Chen and Fan *Fixed Point Theory and Applications* (2015) 2015:13
DOI 10.1186/s13663-015-0261-1

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Weak convergence theorems for inverse-strongly skew-monotone operators and generalized mixed equilibrium problems in Banach spaces

Junmin Chen[1*] and Tiegang Fan[1,2]

*Correspondence: chenjunm01@163.com
[1]College of Mathematics and Computer, Hebei University, Baoding, 071002, China
Full list of author information is available at the end of the article

Abstract

In this paper, we consider an iterative algorithm for finding the common element of the set of solutions for the generalized mixed equilibrium problems, the common fixed points set of two generalized nonexpansive type mappings, and the set of solutions of the variational inequality for an inverse-strongly skew-monotone operator in Banach spaces. Under mild conditions, the weak convergence theorem is established by using the sunny generalized nonexpansive retraction in Banach spaces. Our results refine, supplement, and extend the corresponding results in (Saewan *et al.* in Optim. Lett. 8:501-518, 2014), and other results announced by many other authors.

MSC: 47H05; 47H09; 47J25

Keywords: generalized nonexpansive type mapping; generalized mixed equilibrium problem; maximal monotone operator; inverse-strongly skew-monotone operator

1 Introduction

Let E be a real Banach space with dual space E^*, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty, closed, and convex set in E and J be the duality mapping from E to E^* such that JC is a closed and convex subset of E^*.

Let $A : C \to E^*$ be a monotone operator. The variational inequality problem is to find a point $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of the variational inequality problem (1.1) is denoted by $\mathrm{VI}(C, A)$. Such a problem is connected with the convex minimization problem, the complementarity problem. A well-known method for solving the variational inequality (1.1) is the projection method. Many researchers have studied this algorithm in a Hilbert space and in a Banach space, for instance, [1, 2].

Recall that a mapping $A : C^* \subset E^* \to E$ is said to be *skew-monotone*, if for each $x^*, y^* \in C^*$,

$$\langle Ax^* - Ay^*, x^* - y^*\rangle \geq 0.$$

Recall that a mapping $A : C^* \subset E^* \to E$ is said to be *α-inverse-strongly skew-monotone*, if there exists a positive number α such that

$$\langle Ax^* - Ay^*, x^* - y^* \rangle \geq \alpha \|Ax^* - Ay^*\|^2$$

for all $x^*, y^* \in C^*$. In this case, A is Lipschitz continuous, that is,

$$\|Ax^* - Ay^*\| \leq \frac{1}{\alpha}\|x^* - y^*\|$$

for all $x^*, y^* \in D(A)$.

Recently, Plubtieng and Sriprad [3] considered the variational inequality problem to find $u \in C$ such that

$$\langle AJu, Jv - Ju \rangle \geq 0, \quad \forall v \in C. \tag{1.2}$$

The set of solutions of the variational inequality problem (1.2) is denoted by $\mathrm{VI}(JC, A)$. By using the projection algorithm method with a sunny generalized nonexpansive retraction which was introduced by Ibaraki and Takahashi [4], Plubtieng and Sriprad [3] introduced the following iterative scheme: $x_1 = x \in C$ and

$$x_{n+1} = R_C(x_n - \lambda_n AJx_n) \tag{1.3}$$

for every $n = 1, 2, \ldots$, where R_C is the sunny generalized nonexpansive retraction of E onto C, $A : D(A) \subset E^* \to E$ is an α-inverse-strongly skew-monotone operator, and A satisfies the condition: $\|AJy\| \leq \|AJy - AJu\|$ for all $y \in C$ and $u \in \mathrm{VI}(JC, A)$. They proved that the sequence $\{x_n\}$ generated by (1.3) converges weakly to some element of $\mathrm{VI}(JC, A)$.

The equilibrium problem represents an important area of mathematical sciences such as game theory, financial mathematics, optimization, and so on. Let C be a nonempty, closed, and convex subset of a smooth, strictly convex, and reflexive Banach space E such that JC is closed and convex. In order to solve the equilibrium problem, we define a bifunction $F : JC \times JC \to \mathcal{R}$ satisfying the following conditions:

(A1) $F(x^*, x^*) = 0$, $\forall x^* \in JC$;

(A2) F is monotone, *i.e.*, $F(x^*, y^*) + F(y^*, x^*) \leq 0$, $\forall x^*, y^* \in J(C)$;

(A3) $\lim_{t\downarrow 0} F(tz^* + (1-t)x^*, y^*) \leq F(x^*, y^*)$, $\forall x^*, y^*, z^* \in J(C)$;

(A4) for each $x^* \in J(C)$, $y^* \mapsto F(x^*, y^*)$ is convex and lower semicontinuous.

The *generalized mixed equilibrium problem* is to find $p \in C$ such that

$$F(Jp, Jy) + \langle AJp, Jy - Jp \rangle + \psi(Jy) - \psi(Jp) \geq 0, \quad \forall y \in C, \tag{1.4}$$

where F is a bifunction satisfying suitable conditions, $A : JC \to E$ is a skew-monotone operator, ψ is a real-valued function. The set of solutions of (1.4) is denoted by $\mathrm{GMEP}(F, A, \psi)$, *i.e.*,

$$\mathrm{GMEP}(F, A, \psi) = \bigl\{p \in C : F(Jp, Jy) + \langle AJp, Jy - Jp \rangle + \psi(Jy) - \psi(Jp) \geq 0, \forall y \in C\bigr\}.$$

If $A = 0$, the problem (1.4) reduces to the *mixed equilibrium problem*, which is to find $p \in C$ such that

$$F(Jp, Jy) + \psi(Jy) - \psi(Jp) \geq 0, \quad \forall y \in C. \tag{1.5}$$

The set of solutions of (1.5) is denoted by $\mathrm{MEP}(F, \psi)$, *i.e.*,

$$\mathrm{MEP}(F, \psi) = \{p \in C : F(Jp, Jy) + \psi(Jy) - \psi(Jp) \geq 0, \forall y \in C\}.$$

If $A = 0$, $\psi = 0$, the problem (1.4) reduces to the *equilibrium problem* which is to find $p \in C$ such that

$$F(Jp, Jy) \geq 0, \quad \forall y \in C. \tag{1.6}$$

The set of solutions of (1.6) is denoted by $\mathrm{EP}(F)$. The above formulation (1.6) was considered by Takahashi and Zembayashi [5] and they proved a strong convergence theorem for finding a solution of the equilibrium problem (1.6) in Banach spaces.

There are many authors who studied the problem of finding a common element of the fixed point of nonlinear mappings and the set of solutions of the equilibrium problem in a Hilbert space or in a Banach space, for instance, [6–24]. In [13], Saewan *et al.* introduced a new iterative scheme for finding a common element of the set of solutions of the mixed equilibrium problem and the set of fixed points for a closed ϕ-nonexpansive mapping by using the sunny generalized nonexpansive retraction in Banach spaces. In [18], using the hybrid method, Takahashi and Yao proved a strong convergence theorem for generalized nonexpansive type mappings with equilibrium problems in Banach spaces.

Motivated by [3, 18], and [13], in this paper, using the projection algorithm method with the sunny generalized nonexpansive retraction R_C, we introduce an iterative scheme to find a common element of the set of solutions for the generalized mixed equilibrium problem, the common fixed points for two generalized nonexpansive type mappings and the set of solutions of the variational inequality in Banach spaces. Our results refine, supplement, and extend the corresponding results in [13], and other results announced by many others.

2 Preliminaries

Let E be a real Banach space with dual space E^*, C be a nonempty, closed, and convex subset of E, $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. The normalized duality mapping J from E into 2^{E^*} is given by $J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$, $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if, for each $x, y \in U$, the limit $\lim_{t \to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists. The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \leq \epsilon\right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x \neq y \in E$ with $\|x\| = \|y\| = 1$. It is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. It is well known that E is reflexive if and only if J is surjective; E is smooth if and only if J is single valued; E is strictly convex if and only if J is one-to-one. When E is a reflexive, strictly convex, and smooth space, J^{-1} is also single valued, one-to-one, surjective, and it is also the duality mapping from E^* to E.

