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Sign-changing solutions for asymptotically linear operator equations and applications

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Abstract
In this paper, by using the topological degree and fixed point index theory, the existence of three kinds of solutions (*i.e.*, sign-changing solutions, positive solutions, and negative solutions) for asymptotically linear operator equations is discussed. The abstract results obtained here are applied to nonlinear integral and differential equations.

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Keywords: sign-changing solutions; asymptotically linear operators; topological degree; fixed point index

1 Introduction

In recent years, due to some ecological problems, much attention has been attached to the existence of sign-changing solutions for nonlinear partial differential equations (see [1–3] and the references therein). We note that the proofs of main results in [1–3] depend upon critical point theory. In [4], the authors presented a variational approach to the eigenvalue problem for higher order equations with multipoint boundary conditions. Note that their variational methods were based on properties of the Fenchel conjugate. However, for more general nonlocal problems of ordinary differential equations and dynamic equations on time scales, it is very difficult to find the variational structure of the above problems. To overcome this difficulty, Sun and Cui [5] studied the existence of sign-changing solution for nonlinear operator equations by using the cone theory and combining a uniformly positive condition. Subsequently, Li and Li [6] studied two sign-changing solutions of a class of second-order integral boundary value problems by computing the eigenvalues and the algebraic multiplicities of the corresponding linear problems. Rynne [7] used the global bifurcation theorem to obtain nodal solutions of multipoint boundary value problems, and the author considered the cases where the nonlinear term is asymptotically linear and superlinear. Furthermore, in [8], by using the modification function technique and the Leray-Schauder degree method, some existence and multiplicity results for sign-changing solutions of certain three-point boundary value problems were obtained. A feature of this paper is that the authors gave clear descriptions of the location of the sign-changing solutions. He *et al.* [9] discussed the existence of sign-changing solutions for a class of discrete boundary value problems, and a concrete example was also given. Very recently, Dolbeault *et al.* [10] considered radial solutions of an elliptic equation involving the p-Laplace operator and proved by a shooting method the existence of compactly supported solutions. Zhang and Xie [11] studied sign-changing solutions for the following asymptotically linear three-point boundary value problems:

$$\begin{cases} x''(t) + f(x(t)) = 0, & 0 \le t \le 1, \\ x(0) = 0, & \alpha x(\eta) = x(1), \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, $0 < \eta < 1$. They imposed the following assumptions on f:

($\widetilde{\mathrm{A}}_1$) $f : \mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing function, and $f(0) = 0$;
($\widetilde{\mathrm{A}}_2$) $\lim_{x\to\infty} \frac{f(x)}{x} = \beta_\infty$, where $\lambda_1 < \beta_\infty < \frac{8(1-\alpha\eta)^2}{(1-\alpha\eta^2)^2}$, $\beta_\infty \neq \lambda_n$, $n = 2, 3, \ldots$, and

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1} < \cdots$$

is the sequence of positive solutions for the equation $\sin\sqrt{x} = \alpha \sin\eta\sqrt{x}$;
($\widetilde{\mathrm{A}}_3$) $\lim_{x\to 0} \frac{f(x)}{x} = \beta_0 > \lambda_1$.

Theorem 1.1 (See [11]) *Let* α *and* η *be given numbers with* $0 < \alpha < 1$, $0 < \eta < 1$. *Suppose that* ($\widetilde{\mathrm{A}}_1$)-($\widetilde{\mathrm{A}}_3$) *hold. Then problem* (1.1) *has at least one sign-changing solution. Moreover, problem* (1.1) *has at least two positive solutions and two negative solutions.*

The main purpose of this paper is to abstract more general conditions from ($\widetilde{\mathrm{A}}_1$)-($\widetilde{\mathrm{A}}_3$) of Theorem 1.1 and to obtain some existence theorems of sign-changing solutions for asymptotically linear operator equations. Then, we apply the abstract results obtained in this paper to nonlinear integral equations and elliptic partial differential equations. A feature of this paper is that we weaken the condition ($\widetilde{\mathrm{A}}_2$) of Theorem 1.1 (see Example 4.1). Compared with main results in [12–15], we use a different method consisting of the computation of Leray-Schauder degree for asymptotically linear operators and lower and upper solutions conditions. In addition, the compressive conditions of abstract operator can be removed.

For the discussion of the following sections, we state here preliminary definitions and known results on cones, partial orderings, and topological degree theory, which can be found in [16–18].

