

## RESEARCH

## Open Access



# A Hilbert-type fractal integral inequality and its applications

Qiong Liu and Wenbing Sun\* 

\*Correspondence:  
swb0520@163.com  
Department of Science and  
Information Science, Shaoyang  
University, Shaoyang, 422000,  
P.R. China

## Abstract

By using the fractal theory and the methods of weight function, a Hilbert-type fractal integral inequality and its equivalent form are given. Their constant factors are proved being the best possible, and their applications are discussed briefly.

**Keywords:** fractal set; Hilbert-type fractal integral inequality; weight function

## 1 Introduction

If  $f, g \geq 0$ , satisfying  $0 < \int_0^\infty f^2(x) dx < \infty$ ,  $0 < \int_0^\infty g^2(y) dy < \infty$ , then there is the following basic Hilbert-type integral inequality and its equivalent form [1]

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x, y\}} dx dy < 4 \left\{ \int_0^\infty f^2(x) dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(y) dy \right\}^{\frac{1}{2}}, \quad (1)$$

$$\int_0^\infty \left[ \int_0^\infty \frac{f(x)}{\max\{x, y\}} dx \right]^2 dy < 16 \int_0^\infty f^2(x) dx, \quad (2)$$

where the constants are optimal. Inequalities (1) and (2) are important in the analysis and partial differential equations [1, 2]. In 2004 and 2006, respectively, (1) and (2) were generalized and improved by introducing an independent parameter  $\lambda$  and two parameters  $\lambda_1, \lambda_2$  [3, 4].

In recent years, the fractal theory has been developed rapidly, and it has been widely used in the fields of science and engineering. Some researchers have used the fractal theory to discuss and generalize some classical inequalities on fractal sets [5, 6], but the research into the Hilbert-type integral inequality on the fractal set is still not involved. In this paper, by using the fractal theory and the method of weight function to make a meaningful attempt, a Hilbert-type integral inequality and its equivalent form on a fractal set are established.

## 2 Preliminaries

**Definition 2.1** ([7]) A non-differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}^\alpha$  ( $0 < \alpha \leq 1$ ),  $x \rightarrow f(x)$  is called local fractional continuous at  $x_0$  if for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|f(x) - f(x_0)| < \varepsilon^\alpha$  whenever  $|x - x_0| < \delta$ . If  $f(x)$  is local fractional continuous on the interval  $(a, b)$ , we denote  $f(x) \in C_\alpha(a, b)$ .

**Definition 2.2** ([7]) The local fractional derivative of  $f(x)$  of order  $\alpha$  ( $0 < \alpha \leq 1$ ) at  $x_0$  is defined by

$$f^\alpha(x_0) = \left. \frac{d^\alpha f(x)}{dx^\alpha} \right|_{x=x_0} = \lim_{x \rightarrow x_0} \frac{\Gamma(\alpha + 1)(f(x) - f(x_0))}{(x - x_0)^\alpha},$$

where  $\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} du$  ( $z > 0$ ) [8]. If for all  $x \in I \subseteq \mathbb{R}$ , there exists  $f^{(k+1)\alpha}(x) = \overbrace{D_x^\alpha \cdots D_x^\alpha}^{k+1} f(x)$ , then we denote  $f \in D_{(k+1)\alpha}(I)$ , where  $k = 0, 1, 2, \dots$ .

**Lemma 2.1** ([9]) Suppose that  $f(x) \in C_\alpha(a, b)$  and  $f(x) \in D_\alpha(a, b)$ . Then, for  $0 < \alpha \leq 1$ , we have an  $\alpha$ -differential form

$$d^\alpha f(x) = f^{(\alpha)}(x) dx^\alpha.$$

**Lemma 2.2** ([5]) Let  $I$  be an interval,  $f, g : I \subseteq \mathbb{R} \rightarrow \mathbb{R}^\alpha$  ( $I^0$  is the interior of  $I$ ) such that  $f, g \in D_\alpha(I^0)$ . Then the following differentiation rules are valid:

- (i)  $\frac{d^\alpha(f(x) \pm g(x))}{dx^\alpha} = f^{(\alpha)}(x) \pm g^{(\alpha)}(x)$ ;
- (ii)  $\frac{d^\alpha(f(x)g(x))}{dx^\alpha} = f^{(\alpha)}(x)g(x) + f(x)g^{(\alpha)}(x)$ ;
- (iii)  $\frac{d^\alpha \frac{f(x)}{g(x)}}{dx^\alpha} = \frac{f^{(\alpha)}(x)g(x) - f(x)g^{(\alpha)}(x)}{g^2(x)}$  ( $g(x) \neq 0$ );
- (iv)  $\frac{d^\alpha(Cf(x))}{dx^\alpha} = Cf^{(\alpha)}(x)$ , where  $C$  is a constant;
- (v) If  $\gamma(x) = (f \circ g)(x)$ , then  $\frac{d^\alpha \gamma(x)}{dx^\alpha} = f^{(\alpha)}(g(x))(g^{(1)}(x))^\alpha$ .

**Definition 2.3** ([7]) Let  $f(x) \in C_\alpha(a, b)$ . Then the local fractional integral is defined by

$${}_a I_b^\alpha f(x) = \frac{1}{\Gamma(\alpha + 1)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(\alpha + 1)} \lim_{\lambda \rightarrow 0} \sum_{i=1}^N f(t_i)(\Delta t_i)^\alpha,$$

with  $\Delta t_i = t_i - t_{i-1}$  ( $i = 1, \dots, N$ ) and  $\lambda = \max_{1 \leq i \leq N} \{\Delta t_i\}$ , and  $a = t_0 < t_1 < \dots < t_N = b$  is partition of interval  $[a, b]$ . Here, it follows that  ${}_a I_b^\alpha = 0$  if  $a = b$ ,  ${}_a I_b^\alpha f(x) = -{}_b I_a^\alpha f(x)$  if  $a < b$ .

**Lemma 2.3** ([7])

- (1) Suppose that  $f(x) = g^{(\alpha)}(x) \in C_\alpha(a, b)$ , then we have

$${}_a I_b^\alpha f(x) = g(b) - g(a);$$

- (2) Suppose that  $f(x), g(x) \in D_\alpha(a, b)$ , and  $f^{(\alpha)}(x), g^{(\alpha)}(x) \in C_\alpha(a, b)$ , then we have

$${}_a I_b^\alpha f(x)g^{(\alpha)}(x) = f(x)g(x)|_a^b - {}_a I_b^\alpha f^{(\alpha)}(x)g(x).$$

**Lemma 2.4** ([7]) For  $f(x) = x^\gamma$  ( $\gamma > 0$ ), we have the following equations:

$$\frac{d^\alpha(x^\gamma)}{dx^\alpha} = \frac{\Gamma(1 + \gamma)}{\Gamma(1 + \gamma - \alpha)} x^{\gamma - \alpha};$$

$$\frac{1}{\Gamma(\alpha + 1)} \int_a^b x^\gamma (dx)^\alpha = \frac{\Gamma(1 + \gamma)}{\Gamma(1 + \gamma + \alpha)} (b^{\gamma + \alpha} - a^{\gamma + \alpha}).$$

**Lemma 2.5** ([7, 10]) *If  $f, g (\geq 0) \in C_\alpha(a, b)$ ,  $F, G, h (\geq 0) \in C_\alpha(S^{(\beta)})$ ,  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $S^{(\beta)}$  is a fractal surface, then we have*

(i) *Hölder’s inequality on the fractal set*

$$\begin{aligned} & \frac{1}{\Gamma(\alpha + 1)} \int_a^b f(x)g(x)(dx)^\alpha \\ & \leq \left\{ \frac{1}{\Gamma(\alpha + 1)} \int_a^b f^p(x)(dx)^\alpha \right\}^{\frac{1}{p}} \left\{ \frac{1}{\Gamma(\alpha + 1)} \int_a^b g^q(x)(dx)^\alpha \right\}^{\frac{1}{q}}; \end{aligned}$$

