopic to x_2 iff $\exists$ a continuous $H: [0, 1] \times [0, 1] \longrightarrow S$, $H(0, s_2) = x_1(s_2)$ and $H(1, s_2) = x_2(s_2)\ \forall\ s_2$. We shall sometimes write $H(s_1,s_2) = (\chi(s_1))(s_2)$ where χ is a continuous map from $[0, 1]$ into $P(x_1, x_2)$ in t he derived topology on $P(x_1, x_2)$. Let $f \in C^1(S,\mathbb{R})$, then we can regard the pair (f, x) where $x \in P(x_1, x_2)$ as an element of $C^1([0, 1], \mathbb{R})$ via the definition of the map ϕ^1:- $\phi^1: C^1(S,\mathbb{R}) \times P(x_1,x_2) \longrightarrow C^1([0, 1], \mathbb{R})$; $\phi^1(f, x): s \longmapsto f(x(s))$. In a similar manner if $C^n(S,\mathbb{R}) \equiv C^1(X_{i=1}^n S, \mathbb{R}) \equiv C^1(S^n, \mathbb{R})$; and $X_{i=1}^n(P(x_1, x_2))$ is regarded as a set of maps from $[0, 1]$ to S^n via $(\underline{x}_1,\ldots x_n): s \longmapsto (x_1,(s),\ldots, x_2(s))\ \forall\ (x_1,\ldots x_n) \in X_{i=1}^n(P(x_1, x_2))$; we can define a sequence $\langle\phi^n\rangle$ $n \in Z+$ of mappings

$$\begin{cases} \phi^n: C^n(S, \mathbb{R}) \times X_{i=1}^n(P(x_1,x_2)) \longrightarrow C^1([0,1], \mathbb{R}) \\ \phi^n: (f, (x_1,\ldots,x_n)): s \longmapsto f(x_1(s),\ldots x_n(s)). \end{cases}$$

Recall that $C^n(S, \mathbb{R})$ is a real linear space under the definition $\alpha_1 f_1(x_1,\ldots x_n) + \alpha_2 f_2(x_1,\ldots,x_n) \equiv (\alpha_1 f_1 + \alpha_2 f_2)(x_1,\ldots x_n)\ \forall\ \alpha_1,\alpha_2 \in \mathbb{R}$, $f_1,f_2 \in C^n(S,\mathbb{R})$, $(x_1,\ldots x_n) \in S^n$. Thus $C^n(S,\mathbb{R})$ is a-priori an Abelian group

$\forall\ n \in Z^+$. Using these properties we can then endow the sets $C^n(S,\mathbb{R}) \times \{(x_1,\ldots, x_n)\} \subseteq C^n(S,\mathbb{R}) \times X_{i=1}^{n} P(x_1,x_2)$ with the properties of real linear spaces via:-

$\alpha_1(f_1, (x_1,\ldots x_n)) + \alpha_2(f_2,(x_1,\ldots,x_n)) =$

$(\alpha_1 f_1 + \alpha_2 f_2, (\underline{x},\ldots,x_n))\ \forall\ (\alpha_1,\alpha_2) \in \mathbb{R}$.

Let us assume that some notion of differentiation is defined on S. Then, given an $x \in P(x_1, x_2)$, $x \in P(x_1,x_2) \cap C_\infty([0, 1], S),\ldots,$ we can define paths $d^n(x)$ in S where $d^n(x): s \longmapsto d^n/ds^n(x(s))$ so that the n-tuple $(x, d(x),\ldots,d^n(x))$ can be regarded as a map from $[0, 1]$ into S^n via:-

$(x, d(x), \ldots, d^n(x)): s \longmapsto (x(s), \ldots d^n(x)(s))$. Then the family of sets $\langle A^n(x) \rangle\ n \in Z^+$ where $A^{n+1}(x) = C^{n+1}(S,\mathbb{R}) \times \{(x,..,d^n(x))\}$ is as above a r eal linear space. Consider the linear functional $I \in (C^1([0, 1], \mathbb{R}))^*$ (the vector dual of $C^1([0,1], \mathbb{R})$) where:-

$$I: f \longmapsto \int_{[0,1]} ds\ f(s)\ ds\ \ \forall\ f \in C^1([0, 1], \mathbb{R})$$

Using it, we can define a sequence of linear functionals $\langle \vartheta^n \rangle n \in Z^+$, $\vartheta^n \in A^n(x)^*\ \forall\ n \in Z_+$ via:- $\vartheta^n \equiv I \circ \phi^n$

or:-

$$\vartheta^n((f,(x, \ldots, d^{n-1}(x))) = \int_{[0,1]} ds\ f(x(s), \ldots, d^{n-1}(x)(s))$$

Let us now turn to the variational problem. Consider two paths $x_1, x_2 \in P(x_1, x_2)$ which are homotopic with respect to $\{x_1, x_2\}$ with homotopy H(or χ). We assume $x_1, x_2 \in P(x_1, x_2) \cap C_\infty([0, 1], \mathbb{R})$ and that $\partial H/\partial s_1$ exists. We shall consider the family of real linear spaces $\langle A^n(\mathcal{X}(s_1))\rangle\, s_1 \in [0, 1]$, where of course $\mathcal{X}(s_1)(s_2) \equiv H(s_1,s_2)$. If we define a function $\Phi \in C^1(A^n(S, \mathbb{R}), C^1([0, 1], \mathbb{R}))$ by $\Phi(f): s_1 \longmapsto \mathcal{G}^n(f, (\mathcal{X}(s_1), .. \partial_2^{n-1}\mathcal{X}(s_1))$, we can consider the derivatives $\partial\Phi(f)/\partial s_1$. The vanishing of $\partial\Phi(f)/\partial s_1$ occurs at the extrema of the function $\Phi(f)$. Writing

$$\mathcal{G}^n(f, (\mathcal{X}(s_1)\ldots, \partial_2^{n-1}(\mathcal{X})(s))) = \Phi(f)(s_1) =$$

$$\int_{[0,1]} ds_2\, f(\mathcal{X}(s_1)(s_2), \ldots, (\partial_2^{n-1}\mathcal{X}(s_1))(s_2)) =$$

$$\int_{[0,1]} ds_2\, f(H(s_1,s_2), \ldots, \partial_2^{n-1} H(s_1,s_2))$$

Where $\partial_2 \equiv \partial/\partial s_2$, we have

$$\partial_1(\Phi(f))(s_1) = \int_{[0,1]} ds_2 \quad \partial_1(f)(H(s_1,s_2), \ldots, \partial_2^{n-1}H(s_1,s_2))$$

$$= \int_{[0,1]} ds_2 \sum_{i=0}^{n-1} (\partial_1 \partial_2^i H(s_1,s_2)). \quad \nabla^i(s_1) \quad f(H(s_1,s_2), \ldots \partial_2^{n-1}H(s_1,s_2))$$

$\nabla^i \equiv \partial/\partial\, (\partial_2^i X(s_1))$.

We have $\partial_2^i \partial_1 H(s_1,s_2) = \partial_1 \partial_2^i H(s_1,s_2) = 0$ at $s_2 = (0, 1)$ $\forall$ $s_1 \in [0, 1]$, corresponding to the fixed end points so that, using a partial integration and inte÷rating it, we obtain:- $\partial_1(\Phi(f))(s_1) =$

$$\int_{[0,1]} ds_2 \sum_{i=0}^{n-1} (-1)^{-1}{}_1 H(s_1,s_2) \cdot \partial_2^i \circ \partial^i(s_1)\; f\; (H(s_1,s_2), \ldots \partial_2^{n-1}H(s_1,s_2))$$

Let us fix s_1 momentarilly, then we have a real valued function depending on f and an $x \in P(x_1, x_2)$ $x \equiv \mathcal{X}(s_1)$.

$$\int_{[0,\,1]} ds_2 \sum_{i=o}^{n-1} \partial_1 H(s_1,s_2)\cdot(-)^i \partial_2{}^i o\, \nabla^i(f(x(s_2),\ldots \partial_2{}^{n-1}(x)(s_2)))$$

$$= \int_{[0,1]} ds_2\; \partial_1 H(s_1,\, s_2)\cdot \sum_{i=0}^{n-1} (-)^i \partial_2{}^i\; o\, \nabla^i(f(x(s_2),\ldots \partial_2{}^{n-1}(x)(s_2)))$$

Now the bracketed term in the integrand is a function of $\partial_2{}^n(x)$ as can be shown by expanding the derivative as $\partial_2^n = \sum_{i=o}^{n-1} \partial_2{}^{i+1}(x)(s_2)\cdot \nabla^i$. This fact is important to our discussion. Let us define an endomorphism of $C^1([0,\,1]\,,\mathbb{R})$ by

$$\Delta^{n-1}(\phi^{n-1}(f,\;(x,\ldots,d^{n-1}(x))):s_2 \longmapsto$$

$$\sum_{i=o}^{n-1} (-)^i \partial_2{}^i\; o\; \nabla^i(f(x(s_2),\ldots,\partial_2{}^{n-1}(x)(s_2))$$

Δ^{n-1} is then used to define a linear function α^{n-1}: $A^{n-1}(x) \longrightarrow A^n(x)$ via:- $\Delta^{n-1} \,o\, \phi^{n-1} = \phi^n \,o\, \alpha^{n-1}$. Whence we have $\phi^n(\alpha^{n-1}(f,$ $(x,\ldots,\, d^{n-1}(x)))(s) = \sum_{i=o}^{n-1} (-)^i \partial_2{}^i\; o\, \nabla^i \phi^{n-1}(f,(x_1,..d^{n-1}(x))(s)$

In terms of the linear functionals $\langle \vartheta^n \rangle$ $n \in Z^+$, we have:-

$\partial_1(\Phi(f))(s_1) = \vartheta^n(\alpha^{n-1}(f,\;\; (x,\;\ldots,\; d^{n-1}(x)))$. Let us define a map ∂_{n-1}: $An(x)^* \longrightarrow A_{n-1}(x)^*$ via $\partial_{n-1}(\vartheta^n) = \vartheta^n \;o\; \alpha^{n-1}$. Then we must have:-

$$\partial_{n-1}(\vartheta^n):(f,\;(x,\;\ldots,\;d^{n-1}(x)) = \vartheta^n(\alpha^{n-1}(f,\;(x,..\;\; d^{n-1}(x)))$$

Of course, α_{n-1} completely determines ∂_{n-1} and hence the latter is linear. It is perhaps more transparent to write:-

$\alpha_{n-1}(f,\;(x,\;\ldots\; d^{n-1}(x)) = (\sigma_{n-1}(f),(x,\;\ldots,\; d^n(x)))$ where σ_{n-1} $C^{n-1}(S,\quad) \longrightarrow C^n(S,\quad)$ with:-

$$\delta_{n-1}(f)(x(s),\ldots,\ d^n(x)(s)) = \sum_{i=0}^{n-1} (-)^i \partial_1 H(s_1,s)\cdot\partial^i{}_2 o \nabla^i f(x(s),\ \ldots$$
$d^{n-1}(x)(s))$.

The extrema of the functions $\Phi(f)$ define the kernels of the homomorphisms α_{n-1} and ∂_{n-1}. Thus Ker(α_{n-1}) is the set of (f, (x, ... $d^{n-1}(x)))\in A^{n-1}(x)\ni\alpha_{n-1}(f,\ (x,\ldots,d^{n-1}(x))) = 0$ or

$$\sum_{i=0}^{n-1} \partial_1 H(s_1,\ s)\ \cdot\ (-)^i\partial_2{}^i o\nabla^i f(x(s),\ldots\quad d^{n-1}(x(s))) = 0$$

The set of 'n th' degree Euler-Lagrange functions, which in the familiar case n = 2 is:-

$$\alpha_1(f_1((x,\ \dot{x},\ \ddot{x}))(s)) = 0 = \partial_1 H(s_1,s)\ \cdot(\nabla - \partial_2 o\nabla\ )f(x,\ \dot{x}\ )$$

For n = 1, Ker(δ_0) is the set of (f, x) such that $\alpha_0(f((x,x))(s))$ = $0 \Rightarrow \partial_1 H(s_1,s)\cdot\nabla f(x) = 0$. The arbitariness of the 'variations of path':-

$\frac{\delta(x)(s)}{\delta s_1} \equiv \partial_1(H(s_1,s))$ in both cases means that $(\nabla - \partial_2 o\nabla)(f(x,\dot{x}))$ = 0, and in the second case $\nabla f(x) = 0$, the latter meaning that f(x) is independent of x. Finally the third order case is also fairly familiar $\alpha_3(f)((x,\ldots,\ \dddot{x})(s)) = (\nabla - \partial_2 o\nabla + \partial_2^2\ o\nabla^1)f(x(s),\ ..\ \ddot{x}(s)) = 0$

Let us now consider the maps $\beta_{n-1}:\ A_{n-1}\longrightarrow A_n$

$$\phi^n(\beta_{n-1}(f,\ (x,\ \ldots\ d^{n-1}(x)))(s) \equiv d/ds(\ \phi^{n-1}(f,(x,\ ..\ \ d^{n-1}(x))(s)$$
$$= \sum_{i=0}^{n-1}\ d^{i+1}\ (x)(s)\ \cdot\nabla^i\ f(x(s),..\ \ d^{n-1}(x(s))$$

Or, writing $\beta_{n-1}(f,\ (x,..\ \ ,\ d^{n-1}(x))) = (\gamma_{n-1}(f),(x,\ldots,\ d^n(x)))$ where $\gamma_{n-1}:\ C^{n-1}(S,\mathbb{R})\longrightarrow C^n(S,\mathbb{R})$, then:-

$$\gamma_{n-1}(f)(x(s),\ldots,d^n(x)(s)) = \sum_{i=o}^{n-1} d^{i+1}(x)(s)\cdot\ {}^i f(x(s),\ldots\ {}^{n-1}(x)(s))$$

$$= d/ds\ f(x(s),\ldots\ d^{n-1}(x)(s)).$$

Now the identity $\alpha_n \circ \beta_{n-1} = 0$ is always true. $\forall\ n \in Z_+$; the necessary and sufficient condition for the identical vanishing of an element of $A^{n-1}(x)$ under α_{n-1}, is that it is the total derivative of an element of $A^{n-2}(x)$. Ref (5·2)