Let E be a real Banach space. Given a cone $P \subset E$, we define a partial ordering $\le$ with respect to P by $x \le y$ iff $y - x \in P$. A cone P is said to be normal if there exists a constant $N > 0$ such that $\theta \le x \le y$ implies $\|x\| \le N\|y\|$, the smallest N is called the normal constant of P. P is reproducing if $P - P = E$, *i.e.*, for every $x \in E$, we have that $x = y - z$, where $y \in P$, $z \in P$ and total if $\overline{P - P} = E$. Let $B : E \to E$ be a bounded linear operator. B is said to be positive if $B(P) \subset P$. A fixed point u of operator A is said to be a sign-changing fixed point if $u \notin P \cup (-P)$. If $x_0 \in E\backslash\{\theta\}$ satisfies $\lambda A x_0 = x_0$, where λ is some real number, then λ is called a characteristic value of A, and x_0 is called a characteristic function belonging to the characteristic value λ.

Definition 1.1 (See [19]) Let $A : D \to E$ be an operator, $e \in P\backslash\{\theta\}$, and $x_0 \in D$. If for any given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$-\epsilon e \le Ax - Ax_0 \le \epsilon e$$

for all $x \in D$ with $\|x - x_0\| < \delta$, then A is called e-continuous at x_0. If A is e-continuous at each point $x \in D$, then A is called e-continuous on D.

2 Main results

Theorem 2.1 *Let P be a normal and total cone in E, $A : E \to E$ be a completely continuous increasing operator, $A\theta = \theta$, and e-continuous on E. Suppose that*

(i) *there exist $u_1 \in (-P)\backslash\{\theta\}$ and $v_1 \in P\backslash\{\theta\}$ such that $u_1 \leq Au_1$ and $Av_1 \leq v_1$;*
(ii) *there exist $u_2 \in (-P)\backslash\{\theta\}$, $v_2 \in P\backslash\{\theta\}$, and $\delta > 0$ such that $u_1 < u_2 < \theta < v_2 < v_1$, $Au_2 \leq u_2 - \delta e, v_2 + \delta e \leq Av_2$;*
(iii) *the Fréchet derivative A'_∞ at ∞ exists; A'_∞ is increasing; $r(A'_\infty) > 1$; 1 is not a characteristic value of A'_∞.*

Then A has at least five fixed points, two of which are positive, the two others are negative, and the fifth one is a sign-changing fixed point.

Theorem 2.2 *Let P be a normal and total cone in E, $A = KF$, where $F : E \to E$ is a continuous and bounded increasing operator, $K : E \to E$ is a positive linear completely continuous operator and is also e-continuous on E. Suppose that*

(i) *there exist $u_1 \in (-P)\backslash\{\theta\}$ and $v_1 \in P\backslash\{\theta\}$ such that $u_1 \leq Au_1$ and $Av_1 \leq v_1$, and there exists $\alpha > 0$ such that $u_1 \leq -\alpha e$ and $\alpha e \leq v_1$;*
(ii) *$F(\theta) = \theta$, F is Fréchet differentiable at θ, and KF'_θ has a characteristic value $\lambda_0 < 1$ with a characteristic function ψ satisfying $\mu_1 e \leq \psi \leq \mu_2 e$, where $\mu_1 > 0$, $\mu_2 > 0$.*

Moreover, let condition (iii) *of Theorem* 2.1 *hold. Then A has at least one sign-changing fixed point, two positive fixed points, and two negative fixed points.*

In order to prove Theorems 2.1 and 2.2, we need to establish the following lemmas. In this section, we suppose that $B_R = \{x \in E \mid \|x\| < R\}$.

Lemma 2.1 *Let P be a normal and total cone in E and $A : E \to E$ be a completely continuous increasing operator. Suppose that*

(i) *there exist $u, v \in E$ such that $u \leq Au$, $Av \leq v$;*
(ii) *A is Fréchet differentiable at ∞; A'_∞ is increasing; $r(A'_\infty) > 1$ and 1 is not a characteristic value of A'_∞.*

Then there exists $\overline{R} > 0$ such that

$$i(A, \widetilde{\Omega}_1, S_1) = 0, \qquad i(A, \widetilde{\Omega}_2, S_2) = 0$$

for all $R \geq \overline{R}$, where

$$S_1 = \{x \in E \mid x \geq u\}, \qquad S_2 = \{x \in E \mid x \leq v\},$$
$$\widetilde{\Omega}_1 = \{x \in S_1 \mid \|x\| < R\}, \qquad \widetilde{\Omega}_2 = \{x \in S_2 \mid \|x\| < R\}.$$

Proof We only prove that $i(A, \widetilde{\Omega}_1, S_1) = 0$, the proof of $i(A, \widetilde{\Omega}_2, S_2) = 0$ is similar.

Let $\tilde{A}x = u - A'_\infty(u - x)$, $x \in E$. By the compactness of A'_∞ ([16], Proposition 7.33, p.296), we can know that $\tilde{A}x : E \to E$ is completely continuous. Since A'_∞ is increasing, we can find that $\tilde{A}$ is also increasing, and $\tilde{A} : S_1 \to S_1$.

In the following, we show that

$$\left\|h_n - A'_\infty h_n\right\| < \frac{1}{n}\|h_n\| \quad \text{for each } n \in \mathbb{N} \text{ and } h_n \in P \tag{2.1}$$

is not valid.

