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Existence and nonexistence of positive solutions for fractional integral boundary value problem with two disturbance parameters

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Abstract

In this paper, we consider a class of fractional differential equations with integral boundary conditions which involve two disturbance parameters. By using the Guo-Krasnoselskii fixed point theorem, new results on the existence and nonexistence of positive solutions for the boundary value problem are obtained. And the impact of the disturbance parameters on the existence of positive solutions is also investigated. Finally, we give some examples to illustrate our main results.

MSC: 34B15; 26A33

Keywords: fractional differential equation; Riemann-Liouville fractional derivative; integral boundary problem; positive solution; existence and nonexistence; disturbance parameter

1 Introduction

The theory of boundary value problems for ordinary differential equations and functional differential equations plays an important role in many research fields of science and engineering; for details, see [1–9] and the references therein. Meanwhile, fractional differential equations have also widely appeared in various fields such as physics, mechanics, electricity, biology, control theory, *etc.* Therefore, the study of fractional differential equations has gained prominence and has been growing rapidly, see[10–18]. Last but not least, as an important part of fractional differential equations, the integral boundary value problems have also been extensively researched, see [19–23].

In [21], Jia and Liu investigated the existence and nonexistence of positive solutions for the following integral boundary value problem of fractional differential equations with a disturbance parameter in the boundary conditions and the impact of the disturbance parameter on the existence of positive solutions

$$\begin{cases} -{}^{C}D^{\delta}u(t) = f(t,u(t)), \quad t \in (0,1), \\ m_1 u(0) - n_1 u'(0) = 0, \\ m_2 u(1) + n_2 u'(1) = \int_0^1 g(s)u(s)\,ds + a, \end{cases}$$

where ${}^{C}D^{\delta}$ is the Caputo fractional derivative, $1 < \delta \leq 2$, $f \in C([0,1] \times \mathbb{R}^{+}, \mathbb{R}^{+})$ and $g \in C([0,1], \mathbb{R}^{+})$.

In this paper, we are concerned with the Riemann-Liouville fractional differential equation

$$D_{0+}^{\alpha} x(t) = f\big(t, x(t)\big), \quad t \in (0,1), \tag{1.1}$$

with the integral boundary conditions

$$\begin{cases} x(0) = x'(0) = 0, \\ x(1) = \int_0^1 g_1(s)x(s)\,\mathrm{d}s + a, \\ x'(1) = \int_0^1 g_2(s)x(s)\,\mathrm{d}s - b, \end{cases} \tag{1.2}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order α, $3 < \alpha \leq 4$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is a continuous function, $g_1, g_2 \in L^1[0,1]$, and $a, b \geq 0$. The existence and nonexistence of positive solutions for the integral boundary value problem (1.1)-(1.2) and the impact of the disturbance parameters a, b on the existence of positive solutions is also investigated. Finally, we give two examples to illustrate our results.

2 Preliminaries

In this section, we present some useful definitions and related lemmas.

Definition 2.1 (See [12]) Let $\alpha > 0$. The fractional integral operator of a function $y : (0,+\infty) \to \mathbb{R}$ is given by

$$I^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,\mathrm{d}s$$

provided the integral exists.

Definition 2.2 (See [12]) Let $\alpha > 0$. The Riemann-Liouville fractional derivative of a function $y : (0,+\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} y(t) = D^{z} I_{0+}^{z-\alpha} y(t) = \frac{1}{\Gamma(z-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{z} \int_0^t \frac{y(s)}{(t-s)^{\alpha-z+1}}\,\mathrm{d}s,$$

where $z \in \mathbb{N}$, $z - 1 < \alpha < z$, provided the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 2.1 (See [12]) *For $\alpha > 0$, $z \in \mathbb{N}$ and $z-1 < \alpha < z$, if $x \in L^1[0,1]$ and $I_{0+}^{z-\alpha} x \in AC^{z}[0,1]$, we have the equation*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + \cdots + c_z t^{\alpha-z},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, 3, \ldots, z$.

Lemma 2.2 *The boundary value problem* (1.1)-(1.2) *is equivalent to the following integral equation*:

$$\begin{aligned} x(t) = {} & \int_0^1 G(t,s) f\big(s, x(s)\big)\,\mathrm{d}s + \int_0^1 H(t,s)x(s)\,\mathrm{d}s \\ & + \big((\alpha-1)a + b\big)t^{\alpha-2} - \big((\alpha-2)a + b\big)t^{\alpha-1}, \end{aligned} \tag{2.1}$$

where

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}(t-s)^{\alpha-1}+(1-s)^{\alpha-2}t^{\alpha-2}\\ \quad\times((s-t)+(\alpha-2)(1-t)s), & 0\le s<t\le 1,\\ (1-s)^{\alpha-2}t^{\alpha-2}((s-t)+(\alpha-2)(1-t)s), & 0\le t<s\le 1\end{cases}\tag{2.2}$$

and

$$H(t,s)=t^{\alpha-2}(2t-\alpha t+\alpha-1)g_1(s)+t^{\alpha-2}(t-1)g_2(s).\tag{2.3}$$

Proof Assume that $x=x(t)$ is a solution of (1.1), it follows from Lemma 2.1 that

$$x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\big(s,x(s)\big)\,\mathrm{d}s+c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3}+c_4t^{\alpha-4},$$

where $c_i\in\mathbb{R}$, $i=1,2,3,4$.

