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General uniqueness theorem concerning the stability of additive, quadratic, and cubic functional equations

Yang-Hi Lee[1] and Soon-Mo Jung[2*]

[*]Correspondence:
smjung@hongik.ac.kr
[2]Mathematics Section, College of Science and Technology, Hongik University, Sejong, 30016, Republic of Korea
Full list of author information is available at the end of the article

Abstract
We prove a general uniqueness theorem that can easily be applied to the proof of (generalized) Hyers-Ulam stability of the additive, quadratic, cubic, or the cubic-quadratic-additive type functional equation. By using this uniqueness theorem, we can omit the repeated proof for uniqueness of the relevant solutions of those equations.

MSC: Primary 39B82; secondary 39B52

Keywords: generalized Hyers-Ulam stability; uniqueness; cubic-quadratic-additive type functional equation; cubic-quadratic-additive mapping

1 Introduction

In 1940, Ulam [1] posed a problem concerning the stability of functional equations: *Give conditions in order for a linear function near an approximately linear function to exist.* A year later, Hyers [2] gave an answer to the problem of Ulam for additive functions defined on Banach spaces. After Hyers' result, many mathematicians have extended Ulam's problem to other functional equations and generalized Hyers' result in various directions (see [3–7]).

Let V and W be real vector spaces. For a given mapping $f : V \to W$, we define

$$\begin{aligned}
&Af(x,y) := f(x+y) - f(x) - f(y),\\
&Qf(x,y) := f(x+y) - 2f(x) + f(x-y) - 2f(y),\\
&Cf(x,y) := f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y),\\
&f_o(x) := \frac{f(x) - f(-x)}{2},\\
&f_o^{(1)}(x) := \frac{a^3 f_o(x) - f_o(ax)}{a^3 - a},\\
&f_o^{(2)}(x) := -\frac{af_o(x) - f_o(ax)}{a^3 - a},\\
&f_e(x) := \frac{f(x) + f(-x)}{2}
\end{aligned}$$

for all $x, y \in V$. A mapping $f : V \to W$ is called an additive mapping, a quadratic mapping, and a cubic mapping if f satisfies the functional equation $Af(x,y) = 0$, $Qf(x,y) = 0$, and $Cf(x,y) = 0$ for all $x, y \in V$, respectively. We remark that the mappings $g, h, k : \mathbf{R} \to \mathbf{R}$ given by $g(x) = ax$, $h(x) = ax^2$, and $k(x) = ax^3$ are solutions of $Ag(x,y) = 0$, $Qh(x,y) = 0$, and $Ck(x,y) = 0$, respectively.

A mapping $f : V \to W$ is called a cubic-quadratic-additive mapping if and only if f is represented by the sum of an additive mapping, a quadratic mapping, and a cubic mapping. A functional equation is called a cubic-quadratic-additive type functional equation if and only if each of its solutions is a cubic-quadratic-additive mapping. The mapping $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = ax^3 + bx^2 + cx$ is a solution of the cubic-quadratic-additive type functional equation.

In the study of the stability problems for cubic-quadratic-additive type functional equations, we frequently encounter the cases where we should prove the uniqueness of the cubic-quadratic-additive mappings (see [8–18]). Research in this uniqueness problem still has many untouched possibilities to explore.

In this paper, we prove a general uniqueness theorem that can be easily applied to the stability of the cubic-quadratic-additive type functional equations. Using this uniqueness theorem, we do not need to repeat the proof of uniqueness in studying the stability of functional equations mentioned above.

2 Main results

In this section, let X and Y be real normed spaces and let V and W be real vector spaces. In the following theorem, we prove that if, for any given mapping f, there exists a mapping F (near f) with some properties possessed by cubic-quadratic-additive mappings, then the mapping F must be uniquely determined.

Theorem 2.1 *Let $a > 1$ be a real constant, let $\Phi : V\backslash\{0\} \to [0,\infty)$ be a function satisfying one of the following conditions*:

$$\lim_{n\to\infty} \frac{\Phi(a^n x)}{a^n} = 0, \tag{1}$$

$$\lim_{n\to\infty} a^n \Phi\left(\frac{x}{a^n}\right) = \lim_{n\to\infty} \frac{\Phi(a^n x)}{a^{2n}} = 0, \tag{2}$$

$$\lim_{n\to\infty} a^{2n} \Phi\left(\frac{x}{a^n}\right) = \lim_{n\to\infty} \frac{\Phi(a^n x)}{a^{3n}} = 0, \tag{3}$$

$$\lim_{n\to\infty} a^{3n} \Phi\left(\frac{x}{a^n}\right) = 0 \tag{4}$$

for all $x \in V\backslash\{0\}$, and let $f : V \to Y$ be a given mapping. If there exists a mapping $F : V \to Y$ such that

$$\|f(x) - F(x)\| \le \Phi(x) \tag{5}$$

for all $x \in V\backslash\{0\}$ and

$$F_o^{(1)}(ax) := aF_o^{(1)}(x), \qquad F_e(ax) := a^2 F_e(x), \qquad F_o^{(2)}(ax) := a^3 F_o^{(2)}(x) \tag{6}$$

