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On the existence of fixed points that belong to the zero set of a certain function

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Abstract

Let $T : X \to X$ be a given operator and F_T be the set of its fixed points. For a certain function $\varphi : X \to [0,\infty)$, we say that F_T is φ-admissible if F_T is nonempty and $F_T \subseteq Z_\varphi$, where Z_φ is the zero set of φ. In this paper, we study the φ-admissibility of a new class of operators. As applications, we establish a new homotopy result and we obtain a partial metric version of the Boyd-Wong fixed point theorem.

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1 Introduction

Let (X,d) be a metric space. For a given function $\varphi : X \to [0,\infty)$, we define the set

$$Z_\varphi = \{x \in X : \varphi(x) = 0\}.$$

Let $T : X \to X$ be a given operator. The set of fixed points of T is denoted by F_T, that is,

$$F_T = \{x \in X : Tx = x\}.$$

Definition 1.1 We say that the set F_T is φ-admissible if and only if $F_T \neq \emptyset$ and $F_T \subseteq Z_\varphi$.

Let $\mathcal{F}$ be the set of functions $F : [0,\infty)^3 \to [0,\infty)$ satisfying the following conditions:

(F1) $\max\{a,b\} \le F(a,b,c)$, for all $a,b,c \ge 0$;
(F2) $F(a,0,0) = a$, for all $a \ge 0$;
(F3) F is continuous.

As examples, the following functions belong to $\mathcal{F}$:

1. $F(a,b,c) = a + b + c$,
2. $F(a,b,c) = \max\{a,b\} + \ln(c+1)$,
3. $F(a,b,c) = a + b + c(c+1)$,
4. $F(a,b,c) = (a+b)e^c$,
5. $F(a,b,c) = (a+b)(c+1)^n$, $n \in \mathbb{N}$.

Let Ψ be the set of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(Ψ1) ψ is upper semi-continuous from the right;
(Ψ2) $\psi(t) < t$, for all $t > 0$.

For given functions $\varphi : X \to [0,\infty)$, $F \in \mathcal{F}$, and $\psi \in \Psi$, we denote by $\mathcal{T}(\varphi, F, \psi)$ the class of operators $T : X \to X$ satisfying

$$F\bigl(d(Tx, Ty), \varphi(Tx), \varphi(Ty)\bigr) \leq \psi\bigl(F\bigl(d(x,y), \varphi(x), \varphi(y)\bigr)\bigr), \quad (x,y) \in X \times X. \tag{1.1}$$

The aim of this paper is to study the φ-admissibility of the set F_T, where T belongs to the class of operators $\mathcal{T}(\varphi, F, \psi)$, $(F, \psi) \in \mathcal{F} \times \Psi$. As applications, we obtain an homotopy result and a partial metric version of the Boyd-Wong fixed point theorem.

2 Main result

Our main result is given in the following theorem.

Theorem 2.1 *Let (X,d) be a complete metric space and $T : X \to X$ be a given operator. Suppose that the following conditions hold*:

(i) *there exist $\varphi : X \to [0,\infty)$, $F \in \mathcal{F}$, and $\psi \in \Psi$ such that $T \in \mathcal{T}(\varphi, F, \psi)$;*
(ii) *φ is lower semi-continuous.*

Then the set F_T is φ-admissible. Moreover, the operator T has a unique fixed point.

Proof Let ξ be an arbitrary element of the set F_T. Take $x = y = \xi$ in (1.1), and we get

$$F\bigl(0, \varphi(\xi), \varphi(\xi)\bigr) \leq \psi\bigl(F\bigl(0, \varphi(\xi), \varphi(\xi)\bigr)\bigr). \tag{2.1}$$

