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An extension of the Baum-Katz theorem to i.i.d. random variables with general moment conditions

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Abstract

For a sequence of i.i.d. random variables $\{X, X_n, n \geq 1\}$ and a sequence of positive real numbers $\{a_n, n \geq 1\}$ with $0 < a_n/n^{1/p} \uparrow$ for some $0 < p < 2$, the Baum-Katz complete convergence theorem is extended to the $\{X, X_n, n \geq 1\}$ with the general moment condition $\sum_{n=1}^{\infty} n^{r-1} P\{|X| > a_n\} < \infty$, where $r \geq 1$. The relationship between the complete convergence and the strong law of large numbers is established.

MSC: 60F15

Keywords: Baum-Katz theorem; complete convergence; general moment condition; strong law of large numbers

1 Introduction and main result

The concept of complete convergence was first introduced by Hsu and Robbins [1] and has played a very important role in probability theory. A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant C if $\sum_{n=1}^{\infty} P\{|U_n - C| > \varepsilon\} < \infty$ for any $\varepsilon > 0$. Hsu and Robbins [1] proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Their result has been generalized and extended by many authors.

The following result is well known.

Theorem A *Let $r \geq 1$ and $0 < p < 2$. Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with partial sums $S_n = \sum_{k=1}^{n} X_k$, $n \geq 1$. Then the following statements are equivalent*:

$$E|X|^{rp} < \infty, \tag{1.1}$$

$$\sum_{n=1}^{\infty} n^{r-2} P\{|S_n - nb| > \varepsilon n^{1/p}\} < \infty \quad \forall \varepsilon > 0, \tag{1.2}$$

$$\sum_{n=1}^{\infty} n^{r-2} P\Big\{\max_{1 \leq m \leq n} |S_m - mb| > \varepsilon n^{1/p}\Big\} < \infty \quad \forall \varepsilon > 0, \tag{1.3}$$

where $b = 0$ if $0 < rp < 1$ and $b = EX$ if $rp \geq 1$.

When $r = 1$, each of (1.1)∼(1.3) is equivalent to

$$\frac{S_n - nb}{n^{1/p}} \to 0 \quad \text{a.s.} \tag{1.4}$$

When $r = 1$, the equivalence of (1.1) and (1.4) is known as the Marcinkiewicz and Zygmund strong law of large numbers. Katz [2] proved the equivalence of (1.1) and (1.2) for the case of $p = 1$. Baum and Katz [3] proved the equivalence of (1.1) and (1.2) for the case of $0 < p < 2$. The result of Baum and Katz was generalized and extended in several directions. Some versions of the Baum and Katz theorem under higher-order moment conditions were established by Lanzinger [4], Gut and Stadtmüller [5], and Chen and Sung [6]. When $p = 1$, $1 \le r < 3$, and $\{X_n, n \ge 1\}$ is a sequence of pairwise independent, but not necessarily identically distributed, random variables, Spătaru [7] gave sufficient conditions for (1.2).

It is interesting to find more general moment conditions such that the complete convergence holds. In fact, Li *et al.* [8] and Sung [9] have done something. In particular, it is worth pointing out that Sung [9] obtained the following complete convergence for pairwise i.i.d. random variables $\{X, X_n, n \ge 1\}$:

$$\sum_{n=1}^{\infty} n^{-1} P\left\{\left|\sum_{k=1}^{n} X_k - nEXI\big(|X| \le a_n\big)\right| > \varepsilon a_n\right\} < \infty \quad \forall \varepsilon > 0,$$

provided that $\sum_{n=1}^{\infty} P\{|X| > a_n\} < \infty$, where $0 < a_n/n \uparrow$.

Motivated by the work of Sung [9], the aim of this paper is to obtain the complete convergence under more general moment conditions. Our main result includes the Baum and Katz [3] complete convergence and the Marcinkiewicz and Zygmund strong law of large numbers.

Now we state the main result. Some lemmas and the proof of the main result will be detailed in next section.

