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Fair Apportionment in the View of the Venice Commission's Recommendation

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Abstract

In this paper we analyze the consequences of the fairness recommendation of the Venice Commission in allocating voting districts among larger administrative regions. This recommendation requires the size of any constituency not to differ from the average constituency size by more than a fixed limit. We show that this minimum difference constraint, while attractive per definition, is not compatible with monotonicity and Hare-quota properties, two standard requirements of apportionment rules.
We present an algorithm that efficiently finds an allotment such that the differences from the average district size are lexicographically minimized. This apportionment rule is a well-defined allocation mechanism compatible with and derived from the recommendation of the Venice Commission. Finally, we compare this apportionment rule with mainstream mechanisms using real data from Hungary and the United States.

Keywords: Apportionment, voting, elections, Venice Commission, proportionality, lexicographic ordering

JEL classification: C71, D72

A választókörzetek igazságos elosztása a Velencei Bizottság ajánlásának tükrében

Biró Péter - Kóczy Á. László - Sziklai Balázs

Összefoglaló

Dolgozatunkban a Velencei Bizottság ajánlásának következményeit elemezzük a választókörzetek nagyobb adminisztratív körzetek közötti igazságos elosztásának szempontjából. A javaslat szerint egy választókörzet méretének az átlagos körzetmérettől való eltérése nem haladhat meg egy előre rögzített korlátot. Megmutatjuk, hogy ez a legkisebb eltérésfeltétel, bár definíciójából adódóan ígéretes feltétel, nem összeegyeztethető a monotonitás és a Hare-kvóta tulajdonságokkal. Utóbbi tulajdonságok szokásos elvárásként merülnek fel a kiosztási eljárásokkal kapcsolatban.
Bemutatunk egy hatékony algoritmust, ami megtalálja azt a kiosztást, melyre az átlagos körzetmérettől való eltérések lexikografikusan minimálisak. Ez a kiosztási módszer egy jól definiált elosztási mechanizmus, ami ugyanakkor kompatibilis a Velencei Bizottság ajánlásával. Végül a kiosztási módszerünket magyarországi és Egyesült Államok-beli adatokat vizsgálva összevetjük az általánosan használt mechanizmusokkal.

Tárgyszavak: kiosztás, szavazás, választások, Velencei Bizottság, arányosság, lexikografikus rendezés

JEL kód: C71, D72

Fair Apportionment in the View of the Venice Commission's Recommendation*

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Abstract

In this paper we analyze the consequences of the fairness recommendation of the Venice Commission in allocating voting districts among larger administrative regions. This recommendation requires the size of any constituency not to differ from the average constituency size by more than a fixed limit. We show that this minimum difference constraint, while attractive per definition, is not compatible with monotonicity and Hare-quota properties, two standard requirements of apportionment rules.

We present an algorithm that efficiently finds an allotment such that the differences from the average district size are lexicographically minimized. This apportionment rule is a well-defined allocation mechanism compatible with and derived from the recommendation of the

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1 Introduction

One man – one vote! A properly functioning electoral system is the foundation of any parliamentary democracy. The stakes at the elections are very high and therefore the codification of any electoral law should be done with great care. The Code of Good Practice in Electoral Matters is a comprehensive guidebook published in 2002 by the European Commission for Democracy through Law, better known as the Venice Commission that offers help in this subject (Venice Commission, 2002a). The EU observers who reviewed Albania's and Estonia's electoral law in 2011 consequently used this source to evaluate the result (OSCE/ODIHR, 2011; Venice Commission and OSCE/ODIHR, 2011). In 2012 Hungary also introduced some modifications in its electoral law some of which closely followed the recommendations of the Venice Commission. This paper focuses on one particular issue in this Code, namely the fair apportionment of representatives: How to allocate constituencies among political or administrative units, such as counties, regions or states, so that the proportional representation of voters is least violated?

The apportionment problem generates constant debate even in countries with well-established democracies such as the United States. (For a comprehensive historical overview see Balinski and Young, 1982). Democratic countries are run by bodies of elected representatives. An equal influence on their decisions require equally sized electoral districts, that is, constituencies must have the same number of voters. This alone would not cause any difficulty, but in many cases the boundaries of the constituencies must respect geographical, historical or administrative boundaries.

> "**Equality in voting power**, where the elections are not being

> held in one single constituency, requires constituency boundaries to be drawn in such a way that seats in the lower chambers representing the people are distributed equally among the constituencies, in accordance with a specific apportionment criterion, e.g. the number of residents in the constituency, the number of resident nationals (including minors), the number of registered electors, or possibly the number of people actually voting ... Constituency boundaries may also be determined on the basis of geographical criteria and the administrative or indeed historic boundary lines, which often depend on geography ... The maximum admissible departure from the distribution criterion adopted depends on the individual situation, although it should seldom exceed 10% and never 15%, except in really exceptional circumstances (a demographically weak administrative unit of the same importance as others with at least one lower-chamber representative, or concentration of a specific national minority)."

When these requirements are not met the fairness of the whole election is in jeopardy. In Georgia, where the electoral law of 1999 did not set rules about the sizes of constituencies, the number of voters per constituency ranged from some 3,600 voters in the Lent'ekhi district or 4,200 in the Kazbegi districts to over 138,000 in Kutaisi City (Venice Commission, 2002b). In other words, voters from Lent'ekhi or the Kazbegi district had 30 times more influence than those from Kutaisi City.

