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The global attractor of the 2D Boussinesq system with fractional vertical dissipation

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Abstract

The 2D incompressible Boussinesq system with partial or fractional dissipation have recently attracted considerable attention. In this paper, we study the Cauchy problem for the 2D Boussinesq system in a periodic domain with fractional vertical dissipation in the subcritical case, and we prove the global well-posedness of strong solutions. Based on this, we also discuss the existence of the global attractor.

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Keywords: Boussinesq system; fractional vertical dissipation; global well-posedness; global attractor

1 Introduction

This paper studies the 2D incompressible Boussinesq system with fractional vertical dissipation. The model reads

$$\begin{cases} u_t + \nu \Lambda_V^{2\alpha} u + u \cdot \nabla u + \nabla P = \theta e_2, \quad e_2 = (0,1), \\ \operatorname{div} u = 0, \\ \theta_t + \kappa \Lambda_V^{2\beta} \theta + u \cdot \nabla \theta = F(x), \\ u(x,0) = u_0(x), \qquad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.1}$$

where $\Omega := [0,L]^2$ is the periodic domain. $u = (u_1, u_2)$ is the velocity vector field, $u_i = u_i(x,t)$ $(i = 1,2)$, $(x,t) \in \Omega \times \mathbb{R}_+$; $\theta(x,t)$ and $P(x,t)$ denote the scalar temperature and pressure of the fluid, respectively. The constant $\nu > 0$ is the viscosity, and $\kappa > 0$ is the thermal diffusivity. $e_2 = (0,1)$ is the unit vector in the vertical direction, and the unknown function θe_2 is the buoyancy force. $F(x)$ is a time-independent forcing term. For the sake of simplicity, we denote $\Lambda_V := \sqrt{-\partial_{x_2}^2}$, the vertical dissipation, $\Lambda_H := \sqrt{-\partial_{x_1}^2}$, the horizontal dissipation, $\Lambda := \sqrt{-\Delta} = \sqrt{\Lambda_V^2 + \Lambda_H^2}$, the square root of the negative Laplacian. When $\nu = 0$ and $\kappa = 0$, (1.1) reduces to the inviscid 2D Boussinesq equations. If θ is identically zero, (1.1) degenerates to the 2D incompressible Euler equations. In this paper, we assume that the exponents α and β satisfy

$$\alpha, \beta \in \left(\frac{1}{2}, 1\right). \tag{1.2}$$

As suggested by Jiu *et al.* in [1], we classify the parameters α and β into three categories:

(1) the subcritical case, $\alpha + \beta > 1$;

(2) the critical case, $\alpha + \beta = 1$;

(3) the supercritical case, $\alpha + \beta < 1$.

Throughout this paper, $\|\cdot\|_{L^p}$, $\|\cdot\|_{W^{s,p}}$, and $\|\cdot\|_{L^\infty}$ denote the norms of $L^p(\Omega)$, $W^{s,p}(\Omega)$, and $L^\infty(\Omega)$, respectively. When $p = 2$, we denote the norms $\|\cdot\|_{L^2}$ and $\|\cdot\|_{W^{s,2}}$ by $\|\cdot\|$ and $\|\cdot\|_{H^s}$, respectively.

In the past decades, there were a lot of literature about the mathematical theory of the Boussinesq equations (1.1). In the case when ν and κ are positive constants, Cannon and DiBenedetto [2] studied the Cauchy problem for the Boussinesq system, and further proved the existence of a unique global in time weak solution. Furthermore, they established the regularity of the solutions when initial data are smooth. In [3], the global well-posedness was established (see also [4]) for this case. In contrast, in the case when $\nu = \kappa = 0$, the global regularity problem turns out to be extremely difficult and remains open. For this case, we only have the local existence and uniqueness results due to [5] and [6]. Recently, there were many works devoted to the study of the Boussinesq system with partial viscosity or diffusivity (*i.e.*, the zero diffusivity case, $\nu > 0$ and $\kappa = 0$, or the zero viscosity case, $\nu = 0$ and $\kappa > 0$). Actually, Chae [7], and Hou and Li [8] independently proved the global well-posedness; see also [9, 10] for the global well-posedness in the critical spaces and [11, 12] for the case of bounded domain. More precisely, Chae [7] showed the global existence for $(u_0, \theta_0) \in H^3 \times H^3$ while Hou and Li [8] proved the same result for initial data in $H^3 \times H^2$. Zhao [13] generalized the case in [7] to a bounded domain with typical physical boundary conditions $u \cdot \mathbf{n}|_{\partial\Omega} = 0$ and $\theta|_{\partial\Omega} = \overline{\theta}$, where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$ and $\overline{\theta} > 0$ is a constant, and further proved that there exists a unique global smooth solution with smooth initial data. Jin and Fan [14] proved a global uniform regularity in the vanishing viscosity limit in a domain $\Omega = (-\infty, +\infty) \times (0, 1)$ with a slip boundary condition. Hu, Kukavica and Ziane [15] proved that for the initial data $(u_0, \theta_0) \in H^2 \times H^1$, there exists a global in time solution (u, θ) which is a locally in time bounded function in $H^2 \times H^1$, and in [16], proved the persistence of regularity holds, *i.e.*, the solution (u, θ) exists and belongs to $H^s \times H^{s-1}$ for data $(u_0, \theta_0) \in H^s \times H^{s-1}$, where $1 < s < 2$.

When ν and κ depend on the temperature, Lorca and Boldrini [17] proved the global existence of strong solutions for small initial data, and in [18], obtained the global existence of weak solutions and the local existence of strong solutions for general initial data. Recently, Wang and Zhang [19] proved the global well-posedness. Sun and Zhang [20] obtained the existence of global strong solutions to the initial boundary value problem. Li, Pan and Zhang [21, 22] proved the existence of a unique global smooth solution to the initial boundary value problem of 2D inviscid heat conductive Boussinesq equations with nonlinear heat diffusion over a bounded domain with smooth boundary. Moreover, Huang [23] addressed the well-posedness of the 2D (Euler)-Boussinesq equations with zero viscosity positive diffusivity in the polygonal-like domains with Yudovich's type data and in [24] proved the global well-posedness of strong solutions and existence of the global attractor to the initial and boundary value problem in a periodic channel with non-homogeneous boundary conditions for the temperature and viscosity and thermal diffusivity depending on the temperature.

One main focus of recent research on the 2D Boussinesq equations has been on the global regularity issue when only fractional dissipation is present. Adhikari, Cao and Wu

[25] aimed at the global regularity of classical solutions to the 2D Boussinesq equations with vertical dissipation and vertical thermal diffusion, and in [26], further studied the global regularity issue concerning the 2D Boussinesq equations with vertical dissipation and vertical thermal diffusion. Cao and Wu [27] established the global in time existence of classical solutions to the 2D anisotropic Boussinesq equations with vertical dissipation. Danchin and Paicu [28] studied the 2D Boussinesq system with horizontal viscosity in only one equation, they proved the global existence issue for possibly large initial data. Hmidi, Keraani and Rousset [29] investigated an Euler-Boussinesq system which couples the incompressible Euler equation with velocity and a transport model with fractional diffusion for the temperature. Jia, Peng and Li [30] proved the generalized 2D Boussinesq equation has a global and unique solution in suitable functional space. Jiu, Miao, Wu and Zhang [1, 31] established the global regularity to the 2D incompressible Boussinesq equations with general critical dissipation. Jiu, Wu and Yang [32] studied solutions of the 2D incompressible Boussinesq equations with fractional dissipation in the periodic box $\mathbb{T}^2 = [0, 2\pi]^2$. KC, Regmi, Tao, and Wu [33, 34] studied the global (in time) regularity problem concerning the two-dimensional incompressible Boussinesq equations. Larios, Lunasin, and Titi [35] established the global existence and uniqueness theorems for the two-dimensional non-diffusive Boussinesq system with anisotropic viscosity acting only in the horizontal direction. Miao and Xue [36] proved the global well-posedness results for the rough initial data of a class of Boussinesq-Navier-Stokes systems. Stefanov and Wu [37] proved the global regularity problem on the two-dimensional incompressible Boussinesq equations with fractional dissipation. Using energy methods, the Fourier localization technique, and Bony's paraproduct decomposition, Xiang and Yan [38] showed the global existence of the classical solutions to the Boussinesq equations with fractional diffusion. Wu and Xu [39] are concerned with the global well-posedness and inviscid limits of several systems of Boussinesq equations with fractional dissipation. Xu [40] proved the existence, the uniqueness and the regularity of solutions to the Boussinesq equations for an incompressible fluid in $\mathbb{R}^2$, with diffusion modeled by fractional Laplacian. Xu and Xue [41] considered the Yudovich type solutions of the 2D inviscid Boussinesq system with critical and supercritical dissipation, and gave a refined blowup criterion in the supercritical case. Yang, Jiu and Wu [42] examined the global regularity issue on the 2D Boussinesq equations with fractional Laplacian dissipation and thermal diffusion, and further established the global well-posedness for the 2D Boussinesq equations with a new range of fractional powers of the Laplacian. Ye and Xu [43] studied the Cauchy problem to the 2D incompressible Boussinesq equations with fractional dissipation, and in [44] proved the global regularity of the smooth solutions of the 2D Boussinesq equations with a new range of fractional powers of the Laplacian. Using the Fourier localization method, Fang, Qian and Zhang [45] obtained the local and global well-posedness and gave some blowup criterion with the velocity or the temperature for the 2D incompressible generalized Boussinesq system with the general supercritical dissipation.