Take $g_n = \frac{h_n}{\|h_n\|}$. Then

$$\left\|g_n - A'_\infty g_n\right\| < \frac{1}{n} \to 0 \quad (n \to \infty).$$

Since A'_∞ is completely continuous, we have that every subsequence $A'_\infty g_{n_k}$ of $A'_\infty g_n$ converges to g^*. Obviously, $g_n \to g^*$ $(n \to \infty)$. Hence, $A'_\infty g^* = g^*$, which is a contradiction with that 1 is not a characteristic value of A'_∞. It follows that $\inf_{h\in P,\|h\|=1} \|h - A'_\infty h\| := \alpha > 0$. Therefore, we have

$$\left\|h - A'_\infty h\right\| \geq \alpha \|h\| \quad \text{for all } h \in P. \tag{2.2}$$

For each $x \in S_1$, we have that $u - x \in P$. It follows from (2.2) and $\tilde{A}u = u$ that

$$\|x - \tilde{A}x\| = \left\|A'_\infty(u - x) - (u - x)\right\| \geq \alpha\|u - x\|. \tag{2.3}$$

According to the definition of A'_∞, we obtain

$$\begin{aligned}\lim_{\|x\|\to\infty} \frac{\|\tilde{A}x - Ax\|}{\|u - x\|} &\leq \lim_{\|x\|\to\infty} \frac{\|\tilde{A}x - Ax\|}{\|x\| - \|u\|} \\ &\leq \lim_{\|x\|\to\infty} \frac{\|A'_\infty x - Ax\|}{\|x\|} + \lim_{\|x\|\to\infty} \frac{\|u - A'_\infty u\|}{\|x\|} \\ &= 0.\end{aligned}$$

Hence,

$$\lim_{x\in S_1,\|x\|\to\infty} \frac{\|\tilde{A}x - Ax\|}{\|u - x\|} = 0.$$

This implies that there exists $\overline{R} > \|u\|$ such that, for all $x \in S_1$ with $\|x\| \geq \overline{R}$, we have

$$\|\tilde{A}x - Ax\| \leq \frac{\alpha}{2}\|u - x\|. \tag{2.4}$$

It is obvious that $u \in \widetilde{\Omega}_1$. Therefore, $\widetilde{\Omega}_1$ is a nonempty bounded open set of S_1. For $(t,x) \in [0,1] \times \widetilde{\Omega}_1$. Let $H(t,x) = tAx + (1-t)\tilde{A}x$.

Now, we prove

$$H(t,x) \neq x, \quad \forall t \in [0,1], x \in \partial\widetilde{\Omega}_1. \tag{2.5}$$

If (2.5) is not valid, then there exist $t_0 \in [0,1]$ and $x_0 \in \partial\widetilde{\Omega}_1$ such that

$$t_0 Ax_0 + (1 - t_0)\tilde{A}x_0 = x_0. \tag{2.6}$$

This yields $x_0 \in S_1$ and $\|x_0\| = R \geq R_0 > \|u\|$. Furthermore, in virtue of (2.3) and (2.4), we obtain

$$\begin{aligned} &\big\|x_0 - \big[t_0 A x_0 + (1-t_0)\tilde{A}x_0\big]\big\| \\ &\quad \geq \|x_0 - \tilde{A}x_0\| - \|Ax_0 - \tilde{A}x_0\| \\ &\quad \geq \frac{\alpha}{2}\|u - x_0\| \geq \frac{\alpha}{2}\big(\|x_0\| - \|u\|\big) > 0. \end{aligned}$$

This is a contradiction with (2.6), and so (2.5) holds. An application of (2.5) yields that A and $\tilde{A}$ have no fixed point on $\partial\widetilde{\Omega}_1$. Hence, the fixed point indices $i(A, \widetilde{\Omega}_1, S_1)$, $i(\tilde{A}, \widetilde{\Omega}_1, S_1)$ are well defined. It follows from the homotopy invariance of the fixed point index that

$$i(A, \widetilde{\Omega}_1, S_1) = i(\tilde{A}, \widetilde{\Omega}_1, S_1). \tag{2.7}$$

Notice that $A'_\infty(P) \subset P$ and $r(A'_\infty) > 1$. By the Krein-Rutman theorem [17], there exists $\varphi^* \in P\backslash\{\theta\}$ such that

$$A'_\infty \varphi^* = r\big(A'_\infty\big)\varphi^* > \varphi^*. \tag{2.8}$$

At last, we show

$$\tilde{A}x - x \neq \mu\varphi^*, \quad \forall \mu \geq 0, x \in \partial\widetilde{\Omega}_1.$$