(ii) *Hölder’s weighted inequality on the fractal set*

$$\begin{aligned} & \frac{1}{\Gamma^2(\alpha + 1)} \iint_{S^{(\beta)}} h(x, y)F(x, y)G(x, y)(dx)^\alpha (dy)^\alpha \\ & \leq \left\{ \frac{1}{\Gamma^2(\alpha + 1)} \iint_{S^{(\beta)}} h(x, y)F^p(x, y)(dx)^\alpha (dy)^\alpha \right\}^{\frac{1}{p}} \\ & \quad \times \left\{ \frac{1}{\Gamma^2(\alpha + 1)} \iint_{S^{(\beta)}} h(x, y)G^q(x, y)(dx)^\alpha (dy)^\alpha \right\}^{\frac{1}{q}}. \end{aligned}$$

*The inequality keeps the form of equality, then there exist constants A and B such that they are not all zero and  $AF^p(x, y) = BG^q(x, y)$  a.e. on  $S^{(\beta)}$ .*

**Lemma 2.6** *Suppose that  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $0 < \alpha \leq 1$ , and weight functions are defined by*

$$\begin{aligned} \omega(\alpha, p, x) & := \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty \frac{1}{\max\{x^\alpha, y^\alpha\}} \frac{y^{-\frac{\alpha}{2}}}{x^{-\frac{p\alpha}{2q}}}(dy)^\alpha, \quad x \in (0, +\infty), \\ \omega(\alpha, q, y) & := \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty \frac{1}{\max\{x^\alpha, y^\alpha\}} \frac{x^{-\frac{\alpha}{2}}}{y^{-\frac{q\alpha}{2p}}}(dx)^\alpha, \quad y \in (0, +\infty). \end{aligned}$$

*Then*

$$\omega(\alpha, p, x) = \eta(\alpha)x^{\frac{\alpha}{2}(p-2)}, \quad \omega(\alpha, q, y) = \eta(\alpha)y^{\frac{\alpha}{2}(q-2)},$$

*where*

$$\eta(\alpha) = \frac{2^{\alpha+1}}{\Gamma(1 + \alpha)}. \tag{3}$$

*Proof* Set  $\frac{y}{x} = u$ , then  $(dy)^\alpha = x^\alpha (du)^\alpha$ . Note the following exchange integral, let  $u = \frac{1}{t}$ , and  $\sqrt{u} = s$ , we have  $(du)^\alpha = -t^{-2\alpha}(dt)^\alpha$  and  $u^{-\frac{\alpha}{2}}(du)^\alpha = 2^\alpha(ds)^\alpha$ . Then we have

$$\begin{aligned} \omega(\alpha, p, x) & = \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty \frac{1}{\max\{x^\alpha, y^\alpha\}} \frac{y^{-\frac{\alpha}{2}}}{x^{-\frac{p\alpha}{2q}}}(dy)^\alpha \\ & = x^{\frac{\alpha}{2}(p-2)} \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty \frac{\max\{1, u^\alpha\}}{\max\{x^\alpha, y^\alpha\}} u^{-\frac{\alpha}{2}}(du)^\alpha \\ & = x^{\frac{\alpha}{2}(p-2)} \frac{1}{\Gamma(\alpha + 1)} \left[ \int_0^1 u^{-\frac{\alpha}{2}}(du)^\alpha + \int_1^\infty u^{-\frac{3\alpha}{2}}(du)^\alpha \right] \end{aligned}$$

$$\begin{aligned}
 &= x^{\frac{\alpha}{2}(p-2)} \frac{1}{\Gamma(\alpha + 1)} \left[ \int_0^1 u^{-\frac{\alpha}{2}} (du)^\alpha + \int_0^1 t^{-\frac{\alpha}{2}} (dt)^\alpha \right] \\
 &= x^{\frac{\alpha}{2}(p-2)} \left( \frac{2}{\Gamma(\alpha + 1)} \int_0^1 u^{-\frac{\alpha}{2}} (du)^\alpha \right) \\
 &= x^{\frac{\alpha}{2}(p-2)} \left( \frac{2^{\alpha+1}}{\Gamma(\alpha + 1)} \int_0^1 (ds)^\alpha \right) \\
 &= \frac{2^{\alpha+1}}{\Gamma(1 + \alpha)} x^{\frac{\alpha}{2}(p-2)} \\
 &= \eta(\alpha) x^{\frac{\alpha}{2}(p-2)}.
 \end{aligned}$$