In this paper, we prove the existence of global attractor for the solution operator $S(t)$ to the Boussinesq system (1.1) in the space $H^s \times H^s$, where $s \geq 1$. First, we show the global existence of weak solutions to the Boussinesq system (1.1), that is, solutions satisfying Definition 1.1.

Definition 1.1 Let

$$H_1 = \left\{ u \in L^2(\Omega)^2 : \nabla \cdot u = 0, \int_\Omega u_1\, dx = \int_\Omega u_2\, dx = 0 \right\}$$

and

$$H_2 = \left\{ \theta \in L^2(\Omega)^2 : \int_\Omega \theta\, dx = 0 \right\}.$$

Suppose $F(x) \in H^{-\beta}$ and $(u_0, \theta_0) \in H_1 \times H_2$. Then the pair (u, θ) is said to be a global weak solution of the Boussinesq system (1.1), if for any $T \in (0, +\infty)$, and $u \in C([0,T]; H_1) \cap L^2(0,T; H^\alpha)$, $\theta \in C([0,T]; H_2) \cap L^2(0,T; H^\beta)$, such that for any $\phi = (\phi_1, \phi_2) \in C^\infty(\Omega)^2$, and for any $\psi \in C_0^\infty(\Omega)$, we have

$$\frac{d}{dt}\int_\Omega u\phi\, dx - \int_\Omega u(u \cdot \nabla\phi)\, dx + \nu \int_\Omega \left(\Lambda_V^\alpha u\right)\left(\Lambda_V^\alpha \phi\right) dx = \int_\Omega \theta e_2 u\, dx,$$
$$\frac{d}{dt}\int_\Omega \theta\psi\, dx - \int_\Omega \left(\Lambda_V^\beta \theta\right)\left(\Lambda_V^\beta \psi\right) dx = \int_\Omega F(x)\psi\, dx.$$

We omit the details for this part and refer the reader to [18].

The following is the main result of this paper.

Theorem 1.1 *Let $s \geq 1$, and assume that $(u_0, \theta_0) \in H^s \times H^s$, and $F(x) \in H^{s-\beta} \cap L^p$, where $p \in (2, +\infty)$. Then there exists a unique strong solution $(u(t), \theta(t))$ of the Boussinesq system* (1.1), *such that, for any* $0 < T < +\infty$,

$$\big(u(t), \theta(t)\big) \in C\big([0,T]; H^s\big) \tag{1.3}$$

and

$$(u_t, \theta_t) \in L^2\big(0,T; H^{s-\alpha}\big) \cap L^2\big(0,T; H^{s-\beta}\big). \tag{1.4}$$

The main goal in this paper is to prove the existence of the global attractor for the Boussinesq system (1.1), so we have the following theorem.

Theorem 1.2 *Let $s \geq 1$, and $F(x) \in H^{s-\beta} \cap L^p$, where $p \in [2, +\infty)$. Then, for all $t \in [0, +\infty)$, the solution operator $\{S(t)\}$ of the Boussinesq system* (1.1):

$$S(t)(u_0, \theta_0) = \big(u(t), \theta(t)\big)$$

defines a semigroup in the space $H^s \times H^s$.

Moreover, the following statements are valid:

(1) *for any $(u_0, \theta_0) \in H^s \times H^s$, $t \mapsto S(t)(u_0, \theta_0)$ is a continuous function from $\mathbb{R}_+$ into $H^s \times H^s$;*

(2) *for any fixed $t > 0$, $S(t)$ is a continuous and compact map in $H^s \times H^s$;*

(3) *$\{S(t)\}$ possesses a global attractor $\mathcal{A}$ in the space $H^s \times H^s$. The global attractor $\mathcal{A}$ is compact and connected in $H^s \times H^s$ and is the maximal bounded attractor and the minimal invariant set in $H^s \times H^s$ in the sense of the set inclusion relation.*

2 Preliminaries

In this section, we state some important inequalities and facts which will be used in the proof of Theorem 1.2. First, we introduce the Kate-Ponce and commutator inequalities from [46]; see also [16, 47, 48].

Lemma 2.1 ([46]) *Suppose that* $f, g \in C_c^\infty(\Omega)$. *Let* $s > 0$ *and* $1 < r \le p_1, p_2, q_1, q_2 \le +\infty$ *such that* $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$ *with the restriction* $p_1, q_2 \neq +\infty$. *Then we have*

$$\left\|\Lambda^s(fg)\right\|_{L^r} \le C\left(\left\|\Lambda^s f\right\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{q_1}}\left\|\Lambda^s g\right\|_{L^{q_2}}\right), \tag{2.1}$$

where $C > 0$ *is a constant.*

Lemma 2.2 ([46]) *Suppose that* $f, g \in C_c^\infty(\Omega)$. *Let* $s > 0$ *and* $2 < p_1, p_2, q_1, q_2 \le +\infty$ *such that* $\frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$. *Then we have*

$$\left\|\Lambda^s(f\cdot\nabla g) - f\cdot\left(\Lambda^s\nabla g\right)\right\|_{L^2} \le C\left(\left\|\Lambda^s f\right\|_{L^{p_1}}\|\nabla g\|_{L^{p_2}} + \|\nabla f\|_{L^{q_1}}\left\|\Lambda^s g\right\|_{L^{q_2}}\right), \tag{2.2}$$

where $C > 0$ *is a constant.*

In order to achieve that, the solution operators $\{S(t), \forall t \ge 0\}$ are continuous in the space $H^s \times H^s$ with respect to t. We recall the following lemma, which is a particular case of a general interpolation theorem in [49].

Lemma 2.3 *Let* V, H, V' *be three Hilbert spaces such that*

$$V \subset H = H' \subset V',$$

where H' *is the dual space of* H *and* V' *is the dual space of* V.

If a function u *belong to* $L^2(0, T; V)$ *and its derivative* u' *belongs to* $L^2(0, T; V')$, *then* u *is almost everywhere equal to a function continuous from* $[0, T]$ *into* H.

Lemma 2.3 has been proved in [50]. Therefore, here we omit the proof.

3 Proof of Theorem 1.2

3.1 Uniform estimates

In this subsection, we prove the existence of absorbing ball whose proof will be done in a series of lemmas, and consists of a priori estimates on an arbitrary time interval $[0, T]$. In the following, we denote by C a positive constant which is independent of time t and dependent of the initial data u_0 and θ_0. First of all, we will prove the existence of absorbing ball in $L^2$and L^p for θ.

Lemma 3.1 *Under the assumptions of Theorem* 1.2, *there exists* $T_1 = T_1(\|\theta_0\|) > 0$, *for all* $t \ge T_1$, *we have*

$$\left\|\theta(t)\right\| \le C \tag{3.1}$$

and

$$\int_t^{t+1} \left\| \Lambda_V^\beta \theta \right\|^2 d\tau \le C. \tag{3.2}$$

Proof Throughout this paper, we let λ_1 be the first eigenvalue of Λ_V. By the results from [48], we can get

$$\left\| \theta(t) \right\|^2 \le e^{-\kappa \lambda_1^{2\beta} t} \left(\|\theta_0\|^2 - \frac{\|F(x)\|^2}{\kappa^2 \lambda_1^{4\beta}} \right) + \frac{\|F(x)\|^2}{\kappa^2 \lambda_1^{4\beta}}. \tag{3.3}$$

Hence, (3.3) immediately implies the estimate (3.1) for a uniform bound for some $T_1 > 0$ large enough.

Furthermore, taking the L^2 inner product of $(1.1)_3$ with θ, and integrating in time, we have

$$\left\| \theta(t+1) \right\|^2 + \kappa \int_t^{t+1} \left\| \Lambda_V^\beta \theta \right\|^2 d\tau \le \left\| \theta(t) \right\|^2 + \frac{\|F(x)\|^2}{\kappa \lambda_1^2}. \tag{3.4}$$

Therefore, the time average estimate (3.2) follows from (3.1). For all $p \in [2, +\infty)$, similar to [48], we deduce the following equation:

$$\left\| \theta(t) \right\|_{L^p} \le e^{\frac{-\kappa \lambda_1^{2\beta} t}{p}} \left(\|\theta_0\|_{L^p} - \frac{p\|F(x)\|_{L^p}}{\kappa \lambda_1^{2\beta}} \right) + \frac{p\|F(x)\|_{L^p}}{\kappa \lambda_1^{2\beta}}, \tag{3.5}$$

which gives the uniform L^p estimate and absorbing ball in L^p for θ whenever $\theta_0 \in L^p(\Omega)$ for all $p \in [2, +\infty)$. □

In the next lemma, we shall prove the existence of an absorbing ball in L^2 for u.