In the United States, on the other hand, no deviations are permitted, at least in theory. In Singapore the toleration limit is 30%, and further examples are given in Handley (2007): "Other common thresholds are 5 percent (e.g., New Zealand, Albania, and Yemen); 10 percent (e.g., Australia, Italy, and the Ukraine); 15 percent (e.g., Armenia, Germany, and the Czech Republic) and 20 percent (e.g., Zimbabwe and Papua New Guinea). In Canada, the independent commissions charged with creating federal electoral districts are allowed to deviate by up to 25 percent from the provincial quotas, and even more under 'extraordinary circumstances'." A recent proposal to reform the constituency map of the United Kingdom worked with a 5% permitted deviation from the average size (Balinski, Johnston, McLean, Young, and Cummine, 2010).

In the European Union the recommendation of the Venice Commission is seen as the guideline in electoral matters. For instance, the draft version of the 2012 electoral law of Hungary adopted this recommendation almost word by word, but the 10-15% maximal difference between the population of any two constituency turned out to be infeasible given the actual size of the parliament and the populations of counties, if the constituencies cannot extend over county borders (Biró, Kóczy, and Sziklai, 2012; Bodnár, 2012). Even with the subsequent relaxation allowing a 15% (at most 20%) departure from the average size of constituencies in the final version the requirements were just met.

2 The apportionment problem and its properties

An *apportionment problem* $(\mathbf{p}, H)$ is a pair consisting a vector

$$\mathbf{p} = (p_1, p_2, \ldots, p_n)$$

of state populations, where $P = \sum_i p_i$ is the population of the country and H denotes the number of seats in the legislature or House. Our task is to determine the non-negative integers $a_1, a_2, \ldots, a_n$ with $\sum_i a_i = H$ representing the number of constituencies in states $1, 2, \ldots, n$.

Let $\mathbf{p} \in \mathbb{N}^n$ and $\mathbf{a} \in \mathbb{N}_0^n$ be the n-dimensional vectors that contain the population sizes and the allotted number of seats respectively. An *apportionment method or rule* is a function M that assigns an allotment for each instance of an apportionment problem. In general, apportionment methods need not result in a unique allotment, i.e. M can be set-valued. Throughout the paper we will employ the following notation: let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we say that $\mathbf{x} \geq \mathbf{y}$ if $x_i \geq y_i$ for $i = 1, 2, \ldots, n$.

In the following we introduce several properties of apportionments.

Quota A good apportionment rule is as close to proportionality as possible. The apportionment rule $M(\mathbf{p}, H) = \mathbf{a}$ satisfies *exact quota* when the fractions

$a_i = \frac{p_i}{P}H$ are integers for all $i \in \{1, \ldots, n\}$. Unfortunately it is hardly ever the case that such an ideal situation occurs.

Since states cannot receive fractional districts, taking one of the *nearest* integers to the exactly proportional share is a natural choice. The apportionment rule $M(\mathbf{p}, H) = \mathbf{a}$ satisfies *lower (upper) quotas*, if no state receives less (more) constituencies than the lower (upper) integer part of its respective share, that is $a_i \geq \left\lfloor \frac{p_i}{P}H \right\rfloor$ for all $i \in \{1, \ldots, n\}$ and $a_i \leq \left\lceil \frac{p_i}{P}H \right\rceil$ for all $i \in \{1, \ldots, n\}$, respectively.

An apportionment rule M satisfies the *Hare quota* property if it satisfies both upper and lower quota.

Monotonicity The individual states should receive more seats when more seats are available in the House. Formally:

Definition 1. *An apportionment rule M is* house-monotonic *if $M(\mathbf{p}, H') \geq M(\mathbf{p}, H)$ for any apportionment problem $(\mathbf{p}, H)$ and House sizes $H' > H$.*

A scenario where increasing the House size would decrease the number of seats allotted to a state is often considered undesirable, perhaps even paradoxical. An apportionment rule where this is possible is said to exhibit the *Alabama paradox* referring to a historical occurrence of the phenomenon in the case of state Alabama. A rule is said to be *house-monotonic* if it does not suffer from such weakness.

There is a related monotonicity requirement and the associated paradox when populations are considered. The *population paradox* arises when the population of two states increases at different rates. Then it is possible that the state with more rapid growth actually loses seats to the state with slower growth.

Definition 2. *An apportionment rule M is population-monotonic if $M(\mathbf{p}', H)_i \geq M(\mathbf{p}, H)_i$ for any House size H and population sizes $\mathbf{p}, \mathbf{p}'$ such that $p'_i > p_i$, $p'_j > p_j$ and $\frac{p'_i}{p_i} \geq \frac{p'_j}{p_j}$ while $p'_k = p_k$ for $k \in \{1, 2, \ldots, n\}, k \neq i, j$.*

Balinski and Young (1975) provided a so-called Quota-method that is house-monotonic and fulfills the Hare-quota property as well, but proved that no method satisfies Hare-quota that is free from both Alabama and the population paradoxes (Balinski and Young, 1982).

Maximal difference property The next property characterizes the recommendation[1] made by the Venice Commission (2002a). Let $\bar{a} = \frac{P}{H}$ denote the average size of a constituency and let δ_i be the difference in percentage, displayed by the constituencies of state i and let d_i be its absolute value. Formally

$$\delta_i = \frac{\frac{p_i}{a_i} - \bar{a}}{\bar{a}} \quad \text{and} \quad d_i = |\delta_i| \tag{1}$$

For a given apportionment problem $(\mathbf{p}, H)$ let $\alpha_{(\mathbf{p},H)}$ be the smallest maximal difference that can be achieved with an allotment i.e.

$$\alpha_{(\mathbf{p},H)} = \min_{\mathbf{a}} \max_{i \in \{1,\dots,n\}} \{d_i\} \tag{2}$$

Definition 3. *An apportionment rule M satisfies the* maximal difference property *if* $\left| \frac{\frac{p_i}{M(\mathbf{p},H)_i} - \bar{a}}{\bar{a}} \right| \leq \alpha_{(\mathbf{p},H)}$ *for any* $i \in \{1, \dots, n\}$.