If $F(0, \varphi(\xi), \varphi(\xi)) \neq 0$, from ($\Psi$2), we get

$$\psi\bigl(F\bigl(0, \varphi(\xi), \varphi(\xi)\bigr)\bigr) < F\bigl(0, \varphi(\xi), \varphi(\xi)\bigr),$$

which is impossible from (2.1). Consequently, we have

$$F\bigl(0, \varphi(\xi), \varphi(\xi)\bigr) = 0.$$

Using the above equality and (F1), we obtain

$$\varphi(\xi) \leq F\bigl(0, \varphi(\xi), \varphi(\xi)\bigr) = 0,$$

which yields

$$\varphi(\xi) = 0.$$

Consequently, we have

$$F_T \subseteq Z_\varphi. \tag{2.2}$$

Now, we have to prove that F_T is a nonempty set. Let x_0 be an arbitrary element of X. Consider the Picard sequence $\{x_n\} \subset X$ defined by

$$x_n = T^n x_0, \quad n \in \mathbb{N} = \{0, 1, 2, \ldots\},$$

where T^n is the nth iterate of T. If for some $N \in \mathbb{N}$ we have $x_N = x_{N+1}$, then x_N will be an element of F_T. As a result we can suppose that

$$d(x_n, x_{n+1}) > 0, \quad n \in \mathbb{N}. \tag{2.3}$$

Using (1.1), we have

$$\begin{aligned} &F\bigl(d(Tx_n, Tx_{n-1}), \varphi(Tx_n), \varphi(Tx_{n-1})\bigr) \\ &\quad \leq \psi\bigl(F\bigl(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1})\bigr)\bigr), \quad n \in \mathbb{N}^*; \end{aligned} \tag{2.4}$$

here $\mathbb{N}^* = \{1, 2, \ldots\}$. If for some $N \in \mathbb{N}^*$, we have

$$F\bigl(d(x_N, x_{N-1}), \varphi(x_N), \varphi(x_{N-1})\bigr) = 0,$$

then property (F1) yields

$$d(x_N, x_{N-1}) \leq F\bigl(d(x_N, x_{N-1}), \varphi(x_N), \varphi(x_{N-1})\bigr) = 0,$$

which is a contradiction with (2.3). Thus

$$F\bigl(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1})\bigr) > 0, \quad n \in \mathbb{N}^*. \tag{2.5}$$

Using (2.4), (2.5), the definition of the sequence $\{x_n\}$, and $(\Psi2)$, we have

$$\begin{cases} F(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n)) \leq \psi(F(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1}))), \\ \psi(F(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1}))) < F(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1})), \end{cases} \quad n \in \mathbb{N}^*. \tag{2.6}$$

It follows immediately from (2.6) that there exists some $c \geq 0$ such that

$$\begin{aligned} &\lim_{n\to\infty} F\bigl(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n)\bigr) \\ &\quad = \lim_{n\to\infty} \psi\bigl(F\bigl(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1})\bigr)\bigr) = c. \end{aligned} \tag{2.7}$$

Suppose now that $c > 0$. Using the properties $(\Psi1)$-$(\Psi2)$, we deduce from (2.7) that

$$c = \limsup_{n\to\infty} \psi\bigl(F\bigl(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1})\bigr)\bigr) \leq \psi(c) < c,$$

which is a contradiction. As a consequence, we have $c = 0$, that is,

$$\begin{aligned} &\lim_{n\to\infty} F\bigl(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n)\bigr) \\ &\quad = \lim_{n\to\infty} \psi\bigl(F\bigl(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1})\bigr)\bigr) = 0. \end{aligned} \tag{2.8}$$

Using (F1) and (2.8), we get

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = \lim_{n\to\infty} \varphi(x_n) = 0. \tag{2.9}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d). Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that, for all $k \in \mathbb{N}$,

$$n(k) > m(k) > k, \qquad d(x_{m(k)}, x_{n(k)}) \geq \varepsilon, \qquad d(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{2.10}$$

Using (2.10), for all $k \in \mathbb{N}$ we have

$$\begin{aligned}
\varepsilon &\leq d(x_{m(k)}, x_{n(k)}) \\
&\leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) \\
&< \varepsilon + d(x_{n(k)-1}, x_{n(k)}),
\end{aligned}$$

which yields

$$\varepsilon \leq d(x_{m(k)}, x_{n(k)}) < \varepsilon + d(x_{n(k)-1}, x_{n(k)}), \quad k \in \mathbb{N}. \tag{2.11}$$

Letting $k \to \infty$ in the above inequality and using (2.9), we obtain

$$\begin{aligned}
&\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon^+ \quad \textit{i.e.} \\
&\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon \quad \text{and} \quad d(x_{m(k)}, x_{n(k)}) \geq \epsilon \quad \text{for } k \in \mathbb{N}.
\end{aligned} \tag{2.12}$$