That $\alpha_n \circ \beta_{n-1} = 0$ or $\mathrm{Im}(\beta_{n-1}) < \mathrm{Ker}(\alpha_n)$ also implies that $\delta_n \circ \beta_{n-1} = 0$. In a similar way to the definition of the homo morphism δ_n define a σ_n via β by $\sigma_{n-1}: A_n(x)^* \longrightarrow A_{n-1}(x)^*$ with $\vartheta_n \circ \beta_{n-1} = \sigma_{n-1}(\vartheta_n)$. Then by our definition we have $(\sigma_{n-1} \circ \partial_n)(\vartheta_n) = \partial_n(\vartheta_{n+1}) \circ \beta_{n-1} = (\vartheta_{n+1} \circ \alpha_n) \circ \beta_{n-1}$ or $\vartheta_{n+1} \circ (\alpha_n \circ \beta_{n-1}) = 0$. So that $\sigma_{n-1} \circ \partial_n = 0\ \forall\ n \in Z_+$. Thus our considerations lead us to surmise the existence of the families A and A* of semi-exact sequences:-

$$0 \longrightarrow A_{n-1}(x) \xrightarrow{\beta_{n-1}} A_n(x) \xrightarrow{\alpha_n} A_{n+1}(x) \qquad \begin{matrix}(\underline{A}) \\ \alpha_n \circ \beta_{n-1} = 0\end{matrix}$$

$$A^*_{n-1}(x) \xleftarrow{\sigma_{n-1}} A^*_n(x) \xleftarrow{\partial_n} A^*_{n+1}(x) \longrightarrow 0 \qquad \begin{matrix}(\underline{A}^*) \\ \sigma_{n-1} \circ \partial_n = 0\end{matrix}$$

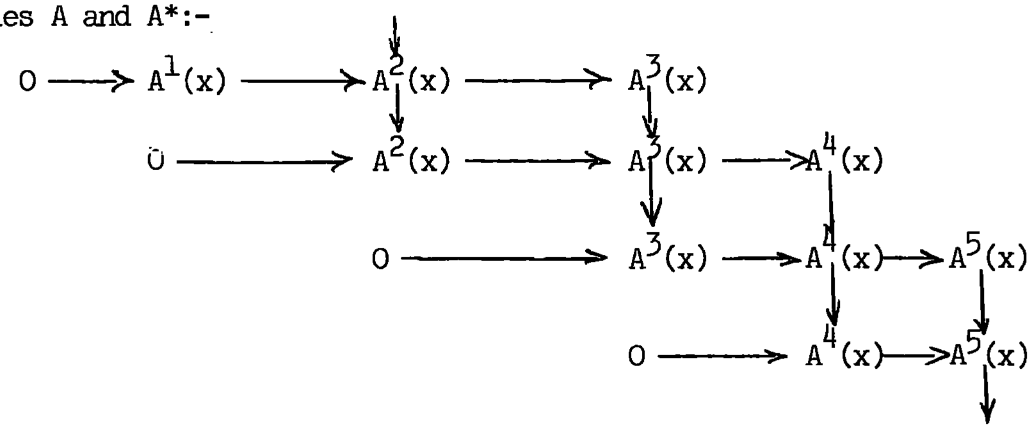
These two families of semi-exact sequences give rise via the cohomology and homology functors on the category of diagraons to the two families of exact sequences:-

$$0 \longrightarrow \mathrm{Im}(\beta_{n-1}) \longrightarrow \mathrm{Ker}(\alpha_n) \longrightarrow H^n(\underline{A}) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Im}(\partial_{n+1}) \longrightarrow \mathrm{Ker}(\sigma_n) \longrightarrow H_n(\underline{A}^*) \longrightarrow 0$$

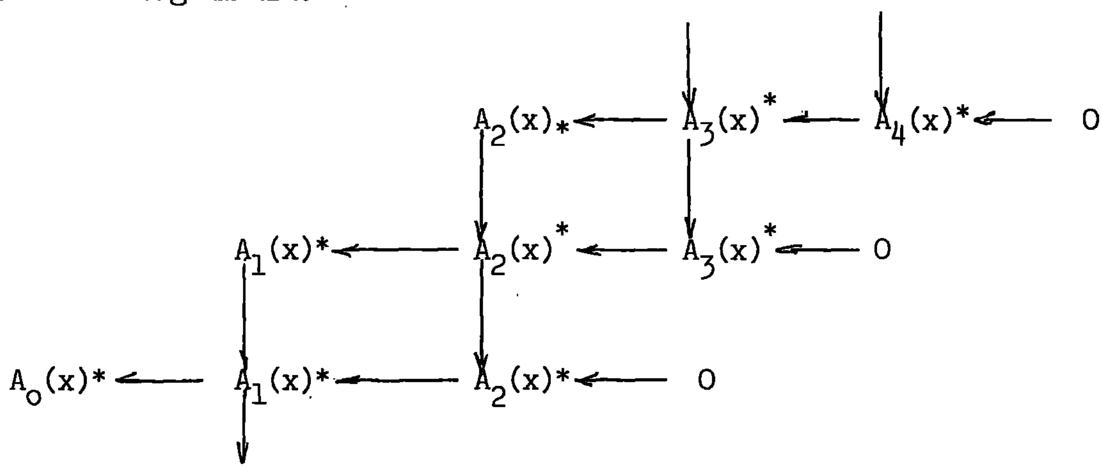
By definition $Ker(\alpha_n)/Im(\rho_{n-1}) \equiv H^n(\underline{A}^{\cdot})$ and $Ker(\sigma n)/Im(\partial_{n+1}) \equiv Hn(\underline{A}^*)$. We will thus be reasonably justified to use the terms cochains, coboundaries, cocycles for elements of $A^n(x)$, $Im(\beta_{n-1})$ and $Ker(\alpha_n)$ and chains, boundaries and cycles for elements of $A^n(x)^*$ $Im(\partial_{n+1})$ and $Ker(\sigma_n)$ respectively. This is of course a rather large abuse of language. The family $\langle \theta^n \rangle n \in Z_+$ of linear functionals on $A^n(X)$ which we have defined, is fixed once and for all, and there is a canonical correspondence, which is the functor from the category of vector spaces into itself $A \longmapsto A^*$; the functor, being contravariant, introduces conceptual difficulties into the use of the 'homology' groups, so we will stick to the 'cohomology' groups, or discard the cumbersome action functionals for the Lagrange formalism! Let us note in passing the rather 'nice' commutative diagrams in the families A and A*:-



The diagram is not commutative except down the steps e.g.:- $A^1 \rightsquigarrow A_2 A_2 \rightsquigarrow A_3 A_3 \rightsquigarrow A_4$ etc. the rows are exact whilst the columns, where defined are identity maps. Some fairly abvious facts can be verified with the aid of this diagram. For instance, note that $Im(\beta_2 o \beta_1) \triangleleft Im(\beta_2) \triangleleft Ker(\alpha_3)$ with $Im(\beta_1) \triangleleft Ker(\alpha_3)$. Using the freshman theorem we can write

$H^n(A) = (\text{Ker}\,(\alpha_n)/\text{Im}(\beta_{n-1}\circ\beta_{n-2}))\big/\text{Im}(\beta_{n-1})/\text{Im}(\beta_{n-1}\circ\beta_{n-2})$. This splits $\text{Im}(\beta_{n-1})$ into $\text{Im}(\beta_{n-1}\circ\beta_{n-2})$ equivalence classes and $\text{Ker}(\alpha_n)$ into $\text{Im}(\beta_{n-1}\circ\beta_{n-2})$ cosets. It is obviously equivalent to dividing functions which are like $\dot{\phi}(x, \ldots d^{n-1}(x))$ into classes where $\dot{\phi}(x, \ldots d^{n-1}(x)) = \ddot{\Phi}(x, \ldots, d^{n-3}(x))$ etc.

The above constructions are rather trivial in that nothing new is obtained except systematics, enabling us to relate the theory of homological algebra obtained in section (3) to the Lagrangian formalism in a more consistent way. Let us just note that for the system A* a diagram like:-



The rows are semi-direct, commutativity holds on the steps.

Part (3). Incorporation of the Principel of Relativity into Hamilton's Principle.

In view of the discussion of part (1) and of part (2), we can restate the principle of relativity in a world model $\mathcal{W}$ as follows. Define $G \equiv \mathcal{a}(\mathcal{W}) \cap I(\mathcal{W})$ then the statement is:-

'Let the cocycle $(L,(x, \dot{x}))$ or the cycle $\mathcal{J}^2(L,(x,\dot{x}))$ describe a law of physics. Then the cocycles $(L^g, (x, x'))$ or the cycles $\mathcal{J}^2(L^g, (x,\dot{x}))$ are cohomologous/homologous to the cocycles/cycles $(L, x, \dot{x}))$, $\mathcal{J}^2(L, (x, x)) \forall g \in G'$.

The transformed cocycle $(L \ , (x, \dot{x}))$ is defined by $L^g o(x, \dot{x})): s \longmapsto L(g^{-1}(x(s)),(g^{-1}(x,x))^{\cdot})$.

Note that $(L,(x, \dot{x})) \stackrel{g}{\longmapsto} (L^g ,(x,\dot{x}))$ always maps a 2 cochain into a two cochain iff $g \in I(\mathcal{W})$. Non-inertial transformations of $\mathcal{a}(\mathcal{W})$ map a given order cochain into higher ones. Let us try to formulate the group theory of our problem, retaining 'n'th order cochains for generality. We identify S with $\mathcal{W}$ on which $\mathcal{a}(\mathcal{W}) \cap I(\mathcal{W}) = G$ has its natural action. Then S^n is also a G space with action g: $(x, \ldots, x_n) \longmapsto (gx, \ldots, gxn) \forall g \in G$. The definition of $C^n(S,\mathbb{R})$ as a G space then follows:- g: $f \longmapsto f^g$, $f^g = f \circ g^{-1}$. Whence $A^n(x)$ for $x \in P(x_1,x_2)$ is a G space, g: $(f, (x, .. d^n(x)))$ $\stackrel{g}{\longmapsto} (f, g^{\cdot}(x,\ldots, d^n(x)) = (f^{g-1},(x, \ldots d^n(x))$ where $g^{\cdot}(x, \ldots d^n(x)) \longmapsto (gx,\ldots, g\, d^n(x)) = (g\, x,.. \ d^n(g\cdot x))$, $g\cdot x$: $S \longmapsto g(x(s)). \forall s \in [0, 1]$. We now define the action g: $(f, (x, \ldots d^n(x))) \longmapsto (f, g^{\cdot}(x,.. d^n x)) = (f^{g-1},(x,.. d^n(x)))$ via

$q \in \mathrm{Hom}(G, \mathrm{Aut}(A^n(x)))$, and a space $C_q^m(G, A^n(x))$ of functions from $X_{i=1}^m(G)$ into $A^n(x)$ where, $\forall (n, m) \in Z_+$ we define $f : (g_1, .. g_m) \longmapsto (\phi'(g_1, \ldots g_m), (x, \ldots, d^{n-1}(x)), \phi' \in C_{q'}^m(G, C^n(S, \mathbb{R})), q'(g): f \longmapsto f^g = f \circ g^{-1} \forall g \in G, f \quad C^n(S, \mathbb{R})$.

Define a homomorphism $\delta_m \in \mathrm{Hom}(C_q^m(G, C^n(S, \mathbb{R})), C_{q'}^{m+1}(G, C^n(S, \mathbb{R}))$ $\forall n, m \in Z_+$ by $\delta_m(\phi)(g_1, \ldots g_{m+1}) \equiv q'(g_1)(\phi(g_2, \ldots g_m)) + (-1)^{m+1} \phi(g_1, \ldots g_m) + \sum_{i=}^m (-1)^i \phi(g_1, .. g_i g_{i+1}, .. g_{m+1})$. Then we saw as in chapter (2) the sequence

$\langle (C_{q'}^m(G, C^n(S, \mathbb{R})), \delta^m) \rangle$ $m \in Z_+$ is a complex. If we define homomorphisms $\Delta^m \in \mathrm{Hom}(C_q^m(G, A^n(x)), C_q^{m+1}(G, A^n(x))$ via:-

$\Delta^m(\Phi)(g_1, .. g_{m+1}) \equiv (\delta^m(\phi)(g_1, .., g_m), (x, .. d^{n-1}(x)) \overset{\equiv}{\wedge} (q(g_1)(\phi'(g_2, \ldots g_{m+1}), (x, \ldots d^n(x))) + (-1)^{m+1}(\phi'(g_1, \ldots g_m), (x, \ldots d^n (x))) + \sum_{i=1}^m (-1)^i (\phi'(g_1, .. g_i g_{i+1}, \ldots g_{m+1}), (x, .. d^{n-1}(x)))$. When $\Phi \equiv (\phi', (x, \ldots, d^{n-1}(x))$ with $\phi' \in C_{q'}^m(G, C^n(S, \mathbb{R}))$.

Let us consider now the case when we have a 'cocycle' $f \in \mathrm{Ker}(\beta_{n-1})$ such that the cochain $q(g)(f)$ is also a cocycle $\forall g \in G$ which is cohomologous to f. We must then have:-

$$q(g)(f) = f + \beta_{n-2}(\phi(g))$$

Where $\phi \in C_q^1(G, A^{n-2}(x))$. Using $q(g_1, g_2) \equiv q(g_1) \circ q(g_2)$ we must have:-

$q(g_1)(f) + q(g_1)(\beta_{n-2}(\phi(g_2)) = f + \beta_{n-2}(\phi(g_1 g_2))$. One can readily show that $q(g_1) \circ \beta_{n-2} \equiv \beta_{n-2} \circ q(g_1)$, so that we must have:-

$$q(g_1)(\phi(g_2)) + \phi(g_1) = \phi(g_1 g_2).$$

So that we must have $\phi \in Z_q^1(G, A^{n-2}(x))$. Let us consider the one coboundaries of $Z^1{}_q(G, A^{n-2})(x))$ where $(q(g_1)(\nu) - \nu) = \phi(g_1)$, ν is an element of $G^o_q(G, A^{n-2}(x))$ viz $A^{n-2}(x)$. We must then have $q(g)(f) = f + \beta_{n-2}(q(g_1)(\nu) - \nu)) = f + \beta_{n-2}(g(g_1)(\nu) - \beta_{n-2}(\nu)$. So that if $f' \equiv f + \beta_{n-2}(\nu)$ then f' is invariant under G:- $f' \circ g^{-1} = f'$ $\forall g \in G$ and f' is cohomologous to f. Whence, given a 'cocycle' $f \in Ker(\alpha_{n-1})$ which transforms under G like :$f \xrightarrow{g} f + \beta_{n-2}(\phi(g))$ where $\phi \in Z^1_q(G, A^{n-2}(x))$, f can be chosen invariant under G iff $\phi \in B^1_q(G, A^{n-2}(x))$. The first cohomology group $H^1_q(G, A^{n-2}(x))$ is then the class of gauge functions modulus the trivial guage functions for the group G for the 'n-1'th Euler-Lagrange problem.

