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Scaling invariant Hardy inequalities of multiple logarithmic type on the whole space

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Abstract

In this paper, we establish Hardy inequalities of logarithmic type involving singularities on spheres in $\mathbb{R}^n$ in terms of the Sobolev-Lorentz-Zygmund spaces. We prove it by absorbing singularities of functions on the spheres by subtracting the corresponding limiting values.

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1 Introduction and the main theorem

The classical Hardy inequalities in one dimension are stated as

$$\int_0^\infty x^{-r-1}\left|\int_0^x f(y)\,dy\right|^p dx \le \left(\frac{p}{r}\right)^p \int_0^\infty x^{p-r-1}\left|f(x)\right|^p dx \tag{1.1}$$

and its dual inequality

$$\int_0^\infty x^{r-1}\left|\int_x^\infty f(y)\,dy\right|^p dx \le \left(\frac{p}{r}\right)^p \int_0^\infty x^{p+r-1}\left|f(x)\right|^p dx, \tag{1.2}$$

where $1 < p < \infty$ and $r > 0$; see [1, 2] for instance. The constant $(\frac{p}{r})^p$ is best-possible in both inequalities (1.1) and (1.2). A higher dimensional variant of (1.1) and (1.2) is

$$\left\|\frac{f}{|x|}\right\|_{L_p(\mathbb{R}^n)} \le \frac{p}{n-p}\|\nabla f\|_{L_p(\mathbb{R}^n)} \tag{1.3}$$

for all $f \in W_p^1(\mathbb{R}^n)$, where $n \ge 2$ and $1 < p < n$, and the constant $\frac{p}{n-p}$ in (1.3) is also optimal. For the critical case $p = n$, the inequality (1.3) makes no sense, and instead the inequality

$$\left\|\frac{f}{|x|(1+|\log|x||)}\right\|_{L_n(B_1)} \le C\|f\|_{W_n^1(\mathbb{R}^n)}$$

holds for all $f \in W_n^1(\mathbb{R}^n)$, where $n \ge 2$, $B_1 := \{x \in \mathbb{R}^n; |x| < 1\}$, and the constant C depends only on n (see [3] for instance). There are a number of both mathematical and physical applications of Hardy type inequalities. Among others, we refer the reader to [3–19].

In a recent paper [12], the authors established the logarithmic Hardy type inequality on the two dimensional ball $B_R := \{x \in \mathbb{R}^2; |x| < R\}$ with $R > 0$, by taking into account the behavior of $W_2^1(B_R)$ functions on the boundary $\partial B_R = \{x \in \mathbb{R}^2; |x| = R\}$. Indeed, the following inequality was proved.

Theorem (Theorem 5 in [12]) *Let $n = 2$ and $R > 0$. Then the inequality*

$$\left(\int_{B_R} \frac{|f(x) - f(R\frac{x}{|x|})|^2}{|\log \frac{R}{|x|}|^2} \frac{dx}{|x|^2}\right)^{\frac{1}{2}} \leq 2\left(\int_{B_R} \left|\frac{x}{|x|} \cdot \nabla f(x)\right|^2 dx\right)^{\frac{1}{2}} \tag{1.4}$$

holds for all $f \in W_2^1(B_R)$.

The purpose of this paper is to extend the inequality (1.4) to the higher dimensional cases $n \geq 1$ in terms of the Lorentz-Zygmund type spaces in $\mathbb{R}^n$. To this end, we first recall the Lorentz-Zygmund spaces.

For $n \in \mathbb{N}$ and $1 \leq p, q \leq \infty$, the Lorentz spaces are defined by

$$L_{p,q}(\mathbb{R}^n) := \left\{f \in L_{1,\mathrm{loc}}(\mathbb{R}^n); \|f\|_{L_{p,q}(\mathbb{R}^n)} < +\infty\right\},$$

where

$$\|f\|_{L_{p,q}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \left(|x|^{\frac{n}{p}} |f(x)|\right)^q \frac{dx}{|x|^n}\right)^{\frac{1}{q}}$$

with the usual modification when $q = \infty$. If a function f is non-negative, radially symmetric and non-increasing with respect to the radial direction, then the norm $\|f\|_{L_{p,q}(\mathbb{R}^n)}$ coincides with the Lorentz norm in terms of the rearrangement of f. In fact, it follows that

$$\left(\int_{\mathbb{R}^n} \left(|x|^{\frac{n}{p}} |f(x)|\right)^q \frac{dx}{|x|^n}\right)^{\frac{1}{q}} = \omega_n^{\frac{1}{q}-\frac{1}{p}} \left(\int_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}},$$

where f^* denotes the symmetric decreasing rearrangement of f, and ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Furthermore, the Lorentz-Zygmund spaces on B_R with $R > 0$ are defined by

$$L_{p,q,\lambda}(B_R) := \left\{f \in L_{1,\mathrm{loc}}(B_R); \|f\|_{L_{p,q,\lambda}(B_R)} < +\infty\right\},$$

where $\lambda \in \mathbb{R}$ and

$$\|f\|_{L_{p,q,\lambda}(B_R)} := \left(\int_{B_R} \left(|x|^{\frac{n}{p}} \left|\log \frac{R}{|x|}\right|^{\lambda} |f(x)|\right)^q \frac{dx}{|x|^n}\right)^{\frac{1}{q}}.$$

We then define the Sobolev-Lorentz-Zygmund spaces by

$$W^1 L_{p,q,\lambda}(B_R) := \left\{f \in L_{p,q,\lambda}(B_R); \nabla f \in L_{p,q,\lambda}(B_R)\right\}$$

endowed with the norm $\|\cdot\|_{W^1L_{p,q,\lambda}(B_R)} := \|\cdot\|_{L_{p,q,\lambda}(B_R)} + \|\nabla\cdot\|_{L_{p,q,\lambda}(B_R)}$, and $W_0^1L_{p,q,\lambda}(B_R) := \overline{C_0^\infty(B_R)}^{\|\cdot\|_{W^1L_{p,q,\lambda}(B_R)}}$. Note that the special case $W^1L_{p,p,0}(B_R)$ coincides with the classical Sobolev space $W_p^1(B_R)$. As a further generalization, the Lorentz-Zygmund spaces involving the double logarithmic weights can be introduced by

$$L_{p,q,\lambda_1,\lambda_2}(B_R) := \{f \in L_{1,\mathrm{loc}}(B_R); \|f\|_{L_{p,q,\lambda_1,\lambda_2}(B_R)} < +\infty\},$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ and

$$\|f\|_{L_{p,q,\lambda_1,\lambda_2}(B_R)} := \left(\int_{B_R}\left(|x|^{\frac{n}{p}}\left|\log\frac{eR}{|x|}\right|^{\lambda_1}\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{\lambda_2}|f(x)|\right)^q\frac{dx}{|x|^n}\right)^{\frac{1}{q}}.$$

The Sobolev-Lorentz-Zygmund spaces $W^1L_{p,q,\lambda_1,\lambda_2}(B_R)$ and $W_0^1L_{p,q,\lambda_1,\lambda_2}(B_R)$ are defined similarly to above.

