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Uniform reconstruction of continuous functions with the RAFU method

Eduardo Corbacho

Department of Mathematics, IES Sáenz de Buruaga, 06800 Mérida, Spain. (ecorbachoc@gmail.com)

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Abstract

The RAFU (radical functions) method can be used to obtain the uniform reconstruction of a continuous function from its values at some of the points of partitions of a closed interval. In this work we will prove that we can reconstruct a continuous function from average samples of these points, from linear combinations of them and from local average samples given by convolution. A uniform error bound of order $\mathcal{O}\left(h^{\frac{3}{2}}\right)+\omega\left(h\right)$ *with the step size* h *will be established. If these data are unknown but approximate values of them are known, uniform reconstruction will be also possible. Error estimates of order* $\mathcal{O}\left(h^{\frac{3}{2}}\right)+\omega\left(h\right)+\eta$ *with noise level* η *will be given. The case of a non-uniform net will be treated. Examples and algorithms will be also shown.*

2010 MSC: *41A30; 37L65; 41A65.*

Keywords: *RAFU method; RAFU approximation; uniform approximation.*

1. Introduction

Suppose that the interval $[a,b]$ is partitioned by the $n+1$ equally spaced points $a=x_0<x_1<...<x_n=b$, such that $x_i=a+ih$, for $i=0,...,n$, with $h=\frac{b-a}{n}$. Consider, for each natural n and $k=1,...,n-1$ the functions $F_n\left(x_k,x\right)=\frac{\sqrt[2n+1]{x_k-a}+\sqrt[2n+1]{x-x_k}}{\sqrt[2n+1]{b-x_k}+\sqrt[2n+1]{x_k-a}}$ defined in $[a,b]$. Then, given $f\in C\left[a,b\right]$, the

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sequence of radical functions $(C_n)_n$ defined in $[a, b]$ as

$$C_n(x) = f(x_0) + \sum_{j=2}^{n} [f(x_j) - f(x_{j-1})] \cdot F_n(x_{j-1}, x) \tag{1.1}$$

converges uniformly to f in $[a, b]$ as $n \to +\infty$. We define the **RAFU method** on approximation to an arbitrary function f to any approximation procedure that uses functions C_n defined as (1.1) to approach the function f. As for the RAFU method, the reader can see [7, 8, 9, 10].

In [8] we proved that the called RAFU linear space is uniformly dense in $C[a, b]$ by using a S-separation condition due to Blasco-Moltó [3] or its equivalent S'-separation condition due to Garrido-Montalvo [13]. Moreover, this linear space can be used as an example of approximation by series in the work of Gassó-Hernández-Rojas [14].

The main goal of this work is to use this method to approach a continuous function f from average samples of the values $f(x_j)$, from linear combinations of $f(x_j)$ and $f(x_{j+1})$ and from local average samples given by $\left(\chi_{[-\frac{h}{2}, \frac{h}{2}]} \star f\right)(x)$. In all these cases we will establish a uniform error bound of order $\mathcal{O}\left(h^{\frac{3}{2}})\right) + \omega(h)$. Moreover, if the data $f(x_j)$ or linear combinations or average samples or local average samples are unknown, but approximate values of them are known, that is to say, for the case of the noise data, we will prove that it is also possible to obtain the reconstruction of the function f. Error bounds of order $\mathcal{O}\left(h^{\frac{3}{2}}\right) + \omega(h) + \eta$, where η is the noise level, will be given. Such problems often occur in environmental science, mathematical statistics, digital image, mechanics, numerical analysis and electricity ; we refer to [1, 5, 11, 12, 15, 16, 17] for more details.

Spline functions have been used to approximate a function f in some of these practical applications by other authors, H. Behforooz [1, 2], E.J.M. Delhez [11], F.G. Lang and X.P. Xu [18] and T. Zhanlav and R. Mijiddorj [19]. In these papers it was necessary to suppose that the function f had several derivatives and error estimations were not given in some of them.

Given approximate integral values of a function f belongs to $H^1(a, b)$, the usual Sobolev space, over subintervals $[x_j, x_{j+1}]$, J. Huang and Y. Chen [16] studied the problem of reconstructing the function f from these data. In this work a regularization method was required and the error bound was established in L^2 norm.

In [4] J. Bustamante, R.C. Castillo and A.F. Collar studied a polynomial approximation of functions from their approximate values at nodes. In this case a regularization method was also required.

In this paper, with the only condition that $f \in C[a, b]$, our purpose will be to employ the RAFU method to demonstrate that it is possible its reconstruction in all the mentioned cases. Moreover, the computational methods involved will be very easy to implement. The paper does not impress with the difficulties it overcomes. It does not contain complicated calculations or reasonings, but

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we think that the importance of this technique to solve all these problems will balance the deficiency of difficulties. This approximation method can rather apply to functions with low smoothness. The uniform stability of this approximation method improves the instability of the interpolation by polynomials; we refer to [7, 8, 9, 10] for more details.

Until now the main drawback of the RAFU method on approximation has been its low accuracy for smooth functions. In Section 2 we will improve the degree of uniform approximation given in [7] and this is an important contribution of this work. In fact, the uniform error estimates that RAFU approximation provides can be better than $\omega\left(f, \frac{\pi}{n+1}\right)$ which is, as far as we know, the best uniform error bound known until now in order to approximate continuous functions by algebraic polynomials in $[-1,1]$ ([6] p. 147). Moreover, in the case of RAFU approximation, the approximating continuous functions are always known. In Section 3, as elementary corollaries, we will solve all our main purposes. By using 4.1.0.0 Mathematica program, we will give in Section 4 some concise algorithms used in this paper. This approximation procedurre can also be used when the set of the points that define the subdivisions of the interval $[a,b]$ is not a uniform net. In Section 5 we will study this case.

2. Improvement of the degree of uniform approximation with the RAFU Method

Maybe, until now the main drawback of the RAFU method on approximation has been the order of the convergence of the sequence $(C_n)_n$ to the function f. Here, we will improve it by using a subsequence of the sequence $(2n+1)_n$ of the index of the roots of the funcions $F_n(x_k, x)$ which appear in (1.1).