Similarly, we obtain  $\omega(\alpha, q, y) = \eta(\alpha) y^{\frac{\alpha}{2}(q-2)}$ . □

**Lemma 2.7** *Suppose that  $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \alpha \leq 1$ , and  $\varepsilon > 0$  is small enough, let us define the real functions as follows:*

$$\bar{f}(x) = \begin{cases} 0, & x \in (0, 1), \\ x^{-\frac{\alpha}{2} - \frac{\alpha\varepsilon}{p}}, & x \in [1, \infty), \end{cases} \quad \bar{g}(y) = \begin{cases} 0, & y \in (0, 1), \\ x^{-\frac{\alpha}{2} - \frac{\alpha\varepsilon}{q}}, & y \in [1, \infty), \end{cases}$$

then we have

$$\bar{J} \cdot \varepsilon^\alpha = [{}_0I_\infty^\alpha (x^{\frac{\alpha}{2}(p-2)} \bar{f}^p(x))]^{\frac{1}{p}} [{}_0I_\infty^\alpha (y^{\frac{\alpha}{2}(q-2)} \bar{g}^q(y))]^{\frac{1}{q}} \cdot \varepsilon^\alpha = \frac{1}{\Gamma(\alpha + 1)}, \tag{4}$$

$$\bar{h} \cdot \varepsilon^\alpha = {}_0I_\infty^\alpha \left[ \frac{\bar{f}(x)\bar{g}(y)}{\max\{x^\alpha, y^\alpha\}} \right] \cdot \varepsilon^\alpha > \frac{\eta(\alpha)}{\Gamma(\alpha + 1)} (1 - o(1)) \quad (\varepsilon \rightarrow 0^+). \tag{5}$$

*Proof* Note the properties of local fractal space [5, 7]:  $(a + b)^\alpha = a^\alpha + b^\alpha$  and  $(-a)^\alpha = -a^\alpha$ , we easily obtain

$$\begin{aligned}
 \bar{J} \cdot \varepsilon^\alpha &= [{}_0I_\infty^\alpha (x^{\frac{\alpha}{2}(p-2)} \bar{f}^p(x))]^{\frac{1}{p}} [{}_0I_\infty^\alpha (y^{\frac{\alpha}{2}(q-2)} \bar{g}^q(y))]^{\frac{1}{q}} \cdot \varepsilon^\alpha \\
 &= [{}_1I_\infty^\alpha (x^{-\alpha(1+\varepsilon)})]^{\frac{1}{p}} [{}_1I_\infty^\alpha (y^{-\alpha(1+\varepsilon)})]^{\frac{1}{q}} \cdot \varepsilon^\alpha = \frac{1}{\Gamma(\alpha + 1)}.
 \end{aligned}$$

Let  $t^{\frac{1}{2} - \frac{\varepsilon}{q}} = u, x^{-\frac{1}{2} + \frac{\varepsilon}{q}} = v$ , and from  $(\frac{1}{2} - \frac{\varepsilon}{q})^\alpha t^{-\frac{\alpha}{2} - \frac{\alpha\varepsilon}{q}} (dt)^\alpha = (du)^\alpha, -(\frac{1}{2} - \frac{\varepsilon}{q})^\alpha x^{-\frac{3\alpha}{2} + \frac{\alpha\varepsilon}{q}} (dx)^\alpha = (dv)^\alpha$ , we write

$$\begin{aligned}
 {}_1I_\infty^\alpha (x^{-\alpha}) \cdot {}_0I_{\frac{1}{x}}^\alpha (t^{-\frac{\alpha}{2} - \frac{\alpha\varepsilon}{q}}) &= \frac{1}{\Gamma^2(\alpha + 1)} \left( \int_1^\infty x^{-\alpha} (dx)^\alpha \right) \left( \int_0^{\frac{1}{x}} t^{-\frac{\alpha}{2} - \frac{\alpha\varepsilon}{q}} (dt)^\alpha \right) \\
 &= \frac{1}{(\frac{1}{2} - \frac{\varepsilon}{q})^\alpha \Gamma^2(\alpha + 1)} \left( \int_1^\infty x^{-\alpha} (dx)^\alpha \right) \left( \int_0^{x^{-\frac{1}{2} + \frac{\varepsilon}{q}}} (dv)^\alpha \right) \\
 &= \frac{1}{(\frac{1}{2} - \frac{\varepsilon}{q})^\alpha \Gamma^2(\alpha + 1)} \int_1^\infty x^{-\frac{3\alpha}{2} + \frac{\alpha\varepsilon}{q}} (dx)^\alpha \\
 &= -\frac{1}{(\frac{1}{2} - \frac{\varepsilon}{q})^{2\alpha} \Gamma^2(\alpha + 1)} \int_1^0 (dv)^\alpha \\
 &= \frac{1}{(\frac{1}{2} - \frac{\varepsilon}{q})^{2\alpha} \Gamma^2(\alpha + 1)}.
 \end{aligned}$$