We next introduce the Lorentz-Zygmund spaces in $\mathbb{R}^n$ having the scaling properties. The Lorentz-Zygmund spaces $L_{p,q,\lambda}(\mathbb{R}^n)$ are defined by

$$L_{p,q,\lambda}(\mathbb{R}^n) := \{f \in L_{1,\mathrm{loc}}(\mathbb{R}^n); \|f\|_{L_{p,q,\lambda}(\mathbb{R}^n)} < +\infty\},$$

where

$$\|f\|_{L_{p,q,\lambda}(\mathbb{R}^n)} := \sup_{R>0}\left(\int_{\mathbb{R}^n}\left(|x|^{\frac{n}{p}}\left|\log\frac{R}{|x|}\right|^{\lambda}|f(x)|\right)^q\frac{dx}{|x|^n}\right)^{\frac{1}{q}}.$$

Similarly, the spaces $L_{p,q,\lambda_1,\lambda_2}(\mathbb{R}^n)$ are defined by

$$L_{p,q,\lambda_1,\lambda_2}(\mathbb{R}^n) := \{f \in L_{1,\mathrm{loc}}(\mathbb{R}^n); \|f\|_{L_{p,q,\lambda_1,\lambda_2}(\mathbb{R}^n)} < +\infty\},$$

where

$$\|f\|_{L_{p,q,\lambda_1,\lambda_2}(\mathbb{R}^n)} := \sup_{R>0}\left(\int_{\mathbb{R}^n}\left(|x|^{\frac{n}{p}}\left|\log\frac{R}{|x|}\right|^{\lambda_1}\left|\log\left|\log\frac{R}{|x|}\right|\right|^{\lambda_2}|f(x)|\right)^q\frac{dx}{|x|^n}\right)^{\frac{1}{q}}.$$

Remark The space $L_{p,q,\lambda_1,\lambda_2}(\mathbb{R}^n)$ extends the spaces $L_{p,q,\lambda}(\mathbb{R}^n)$ and $L_{p,q}(\mathbb{R}^n)$ in the sense that $L_{p,q,\lambda,0}(\mathbb{R}^n) = L_{p,q,\lambda}(\mathbb{R}^n)$ and $L_{p,q,0,0}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)$. Moreover, remark that the space $L_{p,q,\lambda_1,\lambda_2}(\mathbb{R}^n)$ has a scaling property in the sense that $\|\delta_l f\|_{L_{p,q,\lambda_1,\lambda_2}(\mathbb{R}^n)} = l^{\frac{n}{p}}\|f\|_{L_{p,q,\lambda_1,\lambda_2}(\mathbb{R}^n)}$, where $(\delta_l f)(x) := f(\frac{x}{l})$ for $l > 0$.

In addition, the Sobolev-Lorentz-Zygmund spaces $W^1L_{p,q,\lambda_1,\lambda_2}(\mathbb{R}^n)$ are defined in the same manner as above. We refer to [20] for an enlightening exposition concerning these functional spaces.

Finally, in order to state the main theorems in this paper, we need to introduce the Lorentz-Zygmund type spaces $\mathcal{L}_{p,q,\lambda}(\mathbb{R}^n)$ taking into account the behavior of functions on spheres defined by

$$\mathcal{L}_{p,q,\lambda}(\mathbb{R}^n) := \{f \in L_{1,\mathrm{loc}}(\mathbb{R}^n); \|f\|_{\mathcal{L}_{p,q,\lambda}(\mathbb{R}^n)} < +\infty\},$$

where

$$\|f\|_{\mathcal{L}_{p,q,\lambda}(\mathbb{R}^n)} := \sup_{R>0}\left(\int_{\mathbb{R}^n}\left(|x|^{\frac{n}{p}}\left|\log\frac{R}{|x|}\right|^{\lambda}\left|f(x)-f\left(R\frac{x}{|x|}\right)\right|\right)^q\frac{dx}{|x|^n}\right)^{\frac{1}{q}}.$$

Furthermore, we define the Lorentz-Zygmund type spaces $\mathcal{L}_{p,q,\lambda_1,\lambda_2}(\mathbb{R}^n)$ by

$$\mathcal{L}_{p,q,\lambda_1,\lambda_2}(\mathbb{R}^n) := \{f \in L_{1,\mathrm{loc}}(\mathbb{R}^n); \|f\|_{\mathcal{L}_{p,q,\lambda_1,\lambda_2}(\mathbb{R}^n)} < +\infty\},$$

where

$$\begin{aligned}\|f\|_{\mathcal{L}_{p,q,\lambda_1,\lambda_2}(\mathbb{R}^n)} := \sup_{R>0}\Bigg(\int_{\mathbb{R}^n}\Bigg(|x|^{\frac{n}{p}}\left|\log\frac{eR}{|x|}\right|^{\lambda_1}\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{\lambda_2}&\left(\chi_{B_{eR}}(x)\left|f(x)-f\left(R\frac{x}{|x|}\right)\right|\right.\\ &\left.+\chi_{B_{eR}^c}(x)\left|f(x)-f\left(e^2R\frac{x}{|x|}\right)\right|\right)\Bigg)^q\frac{dx}{|x|^n}\Bigg)^{\frac{1}{q}}.\end{aligned}$$

Remark The spaces $\mathcal{L}_{p,q,\lambda}(\mathbb{R}^n)$ and $\mathcal{L}_{p,q,\lambda_1,\lambda_2}(\mathbb{R}^n)$ have the same scaling property as in the space $L_{p,q,\lambda_1,\lambda_2}(\mathbb{R}^n)$. Namely, it follows that $\|\delta_l f\|_{\mathcal{L}_{p,q,\lambda}(\mathbb{R}^n)} = l^{\frac{n}{p}}\|f\|_{\mathcal{L}_{p,q,\lambda}(\mathbb{R}^n)}$ and $\|\delta_l f\|_{\mathcal{L}_{p,q,\lambda_1,\lambda_2}(\mathbb{R}^n)} = l^{\frac{n}{p}}\|f\|_{\mathcal{L}_{p,q,\lambda_1,\lambda_2}(\mathbb{R}^n)}$, where $(\delta_l f)(x) := f(\frac{x}{l})$ for $l > 0$.