In this section we will consider partitions $P_n = \{x_0 = a, x_1, ..., x_n = b\}$ of $[a,b]$ with $x_j = a + j \cdot \frac{b-a}{n}$, $j = 0,...,n$. Moreover, each interval $[x_{k-1}, x_k]$ of length $\frac{b-a}{n}$ will be divided into three equal parts of length $\frac{b-a}{3n}$:

$$\left[x_{k-1}, x_{k-1} + \tfrac{b-a}{3n}\right], \left[x_{k-1} + \tfrac{b-a}{3n}, x_k - \tfrac{b-a}{3n}\right], \left[x_k - \tfrac{b-a}{3n}, x_k\right]$$

Lemma 2.1. *For $n \geq 2$, it follows that:*

1. *Let $1 \leq p \leq n-1$ be, p integer. Then* $\left|\sqrt[2n^2+1]{\frac{p}{n}} - 1\right| \leq \frac{1}{n\sqrt{n}}$
2. $\left|\sqrt[2n^2+1]{\frac{1}{3}} - 1\right| \leq \frac{1}{n\sqrt{n}}$
3. $\left|\sqrt[2n^2+1]{\frac{1}{3n}} - 1\right| \leq \frac{1}{n\sqrt{n}}$
4. *Let $1 \leq p \leq n-1$ be, p integer. Then* $\left|\sqrt[2n^2+1]{n-p} - 1\right| \leq \frac{1}{n\sqrt{n}}$

Lemma 2.2. *Let P_n a partition of $[a,b]$. For each natural n and $k = 1, ..., n-1$, we define in $[a,b]$ the function*

$$F_{n,2}(x_k, x) = \frac{\sqrt[2n^2+1]{x_k - a} + \sqrt[2n^2+1]{x - x_k}}{\sqrt[2n^2+1]{b - x_k} + \sqrt[2n^2+1]{x_k - a}}$$

Then, it satisfies that $0 \leq F_{n,2}(x_k, x) \leq 1$.

The values of the functions $F_{n,2}(x_k, x)$, for any k, do not depend on a and b. In fact, considering $x = a + \alpha_x \frac{b-a}{n}$ for an α_x, it verifies that

$$F_{n,2}(x_k,x) = \frac{\sqrt[2n^2+1]{\left(a + k\frac{b-a}{n}\right) - a} + \sqrt[2n^2+1]{\left(a+\alpha_x\frac{b-a}{n}\right) - \left(a + k\frac{b-a}{n}\right)}}{\sqrt[2n^2+1]{\left(a + n\frac{b-a}{n}\right) - \left(a + k\frac{b-a}{n}\right)} + \sqrt[2n^2+1]{\left(a + k\frac{b-a}{n}\right) - a}} =$$

$$= \frac{\sqrt[2n^2+1]{k\frac{b-a}{n}} + \sqrt[2n^2+1]{(\alpha_x - k)\,\frac{b-a}{n}}}{\sqrt[2n^2+1]{(n-k)\,\frac{b-a}{n}} + \sqrt[2n^2+1]{k\frac{b-a}{n}}} = \frac{\sqrt[2n^2+1]{k} + \sqrt[2n^2+1]{(\alpha_x - k)}}{\sqrt[2n^2+1]{(n-k)} + \sqrt[2n^2+1]{k}}$$

Lemma 2.3. *Let P_n be a partition of $[a,b]$ and $x \in \left[x_{k-1} + \frac{b-a}{3n}, x_k - \frac{b-a}{3n}\right]$. Then, for any $k = 1, \ldots, n-1$, it follows that*

1. *If $x - x_k > 0$ then $\frac{\sqrt[2n^2+1]{\frac{1}{n}} + \sqrt[2n^2+1]{\frac{1}{3n}}}{2} \leq F_{n,2}(x_k, x) \leq 1$*
2. *If $x - x_k < 0$ then $0 \leq F_{n,2}(x_k, x) \leq \frac{\sqrt[2n^2+1]{n-1} - \sqrt[2n^2+1]{\frac{1}{3}}}{2}$*

Moreover, these bounds are valid as $x \in \left[a, x_1 - \frac{b-a}{3n}\right]$, $x \in \left[x_{n-1} + \frac{b-a}{3n}, b\right]$ and $x \in \left(x_j - \frac{b-a}{3n}, x_j + \frac{b-a}{3n}\right)$ with $j \neq k$.

Lemma 2.4. *Let P_n be a partition of $[a,b]$. If $x \in \left[x_{k-1} + \frac{b-a}{3n}, x_k - \frac{b-a}{3n}\right]$ with $k = 1, \ldots, n-1$, $x \in \left[a, x_1 - \frac{b-a}{3n}\right]$, $x \in \left[x_{n-1} + \frac{b-a}{3n}, b\right]$, or $x \in \left(x_j - \frac{b-a}{3n}, x_j + \frac{b-a}{3n}\right)$ where $j \neq k$ then for all $n \geq 2$ it follows that*

1. $\left|\frac{\sqrt[2n^2+1]{\frac{1}{n}} + \sqrt[2n^2+1]{\frac{1}{3n}}}{2} - 1\right| \leq \frac{1}{n\sqrt{n}}$
2. $\left|\frac{\sqrt[2n^2+1]{n-1} - \sqrt[2n^2+1]{\frac{1}{3}}}{2} - 0\right| \leq \frac{1}{n\sqrt{n}}$

Proofs of Lemmas 2.1, 2.2, 2.3 and 2.4 can be obtained by elementary estimates.