If otherwise, then there exist $\mu_0 \geq 0$ and $\overline{x} \in \partial\widetilde{\Omega}_1$ such that

$$\tilde{A}\overline{x} - \overline{x} = \mu_0\varphi^*. \tag{2.9}$$

Since $\tilde{A}$ has no fixed point on $\partial\widetilde{\Omega}_1$, we can find that $\mu_0 > 0$. On account of $\overline{x} \leq u$ and the increasing property of $\tilde{A}$, we arrive at $\tilde{A}\overline{x} \leq \tilde{A}u = u$, which together with (2.9) implies that

$$\overline{x} = \tilde{A}\overline{x} - \mu_0\varphi^* \leq u - \mu_0\varphi^*.$$

Let

$$\mu^* = \sup\big\{\mu > 0 | u - \overline{x} \geq \mu\varphi^*\big\}.$$

Then $\mu^* \geq \mu_0 > 0$ and $u - \overline{x} \leq \mu^*\varphi^*$. Combined with (2.9) and the increasing property of A'_∞, we have

$$A'_\infty(u - \overline{x}) \geq \mu^* A'_\infty\varphi^* > \mu^*\varphi^*. \tag{2.10}$$

It follows from (2.9) and (2.10) that

$$\overline{x} = \tilde{A}\overline{x} - \mu_0\varphi^* = u - A'_\infty(u - \overline{x}) - \mu_0\varphi^* < u - \mu^*\varphi^* - \mu_0\varphi^*,$$

i.e., $u - \overline{x} > (\mu^* + \mu_0)\varphi^*$. This contradicts the definition of u^*. According to the lack of direction property of the fixed point index [16], we have $i(\tilde{A}, \widetilde{\Omega}_1, S_1) = 0$. By (2.7), we get $i(A, \widetilde{\Omega}_1, S_1) = 0$. □

Lemma 2.2 (See [17, 18]) *Suppose that $A : E \to E$ is a completely continuous and asymptotically linear operator. If 1 is not a characteristic value of the linear operator A'_∞, then there exists $R_0 > 0$ such that*

$$\deg(I - A, B_R, \theta) = (-1)^\gamma$$

for all $R \geq R_0$, where γ is the sum of the algebraic multiplicities of the real characteristic value of A'_∞ in $(0,1)$.

Proof of Theorem 2.1 By conditions (i) and (ii), we have

$$\begin{aligned} &u_1 < u_2 < \theta < v_2 < v_1, \\ &u_1 \leq Au_1, \qquad Au_2 < u_2, \qquad v_2 < Av_2, \qquad Av_1 \leq v_1. \end{aligned} \tag{2.11}$$

Let

$$\begin{aligned} &W_1 = \{x \in E | x \leq u_2\}, \qquad W_2 = \{x \in E | x \geq v_2\}, \\ &W_3 = \{x \in E | x \geq u_1\}, \qquad W_4 = \{x \in E | x \leq v_1\}. \end{aligned}$$

From Lemma 2.1, there exists $R_1 > 0$ such that

$$i(A, \overline{\Omega}_i, W_i) = 0, \quad i = 1, 2, 3, 4 \tag{2.12}$$

for all $R \geq R_1$, where $\overline{\Omega}_i = \{x \in W_i | \|x\| < R\}$, $i = 1, 2, 3, 4$.

According to Lemma 2.2, there exists $R_2 > 0$ such that

$$\deg(I - A, B_R, \theta) = (-1)^\gamma \tag{2.13}$$

for all $R \geq R_2$, where γ is the sum of algebraic multiplicities for all characteristic values of A'_∞ in $(0,1)$.

It follows from the definition of fixed point index [16], (2.3.4), p.84 and (2.13) that

$$i(A, B_R, E) = \deg\bigl(I - Ar, B_{\overline{R}} \cap r^{-1}(B_R), \theta\bigr) = \deg(I - A, B_R, \theta) = (-1)^\gamma = \pm 1, \tag{2.14}$$

where $r : E \to E$ is an identity operator, and $B_R \subset B_{\overline{R}} = \{x \in E | \|x\| < \overline{R}\}$.

Choose R_3 such that

$$R_3 > \Bigl\{ \sup_{x \in [u_1, v_1]} \|x\|, R_1, R_2 \Bigr\}. \tag{2.15}$$

Let

$$\Omega_i = \bigl\{x \in W_i | \|x\| < R_3\bigr\}, \quad i = 1, 2, 3, 4.$$

It follows from (2.12), (2.15) and Lemma 2.1 that

$$i(A, \Omega_i, W_i) = 0, \quad i = 1, 2, 3, 4. \tag{2.16}$$

Let

$$\Sigma_1 = \{x \in [u_1, u_2] : \|x\| < R_3, \text{and there exists } \mu > 0 \text{ such that } Ax \leq u_2 - \mu e\},$$
$$\Sigma_2 = \{x \in [v_2, v_1] : \|x\| < R_3, \text{and there exists } \mu > 0 \text{ such that } v_2 + \mu e \leq Ax\}.$$