for all $x \in V$, then F is given by

$$F(x) = \begin{cases} \lim_{n\to\infty}(\frac{f_o^{(1)}(a^n x)}{a^n} + \frac{f_e(a^n x)}{a^{2n}} + \frac{f_o^{(2)}(a^n x)}{a^{3n}}) & \text{if } \Phi \text{ satisfies (1)}, \\ \lim_{n\to\infty}(a^n f_o^{(1)}(\frac{x}{a^n}) + \frac{f_e(a^n x)}{a^{2n}} + \frac{f_o^{(2)}(a^n x)}{a^{3n}}) & \text{if } \Phi \text{ satisfies (2)}, \\ \lim_{n\to\infty}(a^n f_o^{(1)}(\frac{x}{a^n}) + a^{2n} f_e(\frac{x}{a^n}) + \frac{f_o^{(2)}(a^n x)}{a^{3n}}) & \text{if } \Phi \text{ satisfies (3)}, \\ \lim_{n\to\infty}(a^n f_o^{(1)}(\frac{x}{a^n}) + a^{2n} f_e(\frac{x}{a^n}) + a^{3n} f_o^{(2)}(\frac{x}{a^n})) & \text{if } \Phi \text{ satisfies (4)} \end{cases} \tag{7}$$

for all $x \in V\backslash\{0\}$. In other words, F is the unique mapping satisfying the conditions (5) *and* (6).

Proof Assume that F is a mapping satisfying (5) and (6) for a given mapping $f : V \to Y$. First, we consider the mapping $F_o^{(1)}$. If $\Phi : V\backslash\{0\} \to [0,\infty)$ satisfies the condition (1), then it follows from (6) that

$$\begin{aligned} &\left\| F_o^{(1)}(x) - \frac{f_o^{(1)}(a^n x)}{a^n} \right\| \\ &= \frac{1}{a^n} \left\| F_o^{(1)}(a^n x) - f_o^{(1)}(a^n x) \right\| \\ &= \frac{1}{2(a^3-a)a^n} \left\| a^3 F(a^n x) - a^3 f(a^n x) - a^3 F(-a^n x) + a^3 f(-a^n x) \right. \\ &\quad \left. - F(a^{n+1}x) + f(a^{n+1}x) + F(-a^{n+1}x) - f(-a^{n+1}x) \right\| \\ &\leq \frac{1}{2(a^3-a)a^n} \left(a^3 \left\| F(a^n x) - f(a^n x) \right\| + a^3 \left\| F(-a^n x) - f(-a^n x) \right\| \right. \\ &\quad \left. + \left\| F(a^{n+1}x) - f(a^{n+1}x) \right\| + \left\| F(-a^{n+1}x) - f(-a^{n+1}x) \right\| \right) \\ &\leq \frac{a^3\Phi(a^n x) + a^3\Phi(-a^n x) + \Phi(a^{n+1}x) + \Phi(-a^{n+1}x)}{2(a^3-a)a^n} \\ &\to 0, \quad \text{as } n \to \infty \end{aligned}$$

for all $x \in V\backslash\{0\}$; that is, we see that $F_o^{(1)}(x) = \lim_{n\to\infty} \frac{1}{a^n} f_o^{(1)}(a^n x)$ for all $x \in V\backslash\{0\}$.

If $\Phi : V\backslash\{0\} \to [0,\infty)$ satisfies the condition (2), (3), or (4), then it follows from (6) that

$$\begin{aligned} &\left\| F_o^{(1)}(x) - a^n f_o^{(1)}\left(\frac{x}{a^n}\right) \right\| \\ &= a^n \left\| F_o^{(1)}\left(\frac{x}{a^n}\right) - f_o^{(1)}\left(\frac{x}{a^n}\right) \right\| \\ &= \frac{a^n}{2(a^3-a)} \left\| a^3 F\left(\frac{x}{a^n}\right) - a^3 f\left(\frac{x}{a^n}\right) - a^3 F\left(\frac{-x}{a^n}\right) + a^3 f\left(\frac{-x}{a^n}\right) \right. \\ &\quad \left. - F\left(\frac{x}{a^{n-1}}\right) + f\left(\frac{x}{a^{n-1}}\right) + F\left(\frac{-x}{a^{n-1}}\right) - f\left(\frac{-x}{a^{n-1}}\right) \right\| \\ &\leq \frac{a^n}{2(a^3-a)} \left(a^3 \left\| F\left(\frac{x}{a^n}\right) - f\left(\frac{x}{a^n}\right) \right\| + a^3 \left\| F\left(\frac{-x}{a^n}\right) - f\left(\frac{-x}{a^n}\right) \right\| \right. \\ &\quad \left. + \left\| F\left(\frac{x}{a^{n-1}}\right) - f\left(\frac{x}{a^{n-1}}\right) \right\| + \left\| F\left(\frac{-x}{a^{n-1}}\right) - f\left(\frac{-x}{a^{n-1}}\right) \right\| \right) \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2(a^3-a)}\left(a^{n+3}\Phi\left(\frac{x}{a^n}\right)+a^{n+3}\Phi\left(\frac{-x}{a^n}\right)+a^n\Phi\left(\frac{x}{a^{n-1}}\right)+a^n\Phi\left(\frac{-x}{a^{n-1}}\right)\right)\\
&\to 0, \quad \text{as } n\to\infty
\end{aligned}
$$

for all $x \in V\backslash\{0\}$; that is, we see that $F_o^{(1)}(x) = \lim_{n\to\infty} a^n f_o^{(1)}(\frac{x}{a^n})$ for all $x \in V\backslash\{0\}$.