The philosophy behind the Hare-quota and the maximal difference property is very similar, but not quite the same. The Hare-quota specifies how many seats a state should receive at least and at most. If a state gets less than its lower quota, then the allotment can be considered somewhat unfair from the point of view of *that particular state*. The maximal difference property is concerned rather with *the individual voter*. If the population sizes of the constituencies differ too much so does the voters' influence. Not surprisingly, the Hare-quota property plays more central role in the U.S. where the states are large and highly independent. In Europe, where the countries consist of small and in some sense uniform counties, the maximal difference property is more accepted[2].

[1] Although the Venice Commission is flexible on what kind of data should be the distribution criterion based on, it is clear that the difference from the average value is to be minimized. The most common interpretation is that there should be a limit on the allowable departure on the average number of registered voters per constituency (see Handley (2007)). We follow this practice as well, nevertheless our results hold in general, irrespective of the chosen reference data.

[2] Many European electoral laws impose a fixed limit on $\max_{i \in \{1,\dots,n\}} \{d_i\}$ rather than minimizing it. The Venice Commission follows this practice as well. It can happen, however, that, given an apportionment problem, no allotment exists that satisfy a certain limit, while an allotment with minimal difference always exists.

3 The Maximal Difference Property

In this section we review the basic features of the maximal difference property. In the following we will omit the lower index of $\alpha_{(\mathbf{p},H)}$ and write simply α. First let us note that α is not monotone in the House size. To see this consider the allocation problem where $p = (100, 200)$ and let $H = 3$. Then it is possible to distribute the seats according to the exact quota thus $\alpha = 0$. Increasing H by 1 however will spoil both d_1 and d_2.

3.1 Upper bounds on the maximal difference

Obviously d_i is the smallest if state i receives either its lower or upper quota. Let l_i and u_i, respectively denote these quotas of state i respectively and let β_i denote the minimum difference achievable for state i. The maximum of these β_i values, denoted by β, is a natural lower bound for α. Formally:

$$\beta_i = \min\left(\left|\frac{\frac{p_i}{l_i} - \bar{a}}{\bar{a}}\right|, \left|\frac{\frac{p_i}{u_i} - \bar{a}}{\bar{a}}\right|\right), \qquad \beta = \max_{i \in N} \beta_i.$$

Empirical analysis shows that, in general, increasing H results in a lower α *ceteris paribus*. The problem with small House sizes is that they imply a larger average constituency size. Divisibility issues can appear for smaller states that are only a few times as large as $\bar{a}$. It can happen that the average size of the constituencies of state i is equally far away from $\bar{a}$ for both the lower and upper integer part of $\frac{P}{p_i}$, formally

$$\frac{\frac{p_i}{l_i} - \bar{a}}{\bar{a}} = \frac{\bar{a} - \frac{p_i}{u_i}}{\bar{a}}. \tag{3}$$

For instance, if $l_i = 2$ and $u_i = 3$ then $p_i = \frac{12}{5}\bar{a}$ and $d_i = 0.2$. A simple computation shows that, in general, if (3) holds, then $d_i = \frac{1}{2l_i+1}$. The next table summarizes the problematic state population sizes.

In other words, if there is a state with population $\frac{4}{3}$ of the average constituency size then α is at least $\frac{1}{3}$. For this value a lower d cannot be adhered to. One way to overcome this is to increase the house size H and thereby increase the number of constituencies allocated to each state, in particular, to the smallest state. For let i denote the smallest state and

$l_i - u_i$	p_i	β_i
$1-2$	$\frac{4}{3}\bar{a}$	0.333
$2-3$	$\frac{12}{5}\bar{a}$	0.2
$3-4$	$\frac{24}{7}\bar{a}$	0.143
$4-5$	$\frac{40}{9}\bar{a}$	0.111
$5-6$	$\frac{60}{11}\bar{a}$	0.091

Table 1: Critical state population regarding divisibility

$$\gamma \stackrel{def}{=} \begin{cases} \frac{1}{2l_i+1} & \text{if } l_i \neq 0, \\ \infty & \text{if } l_i = 0. \end{cases}$$

As the House size increases, l_i increases, and therefore γ decrases. Note that γ is an upper bound for β but there is no obvious connection between γ and α. For instance, if we are able to distribute the seats according to the exact quota, then α is zero, but γ can be high. However let $p_1, p_2 = 200$ and $p_3 = 600$ and let the House size equal to 7. Then $\gamma = \frac{1}{3}$, but $\alpha \geq 0.4$. We will further analyze the relation of α, β and γ in Section 5 using real data.

3.2 Properties

As we mentioned earlier, the Hare-quota and maximal difference properties have different objectives. An apportionment method that implements the latter will distribute less seats to a state than its lower quota if the maximal difference can be lowered in this way. Large states serve as puffers where superfluous seats can be allocated or seats can be acquired if there are needed elsewhere as these do not change the average size of constituencies dramatically. Table 2 demonstrates this process.

In the above example the total population equals to 201 while the average constituency size is 10.05. If we insist on applying the Hare-quota then State E must receive at least 9 seats. As a result State A - the smallest one - gets only 2. The voters in State B have the greatest influence, more than 44% more than the voters in State A. On the other hand if we apply the maximal difference property the largest bias - 31% - can be observed between State A and E.