We are now in a position to state the main theorems.

Theorem 1.1 *Let $n \in \mathbb{N}$, $1 < \alpha < \infty$ and $\max\{1, \alpha - 1\} < \beta < \infty$. Then the continuous embedding*

$$W^1L_{n,\beta,\frac{\beta-\alpha}{\beta}}(\mathbb{R}^n) \hookrightarrow \mathcal{L}_{\infty,\beta,-\frac{\alpha}{\beta}}(\mathbb{R}^n)$$

holds. In particular, for any $R > 0$, the inequality

$$\begin{aligned}&\left(\int_{\mathbb{R}^n}\frac{|f(x)-f(R\frac{x}{|x|})|^{\beta}}{|\log\frac{R}{|x|}|^{\alpha}}\frac{dx}{|x|^n}\right)^{\frac{1}{\beta}}\\ &\quad\le\frac{\beta}{\alpha-1}\left(\int_{\mathbb{R}^n}|x|^{\beta-n}\left|\log\frac{R}{|x|}\right|^{\beta-\alpha}\left|\frac{x}{|x|}\cdot\nabla f(x)\right|^{\beta}dx\right)^{\frac{1}{\beta}}\end{aligned}\tag{1.5}$$

holds for all $f \in W^1L_{n,\beta,\frac{\beta-\alpha}{\beta}}(\mathbb{R}^n)$, where the embedding constant $\frac{\beta}{\alpha-1}$ in (1.5) is best-possible.

Remark Denoting $W^1L_{2,2,0}(\mathbb{R}^2) = W_2^1(\mathbb{R}^2)$ and restricting the functions in $W_2^1(\mathbb{R}^2)$ on B_R, we see that the special case $n = \alpha = \beta = 2$ in (1.5) yields (1.4) obtained in [12].

Our next aim is to consider the limiting case $\alpha = 1$ in (1.5). However, the inequality (1.5) with $\alpha = 1$ makes no sense since the weight $|\log\frac{1}{|x|}|^{-1}|x|^{-n}$ is not locally integrable at the origin. To overcome this difficulty, we need the aid of a logarithmic weight to recover the corresponding double logarithmic Hardy type inequality. Our next theorem now reads as follows.

Theorem 1.2 *Let $n \in \mathbb{N}$, $1 < \alpha < \infty$ and $\max\{1, \alpha - 1\} < \beta < \infty$. Then the continuous embedding*

$$W^1L_{n,\beta,\frac{\beta-1}{\beta},\frac{\beta-\alpha}{\beta}}(\mathbb{R}^n) \hookrightarrow \mathcal{L}_{\infty,\beta,-\frac{1}{\beta},-\frac{\alpha}{\beta}}(\mathbb{R}^n)$$

holds. In particular, for any $R > 0$, the inequality

$$\left(\int_{\mathbb{R}^n} \frac{\chi_{B_{eR}}(x)|f(x)-f(R\frac{x}{|x|})|^\beta + \chi_{B_{eR}^c}(x)|f(x)-f(e^2R\frac{x}{|x|})|^\beta}{|\log|\log\frac{eR}{|x|}||^\alpha|\log\frac{eR}{|x|}|}\frac{dx}{|x|^n}\right)^{\frac{1}{\beta}}$$
$$\leq \frac{\beta}{\alpha-1}\left(\int_{\mathbb{R}^n}|x|^{\beta-n}\left|\log\frac{eR}{|x|}\right|^{\beta-1}\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{\beta-\alpha}\left|\frac{x}{|x|}\cdot\nabla f(x)\right|^\beta dx\right)^{\frac{1}{\beta}} \tag{1.6}$$

holds for all $f \in W^1L_{n,\beta,\frac{\beta-1}{\beta},\frac{\beta-\alpha}{\beta}}(\mathbb{R}^n)$, where the embedding constant $\frac{\beta}{\alpha-1}$ in (1.6) *is best-possible.*

Remark Remark that we do not need to subtract the boundary value of functions on $|x| = eR$ in the integrand on the left-hand side of (1.6) in spite of the fact that the integrand on the right-hand side has singularities on $|x| = R$, $|x| = eR$, and $|x| = e^2R$.

This paper is organized as follows. Section 2 is devoted to establishing the inequalities (1.5) in Theorem 1.1 and (1.6) in Theorem 1.2. We shall prove the optimality of the embedding constants in the two inequalities (1.5) and (1.6) in Section 3.

2 Proof of inequalities (1.5) and (1.6)

In this section, we shall prove inequalities (1.5) and (1.6).

Proof of (1.5) *in Theorem* 1.1 We first prove (1.5) for $f \in C_0^\infty(\mathbb{R}^n)$. We introduce polar coordinates $(r,\omega) = (|x|, \frac{x}{|x|}) \in (0,\infty) \times S^{n-1}$ and the Lebesgue measure σ on the unit sphere S^{n-1}. We write the integral on the left-hand side of (1.5) restricted on B_R in polar coordinates and then by integration by parts to obtain

$$\begin{aligned}
&\int_{B_R} \frac{|f(x)-f(R\frac{x}{|x|})|^\beta}{|\log\frac{R}{|x|}|^\alpha}\frac{dx}{|x|^n}\\
&= \int_0^R \frac{1}{r(\log\frac{R}{r})^\alpha}\int_{S^{n-1}}\left|f(r\omega)-f(R\omega)\right|^\beta d\sigma(\omega)\,dr\\
&= \frac{1}{\alpha-1}\left[\left(\log\frac{R}{r}\right)^{-\alpha+1}\int_{S^{n-1}}\left|f(r\omega)-f(R\omega)\right|^\beta d\sigma(\omega)\right]_{r=0}^{r=R}\\
&\quad -\frac{1}{\alpha-1}\int_0^R\left(\log\frac{R}{r}\right)^{-\alpha+1}\frac{d}{dr}\int_{S^{n-1}}\left|f(r\omega)-f(R\omega)\right|^\beta d\sigma(\omega)\,dr\\
&= -\frac{\beta}{\alpha-1}\int_0^R\left(\log\frac{R}{r}\right)^{-\alpha+1}\operatorname{Re}\int_{S^{n-1}}\left|f(r\omega)-f(R\omega)\right|^{\beta-2}\\
&\quad\times\overline{\left(f(r\omega)-f(R\omega)\right)}\omega\cdot(\nabla f)(r\omega)\,d\sigma(\omega)\,dr,
\end{aligned}$$