Theorem 1.1 *Let $r \ge 1$ and $0 < p < 2$. Let $\{X, X_n, n \ge 1\}$ be a sequence of i.i.d. random variables with partial sums $S_n = \sum_{k=1}^{n} X_k$, $n \ge 1$, and $\{a_n, n \ge 1\}$ a sequence of positive real numbers with $0 < a_n/n^{1/p} \uparrow$. Then the following statements are equivalent*:

$$\sum_{n=1}^{\infty} n^{r-1} P\big\{|X| > a_n\big\} < \infty, \tag{1.5}$$

$$\sum_{n=1}^{\infty} n^{r-2} P\big\{|S_n - nb_n| > \varepsilon a_n\big\} < \infty \quad \forall \varepsilon > 0, \tag{1.6}$$

$$\sum_{n=1}^{\infty} n^{r-2} P\Big\{\max_{1 \le m \le n} |S_m - mb_n| > \varepsilon a_n\Big\} < \infty \quad \forall \varepsilon > 0, \tag{1.7}$$

where $b_n = 0$ if $0 < p < 1$ and $b_n = EXI(|X| \le a_n)$ if $1 \le p < 2$.

When $r = 1$, each of (1.5)-(1.7) is equivalent to

$$a_n^{-1}(S_n - nb_n) \to 0 \quad \text{a.s.} \tag{1.8}$$

Remark 1.1 When $a_n = n^{1/p}$ for $n \geq 1$, (1.5) is equivalent to (1.1). In this case, $a_n^{-1} \cdot nEXI(|X| > a_n) \to 0$ if $1 \leq p < 2$ and (1.1) holds. Hence, (1.1) $\Rightarrow$ (1.2), (1.1) $\Rightarrow$ (1.3), and (1.1) $\Rightarrow$ (1.4) (in this case, $r = 1$) follow from Theorem 1.1. Although the converses do not follow directly from Theorem 1.1, the proofs can be done easily. When $a_n = n^{1/p}(\ln n)^\alpha$ for $n \geq 1$, where $\alpha > 0$, (1.5) is equivalent to $E|X|^{rp}/(\ln(|X| + 2))^{\alpha rp} < \infty$.

Throughout this paper, the symbol C denotes a positive constant that is not necessarily the same one in each appearance, and $I(A)$ denotes the indicator function of an event A.

2 Lemmas and proofs

To prove the main result, the following lemmas are needed. Lemma 2.1 is the Rosenthal inequality for the sum of independent random variables; see, for example, Petrov [10].

Lemma 2.1 *Let $\{Y_n, n \geq 1\}$ be a sequence of independent random variables with $EY_n = 0$ and $E|Y_n|^s < \infty$ for some $s \geq 2$ and all $n \geq 1$. Then there exists a positive constant C depending only on s such that for all $n \geq 1$,*

$$E \max_{1\leq m\leq n} \left| \sum_{k=1}^{m} Y_k \right|^s \leq C \left\{ \sum_{k=1}^{n} E|Y_k|^s + \left(\sum_{k=1}^{n} EY_k^2 \right)^{s/2} \right\}.$$

Lemma 2.2 *Under the assumptions of Theorem* 1.1, *if* $0 < p < 1$ *and* (1.5) *holds, then*

$$a_n^{-1} \cdot nEXI\bigl(|X| \leq a_n\bigr) \to 0$$

as $n \to \infty$.

Proof Since $0 < p < 1$, by $0 < a_n/n^{1/p} \uparrow$ we have $0 < a_n/n \uparrow \infty$. By (1.5) we have

$$\sum_{n=1}^{\infty} P\bigl\{|X| > a_n\bigr\} < \infty.$$

Therefore, by Lemma 2.4 in Sung [9] we have the desired result. □

Lemma 2.3 *Under the assumptions of Theorem* 1.1, *if* $rp \geq 2$ *and* (1.5) *holds, then*

$$a_n^{-2} \cdot nE|X|^2 I\bigl(|X| \leq a_n\bigr) \leq Cn^{1-2/p}.$$

Proof By $0 < a_n/n^{1/p} \uparrow$ we have $a_k/a_n \leq (k/n)^{1/p}$ for any $1 \leq k \leq n$. Hence,