Let us return to the case n=3, then the group of non-trivial quage functions for the 2nd Euler Lagrange problem:- $\alpha_2(f, (x, \dot{x})) = 0$, is just $H^1{}_q(G, A^1(x))$, or identifying with the group $C^1_{q'}(G, C^1_o(S, \mathbb{R}))$, we see that the group of non-trivial guage functions is:-

$$H^1_q(G, \; C^1_o(S, \mathbb{R})$$

where $Hom(S, \mathbb{R})$ is the underlying Abelian group of the dual space S* of S. We have seen that the principle of relativity requires that if a Lagrangian $f \in Ker(\alpha_2)$ specifies a Law of Physics, then $\forall g \in G$ we must have:-

$$f(g(x(s)), (g(x(s)))^{\cdot}) = f(x(s), \dot{x}(s)) + \beta_1(\phi(g)(x(s)))$$

Where $\phi \in Z^1 q'(G, C^1_o(S, \mathbb{R}))$. Let us consider the guage invariance of f under subgroup $T \triangleleft G$ of spatio-temporal translations. We can write:-

$$f(x+a, \dot{x}) = f(x, \dot{x}) + \beta_1(\phi(a)(x)) \; \forall \; a \in T$$

Choosing a = -x, we can write:-

$$f(x, \dot{x}) = f(o, \dot{x}) - \beta_1(\phi(-x)(x)) = \mathfrak{J}(x) - \beta_1(\Phi(x))$$ where the

functional $\mathfrak{T}$ is $f|S_T$ and Φ is define by $\phi(-x)(x)$. As 'cocycles' $\mathfrak{T}$ and f are thus cohomologous and it will be sufficient to consider the guage variance under G/T of the functional :-

$$\mathfrak{T}((g\,x)^{\cdot}) \longmapsto \mathfrak{T}(x) + \beta_1(\phi(g)(x))$$

with $\phi \in Z^{'}q'(G/T), C_0^1(T,\mathbb{R}))$, the elements of $B'q'(G/T, C_0^1(T,\mathbb{R}))$ allowing $\mathfrak{T}$ to be chosen invariant. Consider the case when G, which is a group extension of G/T by T, is a semi-direct product $T \boxtimes G/T$. Via the 'generalised Mackey theorem' introduced in chapter (4) we can see that the class of guage functions $Z^1_{q'}(G/T, C_0^1(T,\mathbb{R}))$ is isomorphic to those cocycles of $Z_0^2(G,\mathbb{R})$ which vanish when restricted to T or to G/T:- $\not{Z}^2_0(G,\mathbb{R})$. Having made this observation, we proceed to compute the groups $\not{Z}^2_0(G,\mathbb{R})$ when G is the Poincaré, Galilei, Carroll or Static groups. At the end of each calculation relate the cohomology groups to the choice of kinetic energy functional $\mathfrak{T}$ and the notion of 'inertial mass' appropriate to the relativity model $\mathcal{W}$ which is sepcified by $\mathcal{A}(\mathcal{W}) \cap I(\mathcal{W})$. Before performing these calculations, we shall pause to sharpen some group-theoretical tools.

Recall the statement of the 'generalised Mackey theorem' presented in chapter (4):-

'Given a group $K \boxtimes p\, Q$ and an Abelian group A, each $\nu \in Z_0^2$ $(K \boxtimes p\, Q, A)$ is cohomologous to a cocycle ν' of the form:-

$$\nu'((k_1, q_1),(k_2,q_2)) = \xi_1(k_1,q_1 \cdot k_2) + \xi_2(q_1,q_2) + \Phi(q_1^{-1})(k_2)$$

where $\xi_1 \in Z_0^2(K, A)$, $\xi_2 \in Z_0^2(Q, A)$ and $\xi_1 = \nu' | K \times K$; $\xi_2 = \nu' \mid Q \times Q$ and the pair (ξ_1, Φ) are specified by $\Phi \in Z^1p'$

$(Q, C^1_o(K, A))$ with $\delta(\Xi(q))(k_1, k_2) = \xi_1(q^{-1}\cdot k_1, q^{-1}\cdot k_2) - \xi_1(k_1, k_2)\ \forall\ q \in Q$; $\delta(\Xi(q)) \in B^2_o(K, A)$'.

We shall discuss here the special case when K is Abelian and A is the additive group of the reals. Firstly, that K is Abelian implies that we can write

$\delta(\Xi(q))(k_1, k_2) = \delta(\Xi(q))(k_2, k_1)$, since

$\Xi(q)(k_1\ k_2) - \Xi(q)(k_1) - \Xi(q)(k_2) = \delta(\Xi(q))(k_1, k_2)$. Thus we must have:-

$\xi_1(q\cdot k_1, q\ k_2) - \xi_1(k_1\ k_2) = \xi_1(q\cdot k_2, q\cdot k_1) - \xi_1(k_2\cdot k_1)$

or $(\xi_1(q\cdot k_1, q\cdot k_2) - \xi_1(q\cdot k_2, q\ k_1)) = (\xi_1(k_1, k_2) - \xi_1(k_2, k_1))$.

Or if we define $\phi \in C^2_o(K, A)$ via $\phi(k_1, k_2) \equiv \xi_1(k_1, k_2) - \xi_1(k_2, k_1)\ \forall\ k_1, k_2 \in K$. We have $\phi \in Z^2_o(K, A)$ since:-

$\delta^2(\phi)(k_1, k_2, k_3) = \xi_1(k_2, k_3) - \xi_1(k_3, k_2) - \xi_1(k_1\cdot k_2, k_3) + \xi_1(k_3, k_1\cdot k_2) + \xi_1(k_1, k_2\cdot k_3) - \xi_1(k_2\cdot k_3, k_1) - \xi_1(k_1, k_2) + \xi_1(k_2, k_1)$. Which is:-

$\xi_1(k_2, k_3) - \xi_1(k_1\cdot k_2, k_3) + \xi_1(k_1, k_2\cdot k_3) - \xi_1(k_1, k_2) -$

$\xi_1(k_3, k_2) + \xi_1(k_3, k_1\cdot k_2) - \xi_1(k_2\cdot k_3, k_1) + \xi_1(k_2, k_1)$.

The first line is just:- $\delta^2(\xi_1)(k_1, k_2, k_3)$, the second:-

$\xi_1(k_2, k_1) - \xi_1(k_3, k_2, k_1) + \xi_1(k_3, k_2, k_1) - \xi_1(k_3, k_2)$ since K is Abelian and thus $k_2\ k_3 = k_3\ k_2$ and $k_1\ k_2 = k_2\ k_1$; thus the second line is $\delta^2(\xi_1)(k_3, k_2, k_1)$. So we have $\delta^2(\xi)(k_1, k_2, k_3) = \delta^2(\xi_1)(k_1, k_2, k_3) + \delta^2(\xi_1)(k_3, k_2, k_1) = 0$, and $\phi \in Z^2_o(K, A)$ with $\phi(q\cdot k_1, q\cdot k_2) = \phi(k_1, k_2)\ \forall\ q \in Q$ and $\phi(k_1, k_2) = -\phi(k_2, k_1)$.

A result of Bargmann sates that for an 'n parameter' Abelian Lie group, each cocycle of $Z^2{}_0(A(n), \mathbb{R}_{(n,n)})$ can be expressed as the assymetic bilinear form $\xi(a_1, a_2) = \sum_{(\mu,\nu)=(1,1)} \beta_{\mu\nu} a_{1\mu} a_{2\nu}$. For all our cases K will be a Abelian 4 parameter Lie group:- $\mathbb{R}^3 \otimes \mathbb{R}^1$. The only bilinear assymmetric functionals on $\mathbb{R}^3 \otimes \mathbb{R}^1$ which satisfy $\Phi(q \cdot k_1, q \cdot k_2) = \Phi(k_1, k_2) \ \forall \ q \in Q \rightleftharpoons G/ \mathbb{R}^3 \otimes \mathbb{R}^1)$ must vanish identically in the cases we will consider

So we must have $\Phi = 0$ or:-

$\xi_1(k_1, k_2) = \xi_1(k_2, k_1)$, which from our latter assymmetry requirement implies that $\xi_1 = 0$. We have thus proved the theorem that

'When K is Abelian and A $\rightleftharpoons \mathbb{R}$, all cocycles of $Z^2{}_0(K \boxtimes p\ Q, A)$ of the form:-

$\nu((k_1, q_1),(k_2, q_2)) = \Xi(q_1^{-1}(k_2) + \xi_2(q_1, q_2)$ where $\xi_2 = \nu' \restriction Q \times Q$; $\nu' \mid K \times K = 0$ and $\Xi \in Z^1\ p'(Q, \mathrm{Hom}(K, A))$ since $\delta(\Xi(q)) = 0 \ \forall \ q \in Q'$.

In the sequel we will use this corollary with some results of Bargmann which will not be proved here. We shall consider first the two relativity groups of special relativity, the Poincaré and Causality groups.

<u>Case (1) The Poincaré Group</u>.

Here G $\rightleftharpoons \mathbb{R}^4 \boxtimes {}_nL(\mathbb{R})$. Applying the corollary to Mackey's theorem which we discussed above, $\xi_1 = 0$. Since $\Lambda_{\alpha\mu} \Lambda_{\beta\nu} f_{\mu\nu} = f_{\alpha\beta}$ implies f is diagonal where $\xi_1(x_1, x_2) = f_{\mu\nu} x_{1\mu} x_{2\nu}$ and hence must vanish since it is also assymetric, according to Bargmann.

Then, utilising a classical result of Bargmann, $Z^2{}_o(L(\mathbb{R}), \mathbb{R}) = B^2{}_o(L(\mathbb{R}), \mathbb{R})$ which means that we can choose $\xi_2 = 0$. Thus all cocycles of $Z^2{}_o(P(\mathbb{R}), \mathbb{R})$ have the form $\xi((x_1, \Lambda_1), (x_2, \Lambda_2)) = \Phi(\Lambda_2^{-1})(x_1)$ where $\Phi \in Z^1{}_{n'}(L(\mathbb{R}), Z^1{}_o(\mathbb{R}^4, \mathbb{R}))$; which means that:-

(i) $\Phi(\Lambda)(x_1+x_2) = \Phi(\Lambda)(x_1) + \Phi(\Lambda)(x_2) \quad \forall \Lambda \in L(\mathbb{R})$; $x_1, x_2 \in \mathbb{R}^4$

(ii) $\Phi(\Lambda_1 . \Lambda_2)(x) = \Phi(\Lambda_1)(x) + \Phi(\Lambda_2)(\Lambda_1^{-1}x) \; \forall x \in \mathbb{R}^4; \Lambda_1, \Lambda_2 \in L(\mathbb{R})$

We can easily show that $Z^1_{n'}(L(\mathbb{R}), Z^1{}_o(\mathbb{R}^4, \mathbb{R})) = B^1{}_{n'}(L(\mathbb{R}), Z^1{}_o(\mathbb{R}^4, \mathbb{R}))$ or $H^1_{n'}(L(\mathbb{R}), \mathrm{Hom}(\mathbb{R}^4, \mathbb{R})) = 0$. Now $\mathcal{C}(P(\mathbb{R})) \approx Z(z)_{PT}$ where PT: $x \longmapsto -x \; \forall x \in \mathbb{R}^4$, and $(\Lambda . PT) = (PT . \Lambda) \; \forall \Lambda \in L(\mathbb{R})$, which means that:-

$\Phi(\Lambda)(x) + \Phi(PT)(\Lambda^{-1}x) = \Phi(PT)(x) + \Phi(\Lambda)(-x)$ or $\Phi(\Lambda)(x) - \Phi(\Lambda)(-x) = \Phi(PT)(x) - \Phi(PT)(\Lambda^{-1}x)$. Since $\Phi(\Lambda) \in \mathrm{Hom}(\mathbb{R}^4, \mathbb{R})$ $\forall \Lambda \in L(\mathbb{R})$, $\Phi(\Lambda)(-x) = -\Phi(\Lambda)(x)$, so that $-2\Phi(\Lambda)(x) = \Phi(PT)(\Lambda^{-1}x) - \Phi(PT)(x)$, or $\Phi(\Lambda) = -\frac{1}{2}\delta(\Phi(PT))(\Lambda)$ where $\Phi(PT) \in C^o_{n'}(\mathbb{R}^4, \mathbb{R})$ is a constant $f^{\underline{n}}$, an element of the latter. Thus $H^1 n'(L(\mathbb{R}), \mathrm{Hom}(\mathbb{R}^4, \mathbb{R})) = 0$.

From the above considerations, all guage functions for $P(\mathbb{R})$ may be chosen to be trivial, all Lagrangian strictly invariant:- which implies that $\mathcal{J}(\Lambda\dot{x}) = \mathcal{J}(\dot{x}) \; \forall \Lambda \in L(\mathbb{R})$. We thus write $\mathcal{J}(\dot{x}) = \tau(\dot{x}^2)$. That $\mathcal{J}(\dot{x})\, ds$ be a first order infinitesimal requires $\tau(\dot{x}^2) = -\alpha\sqrt{\dot{x}^2}$ where α is a real parameter, thus in terms of space-time we have the free Lagrangian $\mathcal{J}(\dot{x}) = -\alpha(\dot{t}^2 - \dot{x}^2)^{\frac{1}{2}}$.

The Lagrange equations involving the proper time as the evolution parameter $\alpha \ddot{x} = 0$ imply the label of proper inertial mass or 'rest mass' for α . If Newtonian time is used as the evolution parameter the corresponding Newtonian mass involved is given by $\alpha(1 - \dot{x}^2)^{-\frac{1}{2}}$, 'which varies with velocity'. The functional $\mathcal{J}(\dot{x}) = -\alpha\sqrt{x^2}$ is called kinetic energy.

Case (2) The Causality Group.