Second, we consider the mapping F_e. If $\Phi : V\backslash\{0\} \to [0,\infty)$ satisfies the condition (1) or (2), then it follows from (6) that

$$
\begin{aligned}
&\left\| F_e(x) - \frac{f_e(a^n x)}{a^{2n}}\right\|\\
&= \frac{1}{a^{2n}}\left\|F_e(a^n x) - f_e(a^n x)\right\| = \frac{1}{2a^{2n}}\left\|F(a^n x) - f(a^n x) + F(-a^n x) - f(-a^n x)\right\|\\
&\le \frac{1}{2a^{2n}}\left\|F(a^n x) - f(a^n x)\right\| + \frac{1}{2a^{2n}}\left\|F(-a^n x) - f(-a^n x)\right\|\\
&\le \frac{\Phi(a^n x) + \Phi(-a^n x)}{2a^{2n}}\\
&\to 0, \quad \text{as } n\to\infty
\end{aligned}
$$

for all $x \in V\backslash\{0\}$; that is, we see that $F_e(x) = \lim_{n\to\infty} \frac{1}{a^{2n}} f_e(a^n x)$ for all $x \in V\backslash\{0\}$.

If $\Phi : V \to [0,\infty)$ satisfies the condition (3) or (4), we get

$$
\begin{aligned}
&\left\| F_e(x) - a^{2n} f_e\left(\frac{x}{a^n}\right)\right\|\\
&= a^{2n}\left\|F_e\left(\frac{x}{a^n}\right) - f_e\left(\frac{x}{a^n}\right)\right\|\\
&= \frac{a^{2n}}{2}\left\|F\left(\frac{x}{a^n}\right) - f\left(\frac{x}{a^n}\right) + F\left(\frac{-x}{a^n}\right) - f\left(\frac{-x}{a^n}\right)\right\|\\
&\le \frac{a^{2n}}{2}\left\|F\left(\frac{x}{a^n}\right) - f\left(\frac{x}{a^n}\right)\right\| + \frac{a^{2n}}{2}\left\|F\left(\frac{-x}{a^n}\right) - f\left(\frac{-x}{a^n}\right)\right\|\\
&\le \frac{a^{2n}}{2}\left(\Phi\left(\frac{x}{a^n}\right) + \Phi\left(\frac{-x}{a^n}\right)\right)\\
&\to 0, \quad \text{as } n\to\infty
\end{aligned}
$$

for all $x \in V\backslash\{0\}$. Then $F_e(x) = \lim_{n\to\infty} a^{2n} f_e(\frac{x}{a^n})$ for all $x \in V\backslash\{0\}$ holds.

Finally, we consider the mapping $f_o^{(2)}$. If $\Phi : V\backslash\{0\} \to [0,\infty)$ satisfies the condition (1), (2), or (3), then it follows from (6) that

$$
\begin{aligned}
&\left\| F_o^{(2)}(x) - \frac{f_o^{(2)}(a^n x)}{a^{3n}}\right\|\\
&= \frac{1}{a^{3n}}\left\|F_o^{(2)}(a^n x) - f_o^{(2)}(a^n x)\right\|\\
&= \frac{1}{2(a^3-a)a^{3n}}\big\| -aF(a^n x) + af(a^n x) + aF(-a^n x) - af(-a^n x)\\
&\quad + F(a^{n+1}x) - f(a^{n+1}x) - F(-a^{n+1}x) + f(-a^{n+1}x)\big\|\\
&\le \frac{1}{2(a^3-a)a^{3n}}\big(a\left\|F(a^n x) - f(a^n x)\right\| + a\left\|F(-a^n x) - f(-a^n x)\right\|
\end{aligned}
$$

$$\begin{aligned}
&\quad + \| F(a^{n+1}x) - f(a^{n+1}x) \| + \| F(-a^{n+1}x) - f(-a^{n+1}x) \|) \\
&\le \frac{a\Phi(a^n x) + a\Phi(-a^n x) + \Phi(a^{n+1}x) + \Phi(-a^{n+1}x)}{2(a^3 - a)a^{3n}} \\
&\to 0, \quad \text{as } n \to \infty
\end{aligned}$$

for all $x \in V\backslash\{0\}$; that is, we see that $F_o^{(2)}(x) = \lim_{n\to\infty} \frac{1}{a^{3n}} f_o^{(2)}(a^n x)$ for all $x \in V\backslash\{0\}$.