Further, let  $\frac{y}{x} = t$ , and by Lemma 2.6, we have

$$\begin{aligned}
 \bar{h} \cdot \varepsilon^\alpha &= {}_0I_\infty^\alpha \left[ {}_0I_\infty^\alpha \frac{\bar{f}(x)\bar{g}(y)}{\max\{x^\alpha, y^\alpha\}} \right] \cdot \varepsilon^\alpha \\
 &= {}_1I_\infty^\alpha \left[ \left( x^{-\frac{\alpha}{2} - \frac{\alpha\varepsilon}{p}} \right) {}_1I_\infty^\alpha \left( \frac{y^{-\frac{\alpha}{2} - \frac{\alpha\varepsilon}{q}}}{\max\{x^\alpha, y^\alpha\}} \right) \right] \cdot \varepsilon^\alpha \\
 &= [{}_1I_\infty^\alpha (x^{-\alpha(1+\varepsilon)})] \left[ \frac{1}{x} {}_1I_\infty^\alpha \left( \frac{t^{-\frac{\alpha}{2} - \frac{\alpha\varepsilon}{q}}}{\max\{1, t^\alpha\}} \right) \right] \cdot \varepsilon^\alpha \\
 &= [{}_1I_\infty^\alpha (x^{-\alpha(1+\varepsilon)})] \\
 &\quad \times \left[ {}_0I_1^\alpha \left( \frac{t^{-\frac{\alpha}{2} - \frac{\alpha\varepsilon}{q}}}{\max\{1, t^\alpha\}} \right) + {}_1I_\infty^\alpha \left( \frac{t^{-\frac{\alpha}{2} - \frac{\alpha\varepsilon}{q}}}{\max\{1, t^\alpha\}} \right) - {}_0I_{\frac{1}{x}}^\alpha \left( \frac{t^{-\frac{\alpha}{2} - \frac{\alpha\varepsilon}{q}}}{\max\{1, t^\alpha\}} \right) \right] \cdot \varepsilon^\alpha \\
 &= [{}_1I_\infty^\alpha (x^{-\alpha(1+\varepsilon)})] \left[ {}_0I_1^\alpha \left( t^{-\frac{\alpha}{2} - \frac{\alpha\varepsilon}{q}} \right) + {}_1I_\infty^\alpha \left( t^{-\frac{\alpha}{2} - \frac{\alpha\varepsilon}{q}} \right) - {}_0I_{\frac{1}{x}}^\alpha \left( \frac{t^{-\frac{\alpha}{2} - \frac{\alpha\varepsilon}{q}}}{\max\{1, t^\alpha\}} \right) \right] \cdot \varepsilon^\alpha \\
 &= [{}_1I_\infty^\alpha (x^{-\alpha(1+\varepsilon)})] \left[ {}_0I_1^\alpha \left( t^{-\frac{\alpha}{2} - \frac{\alpha\varepsilon}{q}} \right) + {}_0I_1^\alpha \left( t^{-\frac{\alpha}{2} + \frac{\alpha\varepsilon}{q}} \right) - {}_0I_{\frac{1}{x}}^\alpha \left( t^{-\frac{\alpha}{2} - \frac{\alpha\varepsilon}{q}} \right) \right] \cdot \varepsilon^\alpha \\
 &= \frac{1}{\left(\frac{1}{2} - \frac{\varepsilon}{q}\right)^\alpha \Gamma^2(\alpha + 1)} + \frac{1}{\left(\frac{1}{2} + \frac{\varepsilon}{q}\right)^\alpha \Gamma^2(\alpha + 1)} - {}_1I_\infty^\alpha (x^{-\alpha(1+\varepsilon)}) \cdot {}_0I_{\frac{1}{x}}^\alpha \left( t^{-\frac{\alpha}{2} - \frac{\alpha\varepsilon}{q}} \right) \cdot \varepsilon^\alpha \\
 &= \frac{\eta(\alpha)}{\Gamma(\alpha + 1)} + o_1(1) - {}_1I_\infty^\alpha (x^{-\alpha(1+\varepsilon)}) \cdot {}_0I_{\frac{1}{x}}^\alpha \left( t^{-\frac{\alpha}{2} - \frac{\alpha\varepsilon}{q}} \right) \cdot \varepsilon^\alpha \\
 &> \frac{\eta(\alpha)}{\Gamma(\alpha + 1)} + o_1(1) - {}_1I_\infty^\alpha (x^{-\alpha}) \cdot {}_0I_{\frac{1}{x}}^\alpha \left( t^{-\frac{\alpha}{2} - \frac{\alpha\varepsilon}{q}} \right) \cdot \varepsilon^\alpha \\
 &= \frac{\eta(\alpha)}{\Gamma(\alpha + 1)} + o_1(1) - \frac{\varepsilon^\alpha}{\left(\frac{1}{2} - \frac{\varepsilon}{q}\right)^{2\alpha} \Gamma^2(\alpha + 1)} \\
 &= \frac{\eta(\alpha)}{\Gamma(\alpha + 1)} (1 - o(1)) \quad (\varepsilon \rightarrow 0^+). \quad \square
 \end{aligned}$$