Using the properties (F2)-(F3), (2.9), and (2.12), we get

$$\lim_{k\to\infty} F\bigl(d(x_{n(k)}, x_{m(k)}), \varphi(x_{n(k)}), \varphi(x_{m(k)})\bigr) = F(\varepsilon, 0, 0) = \varepsilon^+.$$

Using the above limit and $(\Psi1)$, we obtain

$$\limsup_{k\to\infty} \psi\bigl(F\bigl(d(x_{n(k)}, x_{m(k)}), \varphi(x_{n(k)}), \varphi(x_{m(k)})\bigr)\bigr) \leq \psi(\varepsilon). \tag{2.13}$$

On the other hand, using (1.1) and (F1), for all $k \in \mathbb{N}$ we have

$$\begin{aligned}
\varepsilon \leq d(x_{n(k)}, x_{m(k)}) &\leq d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{m(k)}) \\
&\leq d(x_{n(k)}, x_{n(k)+1}) + F\bigl(d(x_{n(k)+1}, x_{m(k)+1}), \varphi(x_{n(k)+1}), \varphi(x_{m(k)+1})\bigr) + d(x_{m(k)+1}, x_{m(k)}) \\
&\leq d(x_{n(k)}, x_{n(k)+1}) + \psi\bigl(F\bigl(d(x_{n(k)}, x_{m(k)}), \varphi(x_{n(k)}), \varphi(x_{m(k)})\bigr)\bigr) + d(x_{m(k)+1}, x_{m(k)}).
\end{aligned}$$

Passing to the limit superior as $k \to \infty$, using (2.9), (2.13), and $(\Psi2)$, we obtain

$$\varepsilon \leq \psi(\varepsilon) < \varepsilon,$$

which is a contradiction. As a consequence, $\{x_n\}$ is a Cauchy sequence. Since (X,d) is a complete metric space, there is a $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0. \tag{2.14}$$

Since φ is lower semi-continuous, it follows from (2.14) and (2.9) that

$$0 \leq \varphi(z) \leq \liminf_{n\to\infty} \varphi(x_n) = 0,$$

which yields

$$z \in Z_\varphi. \tag{2.15}$$

Now we show that $z \in F_T$. Using (1.1), (F1), and (2.15), we have

$$d(x_{n+1}, Tz) \leq \psi\bigl(F\bigl(d(x_n, z), \varphi(x_n), 0\bigr)\bigr), \quad n \in \mathbb{N}. \tag{2.16}$$

Also using the continuity of F, (F2), (2.14), and (2.9), we have

$$\lim_{n\to\infty} F\bigl(d(x_n, z), \varphi(x_n), 0\bigr) = F(0,0,0) = 0.$$

Note that from (Ψ2), we have

$$\lim_{t\to 0^+} \psi(t) = 0.$$

Then

$$\lim_{n\to\infty} \psi\bigl(F\bigl(d(x_n, z), \varphi(x_n), 0\bigr)\bigr) = \lim_{t\to 0^+} \psi(t) = 0. \tag{2.17}$$

Now, passing $n \to \infty$ in (2.16) and using (2.17), we get

$$\lim_{n\to\infty} d(x_{n+1}, Tz) = 0.$$

The uniqueness of the limit yields $z = Tz$. Thus F_T is a nonempty set, and the φ-admissibility of F_T is proved. Finally, in order to prove the uniqueness of the fixed point, let us assume that $w \in F_T$ with $d(z,w) > 0$. Since F_T is φ-admissible, we know that $z, w \in Z_\varphi$. Now, applying (1.1) with $(x,y) = (z,w)$, we obtain

$$F\bigl(d(z,w), 0, 0\bigr) \leq \psi\bigl(F\bigl(d(z,w), 0, 0\bigr)\bigr).$$

Using the properties (F2) and (Ψ2), we get

$$d(z,w) \leq \psi\bigl(d(z,w)\bigr) < d(z,w),$$

which is a contradiction. Thus T has a unique fixed point. □

Remark 2.2 Take $\varphi \equiv 0$ and $F(a,b,c) = a+b+c$ in Theorem 2.1, and we recover the Boyd-Wong fixed point theorem [1].

Now, we give some examples to illustrate our main result given by Theorem 2.1.