From the boundary conditions $x(0)=x'(0)=0$, we get $c_3=c_4=0$.

And from the boundary conditions $x(1)=\int_0^1 g_1(s)x(s)\,\mathrm{d}s+a$ and $x'(1)=\int_0^1 g_2(s)x(s)\,\mathrm{d}s-b$, we can get

$$\begin{aligned}c_1&=(2-\alpha)\int_0^1 g_1(s)x(s)\,\mathrm{d}s+\int_0^1 g_2(s)x(s)\,\mathrm{d}s\\&\quad-\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-2}(\alpha s-2s+1)f\big(s,x(s)\big)\,\mathrm{d}s-b-(\alpha-2)a,\\c_2&=(\alpha-1)\int_0^1 g_1(s)x(s)\,\mathrm{d}s-\int_0^1 g_2(s)x(s)\,\mathrm{d}s\\&\quad+\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}sf\big(s,x(s)\big)\,\mathrm{d}s+b+(\alpha-1)a.\end{aligned}$$

Then

$$\begin{aligned}x(t)&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\big(s,x(s)\big)\,\mathrm{d}s+t^{\alpha-1}(2-\alpha)\int_0^1 g_1(s)x(s)\,\mathrm{d}s+t^{\alpha-1}\int_0^1 g_2(s)x(s)\,\mathrm{d}s\\&\quad-\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-2}(\alpha s-2s+1)f\big(s,x(s)\big)\,\mathrm{d}s\\&\quad+t^{\alpha-2}(\alpha-1)\int_0^1 g_1(s)x(s)\,\mathrm{d}s-t^{\alpha-2}\int_0^1 g_2(s)x(s)\,\mathrm{d}s\\&\quad+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}sf\big(s,x(s)\big)\,\mathrm{d}s+\big((\alpha-1)a+b\big)t^{\alpha-2}-\big((\alpha-2)a+b\big)t^{\alpha-1}\\&=\frac{1}{\Gamma(\alpha)}\int_0^t\big((t-s)^{\alpha-1}+(1-s)^{\alpha-2}t^{\alpha-2}\big((s-t)+(\alpha-2)(1-t)s\big)\big)f\big(s,x(s)\big)\,\mathrm{d}s\\&\quad+\frac{1}{\Gamma(\alpha)}\int_t^1(1-s)^{\alpha-2}t^{\alpha-2}\big((s-t)+(\alpha-2)(1-t)s\big)f\big(s,x(s)\big)\,\mathrm{d}s\\&\quad+t^{\alpha-2}(2t-\alpha t+\alpha-1)\int_0^1 g_1(s)x(s)\,\mathrm{d}s+t^{\alpha-2}(t-1)\int_0^1 g_2(s)x(s)\,\mathrm{d}s\\&\quad+\big((\alpha-1)a+b\big)t^{\alpha-2}-\big((\alpha-2)a+b\big)t^{\alpha-1}\end{aligned}$$

$$= \int_0^1 G(t,s)f\big(s,x(s)\big)\,\mathrm{d}s + \int_0^1 H(t,s)x(s)\,\mathrm{d}s$$
$$+ \big((\alpha-1)a+b\big)t^{\alpha-2} - \big((\alpha-2)a+b\big)t^{\alpha-1}.$$

Hence, $x = x(t)$ is a solution of the integral equation (2.1) if it is the solution of the boundary value problem (1.1)-(1.2), and *vice versa*.

The proof is completed. □

Lemma 2.3 *Let G be defined by* (2.2), *then*

(1) $G(t,s) \in C[0,1] \times [0,1]$, *and* $G(t,s) > 0$ *for any* $t,s \in (0,1)$,

(2) $(\alpha-2)q(t)k(s)\frac{1}{\Gamma(\alpha)} \leq G(t,s) \leq M_0 k(s)\frac{1}{\Gamma(\alpha)} \leq M_0\frac{1}{\Gamma(\alpha)}$ *for any* $t,s \in [0,1]$, *where*

$$M_0 = \max\{\alpha-1, (\alpha-2)^2\}, \qquad q(t) = t^{\alpha-2}(1-t)^2, \qquad k(s) = s^2(1-s)^{\alpha-2}.$$

Proof (1) It is easy to show that the result holds.

(2) For $s \leq t$, we could use the mean value theorem of differential calculus and get

$$\begin{aligned} G(t,s) &= \frac{1}{\Gamma(\alpha)}\big((t-s)\big((t-s)^{\alpha-2} - (t-ts)^{\alpha-2}\big) + (\alpha-2)(1-t)s(t-ts)^{\alpha-2}\big) \\ &\geq \frac{1}{\Gamma(\alpha)}\big(-(t-s)(\alpha-2)(t-ts)^{\alpha-3}s(1-t) + (\alpha-2)(1-t)s(t-ts)^{\alpha-2}\big) \\ &\geq \frac{1}{\Gamma(\alpha)}(\alpha-2)t^{\alpha-2}(1-s)^{\alpha-2}s^2(1-t)^2 \end{aligned}$$

and

$$\begin{aligned} G(t,s) &\leq \frac{1}{\Gamma(\alpha)}\big(-(t-s)(\alpha-2)(t-s)^{\alpha-3}s(1-t) + (\alpha-2)(1-t)s(t-ts)^{\alpha-2}\big) \\ &= \frac{1}{\Gamma(\alpha)}(\alpha-2)s(1-t)\big((t-ts)^{\alpha-2} - (t-s)^{\alpha-2}\big) \\ &\leq \frac{1}{\Gamma(\alpha)}(\alpha-2)^2 s^2(1-t)^2 t^{\alpha-3}(1-s)^{\alpha-3} \\ &\leq \frac{1}{\Gamma(\alpha)}(\alpha-2)^2 s^2(1-s)^{\alpha-2} \\ &\leq \frac{M_0}{\Gamma(\alpha)}. \end{aligned}$$