$$\begin{aligned} a_n^{-2} \cdot nE|X|^2 I\bigl(|X| \leq a_n\bigr) &= a_n^{-2} \cdot n \sum_{k=1}^{n} E|X|^2 I\bigl(a_{k-1} < |X| \leq a_k\bigr) \quad (\text{set } a_0 = 0) \\ &\leq a_n^{-2} \cdot n \sum_{k=1}^{n} a_k^2 P\bigl\{a_{k-1} < |X| \leq a_k\bigr\} \\ &\leq n^{1-2/p} \sum_{k=1}^{n} k^{2/p} P\bigl\{a_{k-1} < |X| \leq a_k\bigr\} \end{aligned}$$

$$\begin{aligned}
&\le n^{1-2/p}\sum_{k=1}^{n}k^{r}P\{a_{k-1}<|X|\le a_k\} \quad (\text{by } rp\ge 2)\\
&\le n^{1-2/p}\sum_{k=0}^{\infty}\big[(k+1)^r-k^r\big]P\{|X|>a_k\}\\
&\le n^{1-2/p}\cdot\left(1+r2^{r-1}\sum_{k=1}^{\infty}k^{r-1}P\{|X|>a_k\}\right).
\end{aligned}$$

Set $C=1+r2^{r-1}\sum_{k=1}^{\infty}k^{r-1}P\{|X|>a_k\}$. By (1.5), $C<\infty$. So we complete the proof. □

Lemma 2.4 *Under the assumptions of Theorem* 1.1, *if* $s>rp$ *and* (1.5) *holds, then*

$$\sum_{n=1}^{\infty}n^{r-2}\cdot a_n^{-s}nE|X|^sI(|X|\le a_n)<\infty.$$

Proof By $0<a_n/n^{1/p}\uparrow$ we have $a_k/a_n\le(k/n)^{1/p}$ for any $n\ge k$. Hence,

$$\begin{aligned}
\sum_{n=1}^{\infty}n^{r-2}\cdot a_n^{-s}nE|X|^sI(|X|\le a_n)&=\sum_{n=1}^{\infty}n^{r-1}a_n^{-s}\sum_{k=1}^{n}E|X|^sI(a_{k-1}<|X|\le a_k)\\
&\le\sum_{n=1}^{\infty}n^{r-1}a_n^{-s}\sum_{k=1}^{n}a_k^sP\{a_{k-1}<|X|\le a_k\}\\
&=\sum_{k=1}^{\infty}a_k^sP\{a_{k-1}<|X|\le a_k\}\sum_{n=k}^{\infty}n^{r-1}a_n^{-s}\\
&\le\sum_{k=1}^{\infty}k^{s/p}P\{a_{k-1}<|X|\le a_k\}\sum_{n=k}^{\infty}n^{r-1-s/p}\\
&\le C\sum_{k=1}^{\infty}k^{r}P\{a_{k-1}<|X|\le a_k\}<\infty.
\end{aligned}$$

Therefore, the proof is completed. □

Lemma 2.5 *Let* $\{X,X_n\ge 1\}$ *be a sequence of i.i.d. symmetric random variables, and* $\{a_n,n\ge 1\}$ *a sequence of real numbers with* $0<a_n\uparrow\infty$. *Suppose that*

$$\sum_{n=1}^{\infty}n^{-1}P\left\{\left|\sum_{k=1}^{n}X_k\right|>\varepsilon a_n\right\}<\infty \quad \forall\varepsilon>0. \tag{2.1}$$

Then

$$a_n^{-1}\sum_{k=1}^{n}X_k\to 0 \text{ in probability.} \tag{2.2}$$

Proof Set $S_n=\sum_{k=1}^{n}X_k$, $n\ge 1$. Note that for all $\varepsilon>0$,

$$\begin{aligned}
P\{|S_{2n+1}|>\varepsilon a_{2n+1}\}&\le P\{|S_{2n}|>\varepsilon a_{2n+1}/2\}+P\{|X_{2n+1}|>\varepsilon a_{2n+1}/2\}\\
&\le P\{|S_{2n}|>\varepsilon a_{2n}/2\}+P\{|X|>\varepsilon a_{2n+1}/2\}
\end{aligned}$$

and $P\{|X| > \varepsilon a_{2n+1}/2\} \to 0$ as $n \to \infty$. Hence, to prove (2.2), it suffices to prove that

$$a_{2n}^{-1} S_{2n} \to 0 \text{ in probability.} \tag{2.3}$$

We will prove (2.3) by contradiction. Suppose that there exist a constant $\varepsilon > 0$ and a sequence of integers $\{n_i, i \geq 1\}$ with $n_i \uparrow \infty$ such that

$$P\{|S_{2n_i}| > \varepsilon a_{2n_i}\} \geq \varepsilon \quad \text{for all } i \geq 1.$$