Proposition 2.5. *Let P_n be a partition of $[a,b]$ and E_n the step function defined by*

$$E_n(x) = k_1 \cdot \chi_{[a,x_1]} + \sum_{p=2}^{n-1} k_p \cdot \chi_{(x_{p-1},x_p]} + k_n \cdot \chi_{(x_{n-1},b]} \tag{2.1}$$

Let C_n be the radical function associated to E_n defined by

$$C_n(x) = k_1 + \sum_{j=2}^{n} [k_j - k_{j-1}] \cdot F_{n,2}(x_{j-1}, x) \tag{2.2}$$

Then, for all $n \geq 2$ it follows that:

(1) $|C_n(x) - E_n(x)| \leq \frac{2(M_n - m_n)}{n\sqrt{n}}$, $x \in [a,b] \setminus \cup_{k=1}^{n-1} \left(x_k - \frac{b-a}{3n}, x_k + \frac{b-a}{3n}\right)$

(2) $|C_n(x) - [k_j(1-\alpha_x) + k_{j+1}\alpha_x]| \leq \frac{2(M_n - m_n)}{n\sqrt{n}}$, $x \in \left(x_j - \frac{b-a}{3n}, x_j + \frac{b-a}{3n}\right)$, $j = 1, \ldots, n-1$

where M_n and m_n are the maximum and the minimum of the k_j and $\alpha_x \in (0,1)$ is a number which depends upon x.

Proof. The proof is simlar to the proof given in [7].

Part 1. This part is proved considering three possible cases.

Case 1. Suppose that $x \in \left[x_{j-1} + \frac{b-a}{3n}, x_j - \frac{b-a}{3n}\right]$, $j = 2, ..., n-1$ then

$$|C_n(x) - E_n(x)| = |C_n(x) - k_j| = \left|C_n(x) - \left(k_1 + \sum_{p=2}^{j}[k_p - k_{p-1}]\right)\right| =$$

$$\left|\sum_{p=2}^{j}[k_p - k_{p-1}]\left[1 - F_{n,2}(x_p, x)\right] + \sum_{p=j+1}^{n}[k_p - k_{p-1}]\left[0 - F_{n,2}(x_p, x)\right]\right| \leq$$

by Lemmas 2.3 and 2.4

$$\left|\sum_{p=2}^{j}[k_p - k_{p-1}] \cdot \frac{1}{n\sqrt{n}} + \sum_{p=j+1}^{n}[k_p - k_{p-1}] \cdot -\frac{1}{n\sqrt{n}}\right| \leq$$

$$\left|\sum_{p=2}^{j}[k_p - k_{p-1}] \cdot \frac{1}{n\sqrt{n}} + \sum_{p=j+1}^{n}[k_{p-1} - k_p] \cdot \frac{1}{n\sqrt{n}}\right| \leq$$

$$\frac{1}{n\sqrt{n}}\left|[k_j - k_1] + [k_j - k_n]\right| \leq \frac{2(M_n - m_n)}{\sqrt{n}}$$

Case 2. Suppose that $x \in \left[a, x_1 - \frac{b-a}{3n}\right]$. Then $x - x_p < 0$, $p = 1, ..., n-1$ and proceeding as in Case 1 and by using Lemmas 2.3 and 2.4, we obtain

$$|C_n(x) - E_n(x)| = |C_n(x) - k_1| =$$

$$\left|C_n(x) - \left(k_1 + \sum_{p=2}^{j}[k_p - k_{p-1}]\right)\right| = \left|\sum_{p=2}^{n}[k_p - k_{p-1}]\left[0 - F_{n,2}(x_p, x)\right]\right| \leq$$

$$\left|\sum_{p=2}^{n}[k_p - k_{p-1}] \cdot -\frac{1}{n\sqrt{n}}\right| = \left|\sum_{p=2}^{n}[k_{p-1} - k_p] \cdot \frac{1}{n\sqrt{n}}\right| \leq \frac{2(M_n - m_n)}{n\sqrt{n}}$$

Case 3. Suppose that $x \in \left[x_{n-1} + \frac{b-a}{3n}, b\right]$. Then $x - x_p > 0$, $p = 1, ..., n-1$ and proceeding as in Case 1, we can put

$$|C_n(x) - E_n(x)| = |C_n(x) - k_n| = \left|C_n(x) - \left(k_1 + \sum_{p=2}^{j}[k_p - k_{p-1}]\right)\right| =$$

$$\left|\sum_{p=2}^{n}[k_p - k_{p-1}]\left[1 - F_{n,2}(x_p, x)\right]\right| \leq \left|\sum_{p=2}^{n}[k_p - k_{p-1}] \cdot \frac{1}{n\sqrt{n}}\right| \leq \frac{2(M_n - m_n)}{n\sqrt{n}}$$

taking into account Lemmas 2.3 and 2.4.

Part 2. Suppose that $x \in \left(x_j - \frac{b-a}{3n}, x_j + \frac{b-a}{3n}\right)$, $j = 1, ..., n-1$, then

$$[k_j(1-\alpha_x)+k_{j+1}\alpha_x]-C_n(x)=[k_j+(k_{j+1}-k_j)\,\alpha_x]-C_n(x)=$$

$$k_1-k_1+\sum_{p=2}^{j}[k_p-k_{p-1}]\,[1-F_{n,2}\,(x_{p-1},x)]+$$

$$[k_{j+1}-k_j]\,[\alpha_x-F_{n,2}\,(x_j,x)]+\sum_{p=j+1}^{n}[k_{p+1}-k_p]\,[0-F_{n,2}\,(x_p,x)]$$

Since for $x\in\left(x_j-\frac{b-a}{3n},x_j+\frac{b-a}{3n}\right)$ it follows that $0<F_{n,2}\,(x_j,x)<1$ we can put $\alpha_x=F_{n,2}\,(x_j,x)$. So that, from Lemmas 2.3 and 2.4, taking absolute value and proceeding as in Case 1,

$$|C_n(x)-[k_j(1-\alpha_x)+k_{j+1}\alpha_x]|\leq$$

$$\left|\sum_{p=2}^{j}[k_p-k_{p-1}]\cdot\frac{1}{n\sqrt{n}}+\sum_{p=j+1}^{n}[k_p-k_{p+1}]\cdot\frac{1}{n\sqrt{n}}\right|=$$

$$\frac{1}{n\sqrt{n}}\,|[k_j-k_1]+[k_{j+1}-k_n]|\leq\frac{2\,(M_n-m_n)}{n\sqrt{n}}$$

□

Theorem 2.6. *Let f be a continuous function defined in $[a,b]$. Then there exists a sequence of radical functions $(C_n)_n$ defined in $[a,b]$ as in (2.2) such that*

$$|C_n(x)-f(x)|\leq\frac{2\,(M-m)}{n\sqrt{n}}+\omega\left(\frac{b-a}{n}\right)$$

for all $n\geq 2$ and $x\in[a,b]$ being M and m the maximum and the minimum of f in $[a,b]$ respectively and $\omega\left(\frac{b-a}{n}\right)$ its modulus of continuty.