The causality group $C\uparrow(\mathbb{R})$ has the structure $P\uparrow(\mathbb{R}) \boxtimes_k \mathbb{R}^+m$ where $\mathbb{R}^+m$ is the multiplicative group of positive reals and $k(\alpha):(x, \Lambda) \longmapsto (\alpha x, \Lambda)$ $\forall \alpha \in \mathbb{R}^+m; (x, \Lambda) \in P\uparrow(\mathbb{R})$. We shall write $C\uparrow(\mathbb{R}) \rightleftharpoons \mathbb{R}^4 \boxtimes_N (L(\mathbb{R}) \otimes \mathbb{R}^+m)$ where $N \in \mathrm{Hom}(L\uparrow(\mathbb{R}) \otimes \mathbb{R}^+m$, $\mathrm{Aut}(\mathbb{R}^4))$ is defined by $N(\Lambda, \alpha): x \longmapsto \Lambda\alpha x$ $\forall (\Lambda, \alpha) \in L\uparrow \otimes \mathbb{R}^+m$, $x \in \mathbb{R}^4$. Then we compute $Z^2{}_o(C\uparrow(\mathbb{R}), \mathbb{R})$ using our corollary. Each $\nu \in Z^2{}_o(C\uparrow(\mathbb{R}), \mathbb{R})$ have $\nu | \mathbb{R}^4 \times \mathbb{R}^4 = 0$ and hence is written $\nu((x_1, (\Lambda_1, \alpha_1)), (x_2, (\Lambda_2, \alpha_2)) = \xi_2((\Lambda_1, \alpha_1), (\Lambda_2 \alpha_2)) + \Phi(\Lambda_2^{-1}, \alpha_2{}^{-1})(x_1)$ where $\xi_2 = \nu | (\mathbb{R}^+_m \otimes L\uparrow(\mathbb{R}))^2$ and $\Phi \in Z^1{}_o(\mathbb{R}^+_m \otimes L\uparrow(\mathbb{R}), \mathrm{Hom}(\mathbb{R}^4, \mathbb{R}^1))$. In order to compute ξ_2, we use the theorem again in its direct product form. Here, we again have $H^2{}_o(L\uparrow(\mathbb{R}), \mathbb{R}) = 0$, also $H^2{}_o(\mathbb{R}^+_m, \mathbb{R}) = 0$ since $\mathbb{R}^+_m$ 2 cocycles in $\mathbb{R}$ must be assymetric and $\mathbb{R}^+m$ is a one-parameter Lie group. Thus each $\xi \in Z^2{}_o(\mathbb{R}^+m \otimes L\uparrow(\mathbb{R})), \mathbb{R})$ must be of the form $\xi((\Lambda_1, \alpha_1), (\Lambda_2, \alpha_2)) = \chi(\Lambda_2^{-1})(\alpha_1)$ where $\chi \in Z^1{}_o(L\uparrow(\mathbb{R})$ $\mathrm{Hom}(\mathbb{R}^+_m, \mathbb{R}))$, $= \mathrm{Hom}(L\uparrow(\mathbb{R}), \mathrm{Hom}(\mathbb{R}^+_m, \mathbb{R}))$. Since $L\uparrow(\mathbb{R})$ is simple and $\mathrm{Hom}(\mathbb{R}^+_m, \mathbb{R})$ is Abelian, $\mathrm{Hom}(L\uparrow(\mathbb{R}), \mathrm{Hom}(\mathbb{R}^+m, \mathbb{R}) = 0$. Thus

$H^2_o(\mathbb{R}^+m \otimes L\uparrow(\mathbb{R}),\mathbb{R}) = 0$. Whence, each $\wp \in Z^2_o(C\uparrow(\mathbb{R}),\mathbb{R})$ has the form $\wp((x_1, (\Lambda_1,\alpha_1)),(x_2,(\Lambda_2,\alpha_2)) = \Phi(\Lambda_2^{-1}, \alpha_2^{-1})(x_1)$ with $\Phi \in Z^1_{N'}(\mathbb{R}^+_m \otimes L\uparrow(\mathbb{R}), \mathrm{Hom}(\mathbb{R}^4, \mathbb{R}^1))$, which means that

i) $\Phi(\alpha,\Lambda)(x_1, x_2) = \Phi(\alpha,\Lambda)(x_1) + \Phi(\alpha,\Lambda)(x_2)$

ii) $\Phi(\alpha_1\alpha_2,\Lambda_1\Lambda_2)(x) = \Phi(\alpha_1,\Lambda_1)(x) + \Phi(\alpha_2,\Lambda_2)(\alpha_1^{-1}\Lambda_1^{-1} x)$.

Now $\Phi(\alpha,\Lambda)(x) = \Phi((\alpha, e)(0,\Lambda))(x)$ which is just $\Phi(\alpha,\Lambda)(x) = \Phi(\alpha,e)(x) + \Phi(0,\Lambda)(\alpha^{-1}x) = \Phi_1(\alpha)(x) + \Phi_2(\Lambda)(x)$ where $\Phi_1 = \Phi \circ i_1, \Phi_2 = \Phi \circ i_2$, $i_1: \mathbb{R}^+m \to \mathbb{R}^+m \otimes L\uparrow(\mathbb{R})$ $i_2: L\uparrow(\mathbb{R}) \longrightarrow \mathbb{R}^+m \otimes L\uparrow(\mathbb{R})$ are the natural monomorphisms.

$\Phi_1 \in Z^1_{k'}(\mathbb{R}^+m, \mathrm{Hom}(\mathbb{R}^4,\mathbb{R})), \Phi_2 \in Z^1_{n'}(L\uparrow(\mathbb{R}), \mathrm{Hom}(\mathbb{R}^4,\mathbb{R}))$.

From case (i) we can choose $\Phi_2 = 0$, which is that $\Phi(\alpha,\Lambda)(x) = \Phi_1(\alpha)(x) = \Phi_1(\alpha)(\Lambda^{-1}x)$, $\forall \Lambda \in L\uparrow(\mathbb{R})$. This implies that $\Phi(\alpha)(x) = \sigma(\alpha)(x^2)$, but $\Phi(\alpha) \in \mathrm{Hom}(\mathbb{R}^4,\mathbb{R})$ $\forall \alpha \in \mathbb{R}^+_m \Rightarrow \sigma(\alpha) = \Phi_1(\alpha) = 0$. Whence $H^2_o(C\uparrow(\mathbb{R}),\mathbb{R}) = 0$. Also since $H^1_{N'}(\mathbb{R}^+_m \otimes L\uparrow(\mathbb{R}), \mathrm{Hom}(\mathbb{R}^4,\mathbb{R})) = 0$, under $C\uparrow(\mathbb{R})$ Lagrangians must be strictly invariant:- $\mathcal{I}(\dot{x}) = \mathcal{I}(\alpha\Lambda\dot{x})$ $\forall (\alpha,\Lambda) \in \mathbb{R}^+_m \otimes L\uparrow(\mathbb{R})$. With $\alpha = 0$ we must have $\mathcal{I}(\dot{x}) = \mathcal{I}(0)$ a constant. Recall how in case (i) the classical action would be $S(x, \dot{x}) = -\int_{[0,1]} ds\, \alpha\dot{\tau}(x)$ where τ is the proper time.

The integral would be re-expressible as $-\int_{[0,1]} \alpha d\tau(x)$ along the world line x. In the present case, we have $S(x, x) = \int_{[0,1]} ds\, \beta$ where $\beta \equiv \mathcal{I}(\dot{x})$, changing variables to proper-time entails

$$\int_{[0,1]} ds = \int_{[0,1]} (ds/d\tau)d\tau$$

Under $\alpha \in \mathbb{R}^+ m$, $d\tau \mapsto \alpha d\tau$, and $ds/d\tau \mapsto \alpha^{-1} ds/d\tau$. Lagrange's equations imply P = 0 for the Lagrangian $\mathcal{L}$. Thus the requirement of covariance under the causality group leads to no motions at all under the Lagrangian model of calssicam mechanics. In practise, however, one only requires physics to be Poincaré covariant.

Case (3). The Galilei Group

The Galilei group will be the first to allow non-trivial guage functions. We shall express the structure of the group in the form $(\mathbb{R}^3 \otimes \mathbb{R}^1) \rtimes_g E(3,\mathbb{R})_T$ where $E(3,\mathbb{R})_T \rightleftharpoons \mathbb{R}^3_T \rtimes_n O(3,\mathbb{R})$ and $g \in \mathrm{Hom}(E(3,\mathbb{R})_T, \mathrm{Aut}(\mathbb{R}^3 \otimes \mathbb{R}^1))$ is defined via:- $g(\underline{v}, R):(\underline{x}, t) \mapsto (R\,\underline{x} + \underline{v}\,t, t)$ $\forall (\underline{v}, R) \in E(3,\mathbb{R})_T$, $(\underline{x}, t) \in \mathbb{R}^3 \otimes \mathbb{R}^1$. As in the case of the Poincaré group, each $\nu \in Z^2{}_0(G(3,\mathbb{R}),\mathbb{R})$ is zero when restricted to $\mathbb{R}^3 \otimes \mathbb{R}^1$. Recall that in the case of the Poincaré group, the assymetric bilinear form defining $\nu | \mathbb{R}^4 \times \mathbb{R}^4$:- f had to satisfy $\Lambda_{\alpha\mu} \Lambda_{\rho\nu} f_{\mu\nu} = f_{\alpha\rho}$ which implied diagonality and hence nullity for $(f_{\mu\nu})$. In the present case we can show that $\nu | (\mathbb{R}^3 \otimes \mathbb{R}^1) \times (\mathbb{R}^3 \otimes \mathbb{R}^1)$ must vanish. To do so, we map $E(3,\mathbb{R})_T$ into $GL(4,\mathbb{R})$ via the homomorphism M, such that:-
$M(\underline{v}, R)_{oo} \equiv 1$, $M(\underline{v}, R)_{oi} \equiv 0$, $M(\underline{v}, R)_{io} \equiv v_i$ and $M(\underline{v}, R)_{ij} \equiv R_{ij}$ $\forall (\underline{v}, R) \in E(3,\mathbb{R})_T$, $(i, j) \in \{1, 2, 3\}$. That is:-

$$M: (\underline{v}, R) \longmapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ v_1 & & & \\ v_2 & & (R_{ij}) & \\ v_3 & & & \end{bmatrix}$$

Then $M(\underline{v}, R)$ acts on the $GL(4,\mathbb{R})$ module $\mathbb{R}^3 \otimes \mathbb{R}^1 \rightleftharpoons \mathbb{R}^4$ of column-

-wise quadruples in the natural way:-

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ v_1 & & & \\ v_2 & & (R_{ij}) & \\ v_3 & & & \end{bmatrix} \begin{bmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ R_{1j}\, x_j + v_1\, t \\ R_{2j}\, x_j + v_2\, t \\ R_{3j}\, x_j + v_2\, t \end{bmatrix}$$

Every cocycle of $Z^2{}_0(\mathbb{R}^3 \otimes \mathbb{R}^1, \mathbb{R})$ has the form $\xi((\underline{x}_1, t_1), (\underline{x}_2, t_2)) = \sum_{i,j=0} f_{ij}\, x_1^i x_2^j$ where $x_1^0 = t_1$, $x_2^0 = t_2$ and $f_{ij} = -f_{ji}$. Those which are restrictions to $\mathbb{R}^3 \otimes \mathbb{R}^1$ of cocycles of $Z^2{}_0(\mathbb{R}^3 \otimes \mathbb{R}^1 \boxtimes g\, E(3,\mathbb{R})_T, \mathbb{R})$ must also satisfy $f_{ij}\, M_{ik}(\underline{v}, R)\;\; M_{je}(\underline{v}, R) = f_{ke}$ also; or $M.f.\underline{M} = f \;\; \forall (\underline{v}, R) \in E(3,\mathbb{R})_T$. Thus $\underline{f} = 0$ and $\nu \,|\, (\mathbb{R}^3 \otimes \mathbb{R}^1) \times (\mathbb{R}^3 \otimes \mathbb{R}^1) = 0, \forall \nu \in Z^2{}_0(\mathbb{R}^3 \otimes \mathbb{R}^1 \boxtimes g\, E(3,\mathbb{R})_T, \mathbb{R})$.

Using a result of Bargmann that $Z^2{}_0(O(3,\mathbb{R}),\mathbb{R}) = 0$, we can easily see that $H^2{}_0(E(3,\mathbb{R})_T, \mathbb{R}) = 0$, using Mackey's theorem and a construction similar to that used for the Poincaré group to show that $H^1_{n'}\; (O(3,\mathbb{R})\; \mathrm{Hom}(\mathbb{R}^3{}_T, \mathbb{R})) = 0$. Thus there is a one to one and onto correspondence between $H^2{}_0((\mathbb{R}^3 \otimes \mathbb{R}^1), \boxtimes g\, E(3,\mathbb{R})_T, \mathbb{R})$ and the cohomology group $H^1{}_{g'}(E(3,\mathbb{R})_T,\; H^1_0(\mathbb{R}^3 \otimes \mathbb{R}^1, \mathbb{R}))$ via:-

$$\nu(((\underline{x}_1, t_1,), (\underline{v}_1, R_1)), (\underline{x}_2, t_2), (\underline{v}_2, R_2))) \equiv \Phi(\nu)((\underline{v}, R_1)^{-1})(\underline{x}_2, t_2)$$

$$\Phi : \; H^2_0(G(3,\mathbb{R}),\mathbb{R}) \longrightarrow H^1_{g'}(E(3,\mathbb{R})_T,\; \mathrm{Hom}(\mathbb{R}^3 \otimes \mathbb{R}^1, \mathbb{R})).$$

We compute the latter cohomology group as follows.

$\forall\, \Psi \in H^1 g(E(3,\mathbb{R})_T,\; \mathrm{Hom}(\mathbb{R}^3 \otimes \mathbb{R}^1, \mathbb{R}))$ we must have:-

$\Psi(0, e)(\underline{x}, t) = 0 \;\; \forall \;\; (\underline{x}, t) \in \mathbb{R}^3 \otimes \mathbb{R}^1$

$\Psi(\underline{v}, R)((\underline{x}_1, t_1)(\underline{x}_2, t_2)) = \Psi(\underline{v}, R)(\underline{x}_1, t_1) + \Psi(\underline{v}, R)(\underline{x}_2, t2)$ and

$\Phi(\underline{v}_1 + R_1 \underline{v}_2, R_1 R_2)\ (\underline{x}_1, t) = \Phi(\underline{v}_1, R_1)(\underline{x}, t) + \Phi(\underline{v}_2 R_2)$ $g(\underline{v}_1, R_1)^{-1}(\underline{x}, t)$ Define the injections:- $i_1: \mathbb{R}^3 \to \mathbb{R}^3 \otimes \mathbb{R}^1$, $i_2: \mathbb{R}^1 \to \mathbb{R}^3 \otimes \mathbb{R}^1$ $j_1: \mathbb{R}^3_T \to E(3,\mathbb{R})_T$ and $j_2: O(3,\mathbb{R}) \to E(3,\mathbb{R})_T$; and then $\Phi_1(\underline{v}, R) \equiv \Phi(\underline{v}, R) \circ i_1$; $\Phi_2(\underline{v}, R) \equiv \Phi(\underline{v}, R) \circ i_2$. We have $\Phi(\underline{v}, R)(\underline{x}, t) = \Phi(\underline{v}, R)((\underline{x}, 0)(0, t)) = \Phi_1(\underline{v}, R)(\underline{x}) + \Phi_2(\underline{v}, R)(t)$. Clearly $\Phi_1 \in C^1_{g'}(E(3,\mathbb{R})_T, \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}))$, $\Phi_2 \in C^1_{g'}(E(3,\mathbb{R})\ \mathrm{Hom}(\mathbb{R}^1, \mathbb{R}))$. We have