For $s \geq t$, we have that

$$\begin{aligned} G(t,s) &\geq \frac{1}{\Gamma(\alpha)}(\alpha-2)t^{\alpha-2}(1-s)^{\alpha-2}s(1-t) \geq \frac{1}{\Gamma(\alpha)}(\alpha-2)t^{\alpha-2}(1-s)^{\alpha-2}s^2(1-t)^2, \\ G(t,s) &\leq \frac{1}{\Gamma(\alpha)}s^{\alpha-2}(1-s)^{\alpha-2}\big(s+(\alpha-2)s\big) = \frac{1}{\Gamma(\alpha)}s^{\alpha-1}(1-s)^{\alpha-2}(\alpha-1) \leq \frac{M_0}{\Gamma(\alpha)}. \end{aligned}$$

The proof is completed. □

Now we make the following assumption.

(B0) $g_1, g_2 \in L^1[0,1]$ such that $0 \leq \inf_{t,s\in[0,1]} H(t,s) < \sup_{t,s\in[0,1]} H(t,s) := M < 1$.

Let the Banach space $C[0,1]$ be endowed with the norm $\|x\| := \max_{t\in[0,1]} |x(t)|$, and let

$$P = \{x \in C[0,1] : x(t) \geq 0, t \in [0,1]\},$$

then P is a cone in $C[0,1]$.

We define an operator $A : P \to C[0,1]$ by

$$(Ax)(t) = \int_0^1 H(t,s)x(s)\,ds.$$

Lemma 2.4 *Assume* (B0) *holds, then the operator A satisfies the following properties*:

(1) *A is a bounded linear operator*;
(2) $A(P) \subset P$;
(3) *the operator A is reversible*;
(4) $\|(I-A)^{-1}\| \leq \frac{1}{1-M}$.

Proof By (B0), it is obvious that (1), (2) hold.

(3) Since $M < 1$, we get that $\|Ax\| \leq M\|x\| < \|x\|$, then $\|A\| \leq M < 1$, so that $I - A$ is reversible.

(4) Let $y(t) = x(t) - Ax(t)$, that is, $x(t) = y(t) + Ax(t)$, $x(t) = (I-A)^{-1}y(t)$, and $y \in C[0,1]$ for $t \in [0,1]$. From the definition of operator A, we have that

$$x(t) = y(t) + \int_0^1 H(t,s)x(s)\,ds.$$

Let

$$x_0(t) = x(t), \qquad x_m(t) = y(t) + \int_0^1 H(t,s)x_{m-1}(s)\,ds, \quad m = 1,2,\ldots.$$

We apply the method of iteration to solve the above equation.

According to this method, we can get that

$$x(t) = (I-A)^{-1}y(t) = y(t) + \int_0^1 R(t,s)y(s)\,ds,$$

where

$$R(t,s) = \sum_{j=1}^{\infty} H_j(t,s), \qquad H_1(t,s) = H(t,s) \quad \text{and}$$

$$H_j(t,s) = \int_0^1 H(t,\tau)H_{j-1}(\tau,s)\,d\tau, \quad j = 2,3,\ldots.$$

Because of $0 \leq H(t,s) < M < 1$, we have that

$$0 \leq R(t,s) = \sum_{j=1}^{\infty} H_j(t,s) < M + M^2 + \cdots + M^n + \cdots = \frac{M}{1-M}. \tag{2.4}$$

Since $(I-A)^{-1}y(t) = x(t)$, we get

$$\left|(I-A)^{-1}y(t)\right| \leq \left|y(t)\right| + \frac{M}{1-M}\left|\int_0^1 y(s)\,ds\right| \leq \|y\| + \frac{M}{1-M}\|y\| = \frac{\|y\|}{1-M}$$

and

$$\left\|(I-A)^{-1}y\right\| = \max_{t\in[0,1]}\left|(I-A)^{-1}y(t)\right| \le \frac{\|y\|}{1-M}.$$

So

$$\left\|(I-A)^{-1}\right\| \le \frac{1}{1-M}.$$

The proof is completed. □

We define another operator $T: P \to C[0,1]$,

$$(Tx)(t) = \int_0^1 G(t,s)f\big(s,x(s)\big)\,\mathrm{d}s + \big((\alpha-1)a+b\big)t^{\alpha-2} - \big((\alpha-2)a+b\big)t^{\alpha-1}. \tag{2.5}$$