Since $u_2 \in \Sigma_1$, $v_2 \in \Sigma_2$, we have $\Sigma_1 \neq \emptyset$, $\Sigma_2 \neq \emptyset$. According to the e-continuity of A in E, we can know that Σ_1 is a nonempty open subset of $[u_1, u_2]$, and Σ_2 is a nonempty open subset of $[v_2, v_1]$ (see the proof of Theorem 2.1 in [8]). By the homotopy invariance and normalization of the fixed point index, we have

$$i(A, \Sigma_1, [u_1, u_2]) = i(u_2, \Sigma_1, [u_1, u_2]) = 1. \tag{2.17}$$

Similarly, we have

$$i(A, \Sigma_2, [v_2, v_1]) = i(v_2, \Sigma_2, [v_2, v_1]) = 1. \tag{2.18}$$

Equations (2.17) and (2.18) imply that A has at least one negative fixed point $x_1 \in \Sigma_1$ and one positive fixed point $x_2 \in \Sigma_2$.

By the permanence property of the fixed point index, we obtain

$$i(A, \Sigma_1, W_1) = i(A, \Sigma_1 \cap [u_1, u_2], [u_1, u_2]) = i(A, \Sigma_1, [u_1, u_2]) = 1. \tag{2.19}$$

Similarly, we have

$$i(A, \Sigma_2, W_2) = i(A, \Sigma_2 \cap [v_2, v_1], [v_2, v_1]) = i(A, \Sigma_2, [v_2, v_1]) = 1. \tag{2.20}$$

It follows from (2.16), (2.19) and the additivity property of the fixed point index that

$$i(A, \Omega_1 \backslash \overline{\Sigma}_1, W_1) = i(A, \Omega_1, W_1) - i(A, \Sigma_1, W_1) = 0 - 1 = -1.$$

Similarly, we have

$$i(A, \Omega_2 \backslash \overline{\Sigma}_2, W_2) = i(A, \Omega_2, W_2) - i(A, \Sigma_2, W_2) = 0 - 1 = -1.$$

Therefore, A has at least one negative fixed point $x_3 \in \Omega_1 \backslash \overline{\Sigma}_1$ and one positive fixed point $x_4 \in \Omega_2 \backslash \overline{\Sigma}_2$.

Moreover, according to (2.16) and the permanence property of the fixed point index, we have

$$i(A, \Omega_i, E) = i(A, \Omega_i \cap W_i, W_i) = i(A, \Omega_i, W_i) = 0, \quad i = 3, 4. \tag{2.21}$$

It follows from (2.14), (2.21) and the additivity property of the fixed point index that we have

$$i(A, B_{R_3} \backslash \overline{\Omega_3 \cup \Omega_4}, E) = i(A, B_{R_3}, E) - i(A, \Omega_3, E) - i(A, \Omega_4, E) = \pm 1. \tag{2.22}$$

Equation (2.22) implies that A has at least one fixed point $x_5 \in (B_{R_3} \backslash \overline{\Omega_3 \cup \Omega_4}) \subset (E \backslash (P \cup (-P)))$, and x_5 is a sign-changing fixed point. □

Proof of Theorem 2.2 It suffices to verify condition (ii) of Theorem 2.1 is satisfied. According to the chain rule for derivatives of composite operator [17], Proposition 4.10, we have $A'_\theta = KF'_\theta$.

By condition (i) and Lemma 2.4 in [19], we know that there exists $\beta > 0$ such that $t\psi + \delta e \le A(t\psi), A(-t\psi) \le -t\psi - \delta e$ for all $t \in (0, \beta)$, where $\delta = \frac{t(\lambda_0^{-1}-1)\mu_1}{2} > 0$.

Let $\overline{\beta} = \min\{\beta, \frac{\alpha}{2\mu_2}\}$, for all $t \in (0, \overline{\beta})$, we have

$$t\psi \le \frac{\alpha}{2\mu_2}\mu_2 e = \frac{\alpha}{2} e \le \frac{v_1}{2} < v_1$$

and

$$u_1 < \frac{u_1}{2} \le -\frac{\alpha}{2} e = -\frac{\alpha}{2\mu_2}\mu_2 e \le -t\psi.$$

Hence, for $u_2 = -t\psi$ and $v_2 = t\psi$ with $t \in (0, \overline{\beta})$, condition (ii) of Theorem 2.1 is satisfied. The proof is completed. □

3 Applications

The main purpose of this section is to apply our theorems to both integral and differential equations. Firstly, we consider the following integral equation:

$$\varphi(x) = \int_G k(x,y) f\big(y, \varphi(y)\big)\, dy, \quad x \in G, \tag{3.1}$$

where G is a bounded closed domain of $\mathbb{R}^N$, $k : G \times G \to \mathbb{R}^1$ is nonnegative continuous, and $k \not\equiv 0$ on $G \times G$, and $f : G \times \mathbb{R}^1 \to \mathbb{R}^1$ is continuous.