$\Phi(\underline{v}, R)(\underline{x}, t) = \Phi((\underline{v}, e)(0, R))\ (\underline{x}, t) = \Phi(\underline{v}, e)(\underline{x}, t) + \Phi(0, R)(g(\underline{v}, e)^{-1}(\underline{x}, t) = \Phi(\underline{v}, e)(\underline{x}, t) + \Phi(0, R)(\underline{x}, -\underline{v}\,t, t)$ Now $\Phi \circ j_2 \equiv \Phi^2$, $\Phi \circ j_1 \equiv \Phi^1$ and we have $\Phi^2(R)(\underline{x}, t) = \Phi^2(R) \circ j_1(x) + \Phi^2(R) \circ i_2(t)$. Clearly $\Phi^2(R_1 R_2) \circ i_2(t) = \Phi(0, R_1 R_2)(0, t) = \Phi(0, R_1)(0, t) + \Phi(0, R_2)(0, t)$ which means $\Phi^2(R_1 R_2) \circ i_2 = \Phi^2(R_1) \circ i_2 + \Phi^2(R_2) \circ i_2$ implying $\Phi^2_2 = 0$ on $S\,O(3,\mathbb{R}) \triangleleft O(3,\mathbb{R}) \cong S\,O(3,\mathbb{R}) \otimes Z(\mathbb{Z})p$, since $S\,O(3,\mathbb{R})$ is simple. Its value on $Z(\mathbb{Z})p$ is defined by its value at p, the non-trivial element of $Z(\mathbb{Z})p$. We also have:-

$\Phi^2(R_1)(\underline{x}\ 0) + \Phi^2(R_2)(R_1^{-1}\underline{x}\ 0) = \Phi^2(R_1 R_2)(\underline{x}, 0)$ which shows that $\Phi^2_1 \in Z^1_{n'}(O(3,\mathbb{R}), \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}))$. But $Z^1_{n'}(O(3,\mathbb{R}), \mathrm{Hom}(\mathbb{R}^3, \mathbb{R})) = B^1_{n'}(O(3,\mathbb{R}), \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}))$ since, $\forall f \in Z^1_{n'}(O(3,\mathbb{R}), \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}))$ we can write $f(R) = -\frac{1}{2}(f(p) \circ n\,(R)^{-1} - f(p))$, $f(p)$ is a constant element of $\mathrm{Hom}(\mathbb{R}^3, \mathbb{R})$. Thus $\Phi^2(R) \circ i_1 = \Phi^2(R) \circ i_2 = 0$ $\forall R \in O(3,\mathbb{R})$ since if we note $p^2 = 1$ then we must have $2\,\Phi^2(p) \circ i_1 = 0$, or $\Phi^2(p)(\underline{x}, 0) = 0$. The group $\mathbb{R}_{add}$ has no finite cyclic subgroups whence $\Phi(p) = 0$. Since $\Phi^2(R)(\underline{x}, t) = \Phi^2(R) \circ i_1(\underline{x}) + \Phi^2(R) \circ i_2(t)$ we must have $\Phi^2 \equiv \Phi \circ j_2 = 0$.

We can then write:-

$\Phi(\underline{v}, R)(\underline{x}, t) = \Phi(\underline{v}, e)(\underline{x}, t) = \Phi^1(\underline{v})(\underline{x}, t)$ where:-

$\Phi^1 \equiv \Phi o\, j_1$, and we also note that since $\Phi(\underline{v}, R)(\underline{x}, t) =$

$\Phi((0, R)(R^{-1}\underline{v}, e))\ (\underline{x}, t) =$

$\Phi(0, R)(\underline{x}, t) + \Phi(R^{-1}\underline{v}, e)(R^{-1}\underline{x}, t)$ we must have

$\Phi^1(\underline{v})(\underline{x}, t) = \Phi^1(R\,\underline{v})(R\,\underline{x}, t) \quad R \quad O(3, \quad)$. Write

$\Phi^1(\underline{v})(\underline{x}, t) = \Phi^1(\underline{v})\ o\ i_1(x) + \Phi^2(\underline{v})\ o\ i_2(t)$. Then, since we have:-

$$\Phi^1(\underline{v}_1 + \underline{v}_2)(\underline{x}, t) = \Phi^1(\underline{v}_1)\ (\underline{x}\ \ t) + \Phi^1(\underline{v}_2)(\underline{x}, -\underline{v}, t, t)$$

we must have

$$\Phi^1(\underline{v}_1 + \underline{v}_2)\ o\ i_1(x) = \Phi^1(\underline{v}_1)\ o\ i_1(\underline{x}) + \Phi^1(\underline{v}_2)\ o\ i_1(\underline{x})$$

and

$$\Phi^1(\underline{v}_1 + \underline{v}_2) o\ i_2(t) = \Phi^1(\underline{v}_1) o\ i_2(t) + \Phi^1(\underline{v}_2) o\ i_2(t) + \Phi^1(\underline{v}_2) o\ i_1(-\underline{v}_1 t)$$

We first note that $\Phi^1(\underline{v}\ ) o\ i_1 \in \mathrm{Hom}(\mathbb{R}^3, \mathbb{R})\ \forall\ \underline{v} \in \mathbb{R}^3_T$ so that

$\Phi^1(\underline{v}_2)(-\underline{v}_1\ t) = -\Phi^1(\underline{v}_2)(\underline{v}_1\ t)$. Thus we must have:-

$$\Phi^1(\underline{v}_1 + \underline{v}_2)\ o\ i_2(t) = \Phi^1(\underline{v}_1) o\ i_2(t) + \Phi^1(\underline{v}_2) o\ i_2(t) - \Phi^1(\underline{v}_2) o\ i_1(\underline{v}_1 t)$$

with the additional identities:-

$\Phi^1(R\ \underline{v})\ O\ i_1\ (R\ \underline{x}) = \Phi^1(\underline{v})\ o\ i_1(x)$, $\Phi^1(R\underline{v}) o\ i_2(t) \equiv \Phi(\underline{v})\ o\ i_2(t)$

$\forall\ R \in O(3, \mathbb{R})$ and that if $\Phi^1(\underline{v})\ o\ i_1 \equiv \phi_1(\underline{v})$ then

$\phi_1 \in \mathrm{Hom}(\mathbb{R}^3_T, \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}))$. Define a functional on $\mathbb{R}^3 \otimes_{\mathbb{R}} \mathbb{R}^3$ by

$\gamma\ (\underline{x} \otimes \underline{v}) \equiv \phi_1(\underline{x})(\underline{x})\ \forall\ (\underline{v}, x) \in \mathbb{R}^3_T \times \mathbb{R}^3$ then we must have

$\gamma\ (R\ \underline{x} \otimes R\ \underline{v}) = \gamma(\underline{x} \otimes \underline{v})$ so that γ must be the inner product:-

$\gamma\ (\underline{x} \otimes \underline{v}) = \alpha\ \underline{v} \cdot \underline{x}$, where $\alpha \in \mathbb{R}$. If $(\Phi^1(\underline{v}) o\ i_2)(t) \equiv \phi_2(\underline{v})(t)$

we must have $\phi_2(\underline{v})(t) = \phi_2(R\ \underline{v})(t)\ \forall R \in O(3, \mathbb{R})$ so that it is necessary that $\phi_2(\underline{v})(t) \equiv f(\underline{v}^2)(t)$ where $f(\underline{v}^2)(t_1 + t_2) = f(\underline{v}^2)(t_1) +$

$f(\underline{v}^2)(t_2)$. This means that $d^2/dt^2(f(\underline{v}^2)(t)) = 0$ or $f(\underline{v}^2)(t) = f'(\underline{v}^2)t + X(\underline{v}^2)$. Since Φ was a normalised cochain $X(\underline{v}^2) = f(\underline{v}^2)(0) = 0$ whence our condition on f' is that:-

$$f'((\underline{v}_1 + \underline{v}_2)^2)\cdot t = f'(\underline{v}_1{}^2)(+ f'(\underline{v}_2{}^2)t - \alpha\underline{v}_2\cdot\underline{v}_1\ t \text{ or}$$

$$f''((\underline{v}_1 + \underline{v}_2)^2) = f''(\underline{v}_1{}^2) + f''(\underline{v}_2{}^2) - \alpha\underline{v}_2\cdot\underline{v}_1) \quad \text{where}$$

$f'' = (\alpha/2)f'$. Write $\underline{v}_2 = \underline{v}_1 + \delta\underline{v}_1$ then:-

$$f''((\underline{v}_1 + \delta\underline{v})^2) - f''(\underline{v}^2) = -\ 2\delta\underline{v}\cdot\underline{v}_1 + f''(\delta\underline{v}^2).$$

Thus $\text{Lim}(\delta\underline{v} \rightarrow 0)\quad f'((\underline{v} + \delta\underline{v})^2) - f''(\underline{v}^2) = -2d\underline{v}\cdot\underline{v} = d\underline{v}\cdot\nabla f(\underline{v}^2)$ where $\nabla \doteq d/d\underline{v}$. So that, on integration we have:-

$$f''(\underline{v}^2) = \int d\,\underline{v}.\nabla f''(\underline{v}) = -\int 2\underline{v}.d\underline{v} = -\underline{v}^2.$$

We must then have $\phi_2(\underline{v})(t) = -\frac{\alpha}{2}\underline{v}^2 t$ and hence with $\Phi^{-1}(\underline{v})(\underline{x}, t) = \phi_1(\underline{v})(t) + \phi_2(\underline{v})(t) = -\frac{\alpha}{2}\underline{v}^2 t + \alpha\underline{v}.\underline{x} = \frac{\alpha}{2}(-\underline{v}^2 t + 2\underline{v}.\underline{x})$ with the result that:- $\Phi(\underline{v}, R)(\underline{x}, t) = \frac{\alpha}{2}(-\underline{v}^2 t + 2\underline{v}.\underline{x})$. The cocycles of $Z^2{}_0(G(3,\mathbb{R}),\mathbb{R})$ have the form $\xi((\underline{x}_1, t_1),(\underline{v}_1, R_1)),((\underline{x}_2, t_2), (\underline{v}_2, R_2)) = \Phi(^-R_1^{-1}\underline{v}_1, R_1^{-1}))(\underline{x}_2, t_2) =$

$\frac{\alpha}{2}(-\underline{v}_1^2 t_2 - 2R_1^{-1}\underline{v}_1\cdot\underline{x}_2) = -\frac{\alpha}{2}(\underline{v}_1{}^2 t_2 + 2\underline{v}_1\cdot R_1\underline{x}_2)$. We see that the elements of $H^2{}_0(G, (3,\mathbb{R}),\mathbb{R})$ are in one-to-one and onto correspondance with elements $H^1{}_{g'}(E(3,\mathbb{R})_T, \text{Hom}(\mathbb{R}^3\otimes\mathbb{R}^1,\mathbb{R}))$, the equivalence classes in the latter group being in one to one correspondence with the real line:- $(\Phi(\alpha)\ (\underline{v}, R)\ )(\underline{x}, t)) \equiv \frac{\alpha}{2}(-\underline{v}^2 t + 2\underline{v}\cdot\underline{x})$. The map $\xi:\mathbb{R} \longrightarrow H^2{}_0(G(3,\mathbb{R}),\ )$ defined by $\xi(\alpha)((\underline{x}_1,t_1),(\underline{v}_1, R_1)), ((\underline{x}_2, t_2))(\underline{v}_2, R_2)) \equiv \Phi^{\alpha}((\underline{v}_1, R_1)^{-1})(\underline{x}_2, t_2)$ is a group isomorphism from the group of additive reals to the

$Ext^1(G(3,\mathbb{R}), \mathbb{R}) \rightleftarrows H^2_0(G(3,\mathbb{R}),\mathbb{R})$.

We have thus computed $H^1_{g'}(E(3,\mathbb{R})_T, Hom(\mathbb{R}^3 \otimes \mathbb{R}^1, \mathbb{R}))$ which, under $\Phi : \mathbb{R} \longrightarrow H^1 g'(E(3,\mathbb{R})_T, Hom(\mathbb{R}^3 \otimes \mathbb{R}^1, \mathbb{R}))$ $\Phi(\alpha)((\underline{v}, R))(\underline{x}, t) \equiv \frac{\alpha}{2}(-\underline{v}^2 t + 2\underline{v}.\underline{x})$, is isomorphic to $\mathbb{R}_{add}$. The group $H^2_0((\mathbb{R}^3 \otimes \mathbb{R}^1) \otimes q\ E(3,\mathbb{R})_T), \mathbb{R})$ was also a by-product.

Let us consider now the application of the above results to classical mechanics. Under $(\underline{v}, R) \in E(3,\mathbb{R})_T$ the kinetic energy functional $\mathfrak{I}$ transforms like:-

$$\mathfrak{I}((\underline{v}, R)(x(s)^{\cdot})) = \mathfrak{I}(\dot{x}(s)) + \frac{d}{ds}\left(\frac{\alpha}{2}(-\underline{v}^2 t(s)+2\underline{v}.\underline{x}(s))\right) \text{ where}$$

we chose the world line $x(s) \equiv (\underline{x}(s), t(s))$ in $\mathcal{W}$. Thus

$$\mathfrak{I}(R\,\dot{\underline{x}}(s)+ \underline{v}\,\dot{t}(s), \dot{t}(s)) = \mathfrak{I}(\dot{\underline{x}}(s), \dot{t}(s)) + \frac{\alpha}{2}(-\underline{v}^2\dot{t}(s)+\underline{v}.\dot{x}(s))$$

Choose $(\underline{v}, R) = (\dot{\underline{x}}(s)/\dot{t}(s), e)$, we then see

$$\mathfrak{I}(0, \dot{t}(s)) = \mathfrak{I}(\dot{\underline{x}}(s),\dot{t}(s)) - \frac{\alpha}{2}\left(\frac{(\dot{\underline{x}}(s))^2}{\dot{t}(s)} + \frac{2\dot{\underline{x}}(s)^2}{\dot{t}(s)}\right) \quad \text{or}$$

$$\mathfrak{I}(\dot{\underline{x}}(s),\dot{t}(s)) = \mathfrak{I}(0, \dot{t}(s)) - \frac{3\alpha}{2}\frac{(\dot{\underline{x}}(s))^2}{\dot{t}(s)}$$

If we recall the requirement that $\mathfrak{I}\,ds$ be a first order infinitessimal then we must have $\mathfrak{I}(0, \dot{t}(s)) = \beta\,\dot{t}(s)$ where $\beta \in \mathbb{R}$. But then $\mathfrak{I}(0, \dot{t}(s)) \in Im(\beta_1)$ and can be dropped, whence we have:-

$$\mathfrak{I}(\dot{\underline{x}}(s), \dot{t}(s)) = -\frac{3\alpha}{2}(\dot{\underline{x}}(s))^2 / \dot{t}(s).$$

Make the choice of time for the evolution parameter on the world line x, then $\dot{t} = 1$ and we thus obtain the free Lagrangian:-

$$\mathfrak{I}(\dot{\underline{x}}(t)) = -\frac{3\alpha}{2}\dot{\underline{x}}(t)^2 = 3\ f_0(x(s), \dot{x}(s)) \text{ a kinetic energy functional.}$$

Note how vividly the world group here defines the kinetic energy functional. The parameter $-3\alpha \equiv$ 'm' is called inertial mass.

Case (4) The Carroll Group.