It can be easily shown that $T: P \to P$. $x(t)$ is a solution of the boundary value problem (1.1)-(1.2) if and only if it satisfies

$$x(t) = (Tx)(t) + (Ax)(t).$$

Hence,

$$x(t) = (I-A)^{-1}(Tx)(t) = (Tx)(t) + \int_0^1 R(t,s)(Tx)(s)\,\mathrm{d}s$$

and

$$\begin{aligned}(I-A)^{-1}(Tx)(t) = {}& \int_0^1 G(t,s)f\big(s,x(s)\big)\,\mathrm{d}s + \int_0^1 R(t,s)\int_0^1 G(s,\tau)f\big(\tau,x(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}s \\ & + \int_0^1 R(t,s)\Big(\big((\alpha-1)a+b\big)s^{\alpha-2} - \big((\alpha-2)a+b\big)s^{\alpha-1}\Big)\,\mathrm{d}s \\ & + \big((\alpha-1)a+b\big)t^{\alpha-2} - \big((\alpha-2)a+b\big)t^{\alpha-1}. \end{aligned} \tag{2.6}$$

Let

$$P_0 = \left\{x \in P : x(t) \ge \frac{q(t)(1-M)}{M_0}\|x\|, t \in [0,1]\right\}.$$

Lemma 2.5 *Assume that condition* (B0) *holds, then the operator* $(I-A)^{-1}T: P \to P_0$ *is completely continuous.*

Proof From the continuity and the non-negativeness of functions G, R and f, we have that if $x \in P$, then $(I-A)^{-1}(Tx)(t) \ge 0$ and $(I-A)^{-1}(Tx) \in P$.

It follows from (2.4) and Lemma 2.3, for $x \in P$ and $t \in [0,1]$,

$$\begin{aligned}&\left|(I-A)^{-1}(Tx)(t)\right| \\ &\quad\le \left\|(I-A)^{-1}\right\| \cdot \left|Tx(t)\right|\end{aligned}$$

$$\leq \frac{1}{1-M}\left|\int_0^1 \frac{M_0}{\Gamma(\alpha)}k(s)f\bigl(s,x(s)\bigr)\,\mathrm{d}s + \bigl((\alpha-1)a+b\bigr)t^{\alpha-2} - \bigl((\alpha-2)a+b\bigr)t^{\alpha-1}\right|$$
$$\leq \frac{1}{1-M}\left(\int_0^1 \frac{M_0}{\Gamma(\alpha)}k(s)f\bigl(s,x(s)\bigr)\,\mathrm{d}s + t^{\alpha-2}\bigl((\alpha-2)a(1-t)+b(1-t)+a\bigr)\right).$$

Hence,

$$\bigl\|(I-A)^{-1}(Tx)\bigr\| \leq \frac{M_0}{1-M}\left(\frac{1}{\Gamma(\alpha)}\int_0^1 k(s)f\bigl(s,x(s)\bigr)\,\mathrm{d}s + (\alpha-1)a+b\right). \tag{2.7}$$

By (2.4), (2.5) and (2.6), we have

$$\begin{aligned}(I-A)^{-1}(Tx)(t) &\geq (Tx)(t)\\ &\geq \frac{(\alpha-2)q(t)}{\Gamma(\alpha)}\int_0^1 k(s)f\bigl(s,x(s)\bigr)\,\mathrm{d}s\\ &\quad + \bigl((\alpha-1)a+b\bigr)t^{\alpha-2} - \bigl((\alpha-1)a+b\bigr)t^{\alpha-1}\\ &= \frac{(\alpha-2)q(t)}{\Gamma(\alpha)}\int_0^1 k(s)f\bigl(s,x(s)\bigr)\,\mathrm{d}s + \bigl((\alpha-1)a+b\bigr)t^{\alpha-2}(1-t)\\ &\geq \frac{q(t)}{\Gamma(\alpha)}\int_0^1 k(s)f\bigl(s,x(s)\bigr)\,\mathrm{d}s + \bigl((\alpha-1)a+b\bigr)t^{\alpha-2}(1-t)^2\\ &= q(t)\left(\frac{1}{\Gamma(\alpha)}\int_0^1 k(s)f\bigl(s,x(s)\bigr)\,\mathrm{d}s + (\alpha-1)a+b\right).\end{aligned}$$

It follows from (2.7) that

$$(I-A)^{-1}(Tx)(t) \geq \frac{q(t)(1-M)}{M_0}\bigl\|(I-A)^{-1}Tx\bigr\|.$$

Hence $(I-A)^{-1}T(P) \subset P_0$.

Let $\{x_n\} \subset P$, $x \in P$, and $\|x_n - x\| \to 0$ as $n \to +\infty$, there exists a constant $r > 0$ such that $\|x_n\| \leq r$ and $\|x\| \leq r$. We have

$$\lim_{n\to\infty} f\bigl(s,x_n(s)\bigr) = f\bigl(s,x(s)\bigr) \quad \text{for a.e. } s \in [0,1].$$

By the Lebesgue dominated convergence theorem, we get

$$\lim_{n\to\infty}(I-A)^{-1}(Tx_n)(t) = (I-A)^{-1}(Tx)(t).$$

So

$$\lim_{n\to\infty}\bigl\|(I-A)^{-1}Tx_n - (I-A)^{-1}Tx\bigr\| = 0.$$

Then the operator $(I-A)^{-1}T$ is continuous.