Method ⇒		Maximal difference			Hare-quota		
State	Population	seats	$\frac{p_i}{a_i}$	δ_i	seats	$\frac{p_i}{a_i}$	δ_i
A	26	3	8.666	$\mathbf{-0.138}$	2	13	**0.293**
B	27	3	9	−0.104	3	9	−0.104
C	28	3	9.333	−0.071	3	9.333	−0.071
D	29	3	9.666	−0.038	3	9.666	−0.038
E	91	8	11.375	0.131	9	10.111	0.006

Table 2: Hare-quota vs. maximal difference

Finally we note that the maximal difference property is not compatible with house-monotonicity either. An apportionment rule that minimizes the maximal difference can produce the Alabama-paradox.

State	Population	Seats	$\frac{p_i}{a_i}$	d_i	Seats	$\frac{p_i}{a_i}$	d_i
A	69	3	23	0.114	4	17.250	−0.104
B	70	3	23.333	0.130	4	17.500	−0.091
C	150	8	18.750	−0.091	7	21.428	0.112
Total	289	14	20.642		15	19.266	

Table 3: House-monotonicity and maximal difference

Table 3 shows an example where increasing the House size from 14 to 15 causes State C to lose a seat. State C is the largest state hence its average constituency size changes only a little when one of its seats is assigned elsewhere. A House-monotone allotment such as $\mathbf{a} = (3, 4, 8)$ would have a 0.193 as maximal difference almost twice as much as the allotment in the example. This also exceeds the 15% limit of the Venice Commission's recommendation, making it an unfeasible solution.

In summary we can say that neither the Hare-quota nor House-monotonicity is compatible with the maximal difference property. Therefore there is no apportionment rule that is conform with the recommendation made by the Venice Commission and is free from the Alabama paradox or produces allotments according to the Hare-quota. We consider this a conflict between equality among states versus equality among voters. The Venice commission clearly cast its vote in favour of the second.

4 The lexicographically optimal solution

The recommendation of the Venice Commission gives a strong constraint for the solution of an apportionment problem. However, the set of allowable allotments can still be large, that may leave room for gerrymandering by the decision makers. As Balinski and Young (1975) also argue, having a well-defined allotment rule that leads to a unique solution is the best way to avoid political issues in the apportionment process.

In this section we define a new apportionment method where the differences from the average size of constituencies are lexicographically minimized. This uniquely defined rule satisfies the maximal difference property, so it is based on the Venice Commission's recommendation. We also give an efficient algorithm to compute such a solution.

Given an apportionment problem $(\mathbf{p}, H)$ and an allotment $\mathbf{a}$, let $\mathbf{d}(\mathbf{a})$ denote a nonnegative n-dimensional vector, where the differences $d_i(\mathbf{a})$ are contained in a nondecreasing order. A solution $\mathbf{a}$ is said to be *lexicographically minimal*, or simply *leximin*, if there is no other allotment $\mathbf{a}'$ where $\mathbf{d}(\mathbf{a}')$ is lexicographically smaller than $\mathbf{d}(\mathbf{a})$, denoted by $\mathbf{d}(\mathbf{a}') \prec \mathbf{d}(\mathbf{a})$.

Greedy leximin algorithm

Let us refer to $\mathbf{a} \in \mathbb{N}^n$ as a *pre-allotment* if the $\sum_i a_i = H$ condition is relaxed. Let $\mathbf{a}^{i+}$ denote a pre-allotment adjusted from $\mathbf{a}$, where $a_i^{i+} = a_i + 1$ and $a_j^{i+} = a_j$ for each $j \neq i$. Similarly, let $\mathbf{a}^{i-}$ denote a pre-allotment, where $a_i^{i-} = a_i - 1$ and $a_j^{i-} = a_j$ for each $j \neq i$. For simplicity, and to ensure the uniqueness of the solution, we assume that $d_i(\mathbf{a})$ is not equal to $d_j(\mathbf{a})$ for any strictly positive pre-allotment $\mathbf{a}$ and pair of counties i and j. (Note that this condition can be always satisfied if we perturb $\mathbf{p}$, and it does not effect the optimality of the solution.)

Phase 1: Let $\mathbf{a}[0]$ be a pre-allotment such that $d_i(\mathbf{a}[0])$ is minimal for each state i (i.e. equal to β_i). Let the total number of seats allocated in $\mathbf{a}[0]$ be $l = \sum_{i=1}^{n} a_i[0]$. If $l = H$ then STOP, $\mathbf{a}[0]$ is the leximin allotment.

Phase 2: If $l < H$ then for each $t = 0, 1 \ldots H - l - 1$ do the following adjustment. Let $\mathbf{a}[t+1] = \mathbf{a}^{i+}[t]$ for $i \in \{1, \ldots, n\}$ such that $d_i(\mathbf{a}^{i+}[t])$ is minimal.

If $l > H$ then for each $t = 0, 1 \ldots H - l - 1$ do the following adjustment. Let $\mathbf{a}[t+1] = \mathbf{a}^{i-}[t]$ for $i \in \{1, \ldots, n\}$ such that $d_i(\mathbf{a}^{i-}[t])$ is minimal.

That is, we first find a pre-allotment $\mathbf{a}[0]$ that is lexicographically minimal and then we simply increase (or decrease) the number of seats in a greedy way, we add (or remove) a seat to (or from) state i if the increased difference is the smallest for this state. In what follows we show that these greedy adjustments lead to leximin pre-allotments in each step, and therefore a leximin allotment at the end of the process.

Theorem 4. *The greedy leximin algorithm results in the leximin solution for the apportionment problem.*

Proof. If $\sum_{i=1}^{n} a_i[0] = l = H$ then $\mathbf{a}[0]$ is the leximin allotment, obviously. We note that the Hare-quota property holds for pre-allotment $\mathbf{a}[0]$, so the difference $|l - H|$ must be less than or equal to n.