Proof. For each $n\geq 2$, let P_n be a partition of $[a,b]$, let E_n be the step function defined by

$$E_n(x)=\begin{cases}f(a) & x\in[a,x_1]\\ f(x_2) & x\in(x_1,x_2]\\ \cdots & \\ f(b) & x\in(x_{n-1},b]\end{cases}$$

and let C_n be the corresponding radical function defined from E_n as (2.2).

If $x\in[a,b]\setminus\cup_{k=1}^{n-1}\left(x_k-\frac{b-a}{3n},x_k+\frac{b-a}{3n}\right)$ then,

$$|C_n(x)-f(x)|=|C_n(x)-E_n(x)+E_n(x)-f(x)|\leq$$

$$\frac{2\,(M-m)}{n\sqrt{n}}+|E_n(x)-f(x)|=\frac{2\,(M-m)}{n\sqrt{n}}+|f(x_j)-f(x)|\leq$$

$$\frac{2\,(M-m)}{n\sqrt{n}}+\omega\left(\frac{b-a}{n}\right)$$

taking into account that $E_n(x)=f(x_j)$ for some j and Proposition 2.5.

If $x\in\cup_{k=1}^{n-1}\left(x_k-\frac{b-a}{3n},x_k+\frac{b-a}{3n}\right)$, Proposition 2.5 applies and we can choose an appropiate index j to obtain

$$|C_n(x)-f(x)|\leq|C_n(x)-[f(x_j)\,(1-\alpha_x)+f(x_{j+1})\alpha_x]|+$$

$$|[f(x_j)\,(1-\alpha_x)+f(x_{j+1})\alpha_x]-f(x)|\leq$$

$$\frac{2\,(M-m)}{n\sqrt{n}}+|[f(x_j)\,(1-\alpha_x)+f(x_{j+1})\alpha_x]-[f(x)\,(1-\alpha_x)+f(x)\alpha_x]|\leq$$

$$\frac{2\,(M-m)}{n\sqrt{n}}+|f(x_j)-f(x)|\,(1-\alpha_x)+|f(x_{j+1})-f(x)|\,(1-\alpha_x)\leq$$

$$\frac{2\,(M-m)}{n\sqrt{n}}+\omega\left(\frac{b-a}{n}\right)(1-\alpha_x+\alpha_x)=\frac{2\,(M-m)}{n\sqrt{n}}+\omega\left(\frac{b-a}{n}\right)$$

□

Remark 2.7. It is well-known (see for instance [6] pp. 147) that if $f\in C\,[-1,1]$, then there exist an algebraic polynomial P_n of degree $\leq n$ such that for all $x\in C\,[-1,1]$,

$$|P_n(x)-f(x)|\leq\omega\left(\frac{\pi}{n+1}\right)$$

As far as we know, this error estimate is the best possible currently known. By means of Theorem 2.6, we have proved an analogous result by using radical continuous functions. In case of the interval $[-1,1]$, the error bound becomes $\frac{2(M-m)}{n\sqrt{n}}+\omega\left(\frac{2}{n}\right)$. So, depending on the function, this error estimate can be better than error bound in algebraic polynomial approximation. Moreover, RAFU method provides the explicit form of the function which approximate to the funtion f for each n. However in the case of algebraic polynomials this does not happen. Therefore, this is an important contribution of this work.

3. Main results

3.1. **Uniform reconstruction of f from average samples.** The following corollary provides a sequence uniformly convergent to the original function f and a uniform error bound. Observe that the uniform error bound is the same as Theorem 2.6.

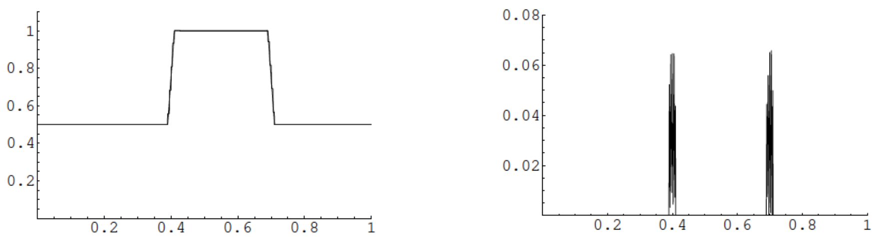
Corollary 3.1. *Under the hypothesis of Theorem 2.6, if the data* k_i *of the step function (2.1) are substituted by* $k_i = \frac{f(x_{i1})n_1+...+f(x_{ip})n_p}{n_1+...+n_p}$, $x_{1q} \in [a, x_1]$ *or* $x_{iq} \in (x_{i-1}, x_i]$, $i = 2, ..., n$, $q = 1, ..., p$, $n_1 + ... + n_q \neq 0$ *then*

$$|C_n(x) - f(x)| \leq \frac{2\,(M-m)}{n\sqrt{n}} + \omega\left(\frac{b-a}{n}\right)$$

for all $x \in [a, b]$, $n \geq 2$ *and where* $C_n(x)$ *is defined as 2.2 but from the new data* k_i.

Proof. In the proof of Proposition 2.5 we can put $k_i = \frac{f(x_{i1})n_1+...+f(x_{ip})n_p}{n_1+...+n_p}$, $i = 1, ..., n$ and the same result holds for M and m. Moreover, if we define the functions E_n in Proposition 2.5 from $k_i = \frac{f(x_{i1})n_1+...+f(x_{ip})n_p}{n_1+...+n_p}$, $i = 1, ..., n$ and we put that $f(x) = \frac{f(x)n_1+...+f(x)n_p}{n_1+...+n_p}$, then we can easily check that Corollary 3.1 is true considering now C_n defined as (2.2) but from k_i, $i = 1, ..., n$. □

Example 3.2. In Figure 1 we show the approximation to the piecewise continuous function $f(x)$ defined by 0.5 if $x \in [0, 0.39)$, $\frac{0.5x-0.185}{0.02}$ if $x \in [0.39, 0.41)$, 1 if $x \in [0.41, 0.69)$, $\frac{-0.5x+0.365}{0.02}$ if $x \in [0.69, 0.71)$ and 0.5 if $x \in [0.71, 1]$ from $k_i = \frac{f(x_{i1})+...+f(x_{i15})}{15}$, $i = 1, ..., 200$ considering $C_{200}(x)$.