In this paper, we denote the *strong convergence*, *weak convergence*, and *weak** *convergence* of a sequence $\{x_n\}$ by $x_n \to x$, $x_n \rightharpoonup x$, and $x_n \rightharpoonup^* x$, respectively.

The duality mapping J is said to be *weakly sequentially continuous* if the weak convergence of a sequence $\{x_n\}$ to x implies the weak* convergence of $\{Jx_n\}$ to Jx in E^*.

The function $\phi : E \times E \to (-\infty, +\infty)$ is defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for all $x, y \in E$; refer to [25]. From $(\|x\|^2 - \|y\|^2)^2 \leq \phi(x,y)$, for all $x, y \in E$, it is easy to see that $\phi(x,y) \geq 0$. From the definition of ϕ, we obtain

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle.$$

If C is a nonempty, closed, and convex subset of E, then for all $x \in E$ there exists a unique $z \in C$ such that

$$\phi(z,x) = \min_{y\in C} \phi(y,x).$$

We denote z by $\Pi_C x$, the mapping Π_C is called a generalized projection from E to C; see [25].

Definition 2.1 Let C be a nonempty closed subset of a real Banach space E,

(1) $T : C \to E$ is said to be *ϕ-nonexpansive* [16] if $F(T) \neq \emptyset$ and $\phi(Tx, Ty) \leq \phi(x,y)$ for all $x, y \in C$;

(2) $T : C \to E$ is said to be *generalized nonexpansive* if $F(T) \neq \emptyset$ and $\phi(Tx, y) \leq \phi(x,y)$ for all $x \in C$, $y \in F(T)$.

Definition 2.2 Let C be a nonempty closed subset of a real Banach space E. A mapping $T : C \to C$ is said to be

(1) *non-spreading* [18], if $\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x)$ for all $x, y \in E$;

(2) *quasi-non-spreading* or *hemi-relatively nonexpansive* [18, 26] if $F(T) \neq \emptyset$ and $\phi(y, Tx) \leq \phi(y, x)$ for all $x \in E$, $y \in F(T)$.

Definition 2.3 Let C be a nonempty closed subset of a real Banach space E. A mapping $T : C \to C$ is said to be *generalized nonexpansive type* if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(x, Ty) + \phi(y, Tx)$$

for all $x, y \in E$.

The asymptotic behavior of non-spreading mappings and generalized nonexpansive type mappings was studied in [18].

We need the following lemmas and theorems for the proofs of our main results.

Lemma 2.1 [27] *Let E be a 2-uniformly smooth Banach space. Then for all $x, y \in E$ there exists a constant $c > 0$ such that*

$$\|Jx - Jy\| \leq c\|x - y\|,$$

where J is the duality mapping of E.

Lemma 2.2 [28] *Let E be a uniformly convex and smooth Banach space, and let $\{x_n\}$, $\{y_n\}$ be sequences in E. If $\{x_n\}$ or $\{y_n\}$ is bounded and $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

We make use of the following mapping V:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2$$

for all $x \in E$ and $x^* \in E^*$. It is obvious that $V(x, x^*) = \phi(x, J^{-1}x^*)$ and $V(x, Jy) = \phi(x, y)$.

Lemma 2.3 *Let E be a strictly convex, smooth, and reflexive Banach space. Then*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

A mapping $R : E \to C$ is called *sunny* if

$$R(Rx + t(x - Rx)) = Rx, \quad \forall x \in E, \forall t \geq 0.$$

A mapping $R : E \to C$ is said to be a *retraction* if $R^2x = Rx$, $\forall x \in E$. If E is smooth and strictly convex, then a sunny generalized nonexpansive retraction of E to C is uniquely determined if it exists; refer to [4]. We also know that if E is reflexive, smooth and strictly convex, C is a nonempty closed subset of E, then there exists a sunny generalized nonexpansive retraction R_C from E onto C if and only if $J(C)$ is closed and convex. In this case, R_C is given by $R_C = J^{-1}\Pi_{J(C)}J$; refer to [29].

The following theorems are in Ibaraki and Takahashi [4].

Theorem 2.4 [4] *Let C be a nonempty closed and a sunny generalized nonexpansive retraction of a smooth, strictly convex Banach space E, and let R be a retraction from E to C. Then the following are equivalent:*

(1) *R is sunny generalized nonexpansive;*
(2) *$\langle x - Rx, Jy - JRx\rangle \leq 0$ for all $x \in E$ and $y \in C$.*

Theorem 2.5 [4] *Let C be a nonempty closed subset and a sunny generalized nonexpansive retraction of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Theorem 2.6 [4] *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C. Let $x \in E$ and $z \in C$. Then the following hold:*

(1) *$z = Rx$ if and only if $\langle x - z, Jy - Jz\rangle \leq 0$ for all $y \in C$*;
(2) $\phi(x, Rx) + \phi(Rx, z) \leq \phi(x, z)$.

Plubtieng and Sriprad [3] proved the following theorems.

Theorem 2.7 [3] *Let E be a Banach space with the dual space E^*. Let C^* be a nonempty, compact, and convex subset of E^* and let A be a skew-monotone and hemicontinuous operator of C^* into E. Then there exists $x_0^* \in C^*$ such that*

$$\langle Ax_0^*, x^* - x_0^*\rangle \geq 0, \quad \forall x^* \in C^*.$$

Theorem 2.8 [3] *Let C be a nonempty, compact, and convex subset of a smooth, strictly convex, and reflexive Banach space E such that JC is a closed and convex set. Let A be a skew-monotone operator of JC into E. Then*

$$u \in \mathrm{VI}(JC, A) \quad \Leftrightarrow \quad u = R_C(u - \lambda Au), \quad \forall \lambda > 0,$$

where R_C is the generalized nonexpansive retraction of E onto C.

Let E be a Banach space with the dual space E^* and let C be a nonempty, compact, and convex subset such that JC is a closed and convex set. Let i_{JC} be indicator of JC. Since $i_{JC} : E^* \to (-\infty, +\infty)$ is proper, lower semicontinuous, and convex, the subdifferential ∂i_{JC} of i_{JC} defined by

$$\partial i_{JC} = \{x \in E : i_{JC}(y^*) \geq i_{JC}(x^*) + \langle x, y^* - x^*\rangle, \forall y^* \in E^*\}, \quad \forall x^* \in E^*,$$

is a *maximal skew-monotone operator* [30]. We denote by $N_{JC}(x^*)$ the *skew normal cone* of JC at a point $x^* \in JC$, that is,

$$N_{JC}(x^*) = \{x \in E : \langle x, x^* - y^*\rangle \geq 0, \forall y^* \in JC\}.$$

From [3, 30], we know that

$$\partial i_{JC} = \begin{cases} N_{JC}(x^*), & x^* \in JC, \\ \emptyset, & x^* \notin JC. \end{cases}$$

An operator $A : D(A) \subset E^* \to E$ is said to be *hemicontinuous* if for all $x^*, y^* \in D(A)$, the mapping f of $[0, 1]$ into E defined by $f(t) = A(tx^* + (1 - t)y^*)$ is continuous.

Theorem 2.9 [3] *Let C be a nonempty, compact, and convex subset of a smooth, strictly convex, and reflexive Banach space E such that JC is closed and convex and let A be a skew-monotone and hemicontinuous operator of JC into E. Let $B \subset E^* \times E$ be an operator defined as follows*:

$$Bv^* = \begin{cases} Av^* + N_{JC}(x^*), & v^* \in JC, \\ \emptyset, & v^* \notin JC. \end{cases}$$

Then B is maximal skew-monotone and $(BJ)^{-1}0 = \mathrm{VI}(JC, A)$.

Let E be a smooth Banach space and let C be a nonempty subset of E. Let $T : C \to C$ be a mapping. Then $p \in C$ is called a *generalized asymptotically fixed point* of T if there exists $\{x_n\} \subset C$ such that $Jx_n \rightharpoonup Jp$ and $\lim_{n\to\infty} \|Jx_n - JTx_n\| = 0$. We denote the set of generalized asymptotically fixed points of T by $\check{F}(T)$.