The structure of $C(3,\mathbb{R})$ the Carroll group will be expressed as $C(3,\mathbb{R}) = (\mathbb{R}^3 \otimes \mathbb{R}^1) ⊠ g' E(3,\mathbb{R})_T$ where $g' \in \mathrm{Hom}(E(3,\mathbb{R})_T, \mathrm{Aut}(\mathbb{R}^3 \otimes \mathbb{R}^1))$ is defined by $g'(\underline{v}, T):(\underline{x}, t) \longmapsto (R\underline{x}, t+\underline{v}.R\underline{x})$. We imbed $C(3,\mathbb{R})$ into $GL(4,\mathbb{R})$ via the monomorphism M':- $M':(\underline{v}, R) \longrightarrow M'(\underline{v}\ e).M'(0, R)$ where $M'oj_2$ is the natural inclusion $O(3,\mathbb{R}) < GL(3,\mathbb{R}) < GL(4,\mathbb{R})$ and $M'oj_1$: $\underline{v} \longmapsto$

$$\begin{bmatrix} 1 & v_1 & v_2 & v_3 \\ 0 & & & \\ 0 & & (\mathbb{1}_3) & \\ 0 & & & \end{bmatrix}$$

Thus
$$M'(\underline{v},R) = \begin{bmatrix} 1 & v_1 & v_2 & v_3 \\ 0 & & & \\ 0 & & (\mathbb{1}_3) & \\ 0 & & & \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & (R_{ij}) & \\ 0 & & & \end{bmatrix} = \begin{bmatrix} 1 & v_j R_{ij} \\ 0 & \\ 0 & (R_{ij}) \\ 0 & \end{bmatrix}$$

We shall need the above morphism later. Firstly however, we compute the group $H'_{g'}(E(3,\mathbb{R})_T, \mathrm{Hom}(\mathbb{R}^3 \otimes \mathbb{R}^1)))$ and later the group $H^2{}_0(C(3,\mathbb{R}), \mathbb{R})$ using our result for the former. Now $\forall\ \Phi \in Z^1{}_{g''}(E(3,\mathbb{R})_T, \mathrm{Hom}(\mathbb{R}^3 \otimes \mathbb{R}^1, \mathbb{R}))$, we must have:-

$\Phi(\underline{v}, R)((\underline{x}_1,t_1)(\underline{x}_2,t_2)) = \Phi(\underline{v},R)(\underline{x}_1,t_1)+\Phi(\underline{v},R)(\underline{x}_2 t_2)$ and

$\Phi(\underline{v}_1+R_1\underline{v}_2, R_1R_2)(x,t) = \Phi(\underline{v}_1,R_1)(\underline{x}, t) + \Phi(\underline{v}_2,R_2)(g'(\underline{v}_1,R_1)^{-1}(\underline{x}, t))$

where $g'(\underline{V}, R)^{-1}:(\underline{x}, t) \longmapsto g(-R^{-1}\underline{v}, R^{-1})(\underline{x},t) = (R^{-1}\underline{x}t-\underline{v}\cdot\underline{x})$. As in case (3) we define the morphism i_1, i_2, j_1 and j_2. We have:-

$\Phi(\underline{v}, R)(\underline{x}, t) = \Phi((\underline{v}, e)(0, R))\ (\underline{x}, t) = \Phi(\underline{v}, e)(\underline{x}, t) + \Phi(0, R)(\underline{x}, t - \underline{v}\cdot\underline{x}) = \Phi^1(\underline{v})(\underline{x}, t) + \Phi^2(R)(\underline{x}, t - \underline{v}.X))$. Where $\Phi^1 \equiv \Phi o\ j_1$, $\Phi^2 \equiv \Phi o j_2$. Let us consider Φ^2. It must satisfy $\Phi^2(R)(\underline{x}, t) =$

$\Phi^2(R) \circ i_1(x) + \Phi^2(R) \circ i_2(t) \; \forall \; R \in O(3,\mathbb{R})$. Now $\Phi_2^2 \in \mathrm{Hom}(O(3,\mathbb{R}),$ $\mathrm{Hom}(\mathbb{R}^1,\mathbb{R}))$ implies $\Phi_2^2 = 0$, where $\Phi_2^2(R) \equiv \Phi^2(R) \circ i_2$. Also $\Phi^2{}_1(R_1R_2)(x) = \Phi_1^2(R_1)(x) + \Phi_1^2(R_1{}^{-1}\underline{x})$ so $\Phi 2_1 \in Z^1{}_{n'}(O(3,\mathbb{R}),$ $\mathrm{Hom}\,(\mathbb{R}^3,\mathbb{R}) = B^1n'(O(3,\mathbb{R}), \mathrm{Hom}(\mathbb{R}^3,\mathbb{R}))$, so we choose $\Phi_1^2 = \Phi_2^2 = 0$ and thus $\Phi^2 = 0$. We must have:-

$\Phi(\underline{v}, R)(\underline{x}, t) = \Phi^1(\underline{v})(\underline{x}, t)) = \Phi^1(R\underline{v})R\underline{x}, t) \; \forall \; R \in O(3,\mathbb{R})$. Write $\Phi^1(v)(\underline{x}, t) = \Phi_1^1(\underline{x})(x) + \Phi_2^1(\underline{v})(t)$ where $\Phi_1^1(\underline{v}) \equiv \Phi^1(\underline{v}) \circ i_1$; $\Phi_2^1(\underline{v}) \equiv$ $\Phi^1(\underline{v}) \circ i, \forall \underline{v} \in \mathbb{R}^3_T$. We see that:-

$\Phi_1^1(\underline{v}_1+\underline{v}_2)(\underline{x}) = \Phi_1^1(\underline{v}_1)(\underline{x}) + \Phi_1^1(\underline{v}_2)(x) + \Phi_2^1(\underline{v}_2)(-\underline{v}_1 \cdot \underline{x})$

and

$\Phi_2^1(\underline{v}_1)(\underline{v}_2)(t) = \Phi_2^1(\underline{v}_1)(t) + \Phi_2^1(\underline{v}_2)(t)$ with the additional requirements that $\Phi_1^1(R\underline{v})(R\underline{x}) = \Phi_1^1(\underline{v})(\underline{x}), \Phi_2^1(\underline{v})(t) = \Phi_2^1(R\underline{v})(t) \; \forall \; R \in O(3,\mathbb{R})$. We must then have $\Phi_2^1(\underline{v})(t) = \Phi_1^1(\underline{v})(x) = 0 \quad \forall \; \underline{v} \in \mathbb{R}^3_T, (\underline{x}, t) \in \mathbb{R}^3 \otimes \mathbb{R}^1$, for the following reason.

The first equation implies $\Phi_2^1(\underline{v}, \underline{x}) = f(\underline{v}, \underline{x})$, that f is linear only on $\underline{x}$. The rotational invariance requires it to be bilinear and hence we must have f = 0. Then that $\Phi_2^1(R\underline{v}) = \Phi_2^1(\underline{v}) \; \forall \; R \in O(3,\mathbb{R})$ implies that $\Phi_2^1(\underline{v}) = \chi(\underline{v}^2)$ say. However χ must be linear and hence vanishes.

Thus we have shown $Z^1_{g'}(E(3,\mathbb{R})_T, \mathrm{Hom}(\mathbb{R}^3 \otimes \mathbb{R}^1,\mathbb{R})) = B^1_{g'}$ $(E(3,\mathbb{R}), \mathrm{Hom}\,(\mathbb{R}^3 \otimes \mathbb{R}^1,\mathbb{R}))$. The mechanics in the Carrollian world must then be invariant:-

$\mathcal{J}(R\dot{\underline{x}}(s), \dot{t}(s)+\underline{v}.\, R\dot{\underline{x}}(s)) = \mathcal{J}(\dot{\underline{x}}(s), \dot{t}(s))$. Choosing $\underline{v} = 0 \Rightarrow$ rotational invariance and with

$R = e\,\mathcal{J}(\dot{x}(s), \dot{t}(s) + \underline{v}.\dot{\underline{x}}(s)) = \mathcal{J}(\dot{x}(s), \dot{t}(s))$.

Take $v_i = -\frac{1}{3}\dot{t}(s)/\dot{x}(s)_i$ then $\underline{v}.\dot{\underline{x}}(s) = -\dot{t}(s)$ and

$\mathcal{J}(\dot{\underline{x}}(s), 0) = \mathcal{J}(\dot{\underline{x}}(s), \dot{t}(s))$ which means that $\mathcal{J}$ doesn't depend on t,

$\mathcal{J}(\dot{\underline{x}}, (s), \dot{t}(s)) \equiv \mathcal{J}'(\dot{\underline{x}}(s))$.. The requirement of rotational invariance and the requirement that $\mathcal{J}'(\dot{\underline{x}}(s))$ be the first order impose

$\mathcal{J}'(\dot{\underline{x}}(s)) = -\alpha\sqrt{\|\dot{\underline{x}}\|^2} \equiv -\alpha\|\dot{\underline{x}}\|$ where $\alpha \in \mathbb{R}$.

The similarity of this kinetic energy functional to the kinetic energy functional of S.R.T. is striking. One can also show that $H^2_0(C(3,\mathbb{R}),\mathbb{R}) = 0$. Since we must have $\nu|(\mathbb{R}^3 \otimes \mathbb{R}^1)^2 = 0$, $H^2_0(E(3,\mathbb{R})_T,\mathbb{R}) = 0$ and $H^1_{g''}(E(3,\mathbb{R})_T, \mathrm{Hom}(\mathbb{R}^3 \otimes \mathbb{R}^1, \mathbb{R})) = 0$.

Case (5). The Static Group

The 'Static' group $S(3,\mathbb{R})$ has the structure:-

$(\mathbb{R}^3 \otimes \mathbb{R}^1) \boxtimes_{g_3} E(3,\mathbb{R})_T$ where $g_3(\underline{v}, R):(\underline{x}, t) \longmapsto (R\underline{x}, t)$

$\forall\, (\underline{v}, R) \in E(3,\mathbb{R})_T$. We wish to compute the group

$H^1_{g_3'}(E(3,\mathbb{R})_T, \mathrm{Hom}(\mathbb{R}^3 \otimes \mathbb{R}^1, \mathbb{R}))$, each element '$\Phi$' of the latter must satisfy:-

$$\Phi(\underline{v}, R)\,(\underline{x}\ \ t_1)(\underline{x}_2, t_2) = \Phi(\underline{v}, R)(\underline{x}_1, t_1) + \Phi(\underline{v}, R)(\underline{x}_2, t_2)$$

and

$$\Phi(\underline{v}_1 + R_1\,\underline{v}_2, R_1\,R_2)(\underline{x}, t) = \Phi(\underline{v}_1, R_1)(\underline{x}, t) + \Phi(\underline{v}_2, R_2)(R_1^{-1}\underline{x}, t)$$

Now we have

$$\Phi(\underline{v}, R)(\underline{x}, t) = \Phi(\underline{v}, e)(\underline{x}, t) + \Phi(0, R)(\underline{x}, t)$$

As we showed before, we must have $(\Phi o j_2) = 0$ so that

$$\Phi(\underline{v}, R)(\underline{x}, t) = \Phi\, o j_1(\underline{v})(\underline{x}, t) \equiv \Phi^1(\underline{v})(\underline{x}, t) = \Phi^1(R\underline{v})(R\underline{x}, t).$$

Write $\Phi^1(\underline{v})(\underline{x}, t) = (\Phi^1(\underline{v}) \circ i_1)(\underline{x}) + (\Phi^1(\underline{v}) \circ i_2)(t) \equiv$ $\Phi^1_1(\underline{v})(\underline{x}) + \Phi^1_2(\underline{v})(t)$, we must then have $\Phi^1_1(\underline{v}_1+\underline{v}_2)(\underline{x}) = \Phi^1_1(\underline{v}_1)(\underline{x}) + \Phi^1_1(\underline{v}_2)(\underline{x})$; $\Phi^1_1(R\,\underline{v})(R\underline{x}) = \Phi^1_1(\underline{v})(\underline{x})$ $\forall R \in O(3,\mathbb{R})$ and $\Phi^1_2(\underline{v}_1 + \underline{v}_2)(t) = \Phi^1_2(\underline{v}_2)(t) + \Phi^1_2(\underline{v}_2)(t)$; $\Phi^1_1(R\underline{v})(t) = \Phi^1_1(\underline{v})(t)$. These four conditions imply $\Phi^1_2 = 0$ and $\Phi^1_1(\underline{v})(\underline{x}) = \beta\underline{v}\cdot\underline{x}$ with $\beta \in \mathbb{R}$.

The kinetic energy functional $\mathcal{T}$ must now satisfy:-

$$\mathcal{T}(\underline{\dot{x}}, \dot{t}) + \frac{d}{ds}(\beta\underline{v}.\underline{\dot{x}}(s)) = \mathcal{T}(R\,\underline{\dot{x}}, \dot{t}) \quad \text{or}$$

$$\mathcal{T}(\dot{x}, \dot{t}) + \beta\underline{v}.\underline{\dot{x}}\,(s) = \mathcal{T}(R\,\underline{\dot{x}}, \dot{t})$$

The only consistent interpretation of this is that obtained as follows, which is due in some part to J.M. Lévy-Leblond. The static group is the limit of the Galilei group obtained in the so-called 'infinite mass' limit, where motion is no-longer defined. The Galilean kinetic energy functional $\mathcal{T}(\underline{\dot{x}}(s))$ is just $\frac{m\underline{\dot{x}}(s)^2}{2}$ if we write $\underline{\dot{y}}(s) \equiv m\,\dot{x}(s)$ $\mathcal{T}(\underline{\dot{y}}(s)) = \frac{1}{2m}\underline{y}(s)^2$ when $m \to \infty\, \mathcal{T}(\underline{\dot{y}}(s)) \longrightarrow 0$, corresponding group theoretically to the map $(\underline{U}, R) \equiv (m\underline{v}, R)$ where with $(\underline{v}, R):(\underline{x},t) \longmapsto (\underline{x}+\underline{v}t, t) = (Rx + \frac{1}{m}\underline{u}\,t, t)$, $m \to \infty$ $(\underline{u}, R):(\underline{x},t) \longmapsto (R\underline{x}, t)$. Thus no motion is possible, we must always have fixed positions as we saw in our dèscription of the Static world. It is easy to show that $H^2_0(S(3,\mathbb{R}),\mathbb{R}) \rightleftharpoons H^1_{g'_3}(E(3,\mathbb{R})_T, \mathrm{Hom}(\mathbb{R}^3 \otimes \mathbb{R}^1, \mathbb{R}))$.

CHAPTER (6)

COHOMOLOGY THEORY AND NON-INERTIAL MOTIONS

COHOMOLOGY THEORY AND NON-INERTIAL MOTIONS.

In this chapter, we shall extend the analysis of chapter (3) to non inertial motions, defining groups analogous to the Galilei group in which constant accelerations act on space time. The approach we adopt is not in the spirit of the latter part of the chapter, but of the first.