Let $\Omega \subset P$ be bounded. Then there exists a positive constant $l > 0$ such that $\|x\| \leq l$ for all $x \in \Omega$. Let $N = \max_{0\leq t\leq 1, 0\leq x\leq l}|f(t,x)| + 1$. By (2.7), for all $x \in \Omega$, we have

$$\bigl\|(I-A)^{-1}(Tx)(t)\bigr\| \leq \frac{M_0}{(1-M)}\left(\frac{N}{\Gamma(\alpha)} + (\alpha-1)a+b\right),$$

which means $(I-A)^{-1}T(\Omega)$ is bounded in P.

In addition, for any given $x \in \Omega$, because $G(t,s)$ is continuous for $(t,s) \in [0,1] \times [0,1]$, then it must be uniformly continuous. So, for any $\varepsilon > 0$, there exists a constant $\delta_0 > 0$ such that for all $s \in [0,1]$, as $|t_1 - t_2| < \delta_0$, we have that

$$\begin{aligned}
&\bigl|G(t_1,s) - G(t_2,s)\bigr| < \varepsilon,\\
&\bigl|t_1^{\alpha-2} - t_2^{\alpha-2}\bigr| < \varepsilon,\\
&\bigl|t_1^{\alpha-1} - t_2^{\alpha-1}\bigr| < \varepsilon.
\end{aligned}$$

For each $x \in \Omega$,

$$\begin{aligned}
&\bigl|(I-A)^{-1}(Tx)(t_1) - (I-A)^{-1}(Tx)(t_2)\bigr|\\
&\quad\le \bigl\|(I-A)^{-1}\bigr\| \cdot \bigl|(Tx)(t_1) - (Tx)(t_2)\bigr|\\
&\quad\le \frac{1}{1-M}\int_0^1 \bigl|G(t_1,s) - G(t_2,s)\bigr|\bigl|f\bigl(s,x(s)\bigr)\bigr|\,\mathrm{d}s\\
&\qquad + \frac{(\alpha-1)a+b}{1-M}\bigl|t_1^{\alpha-2} - t_2^{\alpha-2}\bigr| + \frac{(\alpha-2)a+b}{1-M}\bigl|t_1^{\alpha-1} - t_2^{\alpha-1}\bigr|\\
&\quad< \frac{N + (2\alpha-3)a + 2b}{1-M}\varepsilon.
\end{aligned}$$

We have

$$\bigl\|(I-A)^{-1}(Tx)(t_1) - (I-A)^{-1}(Tx)(t_2)\bigr\| < \frac{N + (2\alpha-3)a + 2b}{1-M}\varepsilon,$$

and $(I-A)^{-1}T(\Omega)$ is equicontinuous in P_0.

Now, according to the Arzela-Ascoli theorem, we conclude that $(I-A)^{-1}T(\Omega)$ is relatively compact.

Therefore, $(I-A)^{-1}T : P \to P_0$ is a completely continuous operator.

The proof is completed. □

By Lemma 2.2, we can easily deduce that the following lemma holds.

Lemma 2.6 *Assume $x \in C[0,1]$, $D_{0^+}^{\alpha}x \in L^1[0,1]$. Then the boundary value problem* (1.1)-(1.2) *has a positive solution if and only if the operator $(I-A)^{-1}T$ has a fixed point in P. Furthermore, if x is a positive solution of the fractional boundary value problem* (1.1)-(1.2), *then $x \in P_0$.*

To prove the existence of positive solution for the boundary value problem (1.1)-(1.2), we state the following Guo-Krasnoselskii fixed point theorem, see [24].

Lemma 2.7 *Let E be a Banach space and $P \subset E$ be a cone. Assume that Ω_1, Ω_2 are bounded open subsets of E with $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that either*

(1) $\|T(x)\| \le \|x\|$, $\forall x \in P \cap \partial\Omega_1$; *and* $\|T(x)\| \ge \|x\|$, $\forall x \in P \cap \partial\Omega_2$, *or*

(2) $\|T(x)\| \ge \|x\|$, $\forall x \in P \cap \partial\Omega_1$; *and* $\|T(x)\| \le \|x\|$, $\forall x \in P \cap \partial\Omega_2$.

Then the operator T has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Existence and nonexistence of positive solutions

Denote

$$f_0=\liminf_{x\to0^+}\inf_{t\in[\frac14,\frac34]}\frac{f(t,x)}{x},\qquad f^0=\limsup_{x\to0^+}\sup_{t\in[0,1]}\frac{f(t,x)}{x},$$

$$f_\infty=\liminf_{x\to\infty}\inf_{t\in[\frac14,\frac34]}\frac{f(t,x)}{x},\qquad f^\infty=\limsup_{x\to\infty}\sup_{t\in[0,1]}\frac{f(t,x)}{x},$$

$$\sigma=\min_{t\in[\frac14,\frac34]}\frac{q(t)(1-M)}{M_0}=\frac{(1-M)3^{\alpha-2}}{M_04^\alpha},\qquad \rho_1=\frac{(1-M)\Gamma(\alpha)}{M_0+\alpha\Gamma(\alpha)},\qquad \rho_2=\frac{7{,}680\Gamma(\alpha)2^\alpha}{203\sigma(\alpha-2)}.$$

Theorem 3.1 *Suppose* (B0) *holds,* $f^0<\rho_1$ *and* $f_\infty>\rho_2$. *Then there exist small enough* a_0 *and* b_0 *such that the boundary value problem* (1.1)-(1.2) *has at least one positive solution for* $0\le a\le a_0$ *and* $0\le b\le b_0$.