(a) Approximating function and f

(b) Approximation error

FIGURE 1. Uniform reconstruction from average samples.

Remark 3.3. If $n_i = 1$, we have the usual average values.

3.2. **Uniform reconstruction of f from approximate values.** In [4] J. Bustamante, R. C. Castillo and A. F. Collar solved this problem by means of a regularization method. In [7] we studied this case but here we give a uniform error bound. The reader can compare our error bound with the estimation of the error shown in [4].

When we do not know the values $f(x_i)$ but the data $f(x_i)+\eta_i$, with $|\eta_i| < \eta$ for a fixed $\eta > 0$ are known, then the following result can be useful to obtain an approximation of the function f.

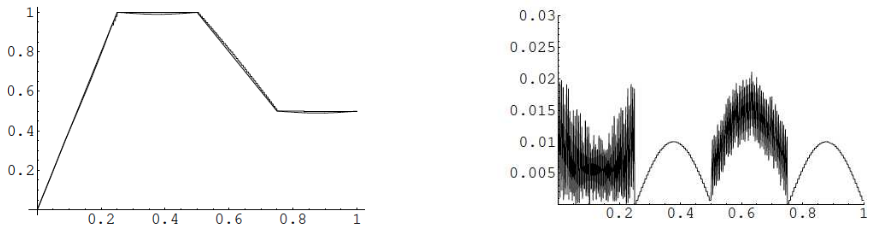
Corollary 3.4. *Under the hypothesis of Theorem 2.6, if the data k_i of the step function (2.1) are changed for $k_i = f(x_i) + \eta_i$, being $|\eta_i| < \eta$, $i = 1,\ldots,n$ then*

$$|C_n(x) - f(x)| \leq \frac{2\,(M - m + \eta)}{n\sqrt{n}} + \omega\left(\frac{b-a}{n}\right) + \eta$$

for all $x \in [a, b]$, $n \geq 2$ and where $C_n(x)$ is defined as (2.2) but from the new data k_i.

Proof. With these data k_i, $i = 1,\ldots, n$ we can obtain the error bound $\frac{2(M-m+\eta)}{n\sqrt{n}}$ in Proposition 2.5. Moreover, if we change $f(x_i)$ for $k_i = f(x_i)+\eta_i$, $i = 1, \ldots, n$ in the proof of Theorem 2.6 then the new error bound becomes $\frac{2(M-m+\eta)}{n\sqrt{n}} + \omega\left(\frac{b-a}{n}\right) + \eta$. □

Example 3.5. Approximation to the piecewise continuous function function $f(x)$ defined by $4x$ if $x \in [0, 0.25)$, 1 if $x \in [0.25, 0.5)$, $\frac{-0.5x+0.5}{0.02}$ if $x \in [0.5, 0.75)$ and 0.5 if $x \in [0.71, 1]$ using the data $k_i = f(x_i)+\eta_i$, $i = 1, \ldots, n$, with $|\eta_i| \leq \frac{1}{100}$ $\left(\eta_i = \frac{1}{100} sin4\pi x_i\right)$ and considering $C_{180}(x)$ (Figure 2).



(a) Approximating function and f

(b) Approximation error

FIGURE 2. Uniform reconstruction from approximate values.

3.3. **Uniform reconstruction of f from local average samples.** In many applications it is more realistic to assume that the available samples are local average samples near a certain x. We consider the special case in which we know data of the type

$$\left(\chi_{[-h,h]} \star f\right)(x) = \int_{-\infty}^{+\infty} \chi_{[-h,h]}(y) f(x-y)dy = \int_{x-h}^{x+h} f(z)dz \tag{3.1}$$

where $\star$ denotes the convolution of the functions $\chi_{[-h,h]}$ and f. Sometimes we deal with phenomena which involve a function and its integral. For example, in mechanics, the velocity $v(t)$ and the displacement $s(t)$, or the acceleration $a(t)$ and the velocity $v(t)$; in statistics, the probability density function and the cumulative distribution function and in electricity, the current function $I(t)$ and the charge function $q(t)$ are some real examples about this consideration. The tasks are to approximate the function f from integral values as (3.1) and to give error bounds for this aproximation. There have been only a few research papers to deal with these problems; see for example, H. Behforooz [1, 2], E.J.M. Delhez [11], F.G. Lang and X.P. Xu [18] and T. Zhanlav and R. Mijiddorj [19]. In these papers it was necessary to suppose that the function f had several derivatives and error estimations were not given in some of them.

Here, with Corollary 3.6, RAFU method solves easily the problem of the reconstruction of the function from the integral values and provides a uniform error bound for this reconstruction with the only condition that $f \in C[a,b]$.

On the other hand, let $\triangle$ be a subdivision of the interval $[a,b]$ with grids $a = x_0 < x_1 < ... < x_n = b$ whose mesh size is denoted by $h = max_{1 \leq i \leq n} h_i$, $h_i = x_i - x_{i-1}$, $1 \leq i \leq n$ and $M_i(f) = \frac{1}{h_i}\int_{x_{i-1}}^{x_i} f(x)dx$. In practice, due to the measurement error, the exact values $M_i(f)$ are unknown but we know approximate average values u_i, $1 \leq i \leq n$ such that $|u_i - M_i(f)| < \delta$ where δ is a positive constant describing the level of error of the data. In [16] J. Huang and Y. Chen proposed a regularization method for solving the problem (P): given the approximate values u_i, $1 \leq i \leq n$ satisfying the previous condition how does one reconstruct the original function f efficiently? They established the rigorous error estimates in L^2 norm for functions $f \in H^1(a,b)$ where $H^1(a,b)$ is the usual Sobolev space consisting of all $L^2(a,b)$-integrable functions whose 1-order weak derivative are also $L^2(a,b)$-integrable. For $f \in H^1(a,b)$. They solved this problem in terms of the Tikhonov regularization method.