Lemma 3.2 *Under the assumptions of Theorem* 1.2, *there exists* $T_2 = T_2(\|u_0\|, \|\theta_0\|) > 0$, *for all* $t \ge T_2$, *we have*

$$\left\| u(t) \right\| \le C \tag{3.6}$$

and

$$\int_t^{t+1} \left\| \Lambda_V^\alpha u \right\|^2 d\tau \le C. \tag{3.7}$$

Proof Taking the L^2 inner product of $(1.1)_1$ with u, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu \left\| \Lambda_V^\alpha u \right\|^2 &= \int \theta e_2 \cdot u \, dx \\ &\le \left| \int \Lambda_V^{-\alpha} \theta \cdot \Lambda_V^\alpha u \, dx \right| \\ &\le \frac{1}{2\nu} \left\| \Lambda_V^{-\alpha} \theta \right\|^2 + \frac{\nu}{2} \left\| \Lambda_V^\alpha u \right\|^2. \end{aligned} \tag{3.8}$$

It follows that

$$\frac{d}{dt}\|u\|^2 + \nu\left\|\Lambda_V^{\alpha}\right\|^2 \le \frac{1}{\nu}\left\|\Lambda_V^{-\alpha}\theta\right\|^2. \tag{3.9}$$

Since u and θ have mean zero, using the Poincaré inequality, we get

$$\frac{d}{dt}\|u\|^2 + \nu\lambda_1^{2\alpha}\|u\|^2 \le \frac{1}{\nu\lambda_1^{2\alpha}}\|\theta\|^2. \tag{3.10}$$

Integrating in time, and using (3.3), we can obtain the following:

(i) in the case that $\nu\lambda_1^{2\alpha} \neq \kappa\lambda_1^{2\beta}$,

$$\begin{aligned}\left\|u(t)\right\|^2 \le{}& e^{-\nu\lambda_1^{2\alpha}t}\|u_0\|^2 + \frac{1}{\nu\lambda_1^{2\alpha}}\left|\frac{e^{-\nu\lambda_1^{2\alpha}t} - e^{-\kappa\lambda_1^{2\beta}t}}{\nu\lambda_1^{2\alpha} - \kappa\lambda_1^{2\beta}}\right|\left(\|\theta_0\|^2 - \frac{\|F(x)\|^2}{\kappa^2\lambda_1^{4\beta}}\right)\\ &+ \frac{\|F(x)\|^2}{\kappa^2\lambda_1^{4\beta}\nu^2\lambda_1^{4\alpha}},\end{aligned} \tag{3.11}$$

(ii) in the case that $\nu\lambda_1^{2\alpha} = \kappa\lambda_1^{2\beta}$,

$$\left\|u(t)\right\|^2 \le e^{-\nu\lambda_1^{2\alpha}t}\|u_0\|^2 + \frac{e^{-\nu\lambda_1^{2\alpha}t}t}{\nu\lambda_1^{2\alpha}}\left(\|\theta_0\|^2 - \frac{\|F(x)\|^2}{\kappa^2\lambda_1^{4\beta}}\right) + \frac{\|F(x)\|^2}{\kappa\lambda_1^{2\beta}\nu^2\lambda_1^{4\alpha}}. \tag{3.12}$$

Hence, (3.11) and (3.12) immediately imply the estimate (3.6) uniform bound for some $T_2 > 0$ large enough.

Furthermore, integrating (3.10) in time, we have

$$\left\|u(t+1)\right\|^2 + \nu\lambda_1^{2\alpha}\int_t^{t+1}\|u\|^2\,d\tau \le \left\|u(t)\right\|^2 + \frac{1}{\nu\lambda_1^{2\alpha}}\|\theta\|^2. \tag{3.13}$$

Therefore, from (3.1) and (3.6), we can get the time average estimate (3.7). This completes the proof. □

Now we will prove the existence of an absorbing ball in L^2 for ω. We are in a position to give the estimates $\|\nabla u\|$ and $\|\Lambda_V^{\alpha}\omega\|$.

Lemma 3.3 *Under the assumptions of Theorem* 1.2, *there exists* $T_3 = T_3(\|\omega_0\|, \|\theta_0\|) > 0$, *for all* $t \ge T_3$, *we have*

$$\left\|\nabla u(t)\right\| \le C \tag{3.14}$$

and

$$\int_t^{t+1}\left\|\Lambda_V^{\alpha}\omega\right\|^2 d\tau \le C. \tag{3.15}$$

Proof In order to complete the proof, we need to use vorticity formulation together with the Biot-Savart law. Taking the curl of equation $(1.1)_1$, we have

$$\omega_t + \nu\Lambda_V^{2\alpha}\omega + u\cdot\nabla\omega = \theta_{x_1} \tag{3.16}$$

and

$$u = \nabla^{\perp}\Delta^{-1}\omega, \tag{3.17}$$

where $\omega = \partial_{x_1}u_2 - \partial_{x_2}u_1$ and $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$. By the Biot-Savart law (see *e.g.* [51, 52]), we have

$$\|u\|_{H^1}^2 \le C\|\omega\|^2, \qquad \|u\|_{H^2}^2 \le C\|\omega\|_{H^1}^2 \le C\|\nabla\omega\|^2, \tag{3.18}$$

where the Poincaré inequality is employed for the last inequality.

Taking the L^2 inner product of (3.16) with ω, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\omega\|^2 + \nu\|\Lambda_V^{\alpha}\omega\|^2 &= \int \Lambda_H\theta \cdot \omega\,dx \\ &\le \left|\int \Lambda_H\Lambda_V^{-\alpha}\theta \cdot \Lambda_V^{\alpha}\omega\,dx\right| \\ &\le \left|\int \Lambda_V^{-\alpha}\theta \cdot \Lambda_H\Lambda_V^{\alpha}\omega\,dx\right| \\ &\le C\|\Lambda_V^{-\alpha}\theta\|^2 + \frac{\varepsilon}{2}\|\Lambda_H\Lambda_V^{\alpha}\omega\|^2. \end{aligned} \tag{3.19}$$

It follows that

$$\frac{d}{dt}\|\omega\|^2 + 2\nu\|\Lambda_V^{\alpha}\omega\|^2 \le C\|\Lambda^{-\alpha}\theta\|^2 + \varepsilon\|\Lambda_H\Lambda_V^{\alpha}\omega\|^2. \tag{3.20}$$

Using the Poincaré inequality, we get

$$\frac{d}{dt}\|\omega\|^2 + \left(2\nu\lambda_1^{2\alpha} - \varepsilon\lambda_1^{2\alpha}\lambda_2^2\right)\|\omega\|^2 \le \frac{1}{\nu\lambda_1^{2\alpha}}\|\theta\|^2, \tag{3.21}$$

where λ_2 be the first eigenvalue of Λ_H. Taking $\varepsilon < \frac{2\nu}{\lambda_2^2}$ and by the variant of uniform Gronwall lemma (see [53], Lemma 2.4), (3.1) and (3.18), it is easy to obtain the uniform estimate (3.14).

Furthermore, integrating (3.21) in time, we have

$$\|\omega(t+1)\|^2 + \left(2\nu\lambda_1^{2\alpha} - \varepsilon\lambda_1^{2\alpha}\lambda_2^2\right)\int_t^{t+1}\|\omega\|^2\,d\tau \le \|\omega(t)\|^2 + \frac{1}{\nu\lambda_1^{2\alpha}}\|\theta\|^2. \tag{3.22}$$

Therefore, we can get (3.15). Similar to Lemma 3.1, and using the Sobolev embedding theorem, we can get a uniform estimate $\|u\|_{L^p}$ and time average estimate of $\|\Lambda_V^{1+\alpha}u\|^2$, that is, for all $p \in (1, +\infty)$ and for any $t \ge T_3$,

$$\|u(t)\|_{L^p} \le C(p), \int_t^{t+1}\|\Lambda_V^{1+\alpha}u\|^2\,d\tau \le C, \tag{3.23}$$

where the constant $C(p) > 0$ only depends on p. Now, we complete the proof. □

Now, let us focus on the existence of an absorbing ball in H^s for (u, θ).

Lemma 3.4 *Under the assumptions of Theorem* 1.2, *there exists* $T_4 = T_4(\|\omega_0\|, \|\theta_0\|) > 0$, *for all* $t \geq T_4$, *we have*

$$\|u(t)\|_{H^s} \leq C, \qquad \|\theta(t)\|_{H^s} \leq C \tag{3.24}$$

and

$$\int_t^{t+1} \|\Lambda_V^{s+\alpha} u\|^2 \, d\tau \leq C, \qquad \int_t^{t+1} \|\Lambda_V^{s+\beta} \theta\|^2 \, d\tau \leq C. \tag{3.25}$$

Proof Taking the L^2 inner product of $(1.1)_3$ with $\Lambda^{2s}\theta$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda^s\theta\|^2 + \kappa\|\Lambda_V^{s+\beta}\theta\|^2 + \kappa\|\Lambda_V^{\beta}\Lambda_H^{s}\theta\|^2 \\
&\quad = \int F(x)\cdot\Lambda^{2s}\theta\,dx - \int u\cdot\nabla\theta\cdot\Lambda^{2s}\theta\,dx \\
&\quad \leq \left|\int \Lambda^s F(x)\cdot\Lambda^s\theta\,dx\right| - \int u\cdot\nabla\theta\cdot\Lambda^{2s}\theta\,dx \\
&\quad \leq \frac{1}{2}\|\Lambda^s F(x)\|^2 + \frac{1}{2}\|\Lambda^s\theta\|^2 - \int u\cdot\nabla\theta\cdot\Lambda^{2s}\theta\,dx.
\end{aligned} \tag{3.26}$$