Lemma 2.10 [18] *Let E be a reflexive smooth Banach space and E^* has a uniformly Gâteaux differentiable norm. Let C be a nonempty closed subset of E such that JC is closed and convex. Let T be a generalized nonexpansive type mapping of C into itself, i.e.,*

$$\phi(Tx, Ty) + \phi(Ty, Tx) \le \phi(x, Ty) + \phi(y, Tx)$$

for all $x, y \in C$. Then the following hold:

(1) $\check{F}(T) = F(T)$;
(2) $JF(T)$ *is closed and convex*;
(3) $F(T)$ *is closed.*

Lemma 2.11 [28] *Let E be a smooth and uniformly convex Banach space and $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathcal{R}$ such that $g(0) = 0$ and $g(\|x - y\|) \le \phi(x, y)$ for all $x, y \in B_r(0)$, where $B_r(0) = \{z \in E : \|z\| \le r\}$.*

Lemma 2.12 [31] *Let E be a uniformly convex Banach space and $r > 0$. Then there exists a strictly increasing, continuous and convex function $h : [0, \infty) \to [0, \infty)$ such that $h(0) = 0$ and*

$$\|\lambda x + \mu y + \gamma z\|^2 \le \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu h(\|x - y\|)$$

for all $x, y, z \in B_r(0)$, where $B_r(0) = \{z \in E : \|z\| \le r\}$, and $\lambda + \mu + \gamma = 1$.

Lemma 2.13 [32] *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le a_n + b_n, \quad \forall n \ge 1.$$

If $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

3 Weak convergence theorems

In this section, we prove a weak convergence theorem for an inverse-strongly skew-monotone operator and two generalized nonexpansive type mappings applying the sunny generalized nonexpansive retraction in Banach spaces.

Lemma 3.1 *Let E be a reflexive, strictly convex, and uniformly smooth Banach space with dual space E^*, and let C be a nonempty, compact, and convex subset of E such that JC is closed and convex, C^* be a nonempty, closed, and convex subset of E^*. $F : JC \times JC \to (-\infty, +\infty)$ that satisfies the conditions* (A1)-(A4). *Let $A : C^* \to E^*$ be a β-inverse-strongly skew-monotone operator and $\psi : JC \to (-\infty, +\infty)$ be a convex and lower semicontinuous, $r > 0$ be a given real number, and $x \in E$ be any point. Then there exists $z \in C$ such that*

$$F(Jz, Jy) + \langle AJz, Jy - Jz\rangle + \psi(Jy) - \psi(Jz) + \frac{1}{r}\langle z - x, Jy - Jz\rangle \ge 0, \quad \forall y \in C,$$

where J is the duality mapping from E to E^*.

Lemma 3.2 *Let C be a nonempty, closed, and convex subset of a uniformly smooth and strictly convex real Banach space E such that JC is closed and convex, let C^* be a nonempty, closed, and convex subset of E^*. $F : JC \times JC \to (-\infty, +\infty)$ that satisfies the conditions* (A1)-(A4). *Let $A : C^* \to E^*$ be a β-inverse-strongly skew-monotone operator and $\psi : JC \to (-\infty, +\infty)$ be a convex and lower semicontinuous function, $r > 0$ be a given real number, and $x \in E$ be any point. Define a mapping $K_r : E \to C$ as follows*:

$$K_r(x) = \Big\{ z \in C : F(Jz, Jy) + \langle AJz, Jy - Jz\rangle + \psi(Jy) - \psi(Jz) + \frac{1}{r}\langle z - x, Jy - Jz\rangle \geq 0, \forall y \in C \Big\}.$$

Then the following hold:

(1) K_r *is single valued*;

(2) K_r *is firmly generalized nonexpansive, i.e.*,

$$\langle K_r x - K_r y, JK_r x - JK_r y\rangle \leq \langle x - y, JK_r x - JK_r y\rangle, \quad \forall x, y \in E;$$

(3) $F(K_r) = \mathrm{GMEP}(F, A, \psi)$;

(4) $J(\mathrm{GMEP}(F, A, \psi))$ *is convex and closed*;

(5) $\phi(x, K_r x) + \phi(K_r x, p) \leq \phi(x, p)$, $\forall x \in E$, $p \in F(K_r)$,

where J is the duality mapping from E to E^*.

Remark 3.1 The proof of Lemmas 3.1 and 3.2 is similar to the proof of Lemma 24 and Theorem 25 in [33]; for details please refer to [33].

Theorem 3.3 *Let E be a uniformly convex and* 2*-uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous, and let C be a closed convex subset of E such that JC is closed and convex. Let $F : JC \times JC \to (-\infty, +\infty)$ be a bifunction satisfying* (A1)-(A4), *$A : JC \to E$ be a β-inverse-strongly skew-monotone operator. Let $B : JC \to E$ be an α-inverse-strongly skew-monotone operator such that $\mathrm{VI}(JC, B) \neq \emptyset$ and $\|BJy\| \leq \|BJy - BJu\|$ for all $y \in C$ and $u \in \mathrm{VI}(JC, B)$. Let S, T be two generalized nonexpansive type mappings of C into itself such that $\Gamma := \mathrm{VI}(JC, B) \cap \mathrm{GMEP}(F, A, \psi) \cap F(S) \cap F(T)$ is not empty. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} u_n = K_{r_n} x_n, \\ z_n = R_C(u_n - \lambda_n BJu_n), \\ y_n = \alpha_n^{(1)} u_n + \alpha_n^{(2)} S z_n + \alpha_n^{(3)} T z_n, \\ x_{n+1} = R_C(\beta_n x + (1 - \beta_n) y_n), \end{cases}$$

where R_C is the sunny generalized nonexpansive retraction of E onto C, J is the duality mapping on E. $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < \frac{\alpha}{c}$, where c is the constant in Lemma 2.1. *The following conditions are satisfied*:

(i) $\beta_n \in (0, 1)$, $\sum_{n=1}^{\infty} \beta_n < +\infty$;

(ii) $\alpha_n^{(1)} + \alpha_n^{(2)} + \alpha_n^{(3)} = 1$, $\limsup_{n\to\infty} \alpha_n^{(1)} < 1$, $\liminf_{n\to\infty} \alpha_n^{(1)} \alpha_n^{(2)} > 0$, *and* $\liminf_{n\to\infty} \alpha_n^{(1)} \alpha_n^{(3)} > 0$;

(iii) $\liminf_{n\to\infty} r_n = \eta > 0$.

Then x_n converges weakly to $u \in \Gamma$, where $u = \lim_{n\to\infty} R_\Gamma x_n$.

Proof First, we show that $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ are bounded.

Let $p \in \Gamma$,

$$\phi(u_n, p) = \phi(K_{r_n} x_n, p) \le \phi(x_n, p). \tag{3.1}$$

Let $v_n = u_n - \lambda_n BJu_n$, from Lemma 2.3 and Theorem 2.6, we have

$$\begin{aligned}
\phi(z_n, p) &= \phi\big(R_C(u_n - \lambda_n BJu_n), p\big) \\
&\le \phi(v_n, p) - \phi(v_n, z_n) \\
&\le \phi(u_n - \lambda_n BJu_n, p) \\
&= V(u_n - \lambda_n BJu_n, Jp) \\
&\le V(u_n - \lambda_n BJu_n + \lambda_n BJu_n, Jp) - 2\big\langle \lambda_n BJu_n, J(u_n - \lambda_n BJu_n) - Jp\big\rangle \\
&= V(u_n, Jp) - 2\lambda_n \big\langle BJu_n, J(u_n - \lambda_n BJu_n) - Jp\big\rangle \\
&= \phi(u_n, p) - 2\lambda_n \langle BJu_n, Ju_n - Jp\rangle \\
&\quad + 2\big\langle -\lambda_n BJu_n, J(u_n - \lambda_n BJu_n) - Ju_n\big\rangle.
\end{aligned} \tag{3.2}$$