### 3 Main results and applications

Introducing the mark:  ${}_0I_\infty^\alpha [{}_0I_\infty^\alpha F(x, y)] = \frac{1}{\Gamma^2(\alpha+1)} \int_0^\infty \int_0^\infty F(x, y) (dx)^\alpha (dy)^\alpha$  (see [7]).

**Theorem 3.1** *If  $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \alpha \leq 1, f, g (> 0) \in C_\alpha(0, \infty)$ , and  $0 < {}_0I_\infty^\alpha (x^{\frac{\alpha}{2}(p-2)} f^p(x)) < \infty, 0 < {}_0I_\infty^\alpha (y^{\frac{\alpha}{2}(q-2)} g^q(y)) < \infty$ , then*

$${}_0I_\infty^\alpha \left[ {}_0I_\infty^\alpha \frac{f(x)g(y)}{\max\{x^\alpha, y^\alpha\}} \right] < \eta(\alpha) \left\{ {}_0I_\infty^\alpha (x^{\frac{\alpha}{2}(p-2)} f^p(x)) \right\}^{\frac{1}{p}} \left\{ {}_0I_\infty^\alpha (y^{\frac{\alpha}{2}(q-2)} g^q(y)) \right\}^{\frac{1}{q}}, \tag{6}$$

where the constant factor  $\eta(\alpha)$  defined in (3) is the best possible.

*Proof* By Hölder’s weighted inequality on the fractal set and Lemma 2.6, we obtain

$$\begin{aligned}
 {}_0I_\infty^\alpha \left[ {}_0I_\infty^\alpha \frac{f(x)g(y)}{\max\{x^\alpha, y^\alpha\}} \right] &= \frac{1}{\Gamma^2(\alpha + 1)} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\alpha, y^\alpha\}} (dx)^\alpha (dy)^\alpha \\
 &= \frac{1}{\Gamma^2(\alpha + 1)} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\alpha, y^\alpha\}} \left[ \frac{y^{-\frac{\alpha}{2p}}}{x^{-\frac{\alpha}{2q}}} \right] \left[ \frac{x^{-\frac{\alpha}{2q}}}{y^{-\frac{\alpha}{2p}}} \right] (dx)^\alpha (dy)^\alpha \\
 &\leq \left\{ \frac{1}{\Gamma^2(\alpha + 1)} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\alpha, y^\alpha\}} \frac{y^{-\frac{\alpha}{2}}}{x^{-\frac{p\alpha}{2q}}} (dx)^\alpha (dy)^\alpha \right\}^{\frac{1}{p}}
 \end{aligned}$$