Example 2.3 We endow the set $X = [0,\infty)$ with the standard metric

$$d(x,y) = |x-y|, \quad (x,y) \in X \times X.$$

Let $T : X \to X$ be the mapping defined by

$$Tx = \begin{cases} 0 & \text{if } 0 \leq x \leq 1, \\ \frac{x}{2} & \text{if } 1 < x. \end{cases}$$

Observe that T is not continuous in X. So, there is no $\psi \in \Psi$ such that

$$d(Tx, Ty) \leq \psi\big(d(x,y)\big), \quad (x,y) \in X \times X.$$

Then the Boyd-Wong fixed point theorem cannot be applied in this case. Let $\varphi : X \to [0,\infty)$ be the function defined by

$$\varphi(x) = x^n, \quad x \in X, \text{ for some } n \in \mathbb{N}^*.$$

Let $F : [0,\infty)^3 \to [0,\infty)$ be the function defined by

$$F(a,b,c) = a + b + c, \quad a,b,c \geq 0.$$

Let $\psi : [0,\infty) \to [0,\infty)$ be the function defined by

$$\psi(t) = \frac{t}{2}, \quad t \geq 0.$$

Observe that F belongs to the set $\mathcal{F}$ and ψ belongs to the set Ψ. We claim that $T \in \mathcal{T}(\varphi, F, \psi)$, that is,

$$d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) \leq \frac{1}{2}\big[d(x,y) + \varphi(x) + \varphi(y)\big], \quad (x,y) \in X \times X. \tag{2.18}$$

In order to prove our claim, we distinguish three cases.

Case 1. $(x,y) \in [0,1] \times [0,1]$.

In this case, we have

$$d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) = 0 \leq \frac{1}{2}\big[d(x,y) + \varphi(x) + \varphi(y)\big].$$

Case 2. $(x,y) \in [0,1] \times (1,\infty)$.

In this case, we have

$$d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) = \frac{y}{2} + \left(\frac{y}{2}\right)^n,$$

while

$$\frac{1}{2}\bigl[d(x,y)+\varphi(x)+\varphi(y)\bigr]=\frac{1}{2}\bigl[y-x+x^n+y^n\bigr].$$

Then we have to prove that

$$y^n\left(\frac{1}{2^{n-1}}-1\right)\leq x^n-x.$$

Observe that the function $h:[0,1]\to\mathbb{R}$ defined by

$$h(x)=x^n-x,\quad x\in[0,1],$$

has a global minimum at $x_n=(\frac{1}{n})^{\frac{1}{n-1}}$ which is equal to $x_n(\frac{1-n}{n})\geq\frac{1-n}{n}$. So, we have just to check that

$$y^n\left(\frac{1}{2^{n-1}}-1\right)\leq\frac{1}{n}-1.$$

Since $y>1$, we have

$$y^n\left(\frac{1}{2^{n-1}}-1\right)\leq\frac{1}{2^{n-1}}-1\leq\frac{1}{n}-1.$$

Then our claim holds in this case.

Case 3. $x,y>1$.

In this case, we have

$$d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)=\frac{|x-y|}{2}+\left(\frac{x}{2}\right)^n+\left(\frac{y}{2}\right)^n$$

and

$$\frac{1}{2}\bigl[d(x,y)+\varphi(x)+\varphi(y)\bigr]=\frac{|x-y|}{2}+\frac{x^n+y^n}{2}.$$

Obviously, we have

$$d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\leq\frac{1}{2}\bigl[d(x,y)+\varphi(x)+\varphi(y)\bigr].$$

Finally, in all cases our claim (2.18) holds, which yields $T\in\mathcal{T}(\varphi,F,\psi)$. By Theorem 2.1, the set F_T is φ-admissible and T has a unique fixed point. In this example, $F_T=\{0\}$ and $\varphi(0)=0$.