In this work, by means of Corollaries 3.4 and 3.6, we establish another solution of problem (P) in the uniform norm for all $f \in C[a,b]$. Note that $H^1(a,b)$ is continuously embedded in $C[a,b]$. Our solution does not need regularization. See Algorithm 4.2 and Figure 5.

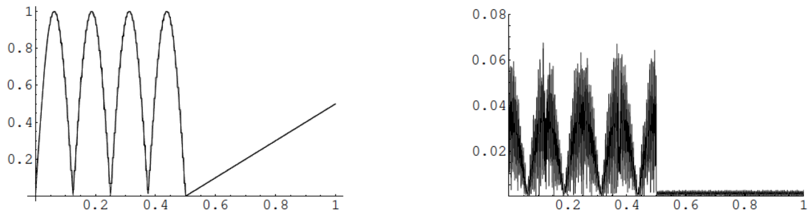
Corollary 3.6. *With the hypothesis of Theorem 2.6, if the data k_i of the step function (2.1) are defined by $k_i = \frac{\int_{\tilde{x}_i-h}^{\tilde{x}_i+h} f(z)dz}{2h}$, with $[\tilde{x}_1 - h, \tilde{x}_1 + h] \subseteq [a, x_1]$ or $[\tilde{x}_i - h, \tilde{x}_i + h] \subseteq (x_{i-1}, x_i]$, $i = 2, ..., n$, then*

$$|C_n(x) - f(x)| \leq \frac{2(M-m)}{n\sqrt{n}} + \omega\left(\frac{b-a}{n}\right)$$

for all $x \in [a,b]$, $n \geq 2$ and where $C_n(x)$ is defined as (2.2) but from the new data k_i.

Proof. We can put that $\int_{\tilde{x}_i-h}^{\tilde{x}_i+h} f(z)dz = f(z_i)2h$ for some value $z_i \in [\tilde{x}_i - h, \tilde{x}_i + h]$ by the integral properties because f is continuous. Then, $k_i = f(z_i)$ for all i and we finish with the same proof of Theorem 2.6. □

Example 3.7. Consider the special case given by $\tilde{x}_i = \frac{x_{i-1}+x_i}{2}$, $i = 1, ..., n$ and $h = \frac{b-a}{n}$ to approximate the continuous function $f(x)$ defined by $|sin 8\pi x|$ if $x \in [0, 0.5)$ and $x - 0.5$ if $x \in [0.5, 1]$ from local average samples $k_i = \frac{\int_{\tilde{x}_i-h}^{\tilde{x}_i+h} f(z)dz}{2h}$, $i = 1, ..., 180$ with $C_{144}(x)$ (Figure 3).



(a) Approximating function and f (b) Approximation error

FIGURE 3. Uniform reconstruction from local average samples.

3.4. **Uniform reconstruction of f from linear combinations.**

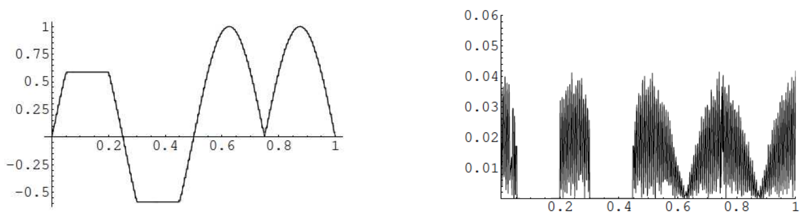
Corollary 3.8. *Under the hypothesis of Theorem 2.6, if the values k_i of the step function (2.1) are defined by* $k_i = \frac{f(\tilde{x}_i)-f(\tilde{x}_{i-1})}{\tilde{x}_i-\tilde{x}_{i-1}} \cdot (x'_i - \tilde{x}_{i-1}) + f(\tilde{x}_{i-1})$ *with* $x'_1 \in [\tilde{x}_0, \tilde{x}_1] \subseteq [a, x_1]$ *or* $x'_i \in [\tilde{x}_{i-1}, \tilde{x}_i] \subseteq (x_{i-1}, x_i]$, $i = 2, ..., n$, *then*

$$|C_n(x) - f(x)| \leq \frac{2\,(M-m)}{n\sqrt{n}} + \omega\left(\frac{b-a}{n}\right)$$

for all $x \in [a, b]$, $n \geq 2$ *and where* $C_n(x)$ *is defined as (2.2) but from the new data* k_i.

Proof. Since $f \in C\,[a, b]$, there exists a point x''_i in each interval $[\tilde{x}_{i-1}, \tilde{x}_i]$ such that $k_i = f(x''_i)$ for all $i = 1, ..., n$. Then, this proof becomes the proof of Theorem 2.6. □

Example 3.9. Consider the special case in which $\tilde{x}_i = x_i$ for all i to approximate the piecewise continuous function $f(x)$ defined by $sin 4\pi x$ if $x \in [0, \frac{1}{20}) \cup [\frac{1}{5}, \frac{3}{10}) \cup [\frac{9}{20}, \frac{1}{2})$, $sin\frac{\pi}{5}$ if $x \in [\frac{1}{20}, \frac{1}{5})$, $sin\frac{2\pi}{5}$ if $x \in [\frac{3}{10}, \frac{9}{20})$ and $|sin 4\pi x|$ if $x \in [\frac{1}{2}, 1]$ from the data $k_i = \frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}} \cdot (x'_i - x_{i-1}) + f(x_{i-1})$ and the values $x'_i = \frac{x_{i-i}+x_i}{2}$ by using $C_{150}(x)$ (Figure 4).



(a) Approximating function and f (b) Approximation error

FIGURE 4. Uniform reconstruction from linear combinations.