Since u is divergence free, $u\cdot\nabla\theta = \nabla\cdot(u\theta)$ and by Lemma 2.1, we have

$$\begin{aligned}
&\left|-\int u\cdot\nabla\theta\cdot\Lambda^{2s}\theta\,dx\right| \\
&\quad = \left|-\int \nabla\cdot(u\theta)\cdot\Lambda^{2s}\theta\,dx\right| \\
&\quad \leq \|\Lambda^{s+\beta_1}\theta\|\,\|\Lambda^{s+1-\beta_1}(u\theta)\| \\
&\quad \leq C\|\Lambda^{s+\beta_1}\theta\|\left(\|\Lambda^{s+1-\beta_1}u\|_{L^{p_1}}\|\theta\|_{L^{q_1}} + \|\Lambda^{s+1-\beta_1}\theta\|_{L^{p_1}}\|u\|_{L^{q_1}}\right),
\end{aligned} \tag{3.27}$$

where $\beta_1 > 0$, $p_1, q_1 > 2$, satisfying $\beta_1 < \beta$ and $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{2}$. In order to determine β_1, p_1, and q_1, we first let $2\max\{1-\alpha, 1-\beta\} < r < 1 \leq s$, by the Sobolev embedding theorem, this implies that

$$\theta_0 \in H^s \subset H^r \subset L^{q_1},$$

where q_1 is chosen such that

$$\frac{1}{q_1} = \frac{1-r}{2} < \min\{\alpha, \beta\} - \frac{1}{2}.$$

By (3.5) and (3.23), for $t > \max\{T_1, T_2\}$, from (3.27) we can infer that

$$\begin{aligned}
\left|-\int u\cdot\nabla\theta\cdot\Lambda^{2s}\theta\,dx\right| &\leq C\|\Lambda^{s+\beta_1}\theta\|\,\|\Lambda^{s+1-\beta_1}u\|_{L^{p_1}} + C\|\Lambda^{s+\beta_1}\theta\|\,\|\Lambda^{s+1-\beta_1}\theta\|_{L^{p_1}} \\
&= I_1 + I_2.
\end{aligned} \tag{3.28}$$

Hence, we set $s+2-\frac{2}{p_1}-\beta_1 = s+\beta_1$, that is, $\beta_1 = \frac{1}{2}+\frac{1}{q_1} < \min\{\alpha,\beta\}$, then, by the Sobolev embedding theorem, we know that $H^{s+2-\frac{2}{p_1}-\beta_1}(\Omega) \hookrightarrow H^{s+1-\beta_1,p_1}(\Omega)$, and I_1 can be estimated as

$$\begin{aligned} I_1 &\le C\|\Lambda^{s+\beta_1}\theta\|\|\Lambda^{s+\beta_1}u\| \\ &\le C\|\Lambda^{s+\beta}\theta\|^{\frac{\beta_1}{\beta}}\|\Lambda^{s}\theta\|^{1-\frac{\beta_1}{\beta}}\|\Lambda^{s+\alpha}u\|^{\frac{\beta_1}{\alpha}}\|\Lambda^{s}u\|^{1-\frac{\beta_1}{\alpha}}, \end{aligned} \tag{3.29}$$

where we have used the interpolation inequality for the tuples $(s, s+\beta_1, s+\beta)$ and $(s, s+\alpha_1, s+\alpha)$. By the Young inequality, we get

$$\begin{aligned} I_1 &\le C\|\Lambda^{s+\beta}\theta\|^{\frac{\beta_1}{\beta}}\|\Lambda^{s}\theta\|^{1-\frac{\beta_1}{\beta}}\|\Lambda^{s+\alpha}u\|^{\frac{\beta_1}{\alpha}}\|\Lambda^{s}u\|^{1-\frac{\beta_1}{\alpha}} \\ &\le \varepsilon\|\Lambda^{s+\beta}\theta\|^2 + C(\varepsilon)\|\Lambda^{s}\theta\|^2 + \varepsilon\|\Lambda^{s+\alpha}u\|^2 + C(\varepsilon)\|\Lambda^{s}u\|^2. \end{aligned} \tag{3.30}$$

Similar to I_1, we can deduce I_2 as follows:

$$I_2 \le \varepsilon\|\Lambda^{s+\beta}\theta\|^2 + C(\varepsilon)\|\Lambda^{s}\theta\|^2. \tag{3.31}$$

Taking the L^2 inner product of $(1.1)_1$ with $\Lambda^{2s}u$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\Lambda^{s}u\|^2 + \nu\|\Lambda_V^{s+\alpha}u\|^2 + \nu\|\Lambda_V^{\alpha}\Lambda_H^{s}u\|^2 \\ &\quad= \int \theta e_2\cdot\Lambda^{2s}u\,dx - \int u\cdot\nabla u\cdot\Lambda^{2s}u\,dx \\ &\quad\le \left|\int \Lambda^{s}\theta\cdot\Lambda^{s}u\,dx\right| - \int u\cdot\nabla u\cdot\Lambda^{2s}u\,dx \\ &\quad\le \frac{1}{2}\|\Lambda^{s}\theta\|^2 + \frac{1}{2}\|\Lambda^{s}u\|^2 - \int u\cdot\nabla u\cdot\Lambda^{2s}u\,dx. \end{aligned} \tag{3.32}$$

Similar to (3.27), for $t > \max\{T_2, T_3\}$, we have

$$\begin{aligned} \left|-\int u\cdot\nabla u\cdot\Lambda^{2s}u\,dx\right| &\le C\|\Lambda^{s+\alpha_1}u\|\|\Lambda^{s+1-\alpha_1}(u\otimes u)\| \\ &\le C\|\Lambda^{s+\alpha_1}u\|\|\Lambda^{s+1-\alpha_1}u\|_{L^{p_2}}\|u\|_{L^{q_2}} \\ &\le C\|\Lambda^{s+\alpha_1}u\|\|\Lambda^{s+1-\alpha_1}u\|_{L^{p_2}}, \end{aligned} \tag{3.33}$$

where $\alpha_1 > 0$, $p_2, q_2 > 2$, satisfying $\alpha_1 < \beta$ and $\frac{1}{p_2}+\frac{1}{q_2} = \frac{1}{2}$. If, we set $s+2-\frac{2}{p_2}-\alpha_1 = s+\alpha_1$, so that $\alpha_1 = \frac{1}{2}+\frac{1}{q_2} = 1-\frac{1}{p_2}$, and we choose q_2 large enough such that $\alpha_1 < \alpha$, then by the Sobolev embedding theorem, we know that $H^{s+2-\frac{2}{p_2}-\alpha_1}(\Omega) \hookrightarrow H^{s+1-\alpha_1,p_2}(\Omega)$, and as we use the interpolation inequality for the tuples $(s, s+\alpha_1, s+\alpha)$ and the Young inequality, we arrive at

$$\begin{aligned} \left|-\int u\cdot\nabla u\cdot\Lambda^{2s}u\,dx\right| &\le C\|\Lambda^{s+\alpha_1}u\|^2 \\ &\le C\|\Lambda^{s+\alpha}u\|^{\frac{2\alpha_1}{\alpha}}\|\Lambda^{s}u\|^{2-\frac{2\alpha_1}{\alpha}} \\ &\le \varepsilon\|\Lambda^{s+\alpha}u\|^2 + C(\varepsilon)\|\Lambda^{s}u\|^2. \end{aligned} \tag{3.34}$$

Finally, summing the differential inequalities (3.26) and (3.32) together, and inserting (3.30)-(3.31) and (3.34) into the resulting inequality, we obtain, for $t > \max\{T_1, T_2, T_3\}$,

$$\begin{aligned} &\frac{d}{dt}\big(\|\Lambda^s u\|^2 + \|\Lambda^s\theta\|^2\big) + \nu\|\Lambda_V^{s+\alpha}u\|^2 + \nu\|\Lambda_H^s\Lambda_V^\alpha u\|^2 + \kappa\|\Lambda_V^{s+\beta}\theta\|^2 + \kappa\|\Lambda_H^s\Lambda_V^\beta\theta\|^2 \\ &\quad\le C\big(\|\Lambda^s F(x)\|^2 + \|\Lambda^s u\|^2 + \|\Lambda^s\theta\|^2\big) + \varepsilon\|\Lambda^{s+\beta}\theta\|^2 + \varepsilon\|\Lambda^{s+\alpha}u\|^2. \end{aligned} \tag{3.35}$$

Using the Poincaré inequality, we get

$$\begin{aligned} &\frac{d}{dt}\big(\|\Lambda^s u\|^2 + \|\Lambda^s\theta\|^2\big) + \big(\nu\lambda_1^{2(s+\alpha)} + \nu\lambda_1^{2\alpha}\lambda_2^{2s} - \varepsilon\lambda_3^{2(s+\alpha)}\big)\|u\|^2 \\ &\qquad + \big(\kappa\lambda_1^{2(s+\beta)} + \kappa\lambda_1^{2\beta}\lambda_2^{2s} - \varepsilon\lambda_3^{2(s+\beta)}\big)\|\theta\|^2 \\ &\quad\le C\big(\|\Lambda^s F(x)\|^2 + \|\Lambda^s u\|^2 + \|\Lambda^s\theta\|^2\big), \end{aligned} \tag{3.36}$$

where λ_3 be the first eigenvalue of Λ.