Example 2.4 We endow the set $X=[\sqrt{2},\infty)$ with the standard metric

$$d(x,y)=|x-y|,\quad (x,y)\in X\times X.$$

Let $T : X \to X$ be the mapping defined by

$$Tx = \begin{cases} \sqrt{2} & \text{if } \sqrt{2} \leq x \leq 2\sqrt{2}, \\ \frac{x}{2} & \text{if } 2\sqrt{2} < x. \end{cases}$$

As in the previous example, the Boyd-Wong fixed point theorem cannot be applied in this case. Let $\varphi : X \to [0, \infty)$ be the function defined by

$$\varphi(x) = x^2 - 2, \quad x \in X.$$

Let $F : [0, \infty)^3 \to [0, \infty)$ be the function defined by

$$F(a, b, c) = a + b + c, \quad a, b, c \geq 0.$$

Let $\psi : [0, \infty) \to [0, \infty)$ be the function defined by

$$\psi(t) = \frac{t}{2}, \quad t \geq 0.$$

We distinguish three cases.

Case 1. $(x, y) \in [\sqrt{2}, 2\sqrt{2}] \times [\sqrt{2}, 2\sqrt{2}]$.

In this case, we have

$$d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) = 0 \leq \frac{1}{2}\bigl[d(x, y) + \varphi(x) + \varphi(y)\bigr].$$

Case 2. $(x, y) \in [\sqrt{2}, 2\sqrt{2}] \times (2\sqrt{2}, \infty)$.

In this case, we have

$$d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) = \frac{y}{2} - \sqrt{2} + \frac{y^2}{4} - 2,$$

while

$$\frac{1}{2}\bigl[d(x, y) + \varphi(x) + \varphi(y)\bigr] = \frac{y}{2} - \frac{x}{2} + \frac{x^2}{2} + \frac{y^2}{2} - 2.$$

Clearly, we have

$$d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) \leq \frac{1}{2}\bigl[d(x, y) + \varphi(x) + \varphi(y)\bigr].$$

Case 3. $(x, y) \in (2\sqrt{2}, \infty)$.

In this case, we have

$$d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) = \frac{|x - y|}{2} + \frac{x^2}{4} + \frac{y^2}{4} - 4,$$

while

$$\frac{1}{2}\bigl[d(x, y) + \varphi(x) + \varphi(y)\bigr] = \frac{|x - y|}{2} + \frac{x^2}{2} + \frac{y^2}{2} - 2.$$

Also, we have

$$d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\leq\frac{1}{2}\big[d(x,y)+\varphi(x)+\varphi(y)\big].$$

As a consequence, the mapping T belongs to $\mathcal{T}(\varphi,F,\psi)$. By Theorem 2.1, the set F_T is φ-admissible and T has a unique fixed point. In this example, $F_T=\{\sqrt{2}\}$ and $\varphi(\sqrt{2})=0$.

Example 2.5 Let (X,d) be the metric space considered in Example 2.4. We take the functions φ, F, and ψ defined in Example 2.4. Let $T:X\to X$ be the mapping defined by

$$Tx=\begin{cases}\sqrt{2} & \text{if } \sqrt{2}\leq x\leq 2\sqrt{2},\\ \frac{\sin x}{2} & \text{if } 2\sqrt{2}<x.\end{cases}$$

Similarly, we have $T\in\mathcal{T}(\varphi,F,\psi)$. By Theorem 2.1, the set F_T is φ-admissible and T has a unique fixed point. In this example, $F_T=\{\sqrt{2}\}$ and $\varphi(\sqrt{2})=0$.

3 Applications

3.1 An homotopy result

Let us denote by $\mathcal{F}^*$ the set of functions $F\in\mathcal{F}$ satisfying the following property:

(F4) for all $a,b,c,d\geq 0$,

$$a\leq c+d \quad\Longrightarrow\quad F(a,b,0)\leq F(c,b,0)+d.$$

As examples, the following functions belong to $\mathcal{F}^*$:

1. $F(a,b,c)=(a+b)e^c$,
2. $F(a,b,c)=(a+b)(c+1)^n$, $n\in\mathbb{N}$.

Observe that $\mathcal{F}^*\subsetneq\mathcal{F}$. To see this, let us consider the function

$$F(a,b,c)=ae^{c+b}+be^{a+c},\quad a,b,c\geq 0.$$

It is not difficult to check that $F\in\mathcal{F}$ but $F\notin\mathcal{F}^*$.

We have the following homotopy result.