Since B is α-inverse-strongly monotone and $p \in \mathrm{VI}(JC, B)$, it follows that

$$\begin{aligned}
&-2\lambda_n \langle BJu_n, Ju_n - Jp\rangle \\
&\quad = -2\lambda_n \langle BJu_n - BJp, Ju_n - Jp\rangle - 2\lambda_n \langle BJp, Ju_n - Jp\rangle \\
&\quad \le -2\lambda_n \alpha \|BJu_n - BJp\|^2.
\end{aligned} \tag{3.3}$$

By Lemma 2.1 and the assumption, we get

$$\begin{aligned}
&2\big\langle -\lambda_n BJu_n, J(u_n - \lambda_n BJu_n) - Ju_n\big\rangle \\
&\quad \le 2\|\lambda_n BJu_n\| \big\|J(u_n - \lambda_n BJu_n) - Ju_n\big\| \\
&\quad \le 2c\lambda_n \|BJu_n\| \|u_n - \lambda_n BJu_n - u_n\| \\
&\quad = 2c\lambda_n^2 \|BJu_n\|^2 \le 2c\lambda_n^2 \|BJu_n - Jp\|^2,
\end{aligned} \tag{3.4}$$

substituting (3.3) and (3.4) into (3.2), and from $b < \frac{\alpha}{c}$, we can get

$$\begin{aligned}
\phi(z_n, p) &\le \phi(v_n, p) - \phi(v_n, z_n) \\
&\le \phi(u_n, p) - 2\lambda_n \alpha \|BJu_n - BJp\|^2 + 2c\lambda_n^2 \|BJu_n - Jp\|^2 \\
&\le \phi(u_n, p) + 2a(bc - \alpha) \|BJu_n - BJp\|^2 \\
&\le \phi(u_n, p) \le \phi(x_n, p).
\end{aligned} \tag{3.5}$$

Since S, T are two generalized nonexpansive type mappings, we have

$$\begin{aligned}
\phi(y_n, p) &= \phi\big(\alpha_n^{(1)} u_n + \alpha_n^{(2)} Sz_n + \alpha_n^{(3)} Tz_n, p\big) \\
&\le \alpha_n^{(1)} \phi(u_n, p) + \alpha_n^{(2)} \phi(Sz_n, p) + \alpha_n^{(3)} \phi(Tz_n, p)
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n^{(1)}\phi(u_n,p) + \alpha_n^{(2)}\phi(z_n,p) + \alpha_n^{(3)}\phi(z_n,p) \\
&\leq \phi(x_n,p).
\end{aligned} \tag{3.6}$$

Using Theorem 2.6, from (3.5) and (3.6), we have

$$\begin{aligned}
\phi(x_{n+1},p) &= \phi\bigl(R_C\bigl(\beta_n x + (1-\beta_n)y_n\bigr),p\bigr) \\
&\leq \phi\bigl(\beta_n x + (1-\beta_n)y_n,p\bigr) \\
&\leq \beta_n\phi(x,p) + (1-\beta_n)\phi(y_n,p) \\
&\leq \beta_n\phi(x,p) + (1-\beta_n)\phi(x_n,p) \\
&\leq \beta_n\phi(x,p) + \phi(x_n,p),
\end{aligned} \tag{3.7}$$

then, from Lemma 2.13, we see that $\lim_{n\to\infty}\phi(x_n,p)$ exists. Therefore, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ are bounded.

Applying (3.4)-(3.6) again, it follows that

$$\begin{aligned}
&\phi(x_{n+1},p) \\
&\quad\leq \beta_n\phi(x,p) + (1-\beta_n)\bigl\{\alpha_n^{(1)}\phi(x_n,p) \\
&\qquad + \bigl(1-\alpha_n^{(1)}\bigr)\bigl[\phi(x_n,p) - 2\lambda_n\alpha\|BJu_n - BJp\|^2 + 2c\lambda_n^2\|BJu_n - Jp\|^2\bigr]\bigr\} \\
&\quad\leq \beta_n\phi(x,p) + (1-\beta_n)\phi(x_n,p) \\
&\qquad + (1-\beta_n)\bigl(1-\alpha_n^{(1)}\bigr)2a(cb-\alpha)\|BJu_n - BJp\|^2.
\end{aligned} \tag{3.8}$$

So, we have

$$\begin{aligned}
&(1-\beta_n)\bigl(1-\alpha_n^{(1)}\bigr)2a(\alpha - cb)\|BJu_n - BJp\|^2 \\
&\quad\leq \beta_n\phi(x,p) + (1-\beta_n)\phi(x_n,p) - \phi(x_{n+1},p) \\
&\quad\leq \beta_n\phi(x,p) + \phi(x_n,p) - \phi(x_{n+1},p),
\end{aligned} \tag{3.9}$$

from $\lim_{n\to\infty}\beta_n = 0$, $\limsup_{n\to\infty}\alpha_n^{(1)} < 1$, and $\lim_{n\to\infty}\phi(x_n,p)$ exists; we have

$$\lim_{n\to\infty}\|BJu_n - BJp\| = 0. \tag{3.10}$$

From Theorem 2.6, Lemma 2.3, and (3.10), we have

$$\begin{aligned}
\phi(z_n,u_n) &= \phi\bigl(R_C(u_n - \lambda_n BJu_n),u_n\bigr) \\
&\leq \phi(u_n - \lambda_n BJu_n,u_n) \\
&= V(u_n - \lambda_n BJu_n,Ju_n) \\
&\leq V(u_n - \lambda_n BJu_n + \lambda_n BJu_n,Ju_n) \\
&\quad - 2\bigl\langle\lambda_n BJu_n,J(u_n - \lambda_n BJu_n) - Ju_n\bigr\rangle \\
&= V(u_n,Ju_n) - 2\lambda_n\bigl\langle BJu_n,J(u_n - \lambda_n BJu_n) - Ju_n\bigr\rangle
\end{aligned}$$

$$\begin{aligned}
&= \phi(u_n,u_n) - 2\lambda_n\langle BJu_n, J(u_n - \lambda_n BJu_n) - Ju_n\rangle \\
&= -2\lambda_n\langle BJu_n, J(u_n - \lambda_n BJu_n) - Ju_n\rangle \\
&\le 2\lambda_n\|BJu_n\|\,\|J(u_n - \lambda_n BJu_n) - Ju_n\| \\
&\le 2c\lambda_n^2\|BJu_n\| \le 2c\lambda_n^2\|BJu_n - BJp\|^2 \to 0,
\end{aligned}$$

then, from Lemma 2.2, we get

$$\lim_{n\to\infty}\|z_n - u_n\| = 0. \tag{3.11}$$

Since $\{x_n\}$, $\{z_n\}$ are bounded, by Lemma 2.12, we get

$$\begin{aligned}
&\phi(y_n,p) \\
&\quad= \phi\bigl(\alpha_n^{(1)}u_n + \alpha_n^{(1)}Sz_n + \alpha_n^{(3)}Tz_n, p\bigr) \\
&\quad= \bigl\|\alpha_n^{(1)}u_n + \alpha_n^{(1)}Sz_n + \alpha_n^{(3)}Tz_n\bigr\|^2 - 2\bigl\langle\alpha_n^{(1)}x_n + \alpha_n^{(2)}Sz_n + \alpha_n^{(3)}Tz_n, Jp\bigr\rangle + \|p\|^2 \\
&\quad\le \alpha_n^{(1)}\|u_n\|^2 + \alpha_n^{(2)}\|Sz_n\|^2 + \alpha_n^{(3)}\|Tz_n\|^2 - \alpha_n^{(1)}\alpha_n^{(2)}g\bigl(\|x_n - Sz_n\|\bigr) \\
&\qquad - 2\alpha_n^{(1)}\langle u_n, Jp\rangle + 2\alpha_n^{(2)}\langle Sz_n, Jp\rangle + 2\alpha_n^{(3)}\langle Tz_n, Jp\rangle + \|p\|^2 \\
&\quad= \alpha_n^{(1)}\phi(u_n,p) + \alpha_n^{(2)}\phi(Sz_n,p) + \alpha_n^{(3)}\phi(Tz_n,p) - \alpha_n^{(1)}\alpha_n^{(2)}g\bigl(\|x_n - Sz_n\|\bigr) \\
&\quad\le \phi(x_n,p) - \alpha_n^{(1)}\alpha_n^{(2)}g\bigl(\|x_n - Sz_n\|\bigr).
\end{aligned} \tag{3.12}$$