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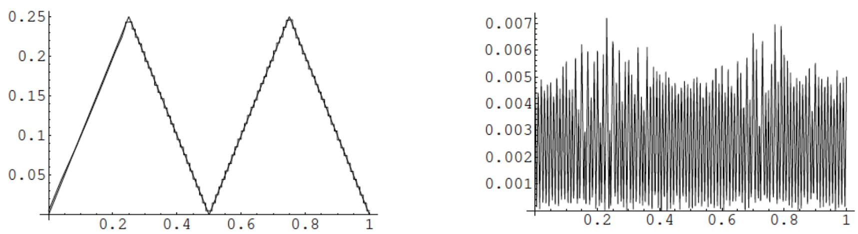
4. Algorithms

We show three algorithms by using the 4.1.0.0 Mathematica program.

Algorithm 4.1. *Uniform reconstruction from average samples.*

$$f[x\_] := As\,Example\,3.2;$$
$$a=0;\ b=1;\ n=3000;\ h=\frac{b-a}{n};\ v=\frac{n}{15};$$
$$t=Table[a+15\cdot h\cdot i,\{i,0,v\}];\ d=Table[f[a+j\cdot h],\{j,0,n-1\}];$$
$$For[i=1,i++,D_i=\frac{\sum_{m=1}^{15} d_{m+15*(i-1)}}{15}];$$
$$k=Table[\frac{\sum_{m=1}^{15} d_{m+15*(i-1)}}{15},\{i,1,v\}];tt=Length[t];\ kk=Length[k];$$
$$For[i=2,i\leq kk,i++,M_i=\frac{(k_i-k_{i-1})\cdot\sqrt[2n^2+1]{t_i-t_1}}{\sqrt[2n^2+1]{t_{tt}-t_i}+\sqrt[2n^2+1]{t_i-t_1}}];$$
$$For[i=2,i\leq kk,i++,N_i=\frac{(k_i-k_{i-1})}{\sqrt[2n^2+1]{t_{tt}-t_i}+\sqrt[2n^2+1]{t_i-t_1}}];$$
$$g[x\_]=k_1+\sum_{i=2}^{kk}\left(M_i+N_i\cdot\sqrt[2n^2+1]{Abs[x-t_i]}\cdot Sign\,(x-t_i)\right);$$
$$Plot[\{f[x],g[x]\},\{x,t_1,t_{tt}\}]$$
$$Plot[Abs[f[x]-g[x]],\{x,t_1,t_{tt}\}]$$

Corollary 3.4 can be used together with Corollaries 3.1, 3.6 or 3.8. For instance, in Algorithm 4.2, we use Corollaries 3.4 and 3.6 to reconstruct uniformly an irregular function f from approximate integral values (Figure 5). Here, Random denotes a random number with uniform distribution on $[-1,1]$ and 0.01 is the considered relative error level of the data. RAFU method provides this easy solution to the Problem (P) suggested by J. Huang and Y. Chen in [16].



(a) Approximating function and f (b) Approximation error

Figure 5. Uniform reconstruction from approximate integral values.

Algorithm 4.2. *Uniform approximation from approximate integral values.*

$$f[x\_] := If[0\leq x<0.25,x,If[0.25\leq x<0.5,-x+0.5,If[0.5\leq x<0.75,x-0,5,If[0.75\leq x\leq 1,-x+1]]]];$$
$$a=0;\ b=1;\ n=100;\ h=\frac{b-a}{n};\ hh=\frac{b-a}{2\cdot n};$$
$$t=Table[a+j\cdot h,\{j,0,n\}];$$

$$k = Table[\frac{N[Integrate[f[x],\{x, \frac{a+j\cdot h+a+(j+1)\cdot h}{2} - hh, \frac{a+j\cdot h+a+(j+1)\cdot h}{2} + hh\}]]}{2\cdot hh}$$
$$\cdot(1 + 0.01 \cdot Random[Real, \{-1, 1\}]), \{j, 0, n-1\}];$$
$$tt = Length[t];\ kk = Length[k];$$
$$For[i = 2, i \leq kk, i++, M_i = \frac{(k_i - k_{i-1})\cdot \sqrt[2n^2+1]{t_i - t_1}}{\sqrt[2n^2+1]{t_{tt} - t_i} + \sqrt[2n^2+1]{t_i - t_1}}];$$
$$For[i = 2, i \leq kk, i++, N_i = \frac{(k_i - k_{i-1})}{\sqrt[2n^2+1]{t_{tt} - t_i} + \sqrt[2n^2+1]{t_i - t_1}}];$$
$$g[x\_] = k_1 + \sum_{i=2}^{kk} \left(M_i + N_i \cdot \sqrt[2n^2+1]{Abs[x - t_i]} \cdot Sign\,(x - t_i) \right);$$
$$Plot[\{f[x], g[x]\}, \{x, t_1, t_{tt}\}]$$
$$Plot[Abs[f[x] - g[x]], \{x, t_1, t_{tt}\}]$$

Algorithm 4.3. *Uniform approximation from linear combinations.*

$$f[x\_] := As\,Example\,3.9;$$
$$a = 0;\ b = 1;\ n = 150;\ h = \frac{b-a}{n};$$
$$t = Table[a + j \cdot h, \{j, 0, n\}];$$
$$k = Table[\frac{f[a+(j+1)\cdot h] - f[a+j\cdot h]}{h} \cdot \frac{h}{2} + f[a + j \cdot h], \{j, 0, n-1\}];$$
$$tt = Length[t];\ kk = Length[k];$$
$$For[i = 2, i \leq kk, i++, M_i = \frac{(k_i - k_{i-1})\cdot \sqrt[2n^2+1]{t_i - t_1}}{\sqrt[2n^2+1]{t_{tt} - t_i} + \sqrt[2n^2+1]{t_i - t_1}}];$$
$$For[i = 2, i \leq kk, i++, N_i = \frac{(k_i - k_{i-1})}{\sqrt[2n^2+1]{t_{tt} - t_i} + \sqrt[2n^2+1]{t_i - t_1}}];$$
$$g[x\_] = k_1 + \sum_{i=2}^{kk} \left(M_i + N_i \cdot \sqrt[2n^2+1]{Abs[x - t_i]} \cdot Sign\,(x - t_i) \right);$$
$$Plot[\{f[x], g[x]\}, \{x, t_1, t_{tt}\}]$$
$$Plot[Abs[f[x] - g[x]], \{x, t_1, t_{tt}\}]$$

5. Uniform reconstruction of f from a non-uniform net

From now on, we will consider partitions $P_n = \{x_0 = a, x_1, ..., x_s = b\}$ of $[a, b]$ with non-uniformly spaced data.