Theorem 3.1 *Let (X,d) be a complete metric space, U be an open subset of X, and V be a closed subset of X with $U\subset V$. Suppose that $H:V\times[0,1]\to X$ has the following properties*:

(C1) $x\neq H(x,\lambda)$ *for every* $x\in V\setminus U$ *and* $\lambda\in[0,1]$;

(C2) *there exist a continuous function* $\varphi:X\to[0,\infty)$, $L\in(0,1)$, *and* $F\in\mathcal{F}^*$ *such that for all* $x,y\in V$ *and* $\lambda\in[0,1]$,

$$F\big(d\big(H(x,\lambda),H(y,\lambda)\big),\varphi\big(H(x,\lambda)\big),\varphi\big(H(y,\lambda)\big)\big)\leq LF\big(d(x,y),\varphi(x),\varphi(y)\big);$$

(C3) *there exists a continuous function* $\eta:[0,1]\to\mathbb{R}$ *such that for all* $x\in V$ *and* $\lambda,\mu\in[0,1]$,

$$F\big(d\big(H(x,\lambda),H(x,\mu)\big),\varphi\big(H(x,\lambda)\big),\varphi\big(H(x,\mu)\big)\big)\leq\big|\eta(\lambda)-\eta(\mu)\big|.$$

Then $H(\cdot,0)$ has a fixed point if and only if $H(\cdot,1)$ has a fixed point.

Proof Suppose that $H(\cdot,0)$ has a fixed point. Consider the set

$$Q = \{t \in [0,1] : x = H(x,t) \text{ for some } x \in U\}.$$

From (C1), clearly 0 is an element of Q, so Q is a nonempty set. We will show that Q is both closed and open in $[0,1]$, and so by the connectedness of $[0,1]$, we are finished since $Q = [0,1]$. First, let us prove that Q is open in $[0,1]$. Let $t_0 \in Q$ and $x_0 \in U$ with $x_0 = H(x_0,t_0)$. Using (C2) with $x = y = x_0$ and $\lambda = t_0$, we obtain

$$F(0,\varphi(x_0),\varphi(x_0)) \leq LF(0,\varphi(x_0),\varphi(x_0)),$$

which implies since $L \in (0,1)$ that

$$F(0,\varphi(x_0),\varphi(x_0)) = 0.$$

Then (F1) yields

$$\varphi(x_0) = 0.$$

Moreover, observe that, for all $t \in [0,1]$, if $x \in U$ is a fixed point of $H(\cdot,t)$, then $\varphi(x) = 0$. On the other hand, since U is open in (X,d), there exists $r > 0$ such that $B(x_0,r) \subseteq U$, where

$$B(x_0,r) = \{z \in X : d(x_0,z) < r\}.$$

Consider the set

$$\Lambda(x_0,\varphi) = \{z \in X : F(d(z,x_0),\varphi(z),0) < r\}.$$

Clearly $\Lambda(x_0,\varphi)$ is nonempty (since $x_0 \in \Lambda(x_0,\varphi)$) and $\Lambda(x_0,\varphi) \subseteq B(x_0,r)$. Let $\varepsilon = (1-L)r > 0$. Since η is continuous on t_0, there exists $\alpha(\varepsilon) > 0$ such that

$$t \in (t_0 - \alpha(\varepsilon), t_0 + \alpha(\varepsilon)) \cap [0,1] \quad \Longrightarrow \quad |\eta(t) - \eta(t_0)| < \varepsilon.$$

Let $t \in (t_0 - \alpha(\varepsilon), t_0 + \alpha(\varepsilon)) \cap [0,1]$. For $x \in \overline{\Lambda(x_0,\varphi)}$ (the closure of $\Lambda(x_0,\varphi)$), we have

$$F(d(H(x,t),x_0),\varphi(H(x,t)),0) = F(d(H(x,t),H(x_0,t_0)),\varphi(H(x,t)),0).$$

Also since

$$d(H(x,t),H(x_0,t_0)) \leq d(H(x,t),H(x,t_0)) + d(H(x,t_0),H(x_0,t_0)),$$

using the properties (F1), (F4) we get

$$\begin{aligned}
&F(d(H(x,t),H(x_0,t_0)),\varphi(H(x,t)),0)\\
&\quad \leq F(d(H(x,t),H(x,t_0)),\varphi(H(x,t)),0) + d(H(x,t_0),H(x_0,t_0))\\
&\quad \leq F(d(H(x,t),H(x,t_0)),\varphi(H(x,t)),0) + F(d(H(x,t_0),H(x_0,t_0)),\varphi(H(x,t_0)),0)
\end{aligned}$$

$$\begin{aligned}
&\leq \bigl|\eta(t)-\eta(t_0)\bigr| + LF\bigl(d(x,x_0),\varphi(x),0\bigr) \\
&< \varepsilon + Lr = r.
\end{aligned}$$