From the definition of R_C and (3.12), we obtain

$$\begin{aligned}
&\phi(x_{n+1},p) \\
&\quad= \phi\bigl(R_C\bigl(\beta_n x + (1-\beta_n)y_n\bigr), p\bigr) \\
&\quad\le \phi\bigl(\beta_n x + (1-\beta_n)y_n, p\bigr) \\
&\quad\le \beta_n\phi(x,p) + \phi(y_n,p) \\
&\quad= \beta_n\phi(x,p) + (1-\beta_n)\bigl[\phi(x_n,p) - \alpha_n^{(1)}\alpha_n^{(2)}g\bigl(\|x_n - Sz_n\|\bigr)\bigr],
\end{aligned} \tag{3.13}$$

this implies that

$$(1-\beta_n)\alpha_n^{(1)}\alpha_n^{(2)}g\bigl(\|x_n - Sz_n\|\bigr) \le \beta_n\phi(x,p) + (1-\beta_n)\phi(x_n,p) - \phi(x_{n+1},p). \tag{3.14}$$

Notice that the limit of $\{\phi(x_n,p)\}$ exists and we have the conditions (i), (ii), so we get

$$\lim_{n\to\infty} g\bigl(\|x_n - Sz_n\|\bigr) = 0.$$

From the property of g, we have

$$\lim_{n\to\infty}\|x_n - Sz_n\| = 0. \tag{3.15}$$

In a similar way, we can conclude

$$\lim_{n\to\infty}\|x_n - Tz_n\| = 0. \tag{3.16}$$

Noticing $u_{n+1} = K_{r_{n+1}}x_{n+1}$, by Lemma 3.2, we get

$$\begin{aligned}\phi(u_{n+1},p) &= \phi(K_{r_{n+1}}x_{n+1},p)\\ &\leq \phi(x_{n+1},p)\\ &\leq \beta_n\phi(x,p) + (1-\beta_n)\phi(u_n,p)\\ &\leq \beta_n\phi(x,p) + \phi(u_n,p),\end{aligned}$$

therefore $\lim_{n\to\infty}\phi(u_n,p)$ exists. Let $t_1 = \sup_{n\geq 1}\{\|x_n\|,\|u_n\|\}$. From Lemma 2.11, there exists a continuous strictly increasing and convex function g_1 with $g_1(0) = 0$ such that $g_1(\|x-y\|) \leq \phi(x,y)$, $\forall x,y \in B_{t_1}(0)$. From the definition of u_n and Lemma 3.2, for $p \in \Gamma$, we have

$$\begin{aligned}g_1\bigl(\|x_n-u_n\|\bigr) &\leq \phi(x_n,u_n)\\ &\leq \phi(x_n,K_{r_n}x_n)\\ &\leq \phi(x_n,p) - \phi(u_n,p)\\ &\leq \beta_{n-1}\phi(x,p) + \phi(u_{n-1},p) - \phi(u_n,p).\end{aligned}$$

Let $n\to\infty$, we obtain $\lim_{n\to\infty} g_1(\|x_n-u_n\|) = 0$. From the property of g_1, we also get

$$\lim_{n\to\infty}\|x_n-u_n\| = 0. \tag{3.17}$$

Since

$$\|z_n-x_n\| \leq \|z_n-u_n\| + \|u_n-x_n\|,$$

we get

$$\lim_{n\to\infty}\|z_n-x_n\| = 0.$$

Since

$$\|z_n-Sz_n\| \leq \|z_n-x_n\| + \|x_n-Sz_n\|,$$

we have

$$\lim_{n\to\infty}\|z_n-Sz_n\| = 0. \tag{3.18}$$

From E is uniformly smooth, J is norm-to-norm continuous. So, we have

$$\|Jx_n-Jz_n\| \to 0$$

and

$$\|Jz_n-JSz_n\| \to 0.$$

From $\{x_n\}$ being bounded, one finds that $\{Jx_n\}$ is bounded, so there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $Jx_{n_i} \rightharpoonup u^*$, and we get $Jz_{n_i} \rightharpoonup u^*$. From (3.18), we have $J^{-1}u^* \in \check{F}(S)$, and we can also obtain $J^{-1}u^* \in \check{F}(T)$. From Lemma 2.10, putting $u = J^{-1}u^*$, we have $u \in F(S)$ and $u \in F(T)$, *i.e.*, $u \in F(S) \cap F(T)$.

Next we show that $u \in \mathrm{VI}(JC, B)$. Let $H \subset E^* \times E$ be an operator as follows:

$$Hv^* = \begin{cases} Bv^* + N_{JC}(v^*), & v^* \in JC, \\ \emptyset, & v^* \notin JC. \end{cases}$$

By Theorem 2.9, H is maximal skew-monotone and $(HJ)^{-1}0 = \mathrm{VI}(JC, B)$. Let $(v^*, w) \in G(H)$. Since $w \in Hv^* = Bv^* + N_{JC}(v^*)$, it follows that $w - Bv^* \in N_{JC}(v^*)$. From $Jz_n \in JC$, we have

$$\langle w - Bv^*, v^* - Jz_n \rangle \geq 0. \tag{3.19}$$

Since $w \in Bv^*$, we get $v^* \in JC$. This implies that there is $v \in C$ such that $Jv = v^*$. Thus it follows from (3.19) that

$$\langle w - BJv, Jv - Jz_n \rangle \geq 0. \tag{3.20}$$

On the other hand, from $z_n = R_C(u_n - \lambda_n BJu_n)$, by Theorem 2.6, we have $\langle (u_n - \lambda_n BJu_n) - z_n, Jz_n - Jv \rangle \geq 0$, and

$$\left\langle \frac{u_n - z_n}{\lambda_n} - BJu_n, Jv - Jz_n \right\rangle \leq 0. \tag{3.21}$$

It follows from (3.20) and (3.21) that

$$\begin{aligned} \langle w, Jv - Jz_n \rangle &\geq \langle BJv, Jv - Jz_n \rangle \\ &\geq \langle BJv, Jv - Jz_n \rangle + \left\langle \frac{u_n - z_n}{\lambda_n} - BJu_n, Jv - Jz_n \right\rangle \\ &= \langle BJv - BJu_n, Jv - Jz_n \rangle + \left\langle \frac{u_n - z_n}{\lambda_n}, Jv - Jz_n \right\rangle \\ &= \langle BJv - BJz_n, Jv - Jz_n \rangle + \langle BJz_n - BJu_n, Jv - Jz_n \rangle \\ &\quad + \left\langle \frac{u_n - z_n}{\lambda_n}, Jv - Jz_n \right\rangle \\ &\geq -\frac{1}{\alpha} \|Jz_n - Ju_n\| \|Jv - Jz_n\| - \frac{1}{a} \|z_n - u_n\| \|Jv - Jz_n\| \\ &\geq -M\left(\frac{1}{\alpha} \|Jz_n - Ju_n\| + \frac{1}{a} \|z_n - u_n\| \right), \end{aligned}$$

where $M = \sup_{n \geq 1}\{\|Jv - Jz_n\|\}$. Taking $n = n_i$, from (3.11) and (3.20) and the weakly sequential continuity of J, we have $\langle w, Jv - Ju \rangle \geq 0$ as $i \to \infty$. So, by the maximality of H, we obtain $Ju \in H^{-1}0$, that is, $u \in (HJ)^{-1}0 = \mathrm{VI}(JC, B)$.