Suppose that $l < H$ (the case of $l > H$ can be proved in a similar way). Let us show by induction for $t = 0, 1, \ldots, H - l$, that $\mathbf{a}[t]$ is the leximin allotment if $l + t$ seats are available, so in particular, $\mathbf{a}[H - l]$ is the leximin allotment for the original problem. The statement is true for $t = 0$, suppose that it is true for an arbitrary $t : 0 < t < H - l$ and let us verify the statement for $t + 1$.

Suppose for a contradiction that there exist an allotment $\mathbf{b}$ where the total number of seats allocated is $l + t$ and $\mathbf{d}(\mathbf{b})$ is lexicographically smaller than $\mathbf{d}(\mathbf{a}[l + t + 1])$.

It is straightforward to see that $\mathbf{a}[0] \leq \mathbf{a} \leq \mathbf{a}'$ and $\mathbf{a} \neq \mathbf{a}'$ implies $\mathbf{d}(\mathbf{a}) \prec \mathbf{d}(\mathbf{a}')$.

First we prove that $\mathbf{d}(\mathbf{a}[t]) \prec \mathbf{d}(\mathbf{b})$. Let i be a state where $b_i > a_i[t]$. Then $\mathbf{a}[0] \leq \mathbf{b}^{i-} \leq \mathbf{b}$ implies $\mathbf{d}(\mathbf{b}^{i-}) \prec \mathbf{d}(\mathbf{b})$. Therefore $\mathbf{d}(\mathbf{b}) \prec \mathbf{d}(\mathbf{a}[t])$ would imply $\mathbf{d}(\mathbf{b}^{i-}) \prec \mathbf{d}(\mathbf{a}[t])$, contradicting with our assumption since $\mathbf{b}^{i-}$ is an allotment with $l + t$ seats.

Let us now assume that when adjusting the pre-allotment $\mathbf{a}[t]$ to $\mathbf{a}[t+1]$ in the greedy algorithm we increased the number of seats in country i. Suppose that the difference $d_i(\mathbf{a}[t])$ is the rth largest, i.e. $d_i(\mathbf{a}[t])$ is the rth entry of vector $\mathbf{d}(\mathbf{a}[t+1])$. The first $r-1$ entries of $\mathbf{d}(\mathbf{a})$ and $\mathbf{d}(\mathbf{a}[t+1])$ are the same, so $\mathbf{d}(\mathbf{a}[t]) \prec \mathbf{d}(\mathbf{b}) \prec \mathbf{d}(\mathbf{a}[t+1])$ implies that the first $r-1$ entries of $\mathbf{b}$ are also the same, so in the corresponding $r-1$ counties these three pre-allotments assign the same number of seats. From $b_i \leq a_i[t+1]$ it follows that among the rest of the $n-r$ counties there must be one, say j, where $b_j > a_j[t+1]$ since both $\mathbf{b}$ and $\mathbf{a}[t+1]$ allocate $l+t+1$ seats, and they are not identical. But $\mathbf{d}(\mathbf{b}) \prec \mathbf{d}(\mathbf{a}[t+1])$ implies $d_j(\mathbf{b}) < d_i(\mathbf{a}[t+1])$, which contradicts with the selection of i in the greedy algorithm. □

Note that both Phase 1 and Phase 2 can be conducted in n^2 steps, if one step means a comparison of two differences.

5 Data Analysis

In this section we first evaluate the 2011 Electoral Law of Hungary that triggered our interest in the recommendations of the Venice Commission at the first place. Then we look at the United States Senate and discuss the allocation of seats according to the leximin method.

5.1 Hungary

The 2011 Electoral Law of Hungary drastically decreased the number of seats in the parliament and fixed the number of constituencies to 106. The law also proposed a seat distribution among the counties. Although the apportionment method was not provided, the law prescribed some principles for subsequent redistribution of seats. These conditions closely followed the directives of the Venice Comission. The law requires that the difference between the population of any constituency and the average constituency size should be within 15%. The only exception is if a constituency would extend over the county border or its connectivity could not be ensured. In this cases higher difference is allowed, but if it ever exceeds 20% then a new allotment should be provided. Table 4 compares the seat distribution proposed by the law with the one that is produced by the leximin algorithm[3].

[3] To calculate δ_i we used the demographic data of the 2010 election.

County	Voters	Seats		Difference (%)	
		law	leximin	law	leximin
Budapest	1 407 470	**18**	**17**	**1**	**6.95**
Baranya	325 943	4	4	5.26	5.26
Bács-Kiskun	438 352	6	6	−5.63	−5.63
Békés	308 471	4	4	−0.38	−0.38
Borsod-Abaúj-Zemplén	567 910	7	7	4.8	4.8
Csongrád	345 945	**4**	**5**	**11.72**	**−10.63**
Fejér	351 237	5	5	−9.26	−9.26
Győr-Moson-Sopron	364 894	5	5	−5.73	−5.73
Hajdú-Bihar	439 618	6	6	−5.35	−5.35
Heves	257 490	3	3	10.87	10.87
Jász-Nagykun-Szolnok	324 869	4	4	4.91	4.91
Komárom-Esztergom	255 396	3	3	9.97	9.97
Nógrád	170 463	2	2	10.1	10.10
Pest	973 668	12	12	4.81	4.81
Somogy	268 844	4	4	−13.18	−13.18
Szabolcs-Szatmár-Bereg	450 556	6	6	−3	−3
Tolna	196 751	3	3	−15.28	−15.28
Vas	215 773	3	3	−7.09	−7.09
Veszprém	300 081	4	4	−3.09	−3.09
Zala	242 236	3	3	4.3	4.3
Total	8 205 967	106	106		

Table 4: The seat distribution and the differences from the average district size by the Electoral Law and by the leximin algorithm

Note that only two out of 20 counties have a different number of seats allotted. The average constituency size in Heves County is 853830 which is 30.87% higher than the average constituency size of Tolna. Therefore voters in Tolna have 30.87% more influence than those living in Heves. If we allow 20% discrepancy from the average constituency size then the difference between voters' influence can be as high as 50%. Interestingly, it is not these counties where the apportionment by law differs from the results of the 7 common methods calculated by Bodnár (2012), but Pest and Somogy.