Proof Since $f^0<\rho_1$, there exists a constant $r_1>0$ such that

$$f(t,x)<\rho_1x\le\rho_1r_1$$

for all $t\in[0,1]$ and $x\in[0,r_1]$.

Let $\Omega_1=\{x\in P_0:\|x\|<r_1\}$, $0\le a\le a_0$, $0\le b\le b_0$ and $\max\{a_0,b_0\}\le\rho_1r_1$.

By Lemma 2.3, for $x\in\partial\Omega_1$, we have $\|x\|=r_1$ and

$$\begin{aligned}0&\le(I-A)^{-1}(Tx)(t)\le\|(I-A)^{-1}\|\cdot|(Tx)(t)|\le\frac{1}{1-M}|(Tx)(t)|\\&\le\frac{1}{1-M}\left(\frac{M_0}{\Gamma(\alpha)}\int_0^1f(s,x(s))\,ds+(\alpha-1)a+b\right)\\&\le\frac{1}{1-M}\left(\frac{M_0}{\Gamma(\alpha)}\int_0^1f(s,x(s))\,ds+(\alpha-1)a_0+b_0\right)\\&\le\frac{M_0}{(1-M)\Gamma(\alpha)}\rho_1\|x\|+\frac{1}{1-M}\big((\alpha-1)\rho_1\|x\|+\rho_1\|x\|\big)\\&=\frac{M_0+\alpha\Gamma(\alpha)}{(1-M)\Gamma(\alpha)}\rho_1\|x\|=\|x\|.\end{aligned}$$

So we get $\|(I-A)^{-1}Tx\|\le\|x\|$, $x\in\partial\Omega_1$.

Since $f_\infty>\rho_2$, there exists a constant $R>0$ such that

$$f(t,x)>\rho_2x$$

for all $t\in[\frac14,\frac34]$ and $x\in[R,+\infty)$.

Let $r_2>\max\{r_1,\frac{R}{\sigma}\}$ and $\Omega_2=\{x\in P_0:\|x\|<r_2\}$.

For all $x\in\partial\Omega_2$, we have that $\|x\|=r_2$ and $x(t)\ge\frac{q(t)(1-M)}{M_0}\|x\|\ge\sigma\|x\|=\sigma r_2>R$ for $t\in[\frac14,\frac34]$. Then

$$\begin{aligned}(I-A)^{-1}(Tx)\left(\frac12\right)\ge(Tx)\left(\frac12\right)&\ge\int_0^1G\left(\frac12,s\right)f(s,x(s))\,ds\\&\ge\frac{(\alpha-2)q(\frac12)}{\Gamma(\alpha)}\int_0^1k(s)f(s,x(s))\,ds\end{aligned}$$

$$\begin{aligned} &\geq \frac{(\alpha-2)q(\frac{1}{2})}{\Gamma(\alpha)} \int_{\frac{1}{4}}^{\frac{3}{4}} k(s) f\bigl(s,x(s)\bigr)\,\mathrm{d}s \\ &\geq \frac{(\alpha-2)\sigma\rho_2}{2^\alpha\Gamma(\alpha)} \int_{\frac{1}{4}}^{\frac{3}{4}} k(s)\,\mathrm{d}s\|x\| \\ &\geq \frac{(\alpha-2)\sigma\rho_2}{2^\alpha\Gamma(\alpha)} \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^2 s^2\,\mathrm{d}s\|x\| \\ &= \frac{203\sigma(\alpha-2)}{7{,}680\Gamma(\alpha)2^\alpha}\rho_2\|x\|. \end{aligned}$$

So $\|(I-A)^{-1}Tx\| \geq \|x\|$, $x \in \partial\Omega_2$.

By Lemma 2.7, we conclude that the operator $(I-A)^{-1}T$ has at least one fixed point in $P_0 \cap (\overline{\Omega}_2 \setminus \Omega_1)$, which implies that the boundary value problem (1.1)-(1.2) has a positive solution.

The proof is completed. □

Theorem 3.2 *Assume* (B0) *holds,* $f^\infty < \rho_1$ *and* $f_0 > \rho_2$. *Then there exist small enough* a_0 *and* b_0 *such that the boundary value problem* (1.1)-(1.2) *has at least one positive solution for* $0 \leq a \leq a_0$ *and* $0 \leq b \leq b_0$.

Proof Since $f^\infty < \rho_1$, for $\varepsilon = \frac{\rho_1 - f^\infty}{2} > 0$, there exists a constant $R_1 > 0$ such that $f(t,x) < (\rho_1 - \varepsilon)x$ for $t \in [0,1]$ and $x \in [R_1,+\infty)$.

Let $L = \max_{(t,x)\in[0,1]\times[0,R_1]} f(t,x)$, so

$$f(t.x) \leq L + (\rho_1 - \varepsilon)x \quad \text{for } t \in [0,1] \text{ and } x \in [0,+\infty).$$

Let $r_3 > \max\{R_1, \frac{L}{\varepsilon}\}$, $\Omega_3 = \{x \in P_0 : \|x\| < r_3\}$, $0 \leq a \leq a_0$, $0 \leq b \leq b_0$ and $\max\{a_0, b_0\} \leq \rho_1 r_3$.