Recall that each world automorphism $\alpha \in \mathcal{A}(W)$ was necessarilly of the form $\alpha : (\underline{x}, t) \longmapsto (R(\alpha)(\underline{x}) + \underline{x}(\alpha) + \beta_2(\alpha)(t)$, $t + T(\alpha))$. Here $\beta_1 = (R, \underline{x})$ was a homomorphism from $\mathcal{A}(W)$ to $E(3,\mathbb{R})$ and T was a homomorphism from $\mathcal{A}(W)$ to the time-translation group $\mathbb{R}^1$. In chapter (3), the map $\beta_2 \in C^1(\mathcal{A}(W), C^1(\mathbb{R}^1,\mathbb{R}^3))$ was discussed when $\alpha \in \mathcal{A}(W) \cap I(W)$ the inertial subgroup of $\mathcal{A}(W)$ which mapped uniform motions onto uniform motions. This condition enabled us to write $\beta_2(\alpha)^{\cdot\cdot}(t) = 0$ which inegrated to $\beta_2(\alpha)(t) = \underline{U}(\alpha)t$, where $\underline{U}$ was a one cocycle of $Z^1 noR\ (\mathcal{A}(W)_I{}', \mathbb{R}^3_T)$. Also, we were able to define the two cocycles $\xi'(\alpha_1,\alpha_2) = \underline{U}(\alpha_1)T(\alpha_2)$, $\xi' \in Z^2{}_{noR}(\mathcal{A}(W)_I{}', \mathbb{R}^3)$ where $\mathcal{A}_I(W)' = \mathcal{A}(W)_I/\mathbb{R}^3$. This leads to the explicit construction of the Galilei Group $I(W) \cap \mathcal{A}(W) = \mathcal{A}_I(W) \Leftarrow \mathbb{R}^3 \boxtimes_{\xi'} \mathcal{A}_I(W)' = G(3,\mathbb{R})$, a non-central non-trivial extension.

In this chapter, we shall relax the constraint $\beta_2(\alpha)^{\cdot\cdot}(t) \equiv 0$ and impose instead, the constraint $\beta_2(\alpha)^{\cdot\cdot\cdot}(t) = 0$ corresponding to a constant acceleration. That is, we construct the group of semi-<u>inertial</u> automorphisms of W, $\mathcal{A}(W) \cap S\,I(W)$, where $S\,I(W)$, the group of semi-inertial functions in $B(W)$, $\equiv \{x^{\cdot\cdot\cdot}(t) = 0 \Rightarrow x^{f\cdot\cdot\cdot}(t) = 0\}$, which maps constant accelerations into constant accelerations. Now $\beta_2(\alpha)^{\cdots}(t) = 0$ integrates to $\beta_2(\alpha)^{\cdot\cdot}(t) = \underline{A}(\alpha)$ where $\underline{A} \in C^1(\mathcal{a}(\mathcal{W}), \mathbb{R}^3_{TT})$, where (for convenience) we call $\mathbb{R}^3_{TT}$ the group of tangent-tangent vectors. Integrating twice again and using the normalised cochain condition we must have:- $\beta_2(\alpha)(t) = \frac{1}{2}\underline{A}(\alpha)t^2 + \underline{B}(\alpha)(t)$ where $\underline{B} \in C^1(\mathcal{a}(\mathcal{W}), \mathbb{R}^3_T)$. As in chapter (3), the group law on $\mathcal{a}(\mathcal{W})$ imposes the condition on β_2 that:-

$$\beta_2(\alpha_1 0 \alpha_2)(t) = \beta_2(\alpha_1)(t + T(\alpha_2)) + R(\alpha_1)(\beta_2(\alpha_2)(t))$$

Imposing this condition means that we must have

$$A(\alpha_1 0 \alpha_2)t^2 + B(\alpha_1 0 \alpha_2)t = \tfrac{1}{2}A(\alpha_1)(t+T(\alpha_2))^2 + B(\alpha_1)(t+T(\alpha_2)) + $$

$$+ \tfrac{1}{2}R(\alpha_1)\cdot A(\alpha_2)t^2 + R(\alpha_1)\cdot B(\alpha_2)t.$$ Which is:-

$$\underline{A}(\alpha_1 0 \alpha_2)t^2 = \underline{A}(\alpha_1)t^2 + R(\alpha_1)\cdot\underline{A}(\alpha_2)t^2$$

$$\underline{B}(\alpha_1 0 \alpha_2)t = \underline{B}(\alpha_1)\, t + R(\alpha_1)\cdot\underline{B}(\alpha_2)t + \underline{A}(\alpha_1)T(\alpha_2)t$$

When we write $\underline{X}(\alpha_1 0 \alpha_2) = \underline{X}(\alpha_1) + \underline{X}(\alpha_2) + \xi'(\alpha_1, \alpha_2)$, where $\xi' \in C^2_{noR}(\mathcal{a}(\mathcal{W})'', \mathbb{R}^3)$ is defined by:-

$\xi'(\alpha_1, \alpha_2) \equiv \frac{1}{2}A(\alpha_1)T(\alpha_2)^2 + B(\alpha_1)T(\alpha_2)$. Similarly we define a two cochain $\xi \in C^2_{noR}(\mathcal{a}(\mathcal{W})^\wedge, \mathbb{R}^3_T)$, $\xi(\alpha_1, \alpha_2) \equiv A(\alpha_1)T(\alpha_2)$.

The groups $\mathcal{a}(\mathcal{W})''$ and $\mathcal{a}(\mathcal{W})^\wedge$ are defined as follows:- $\mathcal{a}(\mathcal{W})'' \equiv \mathcal{a}(\mathcal{W}) \cap SI(\mathcal{W}) / \mathbb{R}^3$ and also $\mathcal{a}(\mathcal{W})^\wedge \equiv \mathcal{a}(\mathcal{W})'' / \mathbb{R}^3 T$. We shall see soon, more explicitly that $\mathbb{R}^3_T \lhd \mathcal{a}(\mathcal{W})''$. It readilly follows that $\xi' \notin Z^2_{noR}(\mathcal{a}(\mathcal{W})'', \mathbb{R}^3)$ and $\xi \in Z^2_{noR}(\mathcal{a}(\mathcal{W})^\wedge, \mathbb{R}^3_T)$ since we must have:-

(1) $\delta(\xi')(\alpha_1,\alpha_2,\alpha_3) = \alpha_1 \cdot \xi'(\alpha_2,\alpha_3) - \xi'(\alpha_1 0 \alpha_2,\alpha_3)$
$+ \xi'(\alpha_1,\alpha_2 0 \alpha_3) - \xi'(\alpha_1,\alpha_2) \equiv R(\alpha_1) \cdot (\frac{1}{2} A(\alpha_2) T(\alpha_3)^2$
$+ B(\alpha_2) T(\alpha_3)) - (\frac{1}{2} A(\alpha_1 0 \alpha_2) T(\alpha_3)^2 + B(\alpha_1 0 \alpha_2) T(\alpha_3))$
$+ (\frac{1}{2} A(\alpha_1)(T(\alpha_2 0 \alpha_3))^2 + B(\alpha_1)(T(\alpha_2 0 \alpha_3)) - (\frac{1}{2} A(\alpha_1) T(\alpha_2)^2$
$+ B(\alpha_1) T(\alpha_2))$. Which is:-

$\delta(\xi')(\alpha_1,\alpha_2,\alpha_3) = R(\alpha_1)\frac{1}{2} A(\alpha_2) T(\alpha_3)^2 + R(\alpha_1)\cdot B(\alpha_2) T(\alpha_3)$
$-(\frac{1}{2}A(\alpha_1) + R(\alpha_1)\cdot A(\alpha_2)) T(\alpha_3)^2 + (B(\alpha_1) + R(\alpha_1)\cdot B(\alpha_2)) T(\alpha_3))$
$+(\frac{1}{2}A(\alpha_1)(T(\alpha_2)^2 + 2T(\alpha_2)T(\alpha_3) + T(\alpha_3)^2) + B(\alpha_1)(T(\alpha_2) + T(\alpha_3))$
$-(\frac{1}{2}A(\alpha_1) T(\alpha_2)^2 + B(\alpha_1) T(\alpha_2))$. Or:-

$$\delta(\xi')(\alpha_1,\alpha_2\alpha_3) = A(\alpha_1) . T(\alpha_2) . T(\alpha_3) \neq 0$$

(2) $\delta(\zeta)(\alpha_1,\alpha_2,\alpha_3) \equiv \alpha_1 \cdot \zeta(\alpha_2 \cdot \alpha_3) - \zeta(\alpha_1 0 \alpha_2,\alpha_3)$
$+ \zeta(\alpha_1, \alpha_2 0 \alpha_3) - \zeta(\alpha_1,\alpha_2)$. Which is just the term

$R(\alpha_1)\cdot A(\alpha_2) T(\alpha_3) - A(\alpha_1 0 \alpha_2) T(\alpha_3) + A(\alpha_1) T(\alpha_2 0 \alpha_3) -$
$A(\alpha_1) T(\alpha_2) = T(\alpha_1) \cdot A(\alpha_2) T(\alpha_3) - (A(\alpha_1) + R(\alpha_1)\cdot A(\alpha_2))$
$T(\alpha_3) + A(\alpha_1)(T(\alpha_2) + T(\alpha_3)) - A(\alpha_1) T(\alpha_2) \equiv 0.$

So that $\xi' \notin Z^2_{noR}(\mathcal{Q}(W)", \mathbb{R}^3)$ but $\zeta \in Z^2_{noR}(\mathcal{Q}(W), \mathbb{R}^2_T)$.

These results have the following immediate consequences. An extension $\mathbb{R}^3{}_T \boxtimes_\zeta \mathcal{Q}(W)^\wedge$ exists which we shall see is similar in many ways to the Galilei group. However, an extension $\mathbb{R}^3 \boxtimes_{\xi'} \mathcal{Q}(W)"$ does not exists, since the associativity condition $\delta(\xi') = 0$ is violated. In fact $\mathfrak{a}(\mathcal{W})''$ forms a loop. We shall discuss this concept after dealing with the group $\mathbb{R}^3_T \boxtimes_\xi \mathfrak{a}(\mathcal{W})^\wedge$. The group $\mathfrak{a}(\mathcal{W})^\wedge$ consists of triples (t, ($\underline{A}$, R)) where $\underline{A}$ is an acceleration, $R \in O(3,\mathbb{R})$ and t is a time translation. They act on $\mathbb{R}^3_T$ via $n' \in \mathrm{Hom}(\mathfrak{a}(\mathcal{W})^\wedge\ \mathrm{Aut}(\mathbb{R}^3_T))$ defined by $n'(t, (\underline{A}, R)): \underline{V} \longmapsto R\underline{V}, \forall\ \underline{V} \in \mathbb{R}^3_T$. Clearly, via $(t,(A, T))(t')\ (\underline{A}',R')) = (t + t', (\underline{A} + R\ \underline{A}'\ RR'))$, we must have $\mathfrak{a}(\mathcal{W}) \cong \mathbb{R}^1 \otimes E(3,\mathbb{R})_{TT}$ where $E(3,\mathbb{R})_{TT}$ is the orthogonal group in the inner product space $\mathbb{R}^3_{TT}$. The group $\mathbb{R}^3_T \boxtimes_{\xi''} (\mathbb{R}^1 \otimes E(3,\mathbb{R})_{TT})$ is thus isomorphic to $\mathbb{R}^3_T \boxtimes_\xi \mathfrak{a}(\mathcal{W})^\wedge$ when $\xi'' \in Z^2_n(\mathbb{R}^1 \otimes E(3,\mathbb{R})_{TT}, \mathbb{R}^3_T))$ is defined by $\xi''((\underline{t}_1, \underline{A}_1\ R_1)), (t_2, (\underline{A}_2, R_2)) \equiv A_1\ t_2\ \forall\ (t_i, (\underline{A}i, Ri)) \in \mathbb{R}^1 \otimes E(3,\mathbb{R})_{TT}$. Let us call this group $G(3,\mathbb{R})_T$ the Galilei group in the world whose locations are velocities. From chapter (3), we can obviously write $G(3,\mathbb{R})_T \cong (\mathbb{R}^3_T \otimes \mathbb{R}^1) \boxtimes_g E(3,\mathbb{R})_{TT}$ with $g \in \mathrm{Hom}(E(3,\mathbb{R})_{TT}, \mathrm{Aut}(\mathbb{R}^3 \otimes \mathbb{R}^1))$ defined by:- $g(\underline{A}, R):(\underline{v}, t) \longmapsto (R\underline{v} + \underline{A}t, t)$. Alternatively we can express $G(3,\mathbb{R})_T$ as $(\mathbb{R}^3_T \otimes_\nu (\mathbb{R}^3_{TT} \otimes \mathbb{R}_1) \boxtimes_m O(3)$ where $m \in \mathrm{Hom}(O(3), \mathrm{Aut}(\mathbb{R}^3_T \otimes_\nu (\mathbb{R}^3_{TT} \otimes \mathbb{R}_1))$ is defined by m(R): $(\underline{v}, (\underline{A}, t)) \longmapsto (R\underline{v}, (R\ \underline{A}, t))\ \forall\ R \in O(3,\mathbb{R})$ with the $\nu \in Z^2_o\ (\mathbb{R}^3_{TT} \otimes \mathbb{R}^1, \mathbb{R}^3_T)$ defined by:-

$\nu((\underline{A}_1, t_1), (\underline{A}_2, t_2)) \equiv \underline{A}_1\ t_2$. The group law on $\mathbb{R}^3_T \otimes_\nu (\mathbb{R}^3_{TT} \otimes \mathbb{R}_1)$ is just the familiar elementary law:-

$(\underline{v}_1, (\underline{A}_1, t_1))(\underline{v}_2, (\underline{A}_2, t_2)) = (\underline{v}_1 + \underline{v}_2 + \underline{A}_1, t_2, (\underline{A}_1 + A_2, t_1 + t_2))$

or $'\ \underline{U} \mapsto \underline{U} + \underline{A}\ t'$!