Upper bounds on the maximal difference

The following figure shows how the maximal difference from the average constituency size (α) changes as we increase the House size from 50 to 180.

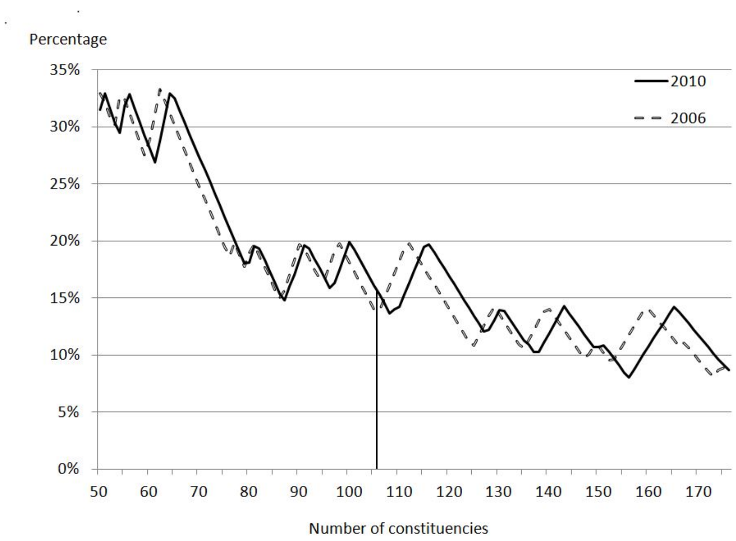

Figure 1: The decline of maximal difference compared to increasing House size using voter data from 2006 and 2010.

Increasing House size indeed implies smaller maximal difference, although α is far from being monotone. The upper bounds imposed by γ are clearly visible. The graph never crosses 33.33%, and for higher H values the upper limits are 20% and 14.28%. This implies that α coincides with β in most of the cases. A deeper analysis shows that $\alpha = \beta$ is true for a broader range of H. From the $[50, 400]$ interval there are only two exceptions, namely, when the House size equals to 87 and 88. But even for these values it is true that $\alpha < \gamma$. Our conjecture is that for real life data α rarely differs from β, therefore γ can be an effective upper bound for both. That means that if one would like to meet the Venice Commission's recommendation, then the House size should be set so high that the lower quota of the smallest county is at least 3 for the strict 15% limit.

Due to the demographic changes the local minimum of α shifted from 106 to 108 in four years. It can easily happen that in the near future 106 seats would mean the local maximum for α. A solution for this issue would be to choose the House size from an interval rather than fixing it. Although this seems to lead to an unpredictable system, in reality it would imply only a minor change from one election to the next as there would be one or two counties that would receive extra seats or have to give up one.

Monotonicity

Figure 2 shows how frequently the Alabama-paradox occurs as the House size changes.

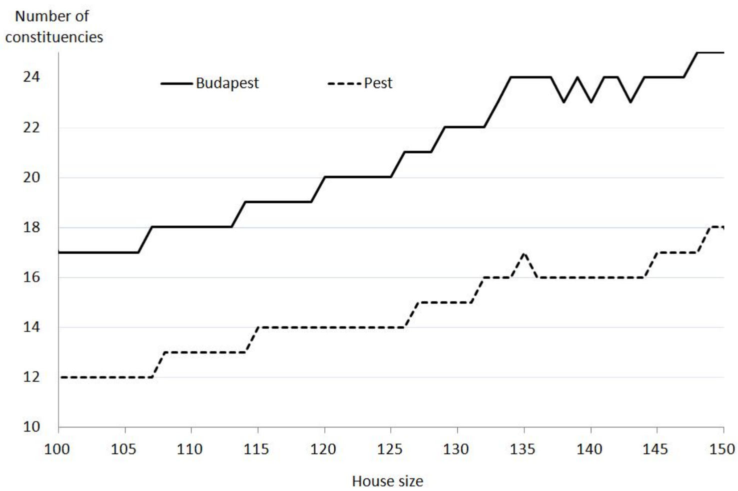

Figure 2: The number of constituencies in Budapest and in Pest county in view of House size

The anomaly occurs only in the two largest counties[4]. As we mentioned earlier, the explanation is simple: large counties behave as puffers. They can store constituencies without affecting the leximin ordering too much and 'borrow' seats for smaller counties that are crucial for the leximin ordering.

[4]For higher House sizes the paradox occurs in the next largest county, Borsod-Abaúj-Zemplén as well.

Changing the size of the regions

Finally, another way to lower the maximal difference is to increase the size of the administrative units that bundles the constituencies. Instead of counties we can use regions requiring only that no constituency extends over the region border. Table 5 summarizes the results for regions.

Region	Voters	Number of seats		Difference. (%)	
		law	leximin	law	leximin
Northern Hungary	995 863	12	13	10.87	1.05
Northern Great Plain	1 215 043	16	16	5.35	1.90
Southern Great Plain	1 092 768	14	14	11.72	0.83
Central Hungary	2 381 138	30	30	4.81	2.53
Central Transdanubia	906 714	12	12	9.97	2.40
Western Transdanubia	822 903	11	11	7.09	3.37
Southern Transdanubia	791 538	11	10	15.28	2.25
Total	8 205 967	106	106		

Table 5: The optimal seat distribution where no constituency extends over the region border

For instance, Northern Hungary consists of Borsod-Abaúj-Zemplén, Heves and Nógrád counties. By the law 7, 3 and 2 seats are assigned to them respectively, altogether 12. Heves produces the highest difference from the average: 10.87%. However if we treat these three counties as one administrative unit then it receives 13 seats and the sizes of its constituencies will be 76605, only 1.05% lower than the average. In this way Western Transdanubia generates the highest average 3.37% which is only a fraction of the 15.28% that Tolna county produces.