If $\Phi : V\backslash\{0\} \to [0,\infty)$ satisfies the condition (4), then it follows from (4) and (6) that

$$\begin{aligned}
&\left\| F_o^{(2)}(x) - a^{3n} f_o^{(2)}\left(\frac{x}{a^n}\right) \right\| \\
&= a^{3n} \left\| F_o^{(2)}\left(\frac{x}{a^n}\right) - f_o^{(2)}\left(\frac{x}{a^n}\right) \right\| \\
&= \frac{a^{3n}}{2(a^3 - a)} \left\| -aF\left(\frac{x}{a^n}\right) + af\left(\frac{x}{a^n}\right) + aF\left(\frac{-x}{a^n}\right) - af\left(\frac{-x}{a^n}\right) \right. \\
&\quad \left. + F\left(\frac{x}{a^{n-1}}\right) - f\left(\frac{x}{a^{n-1}}\right) - F\left(\frac{-x}{a^{n-1}}\right) + f\left(\frac{-x}{a^{n-1}}\right) \right\| \\
&\le \frac{a^{3n}}{2(a^3 - a)} \left(a\left\| F\left(\frac{x}{a^n}\right) - f\left(\frac{x}{a^n}\right) \right\| + a\left\| F\left(\frac{-x}{a^n}\right) - f\left(\frac{-x}{a^n}\right) \right\| \right. \\
&\quad \left. + \left\| F\left(\frac{x}{a^{n-1}}\right) - f\left(\frac{x}{a^{n-1}}\right) \right\| + \left\| F\left(\frac{-x}{a^{n-1}}\right) - f\left(\frac{-x}{a^{n-1}}\right) \right\| \right) \\
&\le \frac{a^{3n}}{2(a^3 - a)} \left(a\Phi\left(\frac{x}{a^n}\right) + a\Phi\left(\frac{-x}{a^n}\right) + \Phi\left(\frac{x}{a^{n-1}}\right) + \Phi\left(\frac{-x}{a^{n-1}}\right) \right) \\
&\to 0, \quad \text{as } n \to \infty
\end{aligned}$$

for all $x \in V\backslash\{0\}$; that is, we see that $F_o^{(2)}(x) = \lim_{n\to\infty} a^{3n} f_o^{(2)}(\frac{x}{a^n})$ for all $x \in V\backslash\{0\}$. Since $F(x) = F_o^{(1)}(x) + F_e(x) + F_o^{(2)}(x)$, F is given by the equalities in (7) and F is uniquely determined for any case. □

In general, it is not easy to apply Theorem 2.1 for practical applications. Hence, we introduce a couple of corollaries which are useful for investigating the uniqueness problems in the stability of the cubic-quadratic-additive functional equations.

Corollary 2.2 *Let $a > 1$ be a real constant and let $\phi : V\backslash\{0\} \to [0,\infty)$ be a function satisfying either*

$$\Phi(x) := \sum_{i=0}^{\infty} \frac{\phi(a^i x)}{a^i} < \infty \tag{8}$$

or

$$\Phi(x) := \sum_{i=0}^{\infty} a^{3i} \phi\left(\frac{x}{a^i}\right) < \infty \tag{9}$$

for all $x \in X\backslash\{0\}$. For any given mapping $f : V \to Y$, if there exists a mapping $F : V \to Y$ satisfying the inequality

$$\| f(x) - F(x) \| \le \Phi(x) \tag{10}$$

for all $x \in V\backslash\{0\}$ and the condition (6) *for all $x \in V$, then F is a unique mapping satisfying the conditions* (6) *and* (10).

Proof If ϕ satisfies (8), then we have

$$\lim_{n\to\infty} \frac{\Phi(a^n x)}{a^n} = \lim_{n\to\infty} \sum_{i=0}^{\infty} \frac{\phi(a^{n+i}x)}{a^{n+i}} = \lim_{n\to\infty} \sum_{i=n}^{\infty} \frac{\phi(a^i x)}{a^i} = 0,$$

i.e., Φ satisfies the condition (1) for all $x \in V\backslash\{0\}$.

For the case when ϕ satisfies (9), it holds that

$$\lim_{n\to\infty} a^{3n}\Phi\left(\frac{x}{a^n}\right) = \lim_{n\to\infty} \sum_{i=0}^{\infty} a^{3n+3i}\phi\left(\frac{x}{a^{n+i}}\right) = \lim_{n\to\infty} \sum_{i=n}^{\infty} a^{3i}\phi\left(\frac{x}{a^i}\right) = 0,$$

i.e., Φ satisfies the condition (4) for all $x \in V\backslash\{0\}$. Hence, our assertion is true in view of Theorem 2.1. □