Without loss of generality, we can assume that $2n_i < n_{i+1}$. By the Lévy inequality (see, for example, formula (2.6) in Ledoux and Talagrand [11]) we have

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{-1} P\{|S_n| > \varepsilon a_n/2\} &\geq \frac{1}{2} \sum_{n=1}^{\infty} n^{-1} P\Big\{\max_{1\leq k\leq n} |S_k| > \varepsilon a_n/2\Big\} \\
&\geq \frac{1}{2} \sum_{i=2}^{\infty} \sum_{n=n_i+1}^{2n_i} n^{-1} P\Big\{\max_{1\leq k\leq n} |S_k| > \varepsilon a_n/2\Big\} \\
&\geq \frac{1}{2} \sum_{i=2}^{\infty} \sum_{n=n_i+1}^{2n_i} n^{-1} P\Big\{\max_{1\leq k\leq n_i} |S_k| > \varepsilon a_{2n_i}/2\Big\} \\
&\geq \frac{1}{2} \sum_{i=2}^{\infty} \sum_{n=n_i+1}^{2n_i} n^{-1} P\{|S_{n_i}| > \varepsilon a_{2n_i}/2\} \\
&= \frac{1}{4} \sum_{i=2}^{\infty} \sum_{n=n_i+1}^{2n_i} n^{-1} \big(P\{|S_{n_i}| > \varepsilon a_{2n_i}/2\} + P\{|S_{2n_i} - S_{n_i}| > \varepsilon a_{2n_i}/2\}\big) \\
&\geq \frac{1}{4} \sum_{i=2}^{\infty} \sum_{n=n_i+1}^{2n_i} n^{-1} P\{|S_{2n_i}| > \varepsilon a_{2n_i}\} \\
&\geq \frac{\varepsilon}{4} \sum_{i=2}^{\infty} \sum_{n=n_i+1}^{2n_i} n^{-1} = \infty,
\end{aligned}$$

which leads a contradiction to (2.1). Hence, (2.3) holds, and so the proof is completed. □

Proof of Theorem 1.1 We first prove that (1.5) implies (1.7). By Lemma 2.2, to prove (1.7), it suffices to prove that

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{\max_{1\leq m\leq n} \left|\sum_{k=1}^{m} \big(X_k - EX_k I(|X_k| \leq a_n)\big)\right| > \varepsilon a_n\right\} < \infty \quad \forall \varepsilon > 0. \tag{2.4}$$

Note that

$$\begin{aligned}
&\left\{\max_{1\leq m\leq n} \left|\sum_{k=1}^{m} \big(X_k - EX_k I(|X_k| \leq a_n)\big)\right| > \varepsilon a_n\right\} \\
&\quad \subset \bigcup_{k=1}^{n} \{|X_k| > a_n\} \cup \left\{\max_{1\leq m\leq n} \left|\sum_{k=1}^{m} \big(X_k I(|X_k| \leq a_n) - EX_k I(|X_k| \leq a_n)\big)\right| > \varepsilon a_n\right\}.
\end{aligned}$$

Hence, by (1.5), to prove (2.4) it suffices to prove that for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{ \max_{1\le m\le n} \left| \sum_{k=1}^{m} \bigl(X_k I(|X_k| \le a_n) - EX_k I(|X_k| \le a_n) \bigr) \right| > \varepsilon a_n \right\} < \infty. \tag{2.5}$$

For any $s \ge 2$, by the Markov inequality and Lemma 2.1,

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{r-2} P\left\{ \max_{1\le m\le n} \left| \sum_{k=1}^{m} \bigl(X_k I(|X_k| \le a_n) - EX_k I(|X_k| \le a_n) \bigr) \right| > \varepsilon a_n \right\} \\ &\quad \le C \sum_{n=1}^{\infty} n^{r-2} a_n^{-s} E \max_{1\le m\le n} \left| \sum_{k=1}^{m} \bigl(X_k I(|X_k| \le a_n) - EX_k I(|X_k| \le a_n) \bigr) \right|^s \\ &\quad \le C \left(\sum_{n=1}^{\infty} n^{r-2} \bigl\{ a_n^{-2} n EX^2 I(|X| \le a_n) \bigr\}^{s/2} + \sum_{n=1}^{\infty} n^{r-1} a_n^{-s} E|X|^s I(|X| \le a_n) \right) \\ &\quad = C(I_1 + I_2). \end{aligned}$$