Starting with $s = s^{(1)} = 1$, and taking $\varepsilon < \min\{\nu\frac{\lambda_1^{2(s+\alpha)}+\lambda_1^{2\alpha}\lambda_2^{2s}}{\lambda_3^{2(s+\alpha)}}, \kappa\frac{\lambda_1^{2(s+\beta)}+\lambda_1^{2\beta}\lambda_2^{2s}}{\lambda_3^{2(s+\beta)}}\}$, then, for $t > T := \max\{T_1, T_2, T_3\}$, by the Poincaré inequality and (3.2), (3.22), we have

$$\int_t^{t+1}\|\Lambda_V^{s^{(1)}}u\|^2\,d\tau \le C, \qquad \int_t^{t+1}\|\Lambda_V^{s^{(1)}}\theta\|^2\,d\tau \le C. \tag{3.37}$$

Hence, applying the uniform Gronwall lemma to (3.36) and using (3.37), for $t > T + 1$, we can obtain

$$\|\Lambda_V^{s^{(1)}}u(t)\|^2\,d\tau \le C, \qquad \|\Lambda_V^{s^{(1)}}\theta(t)\|^2\,d\tau \le C. \tag{3.38}$$

Furthermore, integrating (3.36) and by (3.38), for $t > T + 1$, we find that

$$\int_t^{t+1}\|\Lambda_V^{s^{(1)}+\alpha}u\|^2\,d\tau \le C, \qquad \int_t^{t+1}\|\Lambda_V^{s^{(1)}+\beta}\theta\|^2\,d\tau \le C. \tag{3.39}$$

Then we iterate with $s = s^{(2)} = s^{(1)} + \max\{\alpha, \beta\}$. For $t > T + 1$, from (3.39) we can get

$$\int_t^{t+1}\|\Lambda_V^{s^{(2)}}u\|^2\,d\tau \le C, \qquad \int_t^{t+1}\|\Lambda_V^{s^{(2)}}\theta\|^2\,d\tau \le C. \tag{3.40}$$

Applying the uniform Gronwall lemma to (3.36) again, and (3.40), for $t > T + 2$, we can obtain

$$\|\Lambda_V^{s^{(2)}}u(t)\|^2\,d\tau \le C, \qquad \|\Lambda_V^{s^{(2)}}\theta(t)\|^2\,d\tau \le C, \tag{3.41}$$

and

$$\int_t^{t+1}\|\Lambda_V^{s^{(2)}+\alpha}u\|^2\,d\tau \le C, \qquad \int_t^{t+1}\|\Lambda_V^{s^{(2)}+\beta}\theta\|^2\,d\tau \le C. \tag{3.42}$$

Therefore, with a bootstrapping argument, for any given real number $s \ge 1$, (3.24) and (3.25) are proved.

In addition, for fixed $T > 0$, and taking $\varepsilon < \min\{\nu \frac{\lambda_1^{2(s+\alpha)}+\lambda_1^{2\alpha}\lambda_2^{2s}}{\lambda_3^{2(s+\alpha)}}, \kappa \frac{\lambda_1^{2(s+\beta)}+\lambda_1^{2\beta}\lambda_2^{2s}}{\lambda_3^{2(s+\beta)}}\}$, we also can get

$$\int_0^T \|\Lambda^{s+\alpha}u\|^2\,d\tau < \infty, \qquad \int_0^T \|\Lambda^{s+\beta}\theta\|^2\,d\tau < \infty. \tag{3.43}$$

By (3.40), given $(u_0, \theta_0) \in H^s \times H^s$, for some $t > 0$ large enough, the solution $(u(t), \theta(t))$ of the system (1.1) belongs to the space $H^{s_1} \times H^{s_1}$ for some $s_1 = s + \max\{\alpha, \beta\}$. By the Sobolev compactness embedding theorem in [54], the inclusion map $H^{s_1} \times H^{s_1} \mapsto H^s \times H^s$ is compact. Thus, for any $s \geq 1$, the solution operator $S(t)$ defined by $S(t)(u_0, \theta_0) = (u(t), \theta(t))$ is a compact operator in the space $H^s \times H^s$ for some $t > 0$ large enough. □

3.2 Continuity

This subsection mainly includes two parts, the first part to prove $(u, \theta) \in C(0, T; H^s)^2$, which is to show that the solution operator $\{S(t), \forall t > 0\}$ of Boussinesq system (1.1) are continuous in the space $H^s \times H^s$ with respect to t. In part two, we simultaneously prove uniqueness and continuity of $S(t)$ from $H^s \times H^s$ to itself for any fixed $T > 0$. The proof will be done in a series of lemmas as follows.

Lemma 3.5 *Under the assumptions of Theorem* 1.2, *the solutions of Boussinesq system* (1.1) *satisfy* $(u, \theta) \in C(0, T; H^s)^2$.

Proof For any fixed $T > 0$, from Lemma 3.3 we can ensure that

$$u \in L^2(0, T; H^{s+\alpha}) \tag{3.44}$$

and

$$\theta \in L^2(0, T; H^{s+\beta}). \tag{3.45}$$

Hence, by (3.44) and (3.45) we can obtain

$$(\Lambda^s u, \Lambda^s \theta) \in L^2(0, T; H^\alpha) \times L^2(0, T; H^\beta). \tag{3.46}$$

For any $(\varphi, \psi) \in H^\alpha \times H^\beta$, applying the differential operator Λ^s to $(1.1)_1$ and $(1.1)_3$, and then taking the L^2 inner product with φ and ψ, respectively, we can obtain

$$\int \Lambda^s u_t \cdot \varphi\,dx + \int \Lambda^s(u \cdot \nabla u) \cdot \varphi\,dx + \nu \int \Lambda^s \Lambda_V^{2\alpha} u \cdot \varphi\,dx = \int \Lambda^s \theta e_2 \cdot \varphi\,dx \tag{3.47}$$

and

$$\int \Lambda^s \theta_t \cdot \psi\,dx + \int \Lambda^s(u \cdot \nabla \theta) \cdot \psi\,dx + \kappa \int \Lambda^s \Lambda_V^{2\beta} \theta \cdot \psi\,dx = \int \Lambda^s F(x) \cdot \psi\,dx. \tag{3.48}$$

By the Cauchy-Schwarz inequality, from (3.47) and (3.48) we can know that

$$\int \Lambda^s u_t \cdot \varphi\,dx \leq \|\Lambda^{s-\alpha}(u \cdot \nabla u)\| \|\varphi\|_{H^\alpha} + \nu \|\Lambda^{s+\alpha}u\| \|\varphi\|_{H^\alpha} + \|\Lambda^{s-\alpha}\theta\| \|\varphi\|_{H^\alpha} \tag{3.49}$$

and

$$\int \Lambda^s\theta_t \cdot \psi\, dx \le \big\| \Lambda^{s-\beta}(u\cdot\nabla\theta)\big\| \|\psi\|_{H^\beta} + \nu \big\|\Lambda^{s+\beta}\theta\big\| \|\varphi\|_{H^\beta} + \big\|\Lambda^{s-\beta}F(x)\big\| \|\psi\|_{H^\beta}. \tag{3.50}$$

Since u is divergence free, we now estimate $\|u\cdot\nabla u\|$ and $\|u\cdot\nabla\theta\|$

$$\begin{aligned}\big\|\Lambda^{s-\alpha}(u\cdot\nabla u)\big\| &= \big\|\Lambda^{s-\alpha}\nabla\cdot(u\otimes u)\big\| \\ &\le \big\|\Lambda^{1+s-\alpha}(u\otimes u)\big\| \\ &\le C\|u\|_{L^{p_1}} \big\|\Lambda^{1+s-\alpha}u\big\|_{L^{q_1}},\end{aligned} \tag{3.51}$$

where $\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{2}$ and $q_1=\frac{1}{1-\alpha}$. By Lemma 3.2, we know that, for $p_1=\frac{1}{2\alpha-1}$, $u\in C(0,+\infty;L^{p_1})$. Since

$$\frac{1}{q_1}+\frac{s+\alpha-(1+s-\alpha)}{2}=\frac{1}{2},$$

and by the Sobolev embedding theorem, we get

$$\big\|\Lambda^{1+s-\alpha}u\big\|_{L^{q_1}} \le C\big\|\Lambda^{s+\alpha}u\big\|. \tag{3.52}$$

Inserting (3.52) and (3.51) to (3.49), we obtain

$$\big\|\Lambda^s u_t\big\|_{H^{-\alpha}} \le C\big(\|u\|_{L^{p_1}}+\nu\big)\big\|\Lambda^{s+\alpha}u\big\| + C\big\|\Lambda^{s+\beta}\theta\big\|. \tag{3.53}$$

Similarly, for the term $u\cdot\nabla\theta$, we find that

$$\begin{aligned}\big\|\Lambda^{s-\beta}(u\cdot\nabla\theta)\big\| &= \big\|\Lambda^{s-\beta}\nabla\cdot(u\theta)\big\| \\ &\le \big\|\Lambda^{1+s-\beta}(u\theta)\big\|.\end{aligned} \tag{3.54}$$