Corollary 2.3 *Let $a > 1$ be a real constant, let $\phi, \psi : V\backslash\{0\} \to [0,\infty)$ be functions satisfying each of the following conditions*:

$$\begin{aligned} &\sum_{i=0}^{\infty} a^i \psi\left(\frac{x}{a^i}\right) < \infty, \qquad \sum_{i=0}^{\infty} \frac{\phi(a^i x)}{a^{2i}} < \infty, \\ &\tilde{\Phi}(x) := \sum_{i=0}^{\infty} a^i \phi\left(\frac{x}{a^i}\right) < \infty, \qquad \tilde{\Psi}(x) := \sum_{i=0}^{\infty} \frac{\psi(a^i x)}{a^{2i}} < \infty \end{aligned} \tag{11}$$

for all $x \in V\backslash\{0\}$, and let $f : V \to Y$ be an arbitrarily given mapping. If there exists a mapping $F : V \to Y$ satisfying the inequality

$$\left\| f(x) - F(x) \right\| \le \tilde{\Phi}(x) + \tilde{\Psi}(x) \tag{12}$$

for all $x \in V\backslash\{0\}$ and the condition (6) *for all $x \in V$, then F is a unique mapping satisfying the conditions* (6) *for all $x \in V$ and the inequality* (12) *for all $x \in V\backslash\{0\}$.*

Proof If we put $\Phi(x) = \tilde{\Phi}(x) + \tilde{\Psi}(x)$, then it follows from (12) that

$$\frac{1}{a^{4n}}\Phi\left(a^{2n}x\right) = \sum_{i=0}^{\infty} \frac{1}{a^{4n-i}}\phi\left(a^{2n-i}x\right) + \sum_{i=0}^{\infty} \frac{1}{a^{4n+2i}}\psi\left(a^{2n+i}x\right)$$

for all $x \in V\backslash\{0\}$. We make a change of the summation indices in the preceding equality with $j = i - 2n$ and $k = 2n + i$ to get

$$\begin{aligned} &\frac{1}{a^{4n}}\Phi\left(a^{2n}x\right) \\ &\quad = \frac{1}{a^{2n}} \sum_{j=-2n}^{\infty} a^j \phi\left(\frac{x}{a^j}\right) + \sum_{k=2n}^{\infty} \frac{1}{a^{2k}}\psi\left(a^k x\right) \\ &\quad = \frac{1}{a^{2n}} \sum_{i=1}^{2n} \frac{1}{a^i}\phi\left(a^i x\right) + \frac{1}{a^{2n}} \sum_{i=0}^{\infty} a^i \phi\left(\frac{x}{a^i}\right) + \sum_{i=2n}^{\infty} \frac{1}{a^{2i}}\psi\left(a^i x\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{a^n}\sum_{i=1}^{n-1}\frac{a^i}{a^n}\frac{1}{a^{2i}}\phi\bigl(a^i x\bigr) + \sum_{i=n}^{2n}\frac{a^i}{a^{2n}}\frac{1}{a^{2i}}\phi\bigl(a^i x\bigr) + \frac{1}{a^{2n}}\tilde{\Phi}(x) + \sum_{i=2n}^{\infty}\frac{1}{a^{2i}}\psi\bigl(a^i x\bigr) \\
&\le \frac{1}{a^n}\sum_{i=1}^{\infty}\frac{1}{a^{2i}}\phi\bigl(a^i x\bigr) + \sum_{i=n}^{\infty}\frac{1}{a^{2i}}\phi\bigl(a^i x\bigr) + \frac{1}{a^{2n}}\tilde{\Phi}(x) + \sum_{i=2n}^{\infty}\frac{1}{a^{2i}}\psi\bigl(a^i x\bigr)
\end{aligned}
$$

for any $x \in V\backslash\{0\}$. Hence, it follows from (11) that

$$
\lim_{n\to\infty}\frac{1}{a^{4n}}\Phi\bigl(a^{2n}x\bigr) = 0
$$

for all $x \in V\backslash\{0\}$. On the other hand, we use the above equality to get

$$
\lim_{n\to\infty}\frac{1}{a^{4n+2}}\Phi\bigl(a^{2n+1}x\bigr) = \frac{1}{a^2}\lim_{n\to\infty}\frac{1}{a^{4n}}\Phi\bigl(a^{2n}ax\bigr) = 0
$$

for all $x \in V\backslash\{0\}$. From the above two equalities, we conclude that

$$
\lim_{n\to\infty}\frac{1}{a^{2n}}\Phi\bigl(a^n x\bigr) = 0
$$

for all $x \in V\backslash\{0\}$.

Similarly, we have

$$
a^{2n}\Phi\left(\frac{x}{a^{2n}}\right) = \sum_{i=0}^{\infty}a^{2n+i}\phi\left(\frac{x}{a^{2n+i}}\right) + \sum_{i=0}^{\infty}\frac{1}{a^{2i-2n}}\psi\bigl(a^{i-2n}x\bigr)
$$

for all $x \in V\backslash\{0\}$. If we make a change of the summation indices in the last equality with $j = i + 2n$ and $k = i - 2n$, then we get