Lemma 5.1. *Let K be a positive integer. Then, for $n \geq 2$ it verifies that* $\left| \sqrt[2n^2+1]{n^K} - 1 \right| \leq \frac{2^K - 1}{n\sqrt{n}}$ *and* $\left| \sqrt[2n^2+1]{\frac{1}{n^K}} - 1 \right| \leq \frac{K}{n\sqrt{n}}$

Proof. By induction on K. Case $K = 1$ can be obtained by elementary estimates. Then, we finishes taking into account that

$$\left| \sqrt[2n^2+1]{n^{\pm K}} - 1 \right| = \left| \sqrt[2n^2+1]{n^{\pm K}} - \sqrt[2n^2+1]{n^{\pm 1}} + \sqrt[2n^2+1]{n^{\pm 1}} - 1 \right|$$

□

Lemma 5.2. *Let $P_n = \{a = x_0, x_1, ..., x_s = b\}$ be a partition of $[a, b]$ with $\delta(s) = min_{1 \leq j \leq s} |x_j - x_{j-1}|$. Then, for any $k = 1,\ ...,\ s - 1$ and $x \in [a, b] \setminus \left(x_k - \frac{\delta(s)}{3}, x_k + \frac{\delta(s)}{3} \right)$ it follows that:*

(1) $\sqrt[2n^2+1]{\frac{\delta(s)}{b-a}}\, \frac{1 + \sqrt[2n^2+1]{\frac{1}{3}}}{2} \leq F_{n,2}(x_k, x) \leq 1$ *if* $x - x_k > 0$

(2) $0 \leq F_{n,2}(x_k, x) \leq \frac{\sqrt[2n^2+1]{\frac{b-a}{\delta(s)}} - \sqrt[2n^2+1]{\frac{1}{3}}}{2}$ *if* $x - x_k < 0$

The proof can be obtained by elementary estimates.

Lemma 5.3. *Let $K \geq 2$ be a positive integer such that $\frac{3(b-a)}{n^K} \leq \delta(s)$. Then, for all $n \geq 2$, it verifies that*

(1) $\left|1 - \sqrt[2n^2+1]{\frac{\delta(s)}{b-a}} \frac{1+\sqrt[2n^2+1]{\frac{1}{3}}}{2}\right| \leq \frac{K}{n\sqrt{n}}$

(2) $\left|\frac{\sqrt[2n^2+1]{\frac{b-a}{\delta(s)}} - \sqrt[2n^2+1]{\frac{1}{3}}}{2} - 0\right| \leq \frac{2^{K-1}}{n\sqrt{n}}$

The proof can be obtained easily from Lemmas 2.1 and 5.1.

Proposition 5.4. *Let $P_s = \{a = x_0, x_1, ..., x_s = b\}$ be a partition of $[a, b]$ and let E_s be a step function defined in $[a, b]$ by*

$$E_s(x) = k_1 \cdot \chi_{[x_0, x_1]} + \sum_{i=2}^{s} k_i \cdot \chi_{(x_{i-1}, x_i]}, \quad k_i \quad \textit{real numbers}$$

If $\frac{3(b-a)}{n^K} \leq \delta(s)$, being $\delta(s) = min_{1\leq j\leq s}|x_j - x_{j-1}|$and $K \geq 2$ a positive integer, then for all $n \geq 2$ it follows that:

(1) $|C_n(x) - E_s(x)| \leq \frac{2^K(M_s - m_s)}{n\sqrt{n}}$ *if* $x \in [a, b] \setminus \cup_{j=1}^{s-1}\left(x_j - \frac{\delta(s)}{3}, x_j + \frac{\delta(s)}{3}\right)$

(2) $|C_n(x) - [k_j(1-\alpha_x) + k_{j+1}\alpha_x]| \leq \frac{2^K(M_s - m_s)}{n\sqrt{n}}$ *if* $j = 1, \ ..., \ s-1$ *and* $x \in \left(x_j - \frac{\delta(s)}{3}, x_j + \frac{\delta(s)}{3}\right)$.

where M_s and m_s are the maximum and the minimum of the k_j, $\alpha_x \in (0, 1)$ is a number which depends only on x and $(C_n)_n$ is the sequence of radical functions associated to E_s defined as in (2.2).

Proof. It is analogous to the proof of Proposition 2.5 but now we use Lemmas 5.1, 5.2 and 5.3. □

Theorem 5.5. *Let $P_n = \{a = x_0, x_1, ..., x_{s_n} = b\}$ be a partition of $[a, b]$ with $\delta(s_n) = min_{1\leq j\leq s_n}|x_j - x_{j-1}|$ and $\Delta(s_n) = max_{1\leq j\leq s_n}|x_j - x_{j-1}|$ such that $\frac{3(b-a)}{n^K} \leq \delta(s_n) \leq \Delta(s_n) \leq h$ being $h = \frac{b-a}{n}$ and $K \geq 2$ a positive integer. Let f be a continuous function defined in $[a, b]$. Then there exists a sequence $(C_n)_n$ defined in $[a, b]$ as in (2.2) such that*

$$|C_n(x) - f(x)| \leq \frac{2^K(M - m)}{n\sqrt{n}} + \omega\left(\frac{b-a}{n}\right)$$

for all $n \geq 2$ and $x \in [a, b]$, being M, m and $\omega\left(\frac{b-a}{n}\right)$ as usual.

Proof. Similiar to the proof of Proposition 2.5. Here Proposition 5.4 applies. □

In the same way the results in Section 3 have been obtained from Theorem 2.6, similar results to Section 3 can be derived from Theorem 5.5 for the case of non-uniform net and this is another important contribution of this work.

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