If we chose $2\max\{1-\alpha,1-\beta\}<r_1<1\le s$, we can get

$$u_0\subset H^s\subset H^{r_1}\subset L^{p_2},\qquad \theta_0\subset H^s\subset H^{r_1}\subset L^{p_3},$$

where

$$\frac{1}{p_2}=\frac{1-r_1}{2}<\frac{1}{2},\qquad p_3=\frac{2}{2\beta-1}.$$

By Lemmas 3.1 and 3.2, we know that

$$u\in C\big(0,+\infty;L^{p_2}\big),\qquad \theta\in C\big(0,+\infty;L^{p_3}\big). \tag{3.55}$$

Applying Lemma 2.2, we obtain

$$\big\|\Lambda^{1+s-\beta}(u\theta)\big\| \le C\|u\|_{L^{p_2}}\big\|\Lambda^{1+s-\beta}\theta\big\|_{L^{q_2}} + C\|\theta\|_{L^{p_3}}\big\|\Lambda^{1+s-\beta}u\big\|_{L^{q_3}}, \tag{3.56}$$

where $\frac{1}{p_2}+\frac{1}{q_2}=\frac{1}{2}$ and $\frac{1}{p_3}+\frac{1}{q_3}=\frac{1}{2}$. We now choose $q^*>0$, such that

$$\frac{1}{q^*}+\frac{s+\alpha-(1+s-\beta)}{2}=\frac{1}{2}.$$

Since $\frac{1}{q_3}=\frac{r_1}{2}$, we get

$$\frac{1}{q^*}=\frac{1}{2}(2-\alpha-\beta)<\frac{1}{2}\cdot 2\max\{1-\alpha,1-\beta\}<\frac{1}{2}r_1=\frac{1}{q_3},$$

hence, we know that

$$q_3<q^*.$$

By the Poincaré inequality and the Sobolev embedding theorem, we can get

$$\left\|\Lambda^{1+s-\beta}u\right\|_{L^{q_3}}\le C\left\|\Lambda^{1+s-\beta}u\right\|_{L^{q^*}}\le C\left\|\Lambda^{s+\alpha}u\right\|. \tag{3.57}$$

From $\frac{1}{p_2}+\frac{1}{q_2}=\frac{1}{2}$, we deduce that

$$\frac{1}{q_2}+\frac{2+\beta-(1+s-\beta)}{2}=\frac{1}{2},$$

by the Sobolev embedding theorem, we can easily get

$$\left\|\Lambda^{1+s-\beta}\theta\right\|_{L^{q_2}}\le C\left\|\Lambda^{s+\beta}\theta\right\|. \tag{3.58}$$

Inserting (3.54) and (3.56)-(3.58) into (3.50), we obtain

$$\left\|\Lambda^{s}\theta_t\right\|_{H^{-\beta}}\le C\|\theta\|_{L^{p_3}}\left\|\Lambda^{s+\alpha}u\right\|+C\big(\|u\|_{L^{p_2}}+\kappa\big)\left\|\Lambda^{s+\beta}\theta\right\|+C\left\|\Lambda^{s-\beta}F(x)\right\|. \tag{3.59}$$

By utilizing estimates (3.5), (3.23) and (3.43), from (3.53) and (3.59), we find

$$\int_0^T\left\|\Lambda^{s}\theta_t\right\|_{H^{-\beta}}^2\le+\infty \tag{3.60}$$

and

$$\int_0^T\left\|\Lambda^{s}u_t\right\|_{H^{-\alpha}}^2\le+\infty, \tag{3.61}$$

respectively. By Lemma 2.3, we thus complete the proof. □

Lemma 3.6 *Under the assumptions of Theorem* 1.2, *the solutions of Boussinesq system* (1.1) *is unique and the operator* $S(t):H^s\times H^s\mapsto H^s\times H^s$ *is continuous for any fixed* $T>0$.

Proof Suppose there are two solutions (u_1,θ_1,P_1) and (u_2,θ_2,P_2) to the Boussinesq system (1.1) with two initial data (u_1^0,θ_1^0) and (u_2^0,θ_2^0), respectively. Setting $\tilde{u}=u_1-u_2$, $\tilde{\theta}=\theta_1-\theta_2$, and $\tilde{P} = P_1 - P_2$, then $(\tilde{u}, \tilde{\theta}, \tilde{P})$ satisfies

$$\begin{cases} \tilde{u}_t + u_1 \cdot \nabla\tilde{u} + \tilde{u} \cdot \nabla u_2 + \nu\Lambda_V^{2\alpha}\tilde{u} + \nabla\tilde{P} = \tilde{\theta}e_2, \\ \tilde{\theta}_t + u_1 \cdot \nabla\tilde{\theta} + \tilde{u} \cdot \nabla\theta_2 + \kappa\Lambda_V^{2\beta}\tilde{\theta} = 0. \end{cases} \tag{3.62}$$

Taking the L^2 inner product of $(3.62)_1$ with $\Lambda^{2s}\tilde{u}$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\Lambda^s\tilde{u}\|^2 + \nu\|\Lambda_V^{s+\alpha}\tilde{u}\|^2 + \nu\|\Lambda_V^{\alpha}\Lambda_H^{s}\tilde{u}\|^2 \\ &\quad = \int \tilde{\theta}e_2 \cdot \Lambda^{2s}\tilde{u}\,dx - \int u_1 \cdot \nabla\tilde{u} \cdot \Lambda^{2s}\tilde{u}\,dx - \int \tilde{u} \cdot \nabla u_2 \cdot \Lambda^{2s}\tilde{u}\,dx \\ &\quad = I_1 + I_2 + I_3. \end{aligned} \tag{3.63}$$

Using the interpolation inequality and the Young inequality, we can get

$$I_1 \le \left|\int \Lambda^s\tilde{\theta}e_2 \cdot \Lambda^s\tilde{u}\,dx\right| \le \frac{1}{2}\|\Lambda^s\tilde{\theta}\|^2 + \frac{1}{2}\|\Lambda^s\tilde{u}\|^2. \tag{3.64}$$

To deal with I_2, we can get

$$\begin{aligned} I_2 &\le \left|\int \Lambda^s(u_1 \cdot \nabla\tilde{u}) \cdot \Lambda^s\tilde{u}\,dx\right| \\ &= \left|\int \left(\Lambda^s(u_1 \cdot \nabla\tilde{u}) - u_1 \cdot \nabla\left(\Lambda^s\tilde{u}\right)\right) \cdot \Lambda^s\tilde{u}\,dx\right|, \end{aligned} \tag{3.65}$$

where $\int u_1 \cdot \nabla(\Lambda^s\tilde{u}) \cdot \Lambda^s\tilde{u}\,dx = 0$. Since ∇ and Λ commute, we obtain

$$\begin{aligned} &\left|\int \left(\Lambda^s(u_1 \cdot \nabla\tilde{u}) - u_1 \cdot \nabla\left(\Lambda^s\tilde{u}\right)\right) \cdot \Lambda^s\tilde{u}\,dx\right| \\ &\quad = \left|\int \left(\Lambda^s(u_1 \cdot \nabla\tilde{u}) - u_1 \cdot \left(\Lambda^s\nabla\tilde{u}\right)\right) \cdot \Lambda^s\tilde{u}\,dx\right| \\ &\quad \le C\left\|\left(\Lambda^s(u_1 \cdot \nabla\tilde{u}) - u_1 \cdot \left(\Lambda^s\nabla\tilde{u}\right)\right)\right\| \left\|\Lambda^s\tilde{u}\,dx\right\|. \end{aligned} \tag{3.66}$$

Applying Lemma 2.2, we can obtain

$$\begin{aligned} &\left\|\left(\Lambda^s(u_1 \cdot \nabla\tilde{u}) - u_1 \cdot \left(\Lambda^s\nabla\tilde{u}\right)\right)\right\| \\ &\quad \le C\left(\|\nabla u_1\|_{L^{p_4}}\|\Lambda^s\tilde{u}\|_{L^{p_5}} + \|\Lambda^s u_1\|_{L^{q_4}}\|\nabla\tilde{u}\|_{L^{q_5}}\right) \\ &\quad \le C\left(\|\Lambda u_1\|_{L^{p_4}}\|\Lambda^s\tilde{u}\|_{L^{p_5}} + \|\Lambda^s u_1\|_{L^{q_4}}\|\Lambda\tilde{u}\|_{L^{q_5}}\right). \end{aligned} \tag{3.67}$$

From (1.2), we know that $1 - \alpha < \alpha$, then, choosing $p_4 = \frac{2}{\alpha}$, $p_5 = \frac{2}{1-\alpha}$, $q_4 = \frac{2}{1-\alpha}$, and $q_5 = \frac{2}{\alpha}$, and using the Sobolev embedding inequalities, we can get

$$\|\Lambda u_1\|_{L^{p_4}} \le C\|\Lambda^{2-\alpha}u_1\| \le C\|\Lambda^{s+\alpha}u_1\|, \tag{3.68}$$

$$\|\Lambda^s\tilde{u}\|_{L^{p_5}} \le C\|\Lambda^{s+\alpha}\tilde{u}\|, \tag{3.69}$$

$$\|\Lambda^s u_1\|_{L^{q_4}} \le C\|\Lambda^{s+\alpha}u_1\|, \tag{3.70}$$

and

$$\|\Lambda\tilde{u}\|_{L^{q_5}} \leq C\|\Lambda^{2-\alpha}\tilde{u}\| \leq C\|\Lambda^{s+\alpha}\tilde{u}\|. \tag{3.71}$$