Thus we proved that, for all $t \in (t_0 - \alpha(\varepsilon), t_0 + \alpha(\varepsilon)) \cap [0,1]$, the operator

$$H(\cdot,t) : \overline{\Lambda(x_0,\varphi)} \to \overline{\Lambda(x_0,\varphi)}$$

is well defined. Now, using (C2) and Theorem 2.1, we deduce that, for all $t \in (t_0 - \alpha(\varepsilon), t_0 + \alpha(\varepsilon)) \cap [0,1]$, the operator $H(\cdot,t)$ has a fixed point in V. However, such a fixed point should be in U from (C1). As a consequence,

$$\bigl(t_0 - \alpha(\varepsilon), t_0 + \alpha(\varepsilon)\bigr) \cap [0,1] \subseteq Q,$$

which proves that Q is open in $[0,1]$. Next, we show that Q is closed in $[0,1]$. To see this, let $\{t_n\}$ be a sequence in Q with $t_n \to t \in [0,1]$ as $n \to \infty$. We have to prove that $t \in Q$. From the definition of Q, for all $n \in \mathbb{N}$, there exists $x_n \in U$ with

$$x_n = H(x_n,t_n) \quad \text{and} \quad \varphi(x_n) = 0.$$

Also for all $m,n \in \mathbb{N}$, we have

$$\begin{aligned}
d(x_n,x_m) &= d\bigl(H(x_n,t_n),H(x_m,t_m)\bigr) \\
&\leq d\bigl(H(x_n,t_n),H(x_n,t_m)\bigr) + d\bigl(H(x_n,t_m),H(x_m,t_m)\bigr) \\
&\leq F\bigl(d\bigl(H(x_n,t_n),H(x_n,t_m)\bigr),\varphi\bigl(H(x_n,t_n)\bigr),\varphi\bigl(H(x_n,t_m)\bigr)\bigr) \\
&\quad + F\bigl(d\bigl(H(x_n,t_m),H(x_m,t_m)\bigr),\varphi\bigl(H(x_n,t_m)\bigr),\varphi\bigl(H(x_m,t_m)\bigr)\bigr) \\
&\leq \bigl|\eta(t_n)-\eta(t_m)\bigr| + LF\bigl(d(x_n,x_m),0,0\bigr) \\
&= \bigl|\eta(t_n)-\eta(t_m)\bigr| + Ld(x_n,x_m),
\end{aligned}$$

which yields

$$d(x_n,x_m) \leq \frac{|\eta(t_n)-\eta(t_m)|}{1-L}, \quad m,n \in \mathbb{N}.$$

Letting $m,n \to \infty$ in the above inequality and using the continuity of η, we get $d(x_n,x_m) \to 0$ as $m,n \to \infty$, which implies that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X,d). Then there is some $z \in V$ (since V is closed) such that

$$\lim_{n\to\infty} d(x_n,z) = 0 \quad \text{and} \quad \varphi(z) = 0,$$

since φ is lower semi-continuous. Now, for all $n \in \mathbb{N}$ we have

$$\begin{aligned}
d\bigl(x_n,H(z,t)\bigr) &= d\bigl(H(x_n,t_n),H(z,t)\bigr) \leq d\bigl(H(x_n,t_n),H(x_n,t)\bigr) + d\bigl(H(x_n,t),H(z,t)\bigr) \\
&\leq F\bigl(d\bigl(H(x_n,t_n),H(x_n,t)\bigr),\varphi\bigl(H(x_n,t_n)\bigr),\varphi\bigl(H(x_n,t)\bigr)\bigr) \\
&\quad + F\bigl(d\bigl(H(x_n,t),H(z,t)\bigr),\varphi\bigl(H(x_n,t)\bigr),\varphi\bigl(H(z,t)\bigr)\bigr)
\end{aligned}$$

$$\leq \left|\eta(t_n) - \eta(t)\right| + LF\big(d(x_n, z), 0, 0\big)$$
$$= \left|\eta(t_n) - \eta(t)\right| + Ld(x_n, z).$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} d\big(x_n, H(z,t)\big) = 0.$$

The uniqueness of the limit yields $z = H(z,t)$. Using (C1), we deduce that $z \in U$ and $t \in Q$. Thus Q is closed in $[0,1]$.