5.2 The United States Congress

Much of the literature of apportionment is based on the problems encountered at the regular updates of seat allocation in the United States Congress. In the following we explain how and why our recommended allocation for the US Congress differs and how the current method fares in general when compared with our leximin approach.

5.2.1 The leximin vs. the equal proportions method

To further illustrate the properties of the leximin rule let us compare it with the equal proportion (EP) method, that is used to distribute the seats of the US Congress. The EP is a house-monotone apportionment rule, but it does not satisfy Hare-quota (although it rarely produces a non-quota solution). The table of the apportionment of the 2010 US census compared with the result of the leximin algorithm can be found in the appendix A; Figure 3 provides a visual summary.

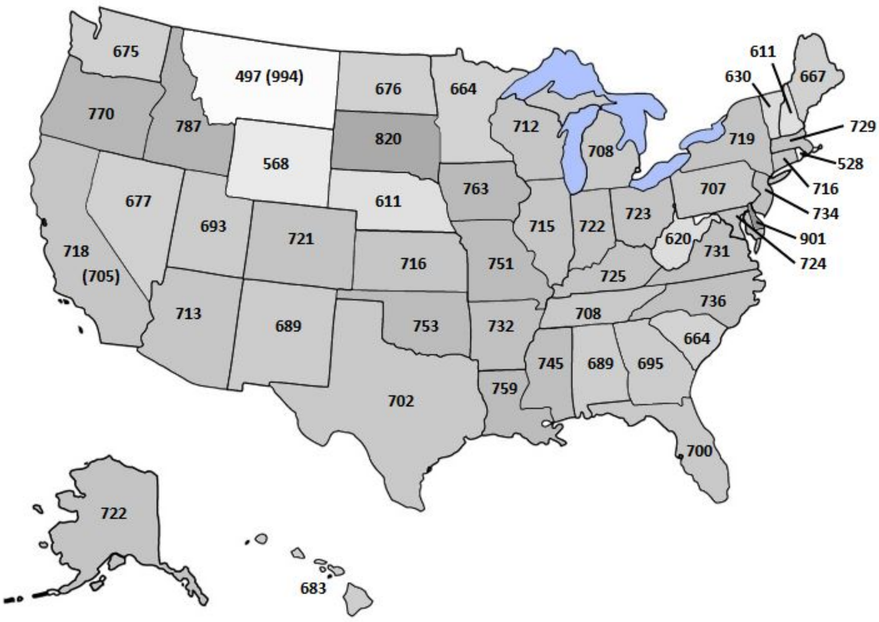

Figure 3: The number of citizens per representative according to the leximin method (in thousands). In parentheses the same figure for the EP method (where different). Note the dramatic swing for Montana.

The two resulting allotments are very similar. In fact there are only two states where the solutions differ: California and Montana. The scenario is the same we have seen before. The largest state lends a seat to one of the smaller ones and the maximal difference drops by almost 10%. It is quite surprising that the voters of Rhode Island - where the average constituency size is the smallest - have 88% more influence than the voters of Montana. Although the leximin allotment reduces this gap somewhat, the only efficient solution would be to drastically increase the House size. As there are 50 states and seven among them end up with only one representative each, the size of the Congress can be considered rather small. Figure 4 shows how the maximal difference changes for higher House sizes.

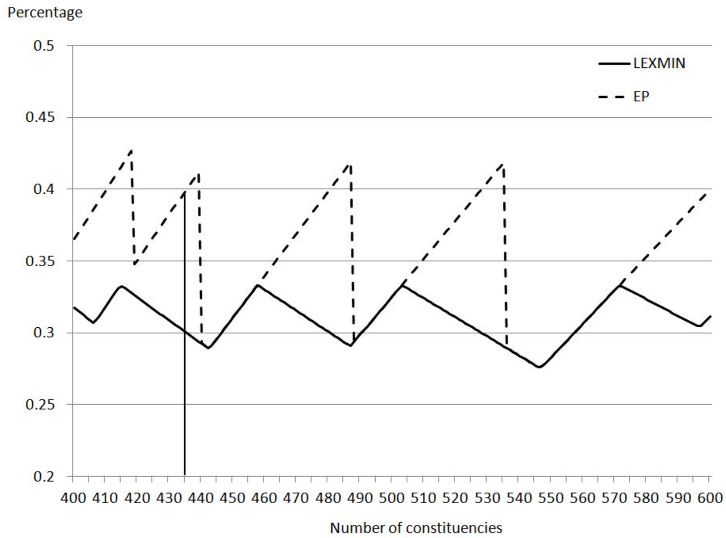

Figure 4: The maximal difference in view of the House size

Maximal differences

As it can be anticipated the maximal difference of the leximin solution never exceeds 33.3% however for the EP there is no such limit. To make certain that the maximal difference is below 20% we have to ensure that the smallest state, Wyoming a) receives at least two representatives and b) the constituency size obtained this way is within 20% of the average. A simple calculation shows that the smallest House size that guarantees these two criteria is 871 - a little more than twice its current size. As it is unlikely that the Congress will be expanded in such fashion the influence of the voters will continue to vary from state to state. A temporary solution would be to increase the number of representatives by seven. The maximal difference for both the leximin and the EP solution meets its minimum at House size 442.