From (3.68)-(3.71), and using the Young inequality, we obtain

$$\begin{aligned} I_2 &\leq C\|\Lambda^{s+\alpha}u_1\|\|\Lambda^{s+\alpha}\tilde{u}\|\|\Lambda^s\tilde{u}\| \\ &\leq C(\varepsilon)\|\Lambda^{s+\alpha}u_1\|^2\|\Lambda^s\tilde{u}\|^2 + \varepsilon\|\Lambda^{s+\alpha}\tilde{u}\|^2. \end{aligned} \tag{3.72}$$

Simply applying the Cauchy-Schwarz inequality, we can get

$$\begin{aligned} I_3 &\leq \left|-\int \tilde{u}\cdot\nabla u_2\cdot\Lambda^{2s}\tilde{u}\,dx\right| \\ &\leq C(\varepsilon)\|\Lambda^{s-\alpha}(\tilde{u}\cdot\nabla u_2)\|^2 + \varepsilon\|\Lambda^{s+\alpha}\tilde{u}\|^2. \end{aligned} \tag{3.73}$$

Applying Lemma 2.1 again, we obtain

$$\|\Lambda^{s-\alpha}(\tilde{u}\cdot\nabla u_2)\| \leq C(\|\Lambda^{s-\alpha}\tilde{u}\|_{Lp_6}\|\Lambda u_2\|_{Lp_7} + \|\tilde{u}\|_{Lq_6}\|\Lambda^{1+s-\alpha}u_2\|_{Lq_7}). \tag{3.74}$$

Similarly, we choose $p_6 = \frac{2}{1-\alpha}$, $p_7 = \frac{2}{\alpha}$, $q_6 = \frac{2}{2\alpha-1}$, and $q_7 = \frac{2}{1-\alpha}$, and we use the Sobolev embedding inequalities, to get

$$\|\Lambda^{s-\alpha}\tilde{u}\|_{L^{p_6}} \leq C\|\Lambda^s\tilde{u}\|, \tag{3.75}$$

$$\|\Lambda u_2\|_{L^{p_7}} \leq C\|\Lambda^{2-\alpha}u_2\| \leq C\|\Lambda^{s+\alpha}u_2\|, \tag{3.76}$$

$$\|\tilde{u}\|_{L^{q_6}} \leq C\|\Lambda^{2-2\alpha}\tilde{u}\| \leq C\|\Lambda^s\tilde{u}\|, \tag{3.77}$$

and

$$\|\Lambda^{1+s-\alpha}u_2\|_{L^{q_7}} \leq C\|\Lambda^{s+\alpha}u_2\|. \tag{3.78}$$

By (3.73)-(3.78) we obtain

$$I_3 \leq C(\varepsilon)\|\Lambda^s\tilde{u}\|^2\|\Lambda^{s+\alpha}u_2\|^2 + \varepsilon\|\Lambda^{s+\alpha}\tilde{u}\|^2. \tag{3.79}$$

Inserting (3.64), (3.72) and (3.79) to (3.63), we can obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\Lambda^s\tilde{u}\|^2 + \nu\|\Lambda_V^{s+\alpha}\tilde{u}\|^2 + \nu\|\Lambda_V^\alpha\Lambda_H^s\tilde{u}\|^2 \\ &\quad\leq C(\|\Lambda^{s+\alpha}u_1\|^2 + \|\Lambda^{s+\alpha}u_2\|^2 + 1)\|\Lambda^s\tilde{u}\|^2 + \frac{1}{2}\|\Lambda^s\tilde{\theta}\|^2 + \varepsilon\|\Lambda^{s+\alpha}\tilde{u}\|^2. \end{aligned} \tag{3.80}$$

Using the Poincaré inequality, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\Lambda^s\tilde{u}\|^2 + (\nu\lambda_1^{2(s+\alpha)} + \nu\lambda_1^{2\alpha}\lambda_3^{2s} - \varepsilon\lambda_2^{2(s+\alpha)})\|\tilde{u}\|^2 \\ &\quad\leq C(\|\Lambda^{s+\alpha}u_1\|^2 + \|\Lambda^{s+\alpha}u_2\|^2 + 1)\|\Lambda^s\tilde{u}\|^2 + \frac{1}{2}\|\Lambda^s\tilde{\theta}\|^2. \end{aligned} \tag{3.81}$$

Taking the L^2 inner product of $(3.62)_2$ with $\Lambda^{2s}\tilde{\theta}$, we get

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s\tilde{\theta}\|^2 + \kappa\|\Lambda_V^{s+\alpha}\tilde{\theta}\|^2 + \kappa\|\Lambda_V^{\alpha}\Lambda_H^{s}\tilde{\theta}\|^2 = -\int u_1\cdot\nabla\tilde{\theta}\cdot\Lambda^{2s}\tilde{\theta}\,dx - \int\tilde{u}\cdot\nabla\theta_2\cdot\Lambda^{2s}\tilde{\theta}\,dx. \tag{3.82}$$

Let $J_1 = -\int u_1\cdot\nabla\tilde{\theta}\cdot\Lambda^{2s}\tilde{\theta}\,dx$ and $J_2 = -\int\tilde{u}\cdot\nabla\theta_2\cdot\Lambda^{2s}\tilde{\theta}\,dx$. Then using the interpolation inequality and the Young inequality, we can get

$$\begin{aligned}
J_1 &\le \left|-\int u_1\cdot\nabla\tilde{\theta}\cdot\Lambda^{2s}\tilde{\theta}\,dx\right| \\
&= \left|\Lambda^s\left(\int u_1\cdot\nabla\tilde{\theta}\right)\cdot\Lambda^s\tilde{\theta}\,dx\right| \\
&= \left|\left(\Lambda^s\left(\int u_1\cdot\nabla\tilde{\theta}\right) - u_1\cdot\nabla\left(\Lambda^s\tilde{\theta}\right)\right)\cdot\Lambda^s\tilde{\theta}\,dx\right| \\
&= \left|\left(\Lambda^s\left(\int u_1\cdot\nabla\tilde{\theta}\right) - u_1\cdot\left(\Lambda^s\nabla\tilde{\theta}\right)\right)\cdot\Lambda^s\tilde{\theta}\,dx\right| \\
&\le C\left\|\Lambda^s\left(\int u_1\cdot\nabla\tilde{\theta}\right) - u_1\cdot\left(\Lambda^s\nabla\tilde{\theta}\right)\right\|\|\Lambda^s\tilde{\theta}\|,
\end{aligned} \tag{3.83}$$

where we have used $\int u_1\cdot\nabla(\Lambda^s\tilde{\theta})\cdot\Lambda^s\tilde{\theta}\,dx = 0$ and also the fact that ∇ and Λ commute. Applying Lemma 2.2, we can obtain

$$\begin{aligned}
&\left\|\Lambda^s\left(\int u_1\cdot\nabla\tilde{\theta}\right) - u_1\cdot\left(\Lambda^s\nabla\tilde{\theta}\right)\right\| \\
&\quad\le C\left(\|\nabla u_1\|_{L^{p_8}}\|\Lambda^s\tilde{\theta}\|_{L^{p_9}} + \|\Lambda^s u_1\|_{L^{q_8}}\|\nabla\tilde{\theta}\|_{L^{q_9}}\right) \\
&\quad\le C\left(\|\Lambda u_1\|_{L^{p_8}}\|\Lambda^s\tilde{\theta}\|_{L^{p_9}} + \|\Lambda^s u_1\|_{L^{q_8}}\|\Lambda\tilde{\theta}\|_{L^{q_9}}\right).
\end{aligned} \tag{3.84}$$

Choosing $p_8 = \frac{2}{\beta}$ and $p_9 = \frac{2}{1-\beta}$, and using the Sobolev embedding inequality, we can obtain

$$\|\Lambda u_1\|_{L^{p_8}} \le C\|\Lambda^{2-\beta}u_1\| \le C\|\Lambda^{s+\alpha}u_1\| \tag{3.85}$$

and

$$\|\Lambda^s\tilde{\theta}\|_{L^{p_9}} \le C\|\Lambda^{s+\beta}\tilde{\theta}\|. \tag{3.86}$$

Let $\gamma = \min\{\alpha, \beta\}$, $q_8 = \frac{2}{1-\gamma}$, and $q_9 = \frac{2}{\gamma}$, we can obtain

$$\|\Lambda^s u_1\|_{L^{q_8}} \le C\|\Lambda^{s+\gamma}u_1\| \le C\|\Lambda^{s+\alpha}u_1\| \tag{3.87}$$

and

$$\|\Lambda\tilde{\theta}\|_{L^{q_9}} \le C\|\Lambda^{2-\gamma}\tilde{\theta}\| \le C\|\Lambda^{s+\gamma}\tilde{\theta}\| \le C\|\Lambda^{s+\beta}\tilde{\theta}\|. \tag{3.88}$$