$$
\begin{aligned}
&a^{2n}\Phi\left(\frac{x}{a^{2n}}\right) \\
&\quad= \sum_{j=2n}^{\infty}a^j\phi\left(\frac{x}{a^j}\right) + \frac{1}{a^{2n}}\sum_{k=-2n}^{\infty}\frac{1}{a^{2k}}\psi\bigl(a^k x\bigr) \\
&\quad= \sum_{i=2n}^{\infty}a^i\phi\left(\frac{x}{a^i}\right) + \frac{1}{a^{2n}}\sum_{i=1}^{2n}a^{2i}\psi\left(\frac{x}{a^i}\right) + \frac{1}{a^{2n}}\sum_{i=0}^{\infty}\frac{1}{a^{2i}}\psi\bigl(a^i x\bigr) \\
&\quad= \sum_{i=2n}^{\infty}a^i\phi\left(\frac{x}{a^i}\right) + \frac{1}{a^n}\sum_{i=1}^{n-1}\frac{a^i}{a^n}a^i\psi\left(\frac{x}{a^i}\right) + \sum_{i=n}^{2n}\frac{a^i}{a^{2n}}a^i\psi\left(\frac{x}{a^i}\right) + \frac{1}{a^{2n}}\tilde{\Psi}(x) \\
&\quad\le \sum_{i=2n}^{\infty}a^i\phi\left(\frac{x}{a^i}\right) + \frac{1}{a^n}\sum_{i=1}^{\infty}a^i\psi\left(\frac{x}{a^i}\right) + \sum_{i=n}^{\infty}a^i\psi\left(\frac{x}{a^i}\right) + \frac{1}{a^{2n}}\tilde{\Psi}(x)
\end{aligned}
$$

for any $x \in V\backslash\{0\}$. Thus, it follows from (11) that

$$
\begin{aligned}
&\lim_{n\to\infty}a^{2n}\Phi\left(\frac{x}{a^{2n}}\right) = 0, \\
&\lim_{n\to\infty}a^{2n+1}\Phi\left(\frac{x}{a^{2n+1}}\right) = a\lim_{n\to\infty}a^{2n}\Phi\left(\frac{1}{a^{2n}}\frac{x}{a}\right) = 0
\end{aligned}
$$

for each $x \in V\backslash\{0\}$. Thus, we see that

$$\lim_{n\to\infty} a^n \Phi\left(\frac{x}{a^n}\right) = 0$$

for each $x \in V\backslash\{0\}$.

Altogether, Φ satisfies (2) for all $x \in V\backslash\{0\}$. Hence, Theorem 2.1 implies that our conclusion of this corollary is true. □

Corollary 2.4 *Let $a > 1$ be a real constant, let $\phi, \psi : V\backslash\{0\} \to [0,\infty)$ be functions satisfying each of the following conditions*:

$$\begin{aligned} &\sum_{i=0}^{\infty} a^{2i}\psi\left(\frac{x}{a^i}\right) < \infty, \qquad \sum_{i=0}^{\infty} \frac{\phi(a^i x)}{a^{3i}} < \infty, \\ &\tilde{\Phi}(x) := \sum_{i=0}^{\infty} a^{2i}\phi\left(\frac{x}{a^i}\right) < \infty, \qquad \tilde{\Psi}(x) := \sum_{i=0}^{\infty} \frac{\psi(a^i x)}{a^{3i}} < \infty \end{aligned} \tag{13}$$

for all $x \in V\backslash\{0\}$, and let $f : V \to Y$ be an arbitrarily given mapping. If there exists a mapping $F : V \to Y$ satisfying the inequality

$$\left\|f(x) - F(x)\right\| \le \tilde{\Phi}(x) + \tilde{\Psi}(x) \tag{14}$$

for all $x \in V\backslash\{0\}$ and the condition (6) *for all $x \in V$, then F is a unique mapping satisfying the conditions* (6) *for all $x \in V$ and* (14) *for all $x \in V\backslash\{0\}$.*

Proof If we put $\Phi(x) = \tilde{\Phi}(x) + \tilde{\Psi}(x)$, then it follows from (13) that

$$\frac{1}{a^{6n}}\Phi\left(a^{2n}x\right) = \sum_{i=0}^{\infty} \frac{1}{a^{6n-2i}}\phi\left(a^{2n-i}x\right) + \sum_{i=0}^{\infty} \frac{1}{a^{6n+3i}}\psi\left(a^{2n+i}x\right)$$

for all $x \in V\backslash\{0\}$. We make a change of the summation indices in the preceding equality with $j = i - 2n$ and $k = 2n + i$ to get

$$\begin{aligned} &\frac{1}{a^{6n}}\Phi\left(a^{2n}x\right) \\ &\quad= \frac{1}{a^{2n}}\sum_{j=-2n}^{\infty} a^{2j}\phi\left(\frac{x}{a^j}\right) + \sum_{k=2n}^{\infty} \frac{1}{a^{3k}}\psi\left(a^k x\right) \\ &\quad= \frac{1}{a^{2n}}\sum_{i=1}^{2n} \frac{1}{a^{2i}}\phi\left(a^i x\right) + \frac{1}{a^{2n}}\sum_{i=0}^{\infty} a^{2i}\phi\left(\frac{x}{a^i}\right) + \sum_{i=2n}^{\infty} \frac{1}{a^{3i}}\psi\left(a^i x\right) \\ &\quad= \frac{1}{a^n}\sum_{i=1}^{n-1} \frac{1}{a^{2i+n}}\phi\left(a^i x\right) + \sum_{i=n}^{2n} \frac{1}{a^{2n+2i}}\phi\left(a^i x\right) + \frac{1}{a^{2n}}\tilde{\Phi}(x) + \sum_{i=2n}^{\infty} \frac{1}{a^{3i}}\psi\left(a^i x\right) \\ &\quad\le \frac{1}{a^n}\sum_{i=1}^{\infty} \frac{1}{a^{3i}}\phi\left(a^i x\right) + \sum_{i=n}^{\infty} \frac{1}{a^{3i}}\phi\left(a^i x\right) + \frac{1}{a^{2n}}\tilde{\Phi}(x) + \sum_{i=2n}^{\infty} \frac{1}{a^{3i}}\psi\left(a^i x\right) \end{aligned}$$