where the boundary value at $r = R$ vanishes since

$$\log\frac{R}{r} = \int_1^{\frac{R}{r}}\frac{dt}{t} \geq \frac{\frac{R}{r}-1}{\frac{R}{r}} = \frac{R-r}{R} \geq 0$$

and

$$\left|f(r\omega)-f(R\omega)\right| \leq \|\nabla f\|_{L_\infty(\mathbb{R}^n)}(R-r)$$

for $0 < r \leq R$, and $\beta - \alpha + 1 > 0$ by assumption. By the Hölder inequality, we have

$$\begin{aligned}
&\int_0^R \frac{1}{r(\log \frac{R}{r})^\alpha} \int_{S^{n-1}} \left| f(r\omega) - f(R\omega) \right|^\beta \, d\sigma(\omega)\, dr \\
&\quad \leq \frac{\beta}{\alpha - 1} \int_0^R \left(\log \frac{R}{r} \right)^{-\alpha+1} \int_{S^{n-1}} \left| f(r\omega) - f(R\omega) \right|^{\beta-1} \left| \omega \cdot (\nabla f)(r\omega) \right| \, d\sigma(\omega)\, dr \\
&\quad \leq \frac{\beta}{\alpha - 1} \left(\int_0^R \int_{S^{n-1}} \frac{1}{r(\log \frac{R}{r})^\alpha} \left| f(r\omega) - f(R\omega) \right|^\beta \, d\sigma(\omega)\, dr \right)^{\frac{\beta-1}{\beta}} \\
&\qquad \times \left(\int_0^R \int_{S^{n-1}} r^{\beta-1} \left(\log \frac{R}{r} \right)^{\beta-\alpha} \left| \omega \cdot (\nabla f)(r\omega) \right|^\beta \, d\sigma(\omega)\, dr \right)^{\frac{1}{\beta}}.
\end{aligned}$$

This implies

$$\begin{aligned}
&\left(\int_{B_R} \frac{|f(x) - f(R\frac{x}{|x|})|^\beta}{|\log \frac{R}{|x|}|^\alpha} \frac{dx}{|x|^n} \right)^{\frac{1}{\beta}} \\
&\quad \leq \frac{\beta}{\alpha - 1} \left(\int_{B_R} |x|^{\beta-n} \left| \log \frac{R}{|x|} \right|^{\beta-\alpha} \left| \frac{x}{|x|} \cdot \nabla f(x) \right|^\beta dx \right)^{\frac{1}{\beta}}. \qquad (2.1)
\end{aligned}$$

In the same manner as above, we have

$$\begin{aligned}
&\left(\int_{B_R^c} \frac{|f(x) - f(R\frac{x}{|x|})|^\beta}{|\log \frac{R}{|x|}|^\alpha} \frac{dx}{|x|^n} \right)^{\frac{1}{\beta}} \\
&\quad \leq \frac{\beta}{\alpha - 1} \left(\int_{B_R^c} |x|^{\beta-n} \left| \log \frac{R}{|x|} \right|^{\beta-\alpha} \left| \frac{x}{|x|} \cdot \nabla f(x) \right|^\beta dx \right)^{\frac{1}{\beta}}. \qquad (2.2)
\end{aligned}$$

Thus combining (2.1) with (2.2), we obtain (1.5) for $f \in C_0^\infty(\mathbb{R}^n)$.

Now we prove (1.5) for $f \in W^1 L_{n,\beta,\frac{\beta-\alpha}{\beta}}(\mathbb{R}^n)$. For $f \in W^1 L_{n,\beta,\frac{\beta-\alpha}{\beta}}(\mathbb{R}^n)$, we choose a sequence $\{f_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n)$ such that $f_j \to f$ in $W^1 L_{n,\beta,\frac{\beta-\alpha}{\beta}}(\mathbb{R}^n)$ as $j \to \infty$ and almost everywhere by density. Since the inequality (1.5) holds for $f_j - f_k \in C_0^\infty(\mathbb{R}^n)$, we see that $\{(f_j)_R^\#\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $L_\beta(\mathbb{R}^n; \frac{dx}{|x|^n})$, where we define

$$f_R^\#(x) := \frac{f(x) - f(R\frac{x}{|x|})}{|\log \frac{R}{|x|}|^{\frac{\alpha}{\beta}}}$$

for $f \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$. Then there exists a function $g_R \in L_\beta(\mathbb{R}^n; \frac{dx}{|x|^n})$ such that $(f_j)_R^\# \to g_R$ in $L^\beta(\mathbb{R}^n; \frac{dx}{|x|^n})$ as $j \to \infty$. The inclusion relationship

$$\begin{aligned}
\left\{ x \in \mathbb{R}^n \setminus \{0\} : f_j\left(R\frac{x}{|x|} \right) \not\to f\left(R\frac{x}{|x|} \right) \right\} &\subset \bigcup_{r>0} \left\{ x \in \mathbb{R}^n \setminus \{0\} : f_j\left(r\frac{x}{|x|} \right) \not\to f\left(r\frac{x}{|x|} \right) \right\} \\
&= \left\{ x \in \mathbb{R}^n \setminus \{0\} : f_j(x) \not\to f(x) \right\}
\end{aligned}$$

implies that $f_j(R\frac{x}{|x|}) \to f(R\frac{x}{|x|})$ almost everywhere, and that $f_R^\# = g_R$. Therefore, the inequality (1.5) holds for all $f \in W^1 L_{n,\beta,\frac{\beta-\alpha}{\beta}}(\mathbb{R}^n)$. $\square$

In order to prove (1.6) in Theorem 1.2, we first show the following proposition.

Proposition 2.1 *Let $n \in \mathbb{N}$, $1 < \alpha < \infty$ and $\max\{1, \alpha - 1\} < \beta < \infty$. Then, for any $R > 0$, the inequality*

$$\left(\int_{B_{eR}} \frac{|f(x) - f(R\frac{x}{|x|})|^\beta}{|\log|\log\frac{eR}{|x|}||^\alpha |\log\frac{eR}{|x|}|} \frac{dx}{|x|^n}\right)^{\frac{1}{\beta}}$$
$$\leq \frac{\beta}{\alpha - 1}\left(\int_{B_{eR}} |x|^{\beta - n}\left|\log\frac{eR}{|x|}\right|^{\beta-1}\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{\beta-\alpha}\left|\frac{x}{|x|}\cdot\nabla f(x)\right|^\beta dx\right)^{\frac{1}{\beta}} \tag{2.3}$$

holds for all $f \in C_0^\infty(\mathbb{R}^n)$.