Let us consider now the object $\mathfrak{a}(\mathcal{W})'' = \mathfrak{a}(\mathcal{W}) \cap SI(\mathcal{W})$, which we called a 'loop'. The composition defined on $\mathfrak{A}(\mathbb{W})''$ consists of multiplying quintruplets together in the following way:-

$(\underline{x}_1, t_1, \underline{v}_1, \underline{A}_1, R_1)(\underline{x}_2, t_2, \underline{v}_2, \underline{A}_2, R_2) =$
$(\underline{x}_1 + R_1 \underline{x}_2 + \frac{1}{2} \underline{A}_1 t_2^2 + \underline{v}_1 t_2, t_1 + t_2, \underline{A}_1 + R_1 \underline{A}_2, \underline{v}_1 + R_1 \underline{v}_2 + \underline{A}_1 t_2, R_1 R_2)$. As we saw above the 'cochain $\xi' \in C^2(\mathfrak{A}(\mathbb{W})'', \mathbb{R}^3)$ defined by $\xi'((t_1, \underline{v}_1, \underline{A}_1, R_1), (t_2, \underline{v}_2, A_2, R_2)) \equiv \frac{1}{2} A_1 t_2^2 + \underline{v}_1 t_2$ did not satisfy the associativity requirement that $\delta(\xi') = 0$. The object $\mathfrak{A}(\mathbb{W})''$ is thus non-associative whilst there exists an identity:-(0, 0, 0, 0, e), and everywhere defined composision and each elements of $\mathfrak{A}(\mathbb{W})''$ has an inverse e.g.:- $(\underline{x}, t, \underline{v}, \underline{A}, R)^{-1} = (-R^{-1}(\underline{x}-\underline{A} t^2 + v t), (-t, -R^{-1}(\underline{v} + \underline{A} t), - R^{-1} \underline{A}, R^{-1}))$. Note that as a subobject, $G(3,\mathbb{R})_T \subset \mathfrak{A}(\mathbb{W})''$ is a group and $\mathbb{R}^3$ is also a subobject which is a group. In order to study such an object as $\mathfrak{A}(\mathbb{W})''$, we shall have to study S. Eilenberg's theory of Prolongations which will enable us to discuss non-inertial 'groups' with greater ease.

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Part(2) GROUP PROLONGATIONS

A loop is an object which is a group in all respects ((i) existence of idempotent (ii) composition rule (iii) existence of inverse), expect that the associativity condition is no longer necessarily true. One could call a group an associative loop. Let L be a loop and let $x_1, x_2, x_3 \in L$. Define the associator $A(x_1, x_2, x_3)$ via:-
$A(x_1, x_2, x_3) \equiv (x_1(x_2 x_3)) [(x_1, x_2)x_3]^{-1} \forall x_1, x_2, x_3 \in L$.
If L is a group, A = 0. Assume that the associators associate:- viz
$(A(x_1, x_2, x_3)) A(x_4, x_5, x_o) A(x_7, x_8, x_9) = (A(x_1, x_2, x_3) A(x_4, x_5, x_6))A(x_7, x_8, x_9) \forall x_1, \ldots x_9 \in L$
Then we have:-
$x_1(x_2(x_3 X_4)) = A(x_1, x_2, x_3 x_4)(x_1, x_2) (x_3, x_4) = A(x_1, x_2, x_{3\bullet} x_4) A(x_{1\bullet} x_2, x_2, x_3)((x_1, x_2) x_3) x_4$ and $x_1(x_2(x_3, x_4)) = A(x_2, x_3, x_4) x_1((x_2, x_3)x_4) = A(x_2, x_3, x_4))A(x_1, x_{2\bullet} x_3, x_4)[x_1(x_2, x_3)] x_4 = A(x_2, x_3, x_4)A(x_1, x_{2\bullet} x_3, x_4) A (x_1, x_2, x_3) ((x_1 x_2) x_3) x_4$

Thus we must here, (assuming additivity for associators) $A(x_2,x_3,x_3) - A(x_{1\bullet} x_2,x_3,x_4)+A(x_1,x_{2\bullet}x_3,x_4)-A(x_1,x_2,x_{3\bullet}x_4) - A(x_1,x_2,x_3) = 0$.
Which is strongly remminiscent of the three cocycle property (in chapter 2). With motiviation provided by the above, Eilenberg and Maclane defined the following notions of group prolongations.

A prolongation of a group Q by an Abelian group G is a pair (L, Φ) satisfying the following five conditions:-

(1) L is a loop and $K \prec \mathrm{Ker}(\Phi)$ with $G = \mathscr{C}(K)$.

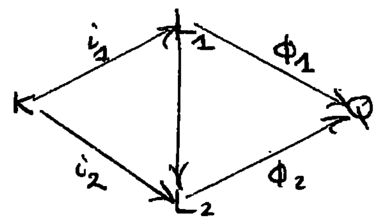
(2) $\Phi \in \mathrm{Hom}(L, Q)$ with $\Phi \circ i = 0$ and $\mathrm{Ker}(\Phi) = \mathrm{Im}(i)$

(3) If A is the associator, $A \in C^3(L, K)$(where Q operators on the Abelian group K via $p \in \mathrm{Hom}(Q)\ \mathrm{Aut}(K)$)), then we require $A(k, q, q_2) = 0 = A(q_1, k, q_2)\ \forall\ q_1, q_2 \in L,\ k \in K$.

(4) $p(\alpha)(g) = \mathrm{In}(\Phi(\alpha))(g))\ \forall\ \alpha \in L,\ g \in G$.

$\mathrm{In}(k)(A) = A\ \forall\ k \in K$ and associator $A = A(x_1, x_2, x_3)$

Teo prolongations (L_1, Φ_1) and (L_2, Φ_2) of Q by G are called equivalent if the below diagram is true



The multiplication of two prolongations is an exactly analogous operation to the multiplication of extensions. We call the product $(L_1, \Phi_1) \wedge (L_2, \Phi_2)$ as in the case of extensions and enlargements. Given an associator $A(x_1, x_2, A(x_3, x_4, x_5))$ we define the cochain $A(x_1, x_2, x_3, x_4, x_5)$ as this quantity. Similarly, we define cochains of $C^{2n+1}(L, K)$ via $A(x_1, \ldots, x_{2n+1}) = A(x_1, \ldots, x_{2n-2}, A(x_{2n-1}, x_{Ln}, x_{2n+1}))\ \forall\ n \in Z_+$. The non-associativity of L in the prolongation (L, Φ) is controlled by the following four classes of prolongations of Q by G:-

Θ^L_n:- $A(x_1, .., x_{2n}, x) = 0$ ⎫ Prolongations for which

Θ^K_n:- $A(x_1, .., x_{2n}, k) = 0$ ⎭ this is true.

G_n^L :- $A(x_1,\ldots,x_{2n}; a) \in G$ ⎧ Prolongations for which

G_n^K :- $A(x_1,\ldots x_{2n,k}) \in G$ ⎩ this is true

$\forall n \geq 0$, $x_1,\ldots,x_{2n} \in L$, $k \in K$. In addition, the class R_n consists of these prolongations (L,Φ) of Q by K which contain a prolongation os Q by K, (where $L' < L$ and $\Phi \mid L' \equiv \Phi'$) of class Θ_n^K. The class G_o^K of prolongations by the condition K = G. Next consider the class $\Theta_n^K \cap G_n^L$, since $G_{n-1}^K \subset G_n^L$ and $\Theta_n^L \subset G_n^L$ with $G_{n-1}^K \subset \Theta_n^K$ and $\Theta_n^L \subset \Theta_n^K$ we have Θ_n^L, $G_{n-1}^K \subset_\wedge \Theta_n^K \cap G_n^L$ and thus, we must have the set of prolongation products $\Theta_n^L \wedge G_{n-1}^K$, $\Theta_n^L \wedge G_{n-1}^K \subset H_n^K \cap G_n^L$. We can thus define the quotient class:-

$\Theta_n^K \cap G_n^L / \Theta_n^L \wedge G_{n-1}^K$ and this is a group which is isomorphic to $H_p^{2n+1}(Q,G)$ $\forall$ $n \in Z_+$ as is shown by Eilenberg-Maclane. Similarly they prove the theorem that $\Theta_{n-1}^L \cap G_n^K / R_n \wedge G_n^L$ is a group isomorphic to $H^{2n+2}(Q, G) \forall n \in Z_+$.

Let us discuss the first few 'n' for clarity. When n = 0, L is associative and we return to the group extension case when G is the central subgroup of K. When n = 1, a prolongation $(L,\Phi) \in H_1^K \cap G_1^L$ if $A(x_1,x_2,k_3) = L$ and $A(x_1,x_2,x_3) \in G \; \forall \; x_1,x_2,x_3 \in L$. The class Θ_1^L consists of those prolongations for which (L,Φ) is associative and hence a group whilst G_o^K is the class of prolongations where K = G. A prolongation in $\Theta_1^L \wedge G_o^K$ is then specified as a prolongation product of a group by a prolongation consisting of pairs $(k, q) = (k, e)(0, q)$ with K associative, Q associative and with an associator which satisfied (i) $A(x_1,x_2,x_3) \in G \;\forall\; x_1,x_2,x_3 \in L$ (ii) $A(((k), x_1,x_2) = A(x_1,k,x_2) = A(x_1,x_2,k) = 0 \;\forall\; x_1,x_2 \in L$ and is hence specified by the associator $A(o, q_1)\,(o_1,q_2)\,(o_1,q_3)) \equiv \delta(\phi)(q_1,q_2,q_3)$ where $(0, q_1)(0, q_2) \equiv (\phi(q_1,q_2), q_1,q_2)$

Thus if $(L,\phi) \in \mathbb{H}^L_1 \wedge G^K_o$, is is also a loop specified by the 3 coboundary $\delta(\phi) \in B^3_p(Q,G)$ since in $\mathbb{H}^L_1$, we must have '$\delta(\phi') = 0$,' . How each prolongation in $\mathbb{H}^K_1 \cap G^L_1$ is specified by a three cocycle of $Z^3_p(Q, G)$ by the following arguments. We saw $(L,\phi) \in \mathbb{H}^K_1 \cap G^L_1$ iff the conditions mentioned above. It trivially follows that, if :- $\Phi(q_1,q_2,q_3) = A(j(q_1),j(q_2), j(q_3))$ where j is a section from Q to L with $\phi \circ j = \mathbb{1}$, then $\Phi \in Z^3_p(Q,G)$. (By an argument similar to our introductory one). Thus it is very plausible that $H^3_p(Q, G) \rightleftarrows \mathbb{H}^K_1 \cap G^L_1 / H^L_1 \wedge G^K_o$. C.f. our quite different characterisation of $H^3_p(Q, G)$ when G is the centre of a Q kernel in chapter (2).

Part(3).

With this theory in mind, let us return to the loop $\mathcal{C}(W)$" which occured earlier. Here, we must replace Q by the Galilei-like group $G(3,\mathbb{R})_T$ and K=G by $\mathbb{R}^3$. Thus $\mathcal{C}(W)$" is a prolongation of $G(3,\mathbb{R})_T$ by $\mathbb{R}^3$. Its associator specifies it as of the class G^K_o since it is a 3 coboundary $\delta(\phi)(\alpha_1,\alpha_2,\alpha_3) \equiv A(\alpha_1)\cdot T(\alpha_2)\cdot T(\alpha_3)$ or :- $\delta(\phi')\big((\underline{v}_1,t_1,\underline{A}_1,R_1), (\underline{a}_2,t_2,\underline{A}_2,R_2),(\underline{v}_3,t_3,\underline{A}_3,R_3)\big) =$

$\underline{A}_1\ t_1\ t_2$. When a loop L is a prolongation of Q by G we shall write $L = G \boxminus_{\phi} Q$. Thus in our case, we must have:-

$$\mathcal{A}(W) \cap \mathcal{SI}(W) \cong \mathbb{R}^3 \boxminus_{\xi} ((\mathbb{R}^3_T \otimes \mathbb{R}_1) \boxtimes g\ E(3, \mathbb{R})_{TT}))$$

where $p \in \mathrm{Hom}(G(3, \mathbb{R})_{TT}, \mathrm{Aut}(\mathbb{R}^3))$ is defined by $g((\underline{v}, t),(\underline{A},R))$: $\underline{x} \longmapsto R\underline{x}$. $\forall\ R \in O(3, \mathbb{R})$.

In this new notation $L = G \boxminus_{\phi} Q$ is specified by the three cocycle $\phi \in Z^3_p\ (Q, G)$, our loop is specified by the three coboundary $\delta(\phi')$. Of course it is only defined for the class $\mathcal{O}^K_1 \cap G^L_1$ of prolongations of Q by G.

The generalisation of the results of this chapter are in progress at the moment. In analogy with the above construction it would seem that the 'group' replacing $G(3, \mathbb{R})_T$ would be a loop containing the Group $G(3, \mathbb{R})_{TT}$ and that hence the theory would break down and one cannot, it seems, prolong loops by loops!

To conclude this chapter we will discuss the role of cup products in the present context in the manner of section (h) of chapter (3).

Recall the existence of the 2 cocycle $\xi \in Z^2_n(\mathbb{R}^1 \otimes E(3, \mathbb{R})_{TT}, \mathbb{R}^3_T)$ defining the group $G(3, \mathbb{R})_T \cong \mathbb{R}^3_T \boxtimes_{\xi} (\mathbb{R}^1 \otimes E(3, \mathbb{R})_{TT})$, given by $\xi((t_1,(\underline{A}_1, R_1),(t_2,(\underline{A}_2,R_2)) = A_1 t_2$. Now let us note that if we call G the group $R' \otimes E(3, \mathbb{R})_{TT}$, Π_1 the group $\mathbb{R}^3_{TT}$ and Π_2 the group $\mathbb{R}^1$ with $\Pi \equiv \mathbb{R}^3_T$, then Π_1 and Π_2 are paired to Π via the cup product $\underline{A} \cup t \equiv \underline{A}\ t$. Also G is a group of automorphisms of Π_1, Π_2 and Π via $p_i \in \mathrm{Hom}(G, A(\Pi_i))$ i = 1, 2, 0 defined by $P_1((t,(\underline{A},R)):\underline{A}' \longmapsto$

$R \cdot \underline{A}'$, $p_2((t, (\underline{A}, R)) : t' \longmapsto t'$ and $p((t, (A, R)) : \underline{V} \longmapsto R \cdot \underline{V}$ $\forall (t, (\underline{A}, R)) \in \mathbb{R}^1 \otimes E(3, \mathbb{R})_{\pi}$ and $(\underline{A}', t'. \underline{V}) \in \mathbb{R}^3_{\pi} \times \mathbb{R}^1 \times \mathbb{R}^3_{\pi}$. We also have $(t, (\underline{A}, B)) \cdot (\underline{A}' \cup t') \equiv (t, (\underline{A}, R)) \cdot \underline{A}' \cup (t, (\underline{A}, R)) \cdot t$. Define one cocycles of $Z^1_{p1}(G, \pi_1)$ and $Z^1_{p2}(G, \pi_2)$ via $f_1 : ((t, (\underline{A}, R)) \longmapsto \underline{A}$ and $f_2 : ((t, (\underline{A}, R)) \longmapsto t$. Then immediately, $f_1 \cup f_2((t_1, (\underline{v}_1, R_1)), \quad (t_2, (\underline{V}_2, R_2))) = \underline{A}_1 \cup t_2 = \underline{A}_1 t_2$ Whence $\delta(f_1 \cup f_2) \neq 0$, by taking $\underline{V} \longmapsto A$ in the calculation of section (h) of part (3)of chapter (3). Whence $\zeta = f_1 \cup f_2$.

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