For the reverse implication, we use the same technique. □

3.2 A partial metric version of Boyd-Wong fixed point theorem

In this part, using Theorem 2.1, we obtain a partial metric version of the Boyd-Wong fixed point theorem.

We start by recalling some basic definitions and properties of partial metric spaces. For more details of such spaces, we refer the reader to [2–20].

A partial metric on a nonempty set X is a function $p : X \to X \to [0,\infty)$ such that for all $x, y, z \in X$, we have

(i) $p(x,x) = p(y,y) = p(x,y) \Longleftrightarrow x = y$;

(ii) $p(x,x) \leq p(x,y)$;

(iii) $p(x,y) = p(y,x)$;

(iv) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x,y) = 0$, then from (i)-(ii), $x = y$; but if $x = y$, $p(x,y)$ may not be 0. A basic example of a partial metric space is the pair $([0,\infty), p)$, where $p(x,y) = \max\{x,y\}$.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where

$$B_p(x,\varepsilon) := \big\{y \in X : p(x,y) < p(x,x) + \varepsilon\big\}.$$

Let (X,p) be a partial metric space. A sequence $\{x_n\} \subset X$ converges to some $x \in X$ with respect to p if and only if

$$\lim_{n\to\infty} p(x_n, x) = p(x,x).$$

A sequence $\{x_n\} \subset X$ is said to be a Cauchy sequence if and only if $\lim_{m,n\to\infty} p(x_n, x_m)$ exists and is finite. The partial metric space (X,p) is said to be complete if and only if every Cauchy sequence $\{x_n\}$ in X converges to some $x \in X$ such that $\lim_{n,m\to\infty} p(x_n, x_m) = p(x,x)$.

If p is a partial metric on X, then the function $d_p : X \to X \to [0,\infty)$ defined by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y), \quad (x,y) \in X^2, \tag{3.1}$$

is a metric on X.

Lemma 3.2 *Let (X,p) be a partial metric space. Then*:

(i) *$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if $\{x_n\}$ is a Cauchy sequence in the metric space (X,d_p)*;

(ii) *the partial metric space* (X,p) *is complete if and only if the metric space* (X,d_p) *is complete. Furthermore,* $\lim_{n\to\infty} d_p(x_n,x) = 0$ *if and only if*

$$\lim_{n\to\infty} p(x_n,x) = p(x,x) = \lim_{m,n\to\infty} p(x_n,x_m).$$

We have the following result.

Corollary 3.3 *Let* (X,p) *be a complete partial metric space and let* $T : X \to X$ *be an operator such that*

$$p(Tx,Ty) \le \psi\bigl(p(x,y)\bigr), \quad (x,y) \in X \times X, \tag{3.2}$$

where $\psi \in \Psi$. *We have the following results*:

(i) *if* $z \in X$ *is a fixed point of* T *then* $p(z,z) = 0$;

(ii) T *has a unique fixed point.*

Proof Let d_p be the metric on X defined by (3.1). We have

$$p(x,y) = d(x,y) + \varphi(x) + \varphi(y), \quad (x,y) \in X \times X,$$

where

$$d(x,y) = \frac{d_p(x,y)}{2}, \qquad \varphi(x) = \frac{p(x,x)}{2}.$$

Then (3.2) yields

$$F\bigl(d(Tx,Ty),\varphi(Tx),\varphi(Ty)\bigr) \le \psi\bigl(F\bigl(d(x,y),\varphi(x),\varphi(y)\bigr)\bigr), \quad (x,y) \in X \times X,$$

where

$$F(a,b,c) = a + b + c, \quad a,b,c \ge 0.$$

From (ii) Lemma 3.2, the metric space (X,d) is complete and the function φ is continuous with respect to the topology of d. Finally the desired result follows from Theorem 2.1. □

Remark 3.4 Take in Corollary 3.3, $\psi(t) = kt$ with $k \in (0,1)$, and we recover Matthews fixed point theorem [9].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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