Proof We first consider the integrals in (2.3) restricted on B_R. Using polar coordinates and integration by parts, we see

$$\int_{B_R} \frac{|f(x) - f(R\frac{x}{|x|})|^\beta}{|\log|\log\frac{eR}{|x|}||^\alpha |\log\frac{eR}{|x|}|} \frac{dx}{|x|^n}$$
$$= \int_0^R \frac{1}{r(\log\frac{eR}{r})(\log(\log\frac{eR}{r}))^\alpha}\int_{S^{n-1}} \left|f(r\omega) - f(R\omega)\right|^\beta d\sigma(\omega)\,dr$$
$$= \frac{1}{\alpha-1}\left[\left(\log\left(\log\frac{eR}{r}\right)\right)^{-\alpha+1}\int_{S^{n-1}}\left|f(r\omega) - f(R\omega)\right|^\beta d\sigma(\omega)\right]_{r=0}^{r=R}$$
$$- \frac{1}{\alpha-1}\int_0^R \left(\log\left(\log\frac{eR}{r}\right)\right)^{-\alpha+1}\frac{d}{dr}\int_{S^{n-1}}\left|f(r\omega) - f(R\omega)\right|^\beta d\sigma(\omega)\,dr$$
$$= -\frac{\beta}{\alpha-1}\int_0^R \left(\log\left(\log\frac{eR}{r}\right)\right)^{-\alpha+1}\operatorname{Re}\int_{S^{n-1}}\left|f(r\omega) - f(R\omega)\right|^{\beta-2}$$
$$\times \overline{\left(f(r\omega) - f(R\omega)\right)}\omega\cdot(\nabla f)(r\omega)\,d\sigma(\omega)\,dr,$$

where the boundary value at $r = R$ vanishes since

$$\log\left(\log\frac{eR}{r}\right) = \int_1^{\log\frac{eR}{r}} \frac{dt}{t} \geq \frac{\log\frac{eR}{r} - 1}{\log\frac{eR}{r}} \geq \frac{R - r}{R\log\frac{eR}{r}}$$

and

$$\left|f(r\omega) - f(R\omega)\right| \leq \|\nabla f\|_{L_\infty(\mathbb{R}^n)}(R - r)$$

for $0 < r \leq R$, and $\beta - \alpha + 1 > 0$ by the assumption. By the Hölder inequality, we have

$$\int_0^R \frac{1}{r(\log\frac{eR}{r})(\log(\log\frac{eR}{r}))^\alpha}\int_{S^{n-1}}\left|f(r\omega) - f(R\omega)\right|^\beta d\sigma(\omega)\,dr$$
$$\leq \frac{\beta}{\alpha-1}\int_0^R \frac{1}{(\log(\log\frac{eR}{r}))^{\alpha-1}}\int_{S^{n-1}}\left|f(r\omega) - f(R\omega)\right|^{\beta-1}\left|\omega\cdot(\nabla f)(r\omega)\right| d\sigma(\omega)\,dr$$
$$= \frac{\beta}{\alpha-1}\int_0^R \frac{1}{r^{\frac{\beta-1}{\beta}}(\log\frac{eR}{r})^{\frac{\beta-1}{\beta}}r^{-\frac{\beta-1}{\beta}}(\log\frac{eR}{r})^{-\frac{\beta-1}{\beta}}(\log(\log\frac{eR}{r}))^{\frac{(\beta-1)\alpha}{\beta}}(\log(\log\frac{eR}{r}))^{\frac{\alpha-\beta}{\beta}}}$$

$$\begin{aligned}
&\times \int_{S^{n-1}} \left|f(r\omega)-f(R\omega)\right|^{\beta-1}\left|\omega\cdot(\nabla f)(r\omega)\right|\,d\sigma(\omega)\,dr \\
&\le \frac{\beta}{\alpha-1}\left(\int_0^R\int_{S^{n-1}}\frac{|f(r\omega)-f(R\omega)|^\beta}{r(\log\frac{eR}{r})(\log(\log\frac{eR}{r}))^\alpha}\,d\sigma(\omega)\,dr\right)^{\frac{\beta-1}{\beta}} \\
&\quad\times\left(\int_0^R\int_{S^{n-1}} r^{\beta-1}\left(\log\frac{eR}{r}\right)^{\beta-1}\left(\log\left(\log\frac{eR}{r}\right)\right)^{\beta-\alpha}\left|\omega\cdot(\nabla f)(r\omega)\right|^\beta\,d\sigma(\omega)\,dr\right)^{\frac{1}{\beta}},
\end{aligned}$$

which implies

$$\begin{aligned}
&\left(\int_{B_R}\frac{|f(x)-f(R\frac{x}{|x|})|^\beta}{|\log|\log\frac{eR}{|x|}||^\alpha|\log\frac{eR}{|x|}|}\frac{dx}{|x|^n}\right)^{\frac{1}{\beta}} \\
&\le\frac{\beta}{\alpha-1}\left(\int_{B_R}|x|^{\beta-n}\left|\log\frac{eR}{|x|}\right|^{\beta-1}\left|\log\left|\log\frac{eR}{|x|}\right|\right|^{\beta-\alpha}\left|\frac{x}{|x|}\cdot\nabla f(x)\right|^\beta dx\right)^{\frac{1}{\beta}}. \qquad (2.4)
\end{aligned}$$

Next, we consider the integrals in (2.3) restricted on $B_{eR}\setminus B_R$. Using polar coordinates and integration by parts, we see

$$\begin{aligned}
&\int_{B_{eR}\setminus B_R}\frac{|f(x)-f(R\frac{x}{|x|})|^\beta}{|\log|\log\frac{eR}{|x|}||^\alpha|\log\frac{eR}{|x|}|}\frac{dx}{|x|^n} \\
&=\int_R^{eR}\frac{1}{r(\log\frac{eR}{r})(\log((\log\frac{eR}{r})^{-1}))^\alpha}\int_{S^{n-1}}\left|f(r\omega)-f(R\omega)\right|^\beta d\sigma(\omega)\,dr \\
&=-\frac{1}{\alpha-1}\left[\left(\log\left(\left(\log\frac{eR}{r}\right)^{-1}\right)\right)^{-\alpha+1}\int_{S^{n-1}}\left|f(r\omega)-f(R\omega)\right|^\beta d\sigma(\omega)\right]_{r=R}^{r=eR} \\
&\quad+\frac{1}{\alpha-1}\int_R^{eR}\left(\log\left(\left(\log\frac{eR}{r}\right)^{-1}\right)\right)^{-\alpha+1}\frac{d}{dr}\int_{S^{n-1}}\left|f(r\omega)-f(R\omega)\right|^\beta d\sigma(\omega)\,dr \\
&=\frac{\beta}{\alpha-1}\int_R^{eR}\left(\log\left(\left(\log\frac{eR}{r}\right)^{-1}\right)\right)^{-\alpha+1}\operatorname{Re}\int_{S^{n-1}}\left|f(r\omega)-f(R\omega)\right|^{\beta-2} \\
&\quad\times\overline{\left(f(r\omega)-f(R\omega)\right)}\omega\cdot(\nabla f)(r\omega)\,d\sigma(\omega)\,dr,
\end{aligned}$$