If $rp \ge 2$, taking s large enough such that $r - 2 - s/p + s/2 < -1$, by Lemma 2.3 we have

$$I_1 \le C \sum_{n=1}^{\infty} n^{r-2-s/p+s/2} < \infty.$$

Since $s > rp$, $I_2 < \infty$ by Lemma 2.4. If $0 < rp < 2$, taking $s = 2$ (in this case $I_1 = I_2$), we have $I_1 = I_2 < \infty$ by Lemma 2.4 again. Hence, (2.5) holds for all $\varepsilon > 0$.

It is trivial that (1.7) implies (1.6). Now we prove that (1.6) implies (1.5). Let $\{X', X'_n, n \ge 1\}$ be an independent copy of $\{X, X_n, n \ge 1\}$. Then we also have

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{ \left| \sum_{k=1}^{n} X'_k - nb_n \right| > \varepsilon a_n \right\} < \infty \quad \forall \varepsilon > 0.$$

Hence,

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{ \left| \sum_{k=1}^{n} (X_k - X'_k) \right| > \varepsilon a_n \right\} < \infty \quad \forall \varepsilon > 0,$$

from which it follows that

$$\sum_{n=1}^{\infty} n^{-1} P\left\{ \left| \sum_{k=1}^{n} (X_k - X'_k) \right| > \varepsilon a_n \right\} < \infty \quad \forall \varepsilon > 0.$$

Then, by Lemma 2.5,

$$a_n^{-1} \sum_{k=1}^{n} (X_k - X'_k) \to 0 \text{ in probability.}$$

By the Lévy inequality (see, for example, formula (2.7) in Ledoux and Talagrand [11]), for any fixed $\varepsilon > 0$,

$$P\Big\{\max_{1\le k\le n}\big|X_k - X_k'\big| > \varepsilon a_n\Big\} \le 2P\left\{\left|\sum_{k=1}^{n}(X_k - X_k')\right| > \varepsilon a_n\right\} \to 0$$

as $n \to \infty$. Then, for all n large enough,

$$P\Big\{\max_{1\le k\le n}\big|X_k - X_k'\big| > \varepsilon a_n\Big\} \le 1/2.$$

Therefore, by Lemma 2.6 in Ledoux and Talagrand [11] and the Lévy inequality (see formula (2.7) in Ledoux and Talagrand [11]) we have that for all n large enough,

$$\begin{aligned} nP\big\{\big|X - X'\big| > \varepsilon a_n\big\} &= \sum_{k=1}^{n} P\big\{\big|X_k - X_k'\big| > \varepsilon a_n\big\} \\ &\le 2P\Big\{\max_{1\le k\le n}\big|X_k - X_k'\big| > \varepsilon a_n\Big\} \le 4P\left\{\left|\sum_{k=1}^{n}(X_k - X_k')\right| > \varepsilon a_n\right\}. \end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} n^{r-1}P\big\{\big|X - X'\big| > \varepsilon a_n\big\} < \infty \quad \forall \varepsilon > 0. \tag{2.6}$$

Since $P\{|X| > a_n/2\} \to 0$ as $n \to \infty$, $|\mathrm{med}(X)/(a_n/2)| \le 1$ for all n large enough. By the weak symmetrization inequality we have that for all n large enough,

$$P\big\{|X| > a_n\big\} \le P\big\{\big|X - \mathrm{med}(X)\big| > a_n/2\big\} \le 2P\big\{\big|X - X'\big| > a_n/2\big\}, \tag{2.7}$$

which, together with (2.6), implies that (1.5) holds.