for any $x \in V\backslash\{0\}$. Hence, by (13), we get

$$\lim_{n\to\infty} \frac{1}{a^{6n}} \Phi\left(a^{2n}x\right) = 0$$

for all $x \in V\backslash\{0\}$. On the other hand, we use the above equality to get

$$\lim_{n\to\infty} \frac{1}{a^{6n+3}} \Phi\left(a^{2n+1}x\right) = \frac{1}{a^3} \lim_{n\to\infty} \frac{1}{a^{6n}} \Phi\left(a^{2n}ax\right) = 0$$

for all $x \in V\backslash\{0\}$.

From the above two equalities, we conclude that

$$\lim_{n\to\infty} \frac{1}{a^{3n}} \Phi\left(a^{n}x\right) = 0$$

for all $x \in V\backslash\{0\}$.

Similarly, we have

$$a^{4n}\Phi\left(\frac{x}{a^{2n}}\right) = \sum_{i=0}^{\infty} a^{4n+2i}\phi\left(\frac{x}{a^{2n+i}}\right) + \sum_{i=0}^{\infty} \frac{1}{a^{3i-4n}}\psi\left(a^{i-2n}x\right)$$

for all $x \in V\backslash\{0\}$. If we make a change of the summation indices in the last equality with $j = i + 2n$ and $k = i - 2n$, then we get

$$\begin{aligned}
&a^{4n}\Phi\left(\frac{x}{a^{2n}}\right) \\
&\quad= \sum_{j=2n}^{\infty} a^{2j}\phi\left(\frac{x}{a^{j}}\right) + \frac{1}{a^{2n}} \sum_{k=-2n}^{\infty} \frac{1}{a^{3k}}\psi\left(a^{k}x\right) \\
&\quad= \sum_{i=2n}^{\infty} a^{2i}\phi\left(\frac{x}{a^{i}}\right) + \frac{1}{a^{2n}} \sum_{i=1}^{2n} a^{3i}\psi\left(\frac{x}{a^{i}}\right) + \frac{1}{a^{2n}} \sum_{i=0}^{\infty} \frac{1}{a^{3i}}\psi\left(a^{i}x\right) \\
&\quad= \sum_{i=2n}^{\infty} a^{2i}\phi\left(\frac{x}{a^{i}}\right) + \frac{1}{a^{n}} \sum_{i=1}^{n-1} \frac{a^{i}}{a^{n}} a^{2i}\psi\left(\frac{x}{a^{i}}\right) + \sum_{i=n}^{2n} \frac{a^{i}}{a^{2n}} a^{2i}\psi\left(\frac{x}{a^{i}}\right) + \frac{1}{a^{2n}}\tilde{\Psi}(x) \\
&\quad\le \sum_{i=2n}^{\infty} a^{2i}\phi\left(\frac{x}{a^{i}}\right) + \frac{1}{a^{n}} \sum_{i=1}^{\infty} a^{2i}\psi\left(\frac{x}{a^{i}}\right) + \sum_{i=n}^{\infty} a^{2i}\psi\left(\frac{x}{a^{i}}\right) + \frac{1}{a^{2n}}\tilde{\Psi}(x)
\end{aligned}$$

for any $x \in V\backslash\{0\}$. Thus, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} a^{4n}\Phi\left(\frac{x}{a^{2n}}\right) = 0, \\
&\lim_{n\to\infty} a^{4n+2}\Phi\left(\frac{x}{a^{2n+1}}\right) = a^2 \lim_{n\to\infty} a^{4n}\Phi\left(\frac{1}{a^{2n}}\frac{x}{a}\right) = 0
\end{aligned}$$

for each $x \in V\backslash\{0\}$. Thus, we see that

$$\lim_{n\to\infty} a^{2n}\Phi\left(\frac{x}{a^{n}}\right) = 0$$

for each $x \in V\backslash\{0\}$.

Altogether, Φ satisfies (3) for all $x \in V\backslash\{0\}$. Hence, Theorem 2.1 implies that our conclusion of this corollary is true. □

3 Applications

In this section, we apply the theorem and corollaries of the last section to show that if for any given mapping f, there exists an additive, a quadratic, a cubic, a quadratic-additive, a cubic-additive, a cubic-quadratic, or a cubic-quadratic-additive mapping F near f, then the mapping F is uniquely determined.