Let $E = C(G)$ denote the space consisting of all continuous functions on G. Then E is a real Banach space with the norm $\|\varphi\| = \max_{x\in G} |\varphi(x)|$ for all $\varphi \in E$. And let $P = \{\varphi \in E : \varphi(x) \ge 0, x \in G\}$. Then P is a normal and total cone in E. Let $e(x) = \int_G k(x,y)\, dy$, $x \in G$. Then $e > 0$.

Define operators $F, K, A : E \to E$ respectively by

$$(F\varphi)(x) = f\big(x, \varphi(x)\big), \quad x \in G, \forall \varphi \in E,$$

$$(K\varphi)(x) = \int_G k(x,y)\varphi(y)\, dy, \quad x \in G, \forall \varphi \in E,$$

and $A = KF$. Obviously, $F : E \to E$ is a continuous and bounded operator. Since $k : G \times G \to \mathbb{R}^1$ is nonnegative continuous, we know that $K : E \to E$ is a linear completely continuous operator and $K(P) \subset P$. Therefore, A is also completely continuous on E. By the Riesz-Schauder theorem, we can suppose that the sequence of all positive characteristic values of K is $\{\lambda_n\}$ and

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots.$$

For convenience, we list some assumptions as follows.

(C_1) $f(\cdot,0)=0$ on G, and for every $x\in G$, $f(x,\varphi)$ is nondecreasing in φ;
(C_2) there exists h with $\mu e\le h$, where μ is a positive number such that

$$k(x,y)\ge h(x)k(z,y),\quad x,y,z\in G;$$

(C_3) $\lim_{\varphi\to 0}\frac{f(x,\varphi)}{\varphi}=f_0$ uniformly for $x\in G$, and $f_0>\lambda_1$;
(C_4) $\lim_{\varphi\to\infty}\frac{f(x,\varphi)}{\varphi}=f_\infty$ uniformly for $x\in G$, $f_\infty>\lambda_1$, $f_\infty\ne\lambda_k$, $k=2,3,\ldots$;
(C_5) for every $x\in G$, there exist $M,N>0$ such that

$$\frac{f(x,M)}{M}<\frac{1}{\|e\|}\quad\text{and}\quad\frac{f(x,-N)}{-N}<\frac{1}{\|e\|}.$$

Lemma 3.1 (See [19]) *Operators $K,A:E\to E$ are e-continuous on E.*

Lemma 3.2 (See [19]) *Suppose that $f(\cdot,0)=0$ on G, and $\lim_{\varphi\to 0}\frac{f(x,\varphi)}{\varphi}=f_0$ uniformly for $x\in G$. Then A is Fréchet differentiable at θ, and $A'_\theta=f_0K$.*

Lemma 3.3 *Suppose that $\lim_{\varphi\to\infty}\frac{f(x,\varphi)}{\varphi}=f_\infty$ uniformly for $x\in G$. Then A is asymptotically linear, and $A'_\infty=f_\infty K$.*

Proof Since $\lim_{\varphi\to\infty}\frac{f(x,\varphi)}{\varphi}=f_\infty$ uniformly for $x\in G$, we have that for every given $\epsilon>0$, there exists $R>0$ such that

$$\left|\frac{f(x,\varphi)}{\varphi}-f_\infty\right|<\epsilon,\quad x\in G,|\varphi|>R. \tag{3.2}$$

Set $M=\sup_{x\in G,|\varphi|\le R}|f(x,\varphi)-f_\infty\varphi|$, which together with (3.2) implies

$$\left|f(x,\varphi)-f_\infty\varphi\right|\le M+\epsilon|\varphi|,\quad\forall x\in G,\varphi\in(-\infty,+\infty).$$

Thus

$$\|A\varphi-f_\infty K\varphi\|=\left\|K(F\varphi)-f_\infty K\varphi\right\|\le\|K\|\big(M+\epsilon\|\varphi\|\big),\quad\forall\varphi\in E.$$

Therefore

$$\lim_{\|\varphi\|\to\infty}\frac{\|A\varphi-f_\infty K\varphi\|}{\|\varphi\|}=0.$$

This implies that A is asymptotically linear, and $A'_\infty=f_\infty K$. □

Theorem 3.1 *Suppose that* (C_1)-(C_5) *are satisfied. Then the integral equation* (3.1) *has at least five nontrivial solutions, two of which are positive, the two others are negative, the fifth one is sign-changing.*

Proof By (C_5), we know that condition (i) of Theorem 2.2 holds. We only need to check condition (ii) of Theorem 2.2 and condition (iii) of Theorem 2.1.

By (C_3) and Lemma 3.2, we have that $A'_\theta=f_0K$, and the characteristic values of the operator f_0K are $\frac{\lambda_1}{f_0},\frac{\lambda_2}{f_0},\ldots$. Since $f_0>\lambda_1$, we know that A'_θ has a characteristic value $\frac{\lambda_1}{f_0}<1$.