Finally, we prove that (1.5) and (1.8) are equivalent when $r = 1$. Assume that (1.5) holds for $r = 1$. Since $\sum_{i=1}^{\infty} iP\{a_i < |X| \le a_{i+1}\} = \sum_{n=1}^{\infty} P\{|X| > a_n\} < \infty$, for any fixed $\varepsilon > 0$, there exists a positive integer N such that $\sum_{i=N+1}^{\infty} iP\{a_i < |X| \le a_{i+1}\} < \varepsilon$. Then, for $n > N + 1$,

$$\begin{aligned} &\left|a_n^{-1}\cdot nEXI\big(|X| \le a_n\big) - a_n^{-1}\sum_{k=1}^{n} EXI\big(|X| \le a_k\big)\right| \\ &\quad\le a_n^{-1}\sum_{k=1}^{n-1} E|X|I\big(a_k < |X| \le a_n\big) \\ &\quad= a_n^{-1}\sum_{k=1}^{n-1}\sum_{i=k}^{n-1} E|X|I\big(a_i < |X| \le a_{i+1}\big) \\ &\quad= a_n^{-1}\sum_{i=1}^{n-1} iE|X|I\big(a_i < |X| \le a_{i+1}\big) \\ &\quad\le a_n^{-1}\sum_{i=1}^{N} iE|X|I\big(a_i < |X| < a_{i+1}\big) + \sum_{i=N+1}^{n-1} iP\big\{a_i < |X| \le a_{i+1}\big\} \end{aligned}$$

$$\begin{aligned}
&\le a_n^{-1}\sum_{i=1}^{N} iE|X|I\big(a_i < |X| < a_{i+1}\big) + \varepsilon \\
&\to \varepsilon \quad \text{as } n \to \infty.
\end{aligned}$$

It follows that

$$\left| a_n^{-1} \cdot nEXI\big(|X| \le a_n\big) - a_n^{-1}\sum_{k=1}^{n} EXI\big(|X| \le a_k\big) \right| \to 0$$

as $n \to \infty$. Hence, to prove (1.8), by Lemma 2.2 it suffices to prove that

$$a_n^{-1}\sum_{k=1}^{n}\big(X_kI\big(|X| \le a_k\big) - EX_kI\big(|X| \le a_k\big)\big) \to 0 \quad \text{a.s.} \tag{2.8}$$

Since $0 < a_n/n^{1/p} \uparrow$ and $0 < p < 2$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} a_n^{-2} Var\big(X_nI\big(|X| \le a_n\big)\big) \\
&\quad \le \sum_{n=1}^{\infty} a_n^{-2}EX^2I\big(|X| \le a_n\big) \\
&\quad = \sum_{n=1}^{\infty} a_n^{-2}\sum_{k=1}^{n} EX^2I\big(a_{k-1} < |X| \le a_k\big) \quad (\text{set } a_0 = 0) \\
&\quad \le \sum_{k=1}^{\infty} a_k^2P\big\{a_{k-1} < |X| \le a_k\big\}\sum_{n=k}^{\infty} a_n^{-2} \\
&\quad \le C\sum_{k=1}^{\infty} kP\big\{a_{k-1} < |X| \le a_k\big\} < \infty.
\end{aligned}$$

Then by the Kolmogorov convergence criterion and the Kronecker lemma, (2.8) holds, and so (1.8) also holds.

Conversely, assume that (1.8) holds. Let $\{X', X'_n, n \ge 1\}$ be an independent copy of $\{X, X_n, n \ge 1\}$. Then we also have

$$a_n^{-1}\left(\sum_{k=1}^{n} X'_k - nb_n\right) \to 0 \quad \text{a.s.}$$

Hence, we have

$$a_n^{-1}\sum_{k=1}^{n}\big(X_k - X'_k\big) \to 0 \quad \text{a.s.}$$

So, we have by $0 < a_n \uparrow$ that

$$a_n^{-1}\big(X_n - X'_n\big) = a_n^{-1}\sum_{k=1}^{n}\big(X_k - X'_k\big) - \big(a_{n-1}a_n^{-1}\big)a_{n-1}^{-1}\sum_{k=1}^{n-1}\big(X_k - X'_k\big) \to 0 \quad \text{a.s.}$$

By the Borel-Cantelli lemma,

$$\sum_{n=1}^{\infty} P\{|X_n - X_n'| > \varepsilon a_n\} < \infty \quad \forall \varepsilon > 0,$$

which, together with (2.7), implies that (1.5) holds for $r = 1$. So we complete the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the manuscript.

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