where the boundary value at $r=R$ vanishes since

$$\log\left(\left(\log\frac{eR}{r}\right)^{-1}\right)=\int_1^{(\log\frac{eR}{r})^{-1}}\frac{dt}{t}\ge\left(\log\frac{eR}{r}\right)\left(\left(\log\frac{eR}{r}\right)^{-1}-1\right)=1-\log\frac{eR}{r}\ge\frac{r-R}{R}$$

and

$$\left|f(r\omega)-f(R\omega)\right|\le\|\nabla f\|_{L^\infty}(r-R)$$

for $R\le r<eR$, and $\beta-\alpha+1>0$ by the assumption. By the Hölder inequality, we have

$$\begin{aligned}
&\int_R^{eR}\frac{1}{r(\log\frac{eR}{r})(\log((\log\frac{eR}{r})^{-1}))^\alpha}\int_{S^{n-1}}\left|f(r\omega)-f(R\omega)\right|^\beta d\sigma(\omega)\,dr \\
&\le\frac{\beta}{\alpha-1}\int_R^{eR}\left(\log\left(\left(\log\frac{eR}{r}\right)^{-1}\right)\right)^{-\alpha+1}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_{S^{n-1}} \bigl|f(r\omega)-f(R\omega)\bigr|^{\beta-1}\bigl|\omega\cdot(\nabla f)(r\omega)\bigr|\,d\sigma(\omega)\,dr \\
&= \frac{\beta}{\alpha-1} \\
&\quad\times \int_R^{eR} \frac{1}{r^{\frac{\beta-1}{\beta}}(\log\frac{eR}{r})^{\frac{\beta-1}{\beta}} r^{-\frac{\beta-1}{\beta}}(\log\frac{eR}{r})^{-\frac{\beta-1}{\beta}}(\log((\log\frac{eR}{r})^{-1}))^{\frac{(\beta-1)\alpha}{\beta}}(\log((\log\frac{eR}{r})^{-1}))^{\frac{\alpha-\beta}{\beta}}} \\
&\quad\times \int_{S^{n-1}} \bigl|f(r\omega)-f(R\omega)\bigr|^{\beta-1}\bigl|\omega\cdot(\nabla f)(r\omega)\bigr|\,d\sigma(\omega)\,dr \\
&\le \frac{\beta}{\alpha-1}\biggl(\int_R^{eR}\int_{S^{n-1}} \frac{|f(r\omega)-f(R\omega)|^\beta}{r(\log\frac{eR}{r})(\log((\log\frac{eR}{r})^{-1}))^\alpha}\,d\sigma(\omega)\,dr\biggr)^{\frac{\beta-1}{\beta}} \\
&\quad\times \biggl(\int_R^{eR}\int_{S^{n-1}} r^{\beta-1}\biggl(\log\frac{eR}{r}\biggr)^{\beta-1}\biggl(\log\biggl(\biggl(\log\frac{eR}{r}\biggr)^{-1}\biggr)\biggr)^{\beta-\alpha} \\
&\quad\times \bigl|\omega\cdot(\nabla f)(r\omega)\bigr|^\beta\,d\sigma(\omega)\,dr\biggr)^{\frac{1}{\beta}},
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\biggl(\int_{B_{eR}\setminus B_R} \frac{|f(x)-f(R\frac{x}{|x|})|^\beta}{|\log|\log\frac{eR}{|x|}||^\alpha|\log\frac{eR}{|x|}|}\frac{dx}{|x|^n}\biggr)^{\frac{1}{\beta}} \\
&\quad\le \frac{\beta}{\alpha-1}\biggl(\int_{B_{eR}\setminus B_R}|x|^{\beta-n}\biggl|\log\frac{eR}{|x|}\biggr|^{\beta-1}\biggl|\log\biggl|\log\frac{eR}{|x|}\biggr|\biggr|^{\beta-\alpha}\biggl|\frac{x}{|x|}\cdot\nabla f(x)\biggr|^\beta dx\biggr)^{\frac{1}{\beta}}.
\end{aligned}
\tag{2.5}
$$

Thus combining (2.4) with (2.5), we obtain (2.3). □

We can prove a dual inequality of (2.3) in a similar way as in Proposition 2.1 stated as follows.

Proposition 2.2 *Let* $n \in \mathbb{N}$, $1 < \alpha < \infty$ *and* $\max\{1, \alpha - 1\} < \beta < \infty$. *Then, for any* $R > 0$, *the inequality*

$$
\begin{aligned}
&\biggl(\int_{B_R^c} \frac{|f(x)-f(eR\frac{x}{|x|})|^\beta}{|\log|\log\frac{R}{|x|}||^\alpha|\log\frac{R}{|x|}|}\frac{dx}{|x|^n}\biggr)^{\frac{1}{\beta}} \\
&\quad\le \frac{\beta}{\alpha-1}\biggl(\int_{B_R^c}|x|^{\beta-n}\biggl|\log\frac{R}{|x|}\biggr|^{\beta-1}\biggl|\log\biggl|\log\frac{R}{|x|}\biggr|\biggr|^{\beta-\alpha}\biggl|\frac{x}{|x|}\cdot\nabla f(x)\biggr|^\beta dx\biggr)^{\frac{1}{\beta}}
\end{aligned}
$$

holds for all $f \in C_0^\infty(\mathbb{R}^n)$.

We shall show (1.6) in Theorem 1.2 by combining Proposition 2.1 with Proposition 2.2.