For all $x \in \partial\Omega_3$, we have $\|x\| = r_3$ and

$$\begin{aligned} \bigl|(I-A)^{-1}(Tx)(t)\bigr| &\leq \bigl\|(I-A)^{-1}\bigr\| \cdot \bigl|(Tx)(t)\bigr| \leq \frac{1}{1-M}\bigl|(Tx)(t)\bigr| \\ &\leq \frac{1}{1-M}\biggl(\int_0^1 \frac{M_0}{\Gamma(\alpha)} f\bigl(s,x(s)\bigr)\,\mathrm{d}s + (\alpha-1)a + b\biggr) \\ &\leq \frac{1}{1-M}\biggl(\int_0^1 \frac{M_0}{\Gamma(\alpha)} f\bigl(s,x(s)\bigr)\,\mathrm{d}s + (\alpha-1)a_0 + b_0\biggr) \\ &\leq \frac{M_0}{(1-M)\Gamma(\alpha)} \int_0^1 \bigl(L + (\rho_1-\varepsilon)\|x\|\bigr)\,\mathrm{d}s + \frac{\alpha}{1-M}\rho_1\|x\| \\ &\leq \frac{M_0 + \alpha\Gamma(\alpha)}{(1-M)\Gamma(\alpha)}\rho_1\|x\| = \|x\|. \end{aligned}$$

So $\|(I-A)^{-1}Tx\| \leq \|x\|$, $x \in \partial\Omega_3$.

Since $f_0 > \rho_2$, there exists a constant $0 < r_4 < R_1$ such that $f(t,x) > \rho_2 x$ for $t \in [\frac{1}{4}, \frac{3}{4}]$ and $x \in [0, r_4]$.

Let $\Omega_4 = \{x \in P_0 : \|x\| < r_4\}$. Similar to the proof of Theorem 3.1, we show $\|(I-A)^{-1}Tx\| \geq \|x\|$, $x \in \partial\Omega_4$.

By Lemma 2.7, we conclude that the operator $(I-A)^{-1}T$ has at least one fixed point in $P_0 \cap (\overline{\Omega}_2 \setminus \Omega_1)$, which implies that the boundary value problem (1.1)-(1.2) has at least one positive solution.

The proof is completed. □

Theorem 3.3 *Suppose* (B0) *holds,* $f_\infty > \rho_2$. *Then there exist large enough positive constants* a_1 *and* b_1 *such that the boundary value problem* (1.1)-(1.2) *has no positive solution for* $a > a_1$ *and* $b > b_1$.

Proof Assume that for any large enough $a > 0$ and $b > 0$, the boundary value problem (1.1)-(1.2) has a positive solution $x(t)$.

Since $f_\infty > \rho_2$, there exists a large enough constant $R_0 > 0$ such that

$$f(t,x) > \rho_2 x \quad \text{for } x \in [\sigma R_0, +\infty) \text{ and } t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

Let $\min\{a_1, b_1\} > 2^{\alpha-1}R_0$, $a > a_1$ and $b > b_1$. So $\alpha a + b > \alpha a_1 + b_1 > (\alpha+1)2^{\alpha-1}R_0 > 2^{\alpha-1}R_0$. By (2.1), Lemma 2.3 and (B0), we have

$$x(t) \geq \big((\alpha-1)a+b\big)t^{\alpha-2} - \big((\alpha-2)a+b\big)t^{\alpha-1}.$$

Hence,

$$\begin{aligned} x\left(\frac{1}{2}\right) &\geq \big((\alpha-1)a+b\big)\left(\frac{1}{2}\right)^{\alpha-2} - \big((\alpha-2)a+b\big)\left(\frac{1}{2}\right)^{\alpha-1} \\ &= \left(\frac{1}{2}\right)^{\alpha-1}(\alpha a+b) > \left(\frac{1}{2}\right)^{\alpha-1}(\alpha a_1+b_1) > R_0, \end{aligned}$$

and we get $\|x\| > R_0$.

On the other hand, in view of Lemma 2.6, $x \in P_0$. Then $x(t) \geq \sigma\|x\| > \sigma R_0$ for $t \in [\frac{1}{4}, \frac{3}{4}]$. Therefore,

$$\begin{aligned} x\left(\frac{1}{2}\right) &= (I-A)^{-1}(Tx)\left(\frac{1}{2}\right) \geq (Tx)\left(\frac{1}{2}\right) \\ &= \int_0^1 G\left(\frac{1}{2}, s\right) f\big(s, x(s)\big)\,ds + \big((\alpha-1)a+b\big)\left(\frac{1}{2}\right)^{\alpha-2} - \big((\alpha-2)a+b\big)\left(\frac{1}{2}\right)^{\alpha-1} \\ &\geq \frac{(\alpha-2)\sigma\rho_2}{2^\alpha\Gamma(\alpha)}\int_{\frac{1}{4}}^{\frac{3}{4}} k(s)\,ds\|x\| + \left(\frac{1}{2}\right)^{\alpha-1}\big(2\big((\alpha-1)a+b\big) - \big((\alpha-2)a+b\big)\big) \\ &\geq \frac{203\sigma(\alpha-2)}{7{,}680\Gamma(\alpha)2^\alpha}\rho_2\|x\| + \left(\frac{1}{2}\right)^{\alpha-1}\big(a(2\alpha-2-\alpha+2)+b\big) \\ &\geq \|x\| + \left(\frac{1}{2}\right)^{\alpha-1}(\alpha a+b) \\ &> \|x\| + R_0. \end{aligned}$$

So $\|x\| > \|x\| + R_0$, which is a contradiction. Thus, there exist large enough positive constants a_1 and b_1 such that the boundary value problem (1.1)-(1.2) has no positive solution for $a > a_1$ and $b > b_1$. □

4 Examples

To illustrate our main results, we present the following examples.