The proofs of the first three corollaries immediately follow from Corollaries 2.2, 2.3, and 2.4, respectively, because each cubic-quadratic-additive mapping satisfies the conditions in (6) provided a is a rational number.

Corollary 3.1 *Let $a > 1$ be a rational number and let $\phi : V\backslash\{0\} \to [0,\infty)$ be a function satisfying the condition* (8) *or* (9) *for all $x \in V\backslash\{0\}$. Let $f : V \to Y$ be a given mapping. If there exists a cubic-quadratic-additive mapping $F : V \to Y$ satisfying the inequality* (10), *then F is uniquely determined.*

Corollary 3.2 *Let $a > 1$ be a rational number and let $\phi, \psi : V\backslash\{0\} \to [0,\infty)$ be functions satisfying the conditions in* (11) *for all $x \in V\backslash\{0\}$. Let $f : V \to Y$ be a given mapping. If there exists a cubic-quadratic-additive mapping $F : V \to Y$ satisfying the inequality* (12), *then F is uniquely determined.*

Corollary 3.3 *Let $a > 1$ be a rational number and let $\phi, \psi : V\backslash\{0\} \to [0,\infty)$ be functions satisfying the conditions in* (13) *for all $x \in V\backslash\{0\}$. Let $f : V \to Y$ be a given mapping. If there exists a cubic-quadratic-additive mapping $F : X \to Y$ satisfying the inequality* (14), *then F is uniquely determined.*

If $p < 1$ then $\Phi(x) := K\|x\|^p$ satisfies (1); if $1 < p < 2$ then $\Phi(x)$ satisfies (2); if $2 < p < 3$ then $\Phi(x)$ satisfies (3); and if $p > 3$ then $\Phi(x)$ satisfies (4). Hence, by Theorem 2.1, we get the following corollaries concerning the Hyers-Ulam-Rassias stability. For the detailed concept of the Hyers-Ulam-Rassias stability, we refer to [1, 2, 4, 6, 17].

When we prove the Hyers-Ulam-Rassias stability, Y is usually assumed to be a Banach space. In this paper, however, we only need to assume that Y is a real normed space provided the validity of inequality (5), (10), (12), (14), or (15) is already guaranteed.

Corollary 3.4 *Let $p \notin \{1,2,3\}$ and $\theta > 0$ be real constants, let X, Y be real normed spaces, and let $f : X \to Y$ be an arbitrarily given mapping. If there exists a mapping $F : X \to Y$ satisfying the inequality*

$$\|f(x) - F(x)\| \le \theta \|x\|^p \tag{15}$$

for all $x \in X\backslash\{0\}$ and the conditions in (6) *for all $x \in X$, then F is a unique mapping satisfying the conditions in* (6) *for all $x \in X$ and the inequality* (15) *for all $x \in X\backslash\{0\}$.*

Since each of the cubic, additive, and cubic-additive mappings satisfies the conditions in (6), using Corollary 3.2, we can easily prove the following corollary.

Corollary 3.5 *Let $p \notin \{1, 2, 3\}$ and $\theta > 0$ be real constants, let X, Y be real normed spaces, and let $f : X \to Y$ be an arbitrarily given mapping. If there exists an additive, a quadratic, a cubic, a quadratic-additive, a cubic-additive, a cubic-quadratic, or a cubic-quadratic-additive mapping $F : X \to Y$ satisfying the inequality* (15) *for all $x \in X\backslash\{0\}$, then F is uniquely determined.*

If we set $\phi(x) = \varepsilon$ in Corollary 3.1, then ϕ satisfies the condition (8). Hence, Corollary 3.1 implies the following result.

Corollary 3.6 *Let V be a real vector space, let Y be a real normed space, and let $f : V \to Y$ be an arbitrarily given mapping. If there exists an additive, a quadratic, a cubic, a quadratic-additive, a cubic-additive, a cubic-quadratic, or a cubic-quadratic-additive mapping $F : X \to Y$ satisfying the inequality*

$$\left\| f(x) - F(x) \right\| \leq \varepsilon$$

for all $x \in V\backslash\{0\}$ and for some $\varepsilon > 0$, then F is uniquely determined.

Remark 3.7 In 2005, Baker [3] proved the Hyers-Ulam stability of a large class of functional equations of the form

$$\sum_{k=0}^{m} f_k(\alpha_k x + \beta_k y) = 0, \tag{16}$$

which includes the additive, the quadratic, the cubic, the quadratic-additive, the cubic-additive, the cubic-quadratic, and the cubic-quadratic-additive type functional equations; in fact, he proved the Hyers-Ulam stability of equation (16) without addressing the uniqueness of the relevant solution of that equation, while the main aim of this paper is to prove a general uniqueness theorem for those equations. From this viewpoint, we can say that this paper complements the results of Baker.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics Education, Gongju National University of Education, Gongju, 32553, Republic of Korea.
[2]Mathematics Section, College of Science and Technology, Hongik University, Sejong, 30016, Republic of Korea.

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