Now we show that $u \in \mathrm{GMEP}(F, A, \psi)$. From $u_n = K_{r_n} x_n$, we have

$$\Theta(Ju_n, Jy) + \frac{1}{r_n} \langle u_n - x_n, Jy - Ju_n \rangle \geq 0, \tag{3.22}$$

where $\Theta(Ju_n, Jy) = F(Ju_n, Jy) + \langle AJu_n, Jy - Ju_n\rangle + \phi(Jy) - \phi(Ju_n)$. Replacing n by n_i, we have

$$\frac{1}{r_{n_i}}\langle u_{n_i} - x_{n_i}, Jy - Ju_{n_i}\rangle \geq -\Theta(Ju_{n_i}, y) \geq \Theta(Jy, Ju_{n_i}), \quad \forall y \in C. \tag{3.23}$$

From $0 < \eta \leq r_n$ and (3.17), we have

$$\lim_{n\to\infty} \frac{u_{n_i} - x_{n_i}}{r_{n_i}} = 0.$$

So, we have

$$\Theta\bigl(Jy, u^*\bigr) \leq 0. \tag{3.24}$$

Put $u_t^* = tJy + (1-t)u^*$ for $t \in (0,1]$ and $y \in C$. From JC is convex, we have $u_t^* \in JC$. From (A1), (A4), and (3.24), we have

$$\begin{aligned} 0 &= \Theta\bigl(u_t^*, u_t^*\bigr) \leq t\Theta\bigl(u_t^*, Jy\bigr) + (1-t)\Theta\bigl(u_t^*, u^*\bigr) \\ &\leq t\Theta\bigl(u_t^*, Jy\bigr) \end{aligned}$$

and then

$$\Theta\bigl(u_t^*, Jy\bigr) \geq 0,$$

letting $t \to 0$, from (A3), we have

$$\Theta\bigl(u^*, Jy\bigr) \geq 0, \quad \forall y \in C.$$

This implies that $u \in \mathrm{GMEP}(F, A, \psi)$.
Hence, $u \in \Gamma := \mathrm{VI}(JC, B) \cap \mathrm{GMEP}(F, A, \psi) \cap F(S) \cap F(T)$.
Let $\xi_n = R_\Gamma x_n$ and $p \in \Gamma$. From $\xi_n \in \Gamma$ and Theorem 2.6, we have

$$\begin{aligned} \phi(x_n, \xi_n) &= \phi(x_n, R_\Gamma x_n) \\ &\leq \phi(x_n, p) - \phi(R_\Gamma x_n, p) \\ &\leq \phi(x_n, p), \end{aligned} \tag{3.25}$$

since $\{x_n\}$ is bounded, $\{\xi_n\}$ is also bounded.
Replacing x_n by x in (3.25), we see that $\{\phi(x, \xi_n)\}$ is bounded.
By Theorem 2.6 and (3.6), (3.7), we get

$$\begin{aligned} \phi(x_{n+1}, \xi_{n+1}) &= \phi(x_{n+1}, R_\Gamma x_{n+1}) \\ &\leq \phi(x_{n+1}, \xi_n) - \phi(R_\Gamma x_{n+1}, \xi_n) \\ &\leq \phi(x_{n+1}, \xi_n) \\ &\leq \beta_n\phi(x, \xi_n) + \phi(y_n, \xi_n) \\ &\leq \beta_n\phi(x, \xi_n) + \phi(x_n, \xi_n). \end{aligned}$$

From $\{\phi(x,\xi_n)\}$ being bounded, $\sum_{n=1}^{\infty}\beta_n < +\infty$, and by Lemma 2.13, the limit of $\{\phi(x_n,\xi_n)\}$ exists.

For any $m \in \mathcal{N}$, from (3.25), we have $\phi(x_{n+m},\xi_n) \leq \phi(x_n,\xi_n)$. Noticing $\xi_{n+m} = R_\Gamma x_{n+m}$, from Theorem 2.6, we obtain

$$\phi(\xi_{n+m},\xi_n) + \phi(x_{n+m},\xi_{n+m}) \leq \phi(x_{n+m},\xi_n) \leq \phi(x_n,\xi_n),$$

so

$$\phi(\xi_{n+m},\xi_n) \leq \phi(x_n,\xi_n) - \phi(x_{n+m},\xi_{n+m}).$$

Let $t_2 = \sup_{n\geq 1}\{\xi_n\}$, from Lemma 2.11, there exists a continuous strictly increasing and convex function g_2 with $g_2(0) = 0$ such that $g_2(\|x-y\|) \leq \phi(x,y)$, $\forall x,y \in B_{t_2}(0)$. Therefore, we have

$$g_2\bigl(\|\xi_{n+m}-\xi_n\|\bigr) \leq \phi(\xi_{n+m},\xi_n) \leq \phi(x_n,\xi_n) - \phi(x_{n+m},\xi_{n+m}).$$

Since $\{\phi(x_n,\xi_n)\}$ is a convergent sequence, by the property of g_2, we see that $\{\xi_n\}$ is a Cauchy sequence. Suppose that $\{\xi_n\}$ converges strongly to $w \in \Gamma$. Noticing $u \in \Gamma$, $\xi_n = R_\Gamma x_n$, and from Theorem 2.6, we obtain

$$\langle x_n - \xi_n, Ju - J\xi_n\rangle \leq 0.$$

J is weakly sequentially continuous, and we have

$$\langle u - w, Ju - Jw\rangle \leq 0.$$

On the other hand, J is monotone, and we have

$$\langle u - w, Ju - Jw\rangle \geq 0.$$

So, we get

$$\langle u - w, Ju - Jw\rangle = 0.$$

Therefore $u = w$. The proof of Theorem 3.3 is completed. □

If we put $A \equiv 0$ and $\psi \equiv 0$ in Theorem 3.3, then Theorem 3.3 reduces to the following result.

Corollary 3.4 *Let E be a uniformly convex and 2-uniformly smooth Banach space and let C be a closed subset of E such that JC is closed and convex. Let $F : JC \times JC \to (-\infty,+\infty)$ be a bifunction satisfying* (A1)-(A4). *Let $B : JC \to E$ be an α-inverse-strongly skew-monotone operator such that* $\mathrm{VI}(JC,B) \neq \emptyset$ *and $\|BJy\| \leq \|BJy - BJu\|$ for all $y \in C$ and $u \in$* $\mathrm{VI}(JC,B)$. *Let S, T be two generalized nonexpansive type mappings of C into itself such that $\Gamma :=$*

$\mathrm{VI}(JC,B) \cap \mathrm{EP}(F) \cap F(S) \cap F(T)$ *is not empty. Let* $\{x_n\}$ *be a sequence generated by* $x_1 = x \in C$ *and*

$$\begin{cases} F(Ju_n, Jy) + \frac{1}{r_n}\langle u_n - x_n, Jy - Ju_n\rangle \geq 0, \quad \forall y \in C, \\ z_n = R_C(u_n - \lambda_n BJu_n), \\ y_n = \alpha_n^{(1)} u_n + \alpha_n^{(2)} Sz_n + \alpha_n^{(3)} Tz_n, \\ x_{n+1} = R_C(\beta_n x + (1-\beta_n)y_n), \end{cases}$$

where R_C *is the sunny generalized nonexpansive retraction of E onto C, J is the duality mapping on E.* $\{\lambda_n\} \subset [a,b]$ *for some a, b with* $0 < a < b < \frac{\alpha}{c}$, *where c is the constant in Lemma* 2.1. *The following conditions are satisfied*:

(i) $\sum_{n=1}^{\infty} \beta_n < \infty$;

(ii) $\alpha_n^{(1)} + \alpha_n^{(2)} + \alpha_n^{(3)} = 1$, $\limsup_{n\to\infty} \alpha_n^{(1)} < 1$, $\liminf_{n\to\infty} \alpha_n^{(1)}\alpha_n^{(2)} > 0$, *and* $\liminf_{n\to\infty} \alpha_n^{(1)}\alpha_n^{(3)} > 0$;

(iii) $\liminf_{n\to\infty} r_n = \eta > 0$.