Example 4.1 We consider the boundary value problem

$$\begin{cases} D_{0+}^{\frac{10}{3}}x(t) = x^{\frac{3}{2}} + x^2 \sin t, \\ x(0) = x'(0) = 0, \\ x(1) = \frac{1}{3}\int_0^1 x(s)\,\mathrm{d}s + a, \\ x'(1) = \frac{2}{3}\int_0^1 x(s)\,\mathrm{d}s - b, \end{cases} \tag{4.1}$$

and we can establish the following results:

(1) The boundary value problem (4.1) has at least one positive solution if parameters $a \in [0, 0.001)$ and $b \in [0, 0.001)$.

(2) The boundary value problem (4.1) has no positive solution if parameters $a \in (2.15 \times 10^{10}, +\infty)$ and $b \in (2.15 \times 10^{10}, +\infty)$.

Proof The boundary value problem (4.1) can be regarded as the boundary value problem (1.1)-(1.2), where $\alpha = \frac{10}{3}$, $g_1(s) = \frac{1}{3}$, $g_2(s) = \frac{2}{3}$, $f(t,x) = x^{\frac{3}{2}} + x^2 \sin t$.

Then $M = \frac{1}{3}$, $M_0 = \frac{7}{3}$, $\sigma = \min_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{q(t)(1-M)}{M_0} = \frac{(1-M)3^{\alpha-2}}{M_0 4^{\alpha}} = 0.0122$, $\rho_1 = \frac{(1-M)\Gamma(\alpha)}{M_0+\alpha\Gamma(\alpha)} = 0.16$, $\rho_2 = \frac{7{,}680\Gamma(\alpha)2^{\alpha}}{203\sigma(\alpha-2)} = 65{,}296.4$ and

$$f_\infty = \infty > \rho_2, \qquad f^0 = 0 < \rho_1.$$

(1) Let $r_1 = 0.0064$, we choose $\max\{a_0, b_0\} < \rho_1 r_1 = 0.001$. When $x \in (0, 0.0064]$, $t \in [0,1]$, we have $f(t,x) \leq \rho_1 r_1$. Then, by Theorem 3.1, when $a \in [0, 0.001)$ and $b \in [0, 0.001)$, the boundary value problem (4.1) has a positive solution.

(2) Let $R_0 = 4.26 \times 10^9$, when $x \in [5.2 \times 10^7, +\infty)$ and $t \in [\frac{1}{4}, \frac{3}{4}]$, so we choose $\min\{a_1, b_1\} > 2^{\alpha-1}R_0 = 2.15 \times 10^{10}$. By Theorem 3.3, for $a \in (2.15 \times 10^{10}, +\infty)$ and $b \in (2.15 \times 10^{10}, +\infty)$, the boundary value problem (4.1) has no positive solution. □

Example 4.2 We consider the boundary value problem

$$\begin{cases} D_{0+}^{\frac{13}{4}}x(t) = \frac{(t\sqrt{x}+1)x^{\frac{1}{3}}}{2+\sqrt{x}}, \\ x(0) = x'(0) = 0, \\ x(1) = \frac{1}{2}\int_0^1 x(s)\,\mathrm{d}s + a, \\ x'(1) = \frac{3}{4}\int_0^1 x(s)\,\mathrm{d}s - b. \end{cases} \tag{4.2}$$

The boundary value problem (4.2) has a positive solution for parameters $a \in [0, 8.16)$ and $b \in [0, 8.16)$.

Proof Where $\alpha = \frac{13}{4}$, $g_1(s) = \frac{1}{2}$, $g_2(s) = \frac{3}{4}$, $f(t,x) = \frac{(t\sqrt{x}+1)x^{\frac{1}{3}}}{2+\sqrt{x}}$, we have that $M_0 = \frac{9}{4}$, $M = \frac{1}{2}$, $\sigma = \min_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{q(t)(1-M)}{M_0} = \frac{(1-M)3^{\alpha-2}}{M_0 4^{\alpha}} = 0.0097$, $\rho_1 = \frac{(1-M)\Gamma(\alpha)}{M_0+\alpha\Gamma(\alpha)} = 0.12$ and $\rho_2 = \frac{7{,}680\Gamma(\alpha)2^{\alpha}}{203\sigma(\alpha-2)} = 75{,}722$.

We set $r_3 = 68.04$, when $x \in (68.04, +\infty)$, $t \in [0,1]$, and we choose $\max\{a_0, b_0\} < \rho_1 r_3 = 8.16$ for $a \in [0, 8.16)$ and $b \in [0, 8.16)$. So we have

$$f_0 = \infty > \rho_2, \qquad f^\infty = 0 < \rho_1.$$

By Theorem 3.2, the boundary value problem (4.2) has a positive solution. □

Competing interests
The authors declare that no competing interests exist.

Authors' contributions
The authors contributed equally to this paper. All authors read and approved the final manuscript.

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