In that case the highest gap between voters influence is ‘only’ 55.19%.

6 Conclusion

More and more countries adopt fairness measures in their electoral law that is based on, or similar to the recommendation of the Venice Commission (2002a). The Maximal Difference Property is very natural and provides greater equality among citizens than other apportionment principles. Unfortunately, the property turns out to be incompatible with the Hare-quota, the population- and house-monotonicities over the class of apportionment problems, so that the Alabama and population paradoxes may arise when using it. Based on the Maximal Difference Property we introduce the well-defined Leximin Rule.

Our apportionment method is not the first. The problem of apportionment goes centuries back, the problem has been around ever since the new member states and population changes required a new seat allocation in the US Congress. Balinski and Young (1982) give an illuminating theoretical and historical overview of the problem of apportionment and the political debates that arose due to it. Methods like Hamilton’s (also called the Method of Largest Remainders), Jefferson’s (Method of Greatest Divisors, but in Europe often referenced as the d’Hondt method), or the Huntingdon-Hill or Equal Proportions method, the currently used method in the US House of Representatives have all been developed as responses to practical problems with apportionment such as the emergence of one or another paradox. Lauwers and Van Puyenbroeck (2006) compare some of these methods.

Apportionment problems are most often used for allocating seats among administrative or political regions based on the population size of these regions: states in the US congress, countries in the EU parliament and so on. Our paper focuses on these applications. Apportionment is also used for the allotment of seats to parties based on the outcome of an election, in fact, sometimes both segmentations appear at the same time; the so-called bi-apportionment is used in some European countries and the problem has been studied by Demange (2012) and Serafini and Simeone (2012).

The Lexicographic Rule is, to the best of our knowledge, an original apportionment method, although lexicographic solution concepts have already been proposed by Gambarelli (1999) and Gambarelli and Palestini (2007). The closest model is by Serafini and Simeone (2012), where the relative differences from the target quotas are lexicographically minimized in the bi-apportionment problem. However, their target quotas are not the same as ours (when restricted to a one-dimensional case), and their methods proposed are more complex, since they are designed for the more general bi-apportionment problem.

There are also papers on minimizing the *relative* difference over pairs of constituencies. Burt and Harris (1963) proposed this concept in for the US House of Representatives, but then it got criticized by Gilbert and Schatz (1964). A recent overview on this concept is given by Edelman (2006). Our problem is different from this one, and it is easy to construct an example where the solutions minimizing the relative difference of any two constituencies and the maximum departure from the average size differ. So far, it seems, none of these models are compatible with the recommendation of the Venice Commission.

A The seat distribution of the US State Congress by the equal proportion method and by the leximin algorithm

State	Voters	Number of seats		Difference (%)	
		EP	leximin	EP	leximin
Alabama	4 802 982	7	7	3.46	3.46
Alaska	721 523	1	1	1.51	1.51
Arizona	6 412 700	9	9	0.24	0.24
Arkansas	2 926 229	4	4	2.92	2.92
California	**37 341 989**	**53**	**52**	**0.87**	**1.03**
Colorado	5 044 930	7	7	1.39	1.39
Connecticut	3 581 628	5	5	0.78	0.78
Delaware	900 877	1	1	26.74	26.74

Florida	18 900 773	27	27	1.51	1.51
Georgia	9 727 566	14	14	2.24	2.24
Hawaii	1 366 862	2	2	3.84	3.84
Idaho	1 573 499	2	2	10.69	10.69
Illinois	12 864 380	18	18	0.55	0.55
Indiana	6 501 582	9	9	1.63	1.63
Iowa	3 053 787	4	4	7.41	7.41
Kansas	2 863 813	4	4	0.72	0.72
Kentucky	4 350 606	6	6	2.01	2.01
Louisiana	4 553 962	6	6	6.78	6.78
Maine	1 333 074	2	2	6.22	6.22
Maryland	5 789 929	8	8	1.82	1.82
Massachusetts	6 559 644	9	9	2.54	2.54
Michigan	9 911 626	14	14	0.39	0.39
Minnesota	5 314 879	8	8	6.52	6.52
Mississippi	2 978 240	4	4	4.75	4.75
Missouri	6 011 478	8	8	5.72	5.72
Montana	**994 416**	**1**	**2**	**39.90**	**30.04**
Nebraska	1 831 825	3	3	14.09	14.09
Nevada	2 709 432	4	4	4.70	4.70
New Hampshire	1 321 445	2	2	7.04	7.04
New Jersey	8 807 501	12	12	3.26	3.26
New Mexico	2 067 273	3	3	3.04	3.04
New York	19 421 055	27	27	1.20	1.20
North Carolina	9 565 781	13	13	3.52	3.52
North Dakota	675 905	1	1	4.90	4.90
Ohio	11 568 495	16	16	1.72	1.72
Oklahoma	3 764 882	5	5	5.93	5.93
Oregon	3 848 606	5	5	8.29	8.29
Pennsylvania	12 734 905	18	18	0.46	0.46
Rhode Island	1 055 247	2	2	25.76	25.76
South Carolina	4 645 975	7	7	6.62	6.62

South Dakota	819 761	1	1	15.33	15.33
Tennessee	6 375 431	9	9	0.33	0.33
Texas	25 268 418	36	36	1.24	1.24
Utah	2 770 765	4	4	2.54	2.54
Vermont	630 337	1	1	11.31	11.31
Virginia	8 037 736	11	11	2.80	2.80
Washington	6 753 369	10	10	4.98	4.98
West Virginia	1 859 815	3	3	12.77	12.77
Wisconsin	5 698 230	8	8	0.21	0.21
Wyoming	568 300	1	1	20.04	20.04
Total	309 183 463	435	435	max: 39.9	max: 30.04

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