Moreover, from the proof of Theorem 4.1 in [19], we deduce that the characteristic function corresponding to the characteristic value $\frac{\lambda_1}{f_0}$ satisfies $\mu_1 e \leq \psi \leq \mu_2 e$, where $\mu_1 > 0$ and $\mu_2 > 0$. This implies that condition (ii) of Theorem 2.2 holds.

According to (C_4) and Lemma 3.3, we know that $A'_\infty = f_\infty K$, $r(A'_\infty) = f_\infty r(K) = f_\infty \frac{1}{\lambda_1} > 1$, and the characteristic values of the operator $f_\infty K$ are $\frac{\lambda_1}{f_\infty}, \frac{\lambda_2}{f_\infty}, \ldots$. Noting that $f_\infty \neq \lambda_k$, $k \neq 2, 3, \ldots$, we get that 1 is not a characteristic value of A'_∞. Hence, condition (iii) of Theorem 2.1 holds. The proof is completed. □

Remark 3.1 From the proof of Theorem 11 of [14] and Theorem 4.2 of [19], we know that condition (C_1) can be replaced by the following one: $f(\cdot, 0) = 0$ on G, and $f(\cdot, \varphi)\varphi \geq 0$ for all $\varphi \in \mathbb{R}$. For nonlinear term f, the sublinear and superlinear cases were considered in [20].

Now, we consider the following boundary value problem for elliptic partial differential equations:

$$\begin{cases} L\varphi(x) = f(x, \varphi(x)), & x \in \Omega, \\ B\varphi = 0, & x \in \partial\Omega, \end{cases} \tag{3.3}$$

where Ω is a bounded open domain in $\mathbb{R}^n$, $\partial\Omega \in C^{2+\mu}$, $0 < \mu < 1$; $f(x, \varphi) : \overline{\Omega} \times \mathbb{R}^1 \to \mathbb{R}^1$ is continuous;

$$L\varphi = -\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial \varphi}{\partial x_i} + c(x)\varphi$$

is a uniformly elliptic operator, *i.e.*, $a_{ij}(x) = a_{ji}(x), b_i(x), c(x) \in C^\mu(\overline{\Omega})$, $c(x) > 0$, and there exists a constant number $\mu_0 > 0$ such that $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \mu_0 |\xi|^2$ for all $x \in \overline{\Omega}$, $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$,

$$B\varphi = b(x)\varphi + \delta \sum_{i=1}^{n} \beta_i(x) \frac{\partial \varphi}{\partial x_i}$$

is a boundary operator, where $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ is a vector field on $\partial\Omega$ of $C^{1+\mu}$ satisfying $\beta \cdot \mathbf{n} > 0$ ($\mathbf{n}$ denotes the outer unit normal vector on $\partial\Omega$), $b(x) \in C^{1+\mu}(\partial\Omega)$, and assume that one of the following cases holds:

(i) $\delta = 0$ and $b(x) \equiv 1$;
(ii) $\delta = 1$ and $b(x) \equiv 0$;
(iii) $\delta = 1$ and $b(x) > 0$.

According to the theory of elliptic partial differential equations (see [21, 22]), we know that for each $u \in C(\overline{\Omega})$, the linear boundary value problem

$$\begin{cases} L\varphi(x) = u(x), & x \in \Omega, \\ B\varphi = 0, & x \in \partial\Omega, \end{cases}$$

has a unique solution $\varphi_u \in C^2(\overline{\Omega})$. Define the operator K by

$$(Ku)(x) = \varphi_u(x), \quad x \in \Omega.$$

Then $K : C(\overline{\Omega}) \to C^2(\overline{\Omega})$ is a linear completely continuous operator and has an unbounded sequence of characteristic values

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots, \quad \lambda_n \to +\infty,$$

and the spectral radius $r(K) = \lambda_1^{-1}$.

Let

$$E = C(\overline{\Omega}), \qquad P = \{\varphi \in E | \varphi(x) \ge 0, x \in \overline{\Omega}\}.$$

Then E is an ordered Banach space with the norm $\|\varphi\| = \sup_{x\in\overline{\Omega}} |\varphi(x)|$, and P is a normal cone in E with $K(P) \subset P$.

For $\varphi \in E$, define Nemytskii operator by

$$(F\varphi)(x) = f(x, \varphi(x)), \quad x \in \overline{\Omega}.$$

Clearly, $F : E \to E$ is continuous. Let $A = KF$. Then $A : E \to E$ is completely continuous.

We list some assumptions which will be used in the following theorem.

(D_1) $\lim_{\varphi\to\infty} \frac{f(x,\varphi)}{\varphi} = f_\infty$ uniformly for $x \in \overline{\Omega}$;
(D_2) $f(x, 0) \equiv 0$, $\lim_{\varphi\to 0} \frac{f(x,\varphi)}{\varphi} = f_0$ uniformly for $x \in \overline{\Omega}$.