Proof of Theorem 1.2 By considering a density argument as used in the proof of Theorem 1.1, it suffices to prove (1.6) for $f \in C_0^\infty(\mathbb{R}^n)$. Applying Proposition 2.2 with R replaced by eR, we obtain

$$
\begin{aligned}
&\biggl(\int_{B_{eR}^c} \frac{|f(x)-f(e^2R\frac{x}{|x|})|^\beta}{|\log|\log\frac{eR}{|x|}||^\alpha|\log\frac{eR}{|x|}|}\frac{dx}{|x|^n}\biggr)^{\frac{1}{\beta}} \\
&\quad\le \frac{\beta}{\alpha-1}\biggl(\int_{B_{eR}^c}|x|^{\beta-n}\biggl|\log\frac{eR}{|x|}\biggr|^{\beta-1}\biggl|\log\biggl|\log\frac{eR}{|x|}\biggr|\biggr|^{\beta-\alpha}\biggl|\frac{x}{|x|}\cdot\nabla f(x)\biggr|^\beta dx\biggr)^{\frac{1}{\beta}}.
\end{aligned}
\tag{2.6}
$$

Thus from (2.3) and (2.6), we obtain (1.6) for $f \in C_0^\infty(\mathbb{R}^n)$, and then for $f \in W^1L_{n,\beta,\frac{\beta-1}{\beta},\frac{\beta-\alpha}{\beta}}(\mathbb{R}^n)$. □

3 Optimality of the embedding constant

In this section, we shall prove that the embedding constant $\frac{\beta}{\alpha-1}$ is best-possible in the inequalities (1.5) in Theorem 1.1 and (1.6) in Theorem 1.2.

First, we consider the optimality of $\frac{\beta}{\alpha-1}$ in (1.5). As a direct consequence of (1.5), we obtain

$$\left(\int_{B_R} \frac{|f(x)|^\beta}{|\log\frac{R}{|x|}|^\alpha}\frac{dx}{|x|^n}\right)^{\frac{1}{\beta}} \leq \frac{\beta}{\alpha-1}\left(\int_{B_R}|x|^{\beta-n}\left|\log\frac{R}{|x|}\right|^{\beta-\alpha}\left|\nabla f(x)\right|^\beta\,dx\right)^{\frac{1}{\beta}} \tag{3.1}$$

for all $f \in W_0^1L_{n,\beta,\frac{\beta-\alpha}{\beta}}(B_R)$. Therefore, it suffices to prove the optimality of $\frac{\beta}{\alpha-1}$ in (3.1). Define a sequence of functions $\{f_m\}$ for large $m \in \mathbb{N}$ by

$$f_m(x) := \begin{cases} (\log(mR))^{\frac{\alpha-1}{\beta}} & \text{when } |x| \leq \frac{1}{m}, \\ (\log\frac{R}{|x|})^{\frac{\alpha-1}{\beta}} & \text{when } \frac{1}{m} \leq |x| \leq \frac{R}{2}, \\ (\log 2)^{\frac{\alpha-1}{\beta}}\frac{2}{R}(R-|x|) & \text{when } \frac{R}{2} \leq |x| \leq R. \end{cases}$$

We can easily check $f_m \in W_0^1L_{n,\beta,\frac{\beta-\alpha}{\beta}}(B_R)$. More precisely, we calculate the norm $\|f_m\|_{W^1L_{n,\beta,\frac{\beta-\alpha}{\beta}}(B_R)}$ below. Letting $\tilde{f}_m(r) := f_m(x)$ with $r = |x| \geq 0$, we have

$$\tilde{f}_m'(r) = \begin{cases} 0 & \text{when } r < \frac{1}{m}, \\ -\frac{\alpha-1}{\beta}r^{-1}(\log\frac{R}{r})^{\frac{\alpha-1}{\beta}-1} & \text{when } \frac{1}{m} < r < \frac{R}{2}, \\ -(\log 2)^{\frac{\alpha-1}{\beta}}\frac{2}{R} & \text{when } \frac{R}{2} < r < R. \end{cases}$$

Thus a direct calculation yields

$$\begin{aligned} &\int_{B_R}|x|^{\beta-n}\left|\log\frac{R}{|x|}\right|^{\beta-\alpha}\left|\nabla f_m(x)\right|^\beta\,dx \\ &\quad = n\omega_n\int_0^R\left|\log\frac{R}{r}\right|^{\beta-\alpha}\left|\tilde{f}_m'(r)\right|^\beta r^{\beta-1}\,dr \\ &\quad = n\omega_n\left(\frac{\alpha-1}{\beta}\right)^\beta\int_{\frac{1}{m}}^{\frac{R}{2}}\left(\log\frac{R}{r}\right)^{-1}r^{-1}\,dr \\ &\qquad + n\omega_n(\log 2)^{\alpha-1}\left(\frac{2}{R}\right)^\beta\int_{\frac{R}{2}}^R\left(\log\frac{R}{r}\right)^{\beta-\alpha}r^{\beta-1}\,dr \\ &\quad = n\omega_n\left(\frac{\alpha-1}{\beta}\right)^\beta\left(\log\bigl(\log(mR)\bigr)-\log(\log 2)\right) \\ &\qquad + n\omega_n 2^\beta(\log 2)^{\alpha-1}\int_0^{\log 2}s^{\beta-\alpha}e^{-\beta s}\,ds \\ &\quad =: n\omega_n\left(\frac{\alpha-1}{\beta}\right)^\beta\left(\log\bigl(\log(mR)\bigr)-\log(\log 2)\right) + n\omega_n C_{\alpha,\beta}, \end{aligned} \tag{3.2}$$

where note that the last integral in (3.2) is finite by the assumption $\beta - \alpha > -1$. On the other hand, we see

$$\begin{aligned}
&\int_{B_R} \frac{|f_m(x)|^\beta}{|\log \frac{R}{|x|}|^\alpha} \frac{dx}{|x|^n} \\
&= n\omega_n \int_0^R \frac{|\tilde{f}_m(r)|^\beta}{|\log \frac{R}{r}|^\alpha} \frac{dr}{r} \\
&= n\omega_n \bigl(\log(mR)\bigr)^{\alpha-1} \int_0^{\frac{1}{m}} \left(\log \frac{R}{r}\right)^{-\alpha} r^{-1}\, dr \\
&\quad + n\omega_n \int_{\frac{1}{m}}^{\frac{R}{2}} \left(\log \frac{R}{r}\right)^{-1} r^{-1}\, dr \\
&\quad + n\omega_n (\log 2)^{\alpha-1} \left(\frac{2}{R}\right)^\beta \int_{\frac{R}{2}}^R (R-r)^\beta \left(\log \frac{R}{r}\right)^{-\alpha} r^{-1}\, dr \\
&=: \frac{n\omega_n}{\alpha-1} + n\omega_n \bigl(\log\bigl(\log(mR)\bigr) - \log(\log 2)\bigr) + n\omega_n C_{R,\alpha,\beta}.
\end{aligned} \tag{3.3}$$