If J is weakly sequentially continuous, then x_n *converges weakly to* $u \in \Gamma$, *where* $u = \lim_{n\to\infty} R_\Gamma x_n$.

If we put $A \equiv 0$, $\psi \equiv 0$, and $T = S$ in Theorem 3.3, then Theorem 3.3 reduces to the following corollary.

Corollary 3.5 *Let E be a uniformly convex and* 2*-uniformly smooth Banach space and let C be a closed subset of E such that JC is closed and convex. Let* $F : JC \times JC \to (-\infty, +\infty)$ *be a bifunction satisfying* (A1)-(A4). *Let* $B : JC \to E$ *be an* α*-inverse-strongly skew-monotone operator such that* $\mathrm{VI}(JC,B) \neq \emptyset$ *and* $\|BJy\| \leq \|BJy - BJu\|$ *for all* $y \in C$ *and* $u \in \mathrm{VI}(JC,B)$ *and T be a generalized nonexpansive type mapping of C into itself such that* $\Gamma := \mathrm{VI}(JC,B) \cap \mathrm{EP}(F) \cap F(T)$ *is not empty. Let* $\{x_n\}$ *be a sequence generated by* $x_1 = x \in C$ *and*

$$\begin{cases} F(Ju_n, Jy) + \frac{1}{r_n}\langle u_n - x_n, Jy - Ju_n\rangle \geq 0, \quad \forall y \in C, \\ z_n = R_C(u_n - \lambda_n BJu_n), \\ y_n = \alpha_n^{(1)} u_n + (1-\alpha_n^{(1)}) Tz_n, \\ x_{n+1} = R_C(\beta_n x + (1-\beta_n)y_n), \end{cases}$$

where R_C *is the sunny generalized nonexpansive retraction of E onto C, J is the duality mapping on E.* $\{\lambda_n\} \subset [a,b]$ *for some a, b with* $0 < a < b < \frac{\alpha}{c}$, *where c is the constant in Lemma* 2.1. *The following conditions are satisfied*:

(i) $\sum_{n=1}^{\infty} \beta_n < \infty$;

(ii) $0 < \liminf_{n\to\infty} \alpha_n^{(1)} \leq \limsup_{n\to\infty} \alpha_n^{(1)} < 1$;

(iii) $\liminf_{n\to\infty} r_n = \eta > 0$.

If J is weakly sequentially continuous, then x_n *converges weakly to* $u \in \Gamma$, *where* $u = \lim_{n\to\infty} R_\Gamma x_n$.

Remark 3.2 The results presented in this paper substantially improve and extend the results of others in the following aspects.

(1) Theorem 3.3 and Corollary 3.4 extend the result [18] on the iterative construction of the fixed point of a single generalized nonexpansive type mapping to the case of common fixed points of two generalized nonexpansive type mappings.

(2) Phuangphoo and Kumam [33] considered the fixed point problem of one closed ϕ-nonexpansive mapping; in this paper, we discuss fixed points of two generalized nonexpansive type mappings, and the closeness of the mapping is omitted.

(3) In this paper, the iterative scheme which we introduced is more general because it can be applied to find a common element of the set of solutions for the generalized mixed equilibrium problem, the common fixed points of two generalized nonexpansive type mappings, and the set of solutions of the variational inequality in Banach spaces.

4 Example

In this paper, we consider the convergence of the iteration which we suggest in the general Banach space. In order to make the theoretical results more intuitive, we give an example in the real number field. In a Hilbert space, the duality mapping J is the identity operator and we have the function $\phi(x,y) = \|x-y\|^2$, R_C is the metric projection P_C. The generalized nonexpansive type mapping should be a non-spreading mapping; the inverse-strongly skew-monotone operator should be an inverse-strongly monotone operator.

Example 4.1 Let $A : R \to R$, $B : R \to R$ be defined: for all $x \in R$,

$$Ax = \frac{x}{2}, \qquad Bx = \frac{x-1}{2}, \qquad \psi(x) = \frac{x^2}{3}.$$

The classical variational inequality is as follows: finding a $u \in C = [1,100]$ such that

$$\langle Bu, v-u\rangle \geq 0, \quad \forall v \in [1,100]. \tag{4.1}$$

The solution of (4.1), denoted by $\mathrm{VI}([1,100],B)$, is $\{1\}$, and B is satisfies the condition: $\|By\| \leq \|By - Bp\|$, $\forall y \in C, p \in \mathrm{VI}(C,B)$.

Let $T : [1,100] \to [1,100]$, $S : [1,100] \to [1,100]$, $F : [1,100] \times [1,100] \to R$ be defined as

$$Tx = \frac{3x+4}{7}, \qquad Sx = \frac{2x+5}{7}, \qquad F(x,y) = -\frac{7}{6}y^2 + \frac{xy}{2} + 3x - 3y + \frac{2}{3}y^2.$$

It is easy to verify A, B are inverse-strongly monotone operators with coefficients $\alpha = 1$ and $\beta = 1$, respectively. T and S are non-spreading mappings and $F(T) \cap F(S) = \{1\}$.

It is obvious that $F(x,y)$ satisfies the following conditions:

(1) $F(x,x) = 0$;

(2) $F(x,y)$ is monotone, *i.e.*, $F(x,y) + F(y,x) = 0$;

(3) for each $x, y, w \in [1,100]$, $\lim_{t\to 0} F(tw + (1-t)x, y) \leq F(x,y)$;

(4) for each $x \in [1,100]$, $y \to F(x,y)$ is convex and lower semicontinuous.

Then we have

$$\begin{aligned} &F(x,y) + \langle Ax, y-x\rangle + \psi(y) - \psi(x) \\ &\quad = -\frac{7}{6}y^2 + \frac{xy}{2} + 3x - 3y + \frac{2}{3}y^2 + \frac{x}{2}(y-x) + \frac{y^2}{3} - \frac{x^2}{3} \\ &\quad = -2x^2 + (y+3)x + y^2 - 3y. \end{aligned}$$

So, we get the solution of the generalized mixed equilibrium problem GMEP(F,A,ψ) is $\{1\}$. At the same time, we obtain $F(T)\cap F(S)\cap \mathrm{GMEP}(F,A,\psi)\cap \mathrm{VI}(C,B)=\{1\}$.

Observe that for all $y\in[1,100]$,

$$\begin{aligned}
&0\le F(u_n,y)+\langle Au_n,y-u_n\rangle+\psi(y)-\psi(u_n)+\frac{1}{r_n}\langle u_n-x_n,y-u_n\rangle\\
&=\left[-2u_n^2+(y+3)u_n+y^2-3y\right]+\frac{1}{r_n}(u_n-x_n)(y-u_n)\\
&\quad\Leftrightarrow\quad 0\le r_n\left[-2u_n^2+(y+3)u_n+y^2-3y\right]+\left(u_ny-u_n^2-x_ny+x_nu_n\right)\\
&\qquad\qquad=r_ny^2+(r_nu_n-3r_n+u_n-x_n)y+\left(3r_nu_n-2r_nu_n^2-u_n^2+x_nu_n\right).
\end{aligned}$$

Let $a=r_n$, $b=r_nu_n-3r_n+u_n-x_n$, $c=3r_nu_n-2r_nu_n^2-u_n^2+x_nu_n$,

$$\begin{aligned}
\Delta&=b^2-4ac\\
&=(r_nu_n-3r_n+u_n-x_n)^2-4r_n\left(3r_nu_n-2r_nu_n^2-u_n^2+x_nu_n\right)\\
&=(u_n-3r_n+3r_nu_n-x_n)^2.
\end{aligned}$$