Let $e = e(x)$ be the solution of the following boundary value problem:

$$\begin{cases} L\varphi(x) = 1, & x \in \Omega, \\ B\varphi = 0, & x \in \partial\Omega. \end{cases}$$

It is easy to check that A is e-continuous on E.

Our main result is the following.

Theorem 3.2 *Suppose that f satisfies* (D_1) *and* (D_2). *In addition, assume that*

(i) *$f(x, \varphi)$ is increasing in φ*;
(ii) $f_0 > \lambda_1$;
(iii) $f_\infty > \lambda_1, f_\infty \ne \lambda_k$, $k = 2, 3, \dots$;
(iv) *for every $x \in \overline{\Omega}$, there exist $M, N > 0$ such that*

$$\frac{f(x, M)}{M} < \frac{1}{\|e\|} \quad \text{and} \quad \frac{f(x, -N)}{-N} < \frac{1}{\|e\|}.$$

Then problem (3.3) *has at least two positive solutions, two negative solutions, and one sign-changing solution.*

Proof Since the characteristic values of the operator A'_θ are $\frac{\lambda_1}{f_0}, \frac{\lambda_2}{f_0}, \dots$, it follows from condition (ii) that A'_θ has a characteristic value $\frac{\lambda_1}{f_0} < 1$. Moreover, by condition (iii), we have

$$r(A'_\infty) = f_\infty r(K) = \frac{f_\infty}{\lambda_1} > 1.$$

Since the characteristic values of the operator A'_∞ are $\frac{\lambda_1}{f_\infty}, \frac{\lambda_2}{f_\infty}, \ldots$, noting that $f_\infty \neq \lambda_k$, $k = 2, 3, \ldots$, we know that 1 is not a characteristic value of A'_∞. Therefore, all conditions of Theorem 2.2 are satisfied. The proof is completed. □

4 An example

In the section, we present an example to explain our results.

Example 4.1 Consider the following second order three-point boundary value problem:

$$\begin{cases} -u''(t) = f(u), & t \in [0,1], \\ u(0) = 0, & u(1) = \frac{1}{2}u(\frac{1}{2}), \end{cases} \tag{4.1}$$

where

$$f(u) = \begin{cases} 7(u-27) + \frac{97}{3}\sqrt[3]{u}, & u \in [27, +\infty), \\ 3u + 16, & u \in [1, 27), \\ 30u - 11, & u \in (\frac{1}{2}, 1), \\ 7u + 32u^6, & u \in [-1, \frac{1}{2}], \\ 7u + 32, & u \in (-32, -1), \\ 7(u+32) + 96\sqrt[5]{u}, & u \in (-\infty, -32]. \end{cases}$$

By direct calculation [23], we can obtain that $\lambda_1 = 6.9497$ is a solution of the following equation:

$$\sin\sqrt{x} = \frac{1}{2}\sin\frac{\sqrt{x}}{2}.$$

Note that

$$\lim_{u\to\infty}\frac{f(u)}{u} = 7 > \frac{288}{49} = \frac{8(1 - \frac{1}{2}\cdot\frac{1}{2})^2}{(1 - \frac{1}{2}\cdot(\frac{1}{2})^2)^2}, \qquad \lim_{u\to 0}\frac{f(u)}{u} = 7 > 6.9497 = \lambda_1.$$

Thus, the conditions of Theorem 1.1 and Theorem 13 of [23] are not satisfied. Therefore, the main results of [11, 23] cannot be applied to Example 4.1. However, it is easy to see that

$$\frac{f(20)}{20} = \frac{76}{20} < \frac{288}{49}, \qquad \frac{f(-20)}{-20} = \frac{27}{5} < \frac{288}{49}.$$

The assumption (C_5) of Theorem 3.1 is satisfied, therefore, problem (4.1) has at least five nontrivial solutions, two of which are positive, the two others are negative, the fifth one is sign-changing.

Remark 4.1 Compared with Theorem 1.1, Theorem 13 of [23] and Theorem 2.1 of [24], the main contributions in this paper are that we change the range of β_0 in $(\widetilde{A}_3)$ and β_∞ in $(\widetilde{A}_2)$ and add the numbers of positive solutions and negative solutions. Furthermore, for the lower and upper solutions conditions, we adopt condition (C_5) which is more easy to check than (H_3) in Theorem 2.1 of [24].

5 Conclusions

The conclusion of Theorem 3.1 generalizes the main results of [11]. In this paper, by employing the computation of the fixed point index for asymptotically linear operators, the existence of positive solutions, negative solutions, and sign-changing solutions for asymptotically linear operator equations are established. In the future, we will consider the case that

$$\lim_{x\to+\infty}\frac{f(x)}{x}\neq\lim_{x\to-\infty}\frac{f(x)}{x}$$

by critical point theory.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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