Here, by applying the inequality $\log \frac{R}{r} \geq \frac{R-r}{R}$ for all $r \leq R$, we can estimate $C_{R,\alpha,\beta}$ as follows:

$$\begin{aligned}
C_{R,\alpha,\beta} &\leq 2^{\beta+1}(\log 2)^{\alpha-1} R^{-\beta+\alpha-1} \int_{\frac{R}{2}}^R (R-r)^{\beta-\alpha}\, dr \\
&= 2^{\beta+1}(\log 2)^{\alpha-1} R^{-\beta+\alpha-1} \int_0^{\frac{R}{2}} s^{\beta-\alpha}\, ds = \frac{2^\alpha (\log 2)^{\alpha-1}}{\beta-\alpha+1},
\end{aligned}$$

where we have used $\beta - \alpha + 1 > 0$ by the assumption. Summing up (3.2) and (3.3), we obtain

$$\begin{aligned}
&\int_{B_R} |x|^{\beta-n} \left|\log \frac{eR}{|x|}\right|^{\beta-1} \left|\log\left|\log \frac{eR}{|x|}\right|\right|^{\beta-\alpha} \bigl|\nabla f_m(x)\bigr|^\beta\, dx \\
&\quad \times \left(\int_{B_R} \frac{|f_m(x)|^\beta}{|\log|\log \frac{eR}{|x|}||^\alpha |\log \frac{eR}{|x|}|} \frac{dx}{|x|^n}\right)^{-1} \to \left(\frac{\alpha-1}{\beta}\right)^\beta
\end{aligned}$$

as $m \to \infty$, which implies that the constant $\frac{\beta}{\alpha-1}$ in (3.1) is best-possible.

We next consider the optimality of $\frac{\beta}{\alpha-1}$ in (1.6) in Theorem 1.2. As a direct consequence of (1.6), we obtain

$$\begin{aligned}
&\left(\int_{B_R} \frac{|f(x)|^\beta}{|\log|\log \frac{eR}{|x|}||^\alpha |\log \frac{eR}{|x|}|} \frac{dx}{|x|^n}\right)^{\frac{1}{\beta}} \\
&\leq \frac{\beta}{\alpha-1} \left(\int_{B_R} |x|^{\beta-n} \left|\log \frac{eR}{|x|}\right|^{\beta-1} \left|\log\left|\log \frac{eR}{|x|}\right|\right|^{\beta-\alpha} \bigl|\nabla f(x)\bigr|^\beta\, dx\right)^{\frac{1}{\beta}}
\end{aligned} \tag{3.4}$$

for all $f \in W_0^1 L_{n,\beta,\frac{\beta-1}{\beta},\frac{\beta-\alpha}{\beta}}(B_R)$. In order to prove that the constant $\frac{\beta}{\alpha-1}$ in (3.4) is best-possible, we take a sequence of functions $\{f_m\}$ for large $m \in \mathbb{N}$ defined by

$$f_m(x) := \begin{cases} (\log(\log(meR)))^{\frac{\alpha-1}{\beta}} & \text{when } |x| \leq \frac{1}{m}, \\ (\log(\log \frac{eR}{|x|}))^{\frac{\alpha-1}{\beta}} & \text{when } \frac{1}{m} \leq |x| \leq \frac{R}{2}, \\ (\log(\log(2e)))^{\frac{\alpha-1}{\beta}} \frac{2}{R}(R-|x|) & \text{when } \frac{R}{2} \leq |x| \leq R. \end{cases}$$

Then a direct calculation yields

$$\begin{aligned}
&\int_{B_R} |x|^{\beta-n} \left|\log\frac{eR}{|x|}\right|^{\beta-1} \left|\log\left|\log\frac{eR}{|x|}\right|\right|^{\beta-\alpha} \left|\nabla f_m(x)\right|^{\beta}\,dx \\
&\quad = n\omega_n\left(\frac{\alpha-1}{\beta}\right)^{\beta}\Bigl(\log\bigl(\log\bigl(\log(meR)\bigr)\bigr) - \log\bigl(\log\bigl(\log(2e)\bigr)\bigr)\Bigr) + n\omega_n C_{\alpha,\beta},
\end{aligned} \tag{3.5}$$

where

$$C_{\alpha,\beta} := (2e)^{\beta}\bigl(\log\bigl(\log(2e)\bigr)\bigr)^{\alpha-1}\int_0^{\log(\log(2e))} s^{\beta-\alpha}e^{\beta(s-e^s)}\,ds.$$

Note that the assumption $\beta-\alpha>-1$ implies $C_{\alpha,\beta}<+\infty$. Furthermore, we see

$$\begin{aligned}
&\int_{B_R} \frac{|f_m(x)|^{\beta}}{|\log|\log\frac{eR}{|x|}||^{\alpha}|\log\frac{eR}{|x|}|}\frac{dx}{|x|^n} \\
&\quad = \frac{n\omega_n}{\alpha-1} + n\omega_n\Bigl(\log\bigl(\log\bigl(\log(meR)\bigr)\bigr) - \log\bigl(\log\bigl(\log(2e)\bigr)\bigr)\Bigr) + n\omega_n C_{R,\alpha,\beta},
\end{aligned} \tag{3.6}$$

where

$$C_{R,\alpha,\beta} := \bigl(\log\bigl(\log(2e)\bigr)\bigr)^{\alpha-1}\left(\frac{2}{R}\right)^{\beta}\int_{\frac{R}{2}}^{R}(R-r)^{\beta}\left(\log\left(\log\frac{eR}{r}\right)\right)^{-\alpha}\left(\log\frac{eR}{r}\right)^{-1} r^{-1}\,dr.$$

Utilizing the elementary inequality $\log(\log\frac{eR}{r}) \ge \frac{R-r}{R}$ for all $r\le R$ and the assumption $\beta-\alpha+1>0$, we easily see that $C_{R,\alpha,\beta}<+\infty$. Hence, from (3.5) and (3.6), we obtain

$$\begin{aligned}
&\int_{B_R} |x|^{\beta-n} \left|\log\frac{eR}{|x|}\right|^{\beta-1} \left|\log\left|\log\frac{eR}{|x|}\right|\right|^{\beta-\alpha} \left|\nabla f_m(x)\right|^{\beta}\,dx \\
&\quad \times\left(\int_{B_R} \frac{|f_m(x)|^{\beta}}{|\log|\log\frac{eR}{|x|}||^{\alpha}|\log\frac{eR}{|x|}|}\frac{dx}{|x|^n}\right)^{-1} \to \left(\frac{\alpha-1}{\beta}\right)^{\beta}
\end{aligned}$$

as $m\to\infty$, which implies that the constant $\frac{\beta}{\alpha-1}$ in (3.4) is best-possible.

Competing interests
We declare that none of the authors have any competing interests in the manuscript.

Authors' contributions
SM and TO gave critical inspiration for the establishment of the Hardy type inequality in this paper. HW proved it rigorously and made the draft. All authors read and approved the final manuscript.

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