Let $H(y)=r_ny^2+(r_nu_n-3r_n+u_n-x_n)y+(3r_nu_n-2r_nu_n^2-u_n^2+x_nu_n)$, and we have $H(y)\ge 0$. If $H(y)$ has at most one solution in R, then we get $\Delta\le 0$. This implies that

$$u_n=\frac{x_n+3r_n}{1+3r_n}. \tag{4.2}$$

Put $\alpha_n^{(1)}=\frac{2n}{4n+3}$, $\alpha_n^{(2)}=\frac{n+2}{4n+3}$, $\alpha_n^{(3)}=\frac{n+1}{4n+3}$, $\beta_n=\frac{1}{n^2}$, $r_n=\frac{n}{3n+2}$, $\lambda_n=\frac{n+1}{3n+2}$. It is easy to see that the sequences $\alpha_n^{(1)}$, $\alpha_n^{(2)}$, $\alpha_n^{(3)}$, β_n, r_n, and λ_n satisfy the conditions in Theorem 3.3. We can rewrite the sequence in Theorem 3.3: $x_1=x\in[1,100]$,

$$\begin{cases}
u_n=\frac{x_n+3r_n}{3r_n+1},\\
z_n=P_{[1,100]}(u_n-\lambda_n\frac{1}{2}Bu_n),\\
y_n=\alpha_n^{(1)}u_n+\alpha_n^{(2)}Sz_n+\alpha_n^{(3)}Tz_n,\\
x_{n+1}=P_{[1,100]}(\beta_nx+(1-\beta_n)y_n).
\end{cases}$$

Then from Theorem 3.3 we conclude that the sequence $\{x_n\}$ converges weakly to 1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors read and approved the final manuscript.

Author details
[1]College of Mathematics and Computer, Hebei University, Baoding, 071002, China. [2]College of Mechanical and Electrical Engineering, Hebei Agricultural University, Baoding, 071001, China.

Acknowledgements
This project is supported by the NSF of Hebei province (A2014201033) and the NNSF of China (11101115, 11271106).

Received: 3 May 2014 Accepted: 6 January 2015 Published online: 01 February 2015

References

1. Iiduka, H, Takahashi, W, Toyoda, M: Approximation of solutions of variational inequalities for monotone mappings. Panam. Math. J. **14**, 417-429 (2004)
2. Iiduka, H, Takahashi, W: Weak convergence of a projection algorithm for variational inequalities in a Banach spaces. J. Math. Anal. Appl. **339**, 668-679 (2008)
3. Plubtieng, S, Sriprad, W: Existence and approximation of solution of the variational inequality problem with a skew monotone operator defined on the dual spaces of Banach spaces. J. Nonlinear Anal. Optim. **1**, 23-33 (2010)
4. Ibaraki, T, Takahashi, W: A new projection and convergence theorems for the projections in Banach spaces. J. Approx. Theory **149**, 1-14 (2007)
5. Takahashi, W, Zembayashi, K: A strong convergence theorem for the equilibrium problem with a bifunction defined on the dual space of a Banach space. In: Proceedings of the 8th International Conference on Fixed Point Theory and Its Applications, pp. 197-209. Yokohama Publishers, Yokohama (2008)
6. Takahashi, S, Takahashi, W: Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space. Nonlinear Anal. **69**, 1025-1033 (2008)
7. Peng, JW, Yao, JC: Ishikawa iterative algorithms for a generalized equilibrium problem and fixed point problems of a pseudo-contraction mapping. J. Glob. Optim. **46**, 331-345 (2010)
8. Zeng, LC, Wu, SY, Yao, JC: Generalized KKM theorem with applications to generalized minimax inequalities and generalized equilibrium problems. Taiwan. J. Math. **10**, 1497-1514 (2006)
9. Peng, JW, Yao, JC: A new hybrid-extragradient method for generalized mixed equilibrium problems and fixed point problems and variational inequality problems. Taiwan. J. Math. **12**, 1401-1433 (2008)
10. Peng, JW, Yao, JC: A new extragradient method for mixed equilibrium problems, fixed point problems and variational inequality problems. Math. Comput. Model. **49**, 1816-1828 (2009)
11. Zeng, LC, Ansari, QH, Yao, JC: Viscosity approximation methods for generalized equilibrium problems and fixed point problems. J. Glob. Optim. **43**, 487-502 (2009)
12. Qin, X, Cho, SY, Kang, SM: Strong convergence of shrinking projection methods for quasi-ϕ-nonexpansive mappings and equilibrium problems. J. Comput. Appl. Math. **234**, 750-760 (2010)
13. Saewan, S, Cho, YJ, Kumam, P: Weak and strong convergence theorems for mixed equilibrium problems in Banach spaces. Optim. Lett. **8**, 501-518 (2014)
14. Cho, YJ, Argyros, IK, Petrot, N: Approximation methods for common solutions of generalized equilibrium, systems of nonlinear variational inequalities and fixed point problems. Comput. Math. Appl. **50**, 2292-2301 (2010)
15. Qin, X, Chang, SS, Cho, YJ: Iterative methods for generalized equilibrium problems and fixed point problems with applications. Nonlinear Anal., Real World Appl. **11**, 2963-2972 (2010)
16. Qin, X, Cho, YJ, Kang, SM: Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces. J. Comput. Appl. Math. **225**, 20-30 (2009)
17. Takahashi, W, Zembayashi, K: Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces. Nonlinear Anal. **70**, 45-57 (2009)
18. Takahashi, W, Yao, JC: Nonlinear operators of monotone type and convergence theorems with equilibrium problems in Banach spaces. Taiwan. J. Math. **15**, 787-818 (2011)
19. Yao, Y, Cho, YJ, Chen, R: An iterative algorithm for solving fixed point problems, variational inequality problems and mixed equilibrium problems. Nonlinear Anal. **71**, 3363-3373 (2009)
20. Yao, Y, Cho, YJ, Liou, Y: Algorithms of common solutions for variational inclusions, mixed equilibrium problems and fixed point problems. Eur. J. Oper. Res. **212**, 242-250 (2011)
21. Cai, G, Bu, S: Weak convergence theorems for general equilibrium problems and variational inequality problems and fixed point problems in Banach spaces. Acta Math. Sci. Ser. B **33**, 303-320 (2013)
22. Ceng, LC, Petru, A, Yao, JC: Composite viscosity approximation methods for equilibrium problem, variational inequality and common fixed points. J. Nonlinear Convex Anal. **15**, 219-240 (2014)
23. Al-Mazrooei, AE, Latif, A, Yao, JC: Solving generalized mixed equilibria, variational inequalities and constrained convex minimization. Abstr. Appl. Anal. **2014**, Article ID 587865 (2014)
24. Ceng, LC, Yao, JC: On the triple hierarchical variational inequalities with constraints of mixed equilibria, variational inclusion and system of generalized equilibria. Tamkang J. Math. **45**(3), 297-334 (2014)
25. Alber, YI: Metric and generalized projection in Banach space: properties and applications. In: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, pp. 15-50. Dekker, New York (1996)
26. Su, Y, Xu, HK, Wang, Z: Strong convergence theorem for a common fixed point of two hemi-relatively nonexpansive mappings. Nonlinear Anal. **71**, 5616-5628 (2009)
27. Xu, ZB, Roach, GF: Characteristic inequalities of uniformly convex and uniformly smooth Banach spaces. J. Math. Anal. Appl. **157**, 189-210 (1991)
28. Kamimura, S, Takahashi, W: Strong convergence of a proximal-type algorithm in a Banach space. SIAM J. Optim. **13**, 938-945 (2002)
29. Kohsaha, F, Takahashi, W: Generalized nonexpansive retractions and a proximal-type algorithm in Banach spaces. J. Nonlinear Convex Anal. **8**, 197-209 (2007)
30. Rockafellar, RT: Maximal monotone operators and proximal point algorithm. SIAM J. Control Optim. **14**, 887-898 (1976)
31. Cho, YJ, Zhou, HY, Guo, G: Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings. Comput. Math. Appl. **47**, 707-717 (2004)
32. Tan, KK, Xu, HK: Approximating fixed points of nonexpansive mappings by Ishikawa iteration process. J. Math. Anal. Appl. **178**, 301-308 (1993)
33. Phuangphoo, P, Kumam, P: Existence and modification of Halpern-Mann iterations for fixed point and generalized mixed equilibrium problems with a bifunction defined on the dual space. J. Appl. Math. **2013**, Article ID 753096 (2013)