Using (3.83) and (3.84), and the Young inequality, we can get

$$\begin{aligned} J_1 &\le \|\Lambda^{s+\alpha}u_1\|\|\Lambda^{s+\beta}\tilde{\theta}\|\|\Lambda^s\tilde{\theta}\| \\ &\le C(\varepsilon)\|\Lambda^{s+\alpha}u_1\|^2\|\Lambda^s\tilde{\theta}\|^2+\varepsilon\|\Lambda^{s+\beta}\tilde{\theta}\|^2. \end{aligned} \tag{3.89}$$

Similar to I_3, we can deduce J_2 as follows:

$$\begin{aligned} J_2 &\le \left|-\int \tilde{u}\cdot\nabla\theta_2\cdot\Lambda^{2s}\tilde{\theta}\,dx\right| = \left|\int \Lambda^{s-\beta}(\tilde{u}\cdot\nabla\theta_2)\cdot\Lambda^{s+\beta}\tilde{\theta}\,dx\right| \\ &\le C(\varepsilon)\|\Lambda^{s-\beta}(\tilde{u}\cdot\nabla\theta_2)\|^2+\varepsilon\|\Lambda^{s+\beta}\tilde{\theta}\|. \end{aligned} \tag{3.90}$$

Applying Lemma 2.1 again, we can obtain

$$\|\Lambda^{s-\beta}(\tilde{u}\cdot\nabla\theta_2)\| \le C\left(\|\Lambda^{s-\beta}\tilde{u}\|_{L^{p_{10}}}\|\Lambda\theta_2\|_{L^{p_{11}}}+\|\tilde{u}\|_{L^{q_{10}}}\|\Lambda^{1+s-\beta}\theta_2\|_{L^{q_{11}}}\right). \tag{3.91}$$

Choosing $p_{10}=\frac{2}{1-\beta}$, $p_{11}=\frac{2}{\beta}$, $q_{10}=\frac{2}{2\beta-1}$, $q_{11}=\frac{1}{1-\beta}$, and using the Sobolev embedding inequality, we can get

$$\|\Lambda^{s-\beta}\tilde{u}\|_{L^{p_{10}}} \le C\|\Lambda^s\tilde{u}\|, \tag{3.92}$$

$$\|\Lambda\theta_2\|_{L^{p_{11}}} \le C\|\Lambda^{2-\beta}\theta_2\| \le C\|\Lambda^{s+\beta}\theta_2\|, \tag{3.93}$$

$$\|\tilde{u}\|_{L^{q_{10}}} \le C\|\Lambda^{2-2\beta}\tilde{u}\| \le C\|\Lambda^s\tilde{u}\|, \tag{3.94}$$

and

$$\|\Lambda^{1+s-\beta}\theta_2\|_{L^{q_{11}}} \le C\|\Lambda^{s+\beta}\theta_2\|. \tag{3.95}$$

From (3.92)-(3.95), and using the Young inequality, we obtain

$$\|\Lambda^{s-\beta}(\tilde{u}\cdot\nabla\theta_2)\|^2 \le C\|\Lambda^s\tilde{u}\|^2\|\Lambda^{s+\beta}\theta_2\|^2, \tag{3.96}$$

hence, inserting (3.96) into (3.90), we can get

$$J_2 \le C(\varepsilon)\|\Lambda^s\tilde{u}\|^2\|\Lambda^{s+\beta}\theta_2\|^2+\varepsilon\|\Lambda^{s+\beta}\tilde{\theta}\|^2. \tag{3.97}$$

Then, inserting (3.89) and (3.97) into (3.82), we can obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\Lambda^s\tilde{\theta}\|^2+\kappa\|\Lambda_V^{s+\beta}\tilde{\theta}\|^2+\kappa\|\Lambda_V^{\beta}\Lambda_H^{s}\tilde{\theta}\|^2 \\ &\quad\le C\left(\|\Lambda^s\tilde{u}\|^2\|\Lambda^{s+\beta}\theta_2\|^2+\|\Lambda^s\tilde{\theta}\|^2\|\Lambda^{s+\alpha}u_1\|^2\right)+\varepsilon\|\Lambda^{s+\beta}\tilde{\theta}\|^2. \end{aligned} \tag{3.98}$$

Using the Poincaré inequality, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\Lambda^s\tilde{\theta}\|^2+\left(\kappa\lambda_1^{2(s+\beta)}+\kappa\lambda_1^{2\beta}\lambda_2^{2s}-\varepsilon\lambda_3^{2(s+\beta)}\right)\|\tilde{\theta}\|^2 \\ &\quad\le C\left(\|\Lambda^s\tilde{u}\|^2\|\Lambda^{s+\beta}\theta_2\|^2+\|\Lambda^s\tilde{\theta}\|^2\|\Lambda^{s+\alpha}u_1\|^2\right). \end{aligned} \tag{3.99}$$

Together with (3.81) and (3.99) this enables us to deduce that

$$\begin{aligned}&\frac{d}{dt}\left(\left\|\Lambda^s\tilde{u}\right\|^2+\left\|\Lambda^s\tilde{\theta}\right\|^2\right)+2\left(\nu\lambda_1^{2(s+\alpha)}+\nu\lambda_1^{2\alpha}\lambda_2^{2s}-\varepsilon\lambda_3^{2(s+\alpha)}\right)\|\tilde{u}\|^2\\&\quad+2\left(\kappa\lambda_1^{2(s+\beta)}+\kappa\lambda_1^{2\beta}\lambda_2^{2s}-\varepsilon\lambda_3^{2(s+\beta)}\right)\|\tilde{\theta}\|^2\\&\quad\le C\left(\left\|\Lambda^{s+\alpha}u_1\right\|^2+\left\|\Lambda^{s+\alpha}u_2\right\|^2+\left\|\Lambda^{s+\beta}\theta_2\right\|^2+1\right)\left(\left\|\Lambda^s\tilde{u}\right\|^2+\left\|\Lambda^s\tilde{\theta}\right\|^2\right).\end{aligned}\tag{3.100}$$

Taking $\varepsilon<\min\{\nu\frac{\lambda_1^{2(s+\alpha)}+\lambda_1^{2\alpha}\lambda_2^{2s}}{\lambda_3^{2(s+\alpha)}},\kappa\frac{\lambda_1^{2(s+\beta)}+\lambda_1^{2\beta}\lambda_2^{2s}}{\lambda_3^{2(s+\beta)}}\}$, using the Gronwall inequality, and by (3.43), for all $t\in[0,T]$, we immediately get

$$\begin{aligned}&\left\|\Lambda^s\tilde{u}\right\|^2+\left\|\Lambda^s\tilde{\theta}\right\|^2+2\left(\nu\lambda_1^{2(s+\alpha)}+\nu\lambda_1^{2\alpha}\lambda_2^{2s}-\varepsilon\lambda_3^{2(s+\alpha)}\right)\int_0^t\|\tilde{u}\|^2\,d\tau\\&\quad+2\left(\kappa\lambda_1^{2(s+\beta)}+\kappa\lambda_1^{2\beta}\lambda_2^{2s}-\varepsilon\lambda_3^{2(s+\beta)}\right)\int_0^t\|\tilde{\theta}\|^2\,d\tau\\&\quad\le\left(\left\|\Lambda^s\tilde{u}_0\right\|^2+\left\|\Lambda^s\tilde{\theta}_0\right\|^2\right)\exp\left\{C\int_0^t\left(\left\|\Lambda^{s+\alpha}u_1\right\|^2+\left\|\Lambda^{s+\alpha}u_2\right\|^2+\left\|\Lambda^{s+\beta}\theta_2\right\|^2+1\right)d\tau\right\}\\&\quad\le\left(\left\|\Lambda^s\tilde{u}_0\right\|^2+\left\|\Lambda^s\tilde{\theta}_0\right\|^2\right)\mathrm{e}^{CT}.\end{aligned}\tag{3.101}$$

By the Riesz lemma, since $\|\Lambda^s\tilde{u}_0\|$ and $\|\Lambda^s\tilde{\theta}_0\|$ go to zero, for almost every t, $\|\Lambda^s\tilde{u}\|$ and $\|\Lambda^s\tilde{\theta}\|$ converge to zero. By Lemma 3.5 we know that $\|\Lambda^s\tilde{u}\|$ and $\|\Lambda^s\tilde{\theta}\|$ are continuous in t, then $\|\Lambda^s\tilde{u}_0\|$ and $\|\Lambda^s\tilde{\theta}_0\|$ converge to zero for all t, which implies that, for any $T\ge 0$,

$$\mathrm{e}^{-CT}\left(\left\|\Lambda^s\tilde{u}\right\|^2+\left\|\Lambda^s\tilde{\theta}\right\|^2\right)\le\left\|\Lambda^s\tilde{u}_0\right\|^2+\left\|\Lambda^s\tilde{\theta}_0\right\|^2=0,\tag{3.102}$$

i.e., $\tilde{u}=0$, $\tilde{\theta}=0$, $u_1=u_2$, and $\theta_1=\theta_2$. So we complete the proof. □

Competing interests
The author declares to have no completing interests.

Authors' contributions
The main ides of this paper was proposed by Xing Su, and he prepared all steps of the proofs in this research. He read and approved the final manuscript.

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