$$\begin{aligned}
 & \times \left\{ \frac{1}{\Gamma^2(\alpha + 1)} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\alpha, y^\alpha\}} \frac{x^{-\frac{\alpha}{2}}}{y^{-\frac{q\alpha}{2p}}}(dx)^\alpha (dy)^\alpha \right\}^{\frac{1}{q}} \\
 & = \left\{ \frac{1}{\Gamma^2(\alpha + 1)} \int_0^\infty \omega(\alpha, p, x)f^p(x)(dx)^\alpha \right\}^{\frac{1}{p}} \\
 & \quad \times \left\{ \frac{1}{\Gamma^2(\alpha + 1)} \int_0^\infty \int_0^\infty \omega(\alpha, q, y)g^q(y)(dy)^\alpha \right\}^{\frac{1}{q}} \\
 & = \eta(\alpha) \left\{ \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty x^{\frac{\alpha}{2}(p-2)}f^p(x)(dx)^\alpha \right\}^{\frac{1}{p}} \\
 & \quad \times \left\{ \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty y^{\frac{\alpha}{2}(q-2)}g^q(y)(dy)^\alpha \right\}^{\frac{1}{q}} \\
 & = \eta(\alpha) \left\{ {}_0I_\infty^\alpha \left( x^{\frac{\alpha}{2}(p-2)}f^p(x) \right) \right\}^{\frac{1}{p}} \left\{ {}_0I_\infty^\alpha \left( y^{\frac{\alpha}{2}(q-2)}g^q(y) \right) \right\}^{\frac{1}{q}}. \tag{7}
 \end{aligned}$$

Now assume that equality holds in (7), there exist two nonzero constants  $A$  and  $B$  such that  $A \frac{y^{-\frac{\alpha}{2}}}{x^{-\frac{p\alpha}{2q}}}f^p(x) = B \frac{x^{-\frac{\alpha}{2}}}{y^{-\frac{q\alpha}{2p}}}$  a.e. in  $(0, \infty) \times (0, \infty)$ , then there is constant  $C \neq 0$  such that  $Ax^{\frac{\alpha}{2}(p-2)}f^p(x) = By^{\frac{\alpha}{2}(q-2)}g^q(y) = C$  a.e. in  $(0, \infty) \times (0, \infty)$ . Assuming that  $A \neq 0$ , we have  $x^{\frac{\alpha}{2}(p-2)}f^p(x) = \frac{C}{A}$  a.e. in  $(0, \infty)$ . Because  $\frac{1}{\Gamma(\alpha+1)} \int_0^\infty \frac{C}{A}(dx)^\alpha = \frac{C}{A\Gamma(\alpha+1)}x^\alpha|_0^\infty$  is diffuse, which contradicts the fact that  $0 < {}_0I_\infty^\alpha(x^{\frac{\alpha}{2}(p-2)}f^p(x)) < \infty$ , thus inequality (7) is strict.

If the constant factor  $\eta(\alpha)$  in (6) is not optimal, then there exists positive  $K < \eta(\alpha)$  such that inequality (6) is still valid if we replace  $\eta(\alpha)$  by  $K$ . Hence by (4) and (5), we have  $\eta(\alpha)(1 - o(1)) < K$ .

Letting  $\varepsilon \rightarrow 0^+$ , we get  $K \geq \eta(\alpha)$ , which contradicts the fact that  $K < \eta(\alpha)$ , therefore  $\eta(\alpha)$  in (6) is the best possible. □

**Theorem 3.2** *Under the conditions of Theorem 3.1, we have*

$${}_0I_\infty^\alpha \left\{ y^{\frac{\alpha(2-q)}{2(q-1)}} \left[ {}_0I_\infty^\alpha \frac{f(x)}{\max\{x^\alpha, y^\alpha\}} \right]^p \right\} < \eta^p(\alpha) {}_0I_\infty^\alpha \left( x^{\frac{\alpha}{2}(p-2)}f^p(x) \right), \tag{8}$$

where the constant factor  $\eta^p(\alpha)$  is the best possible, and inequality (8) is equivalent to inequality (6).

*Proof* Define  $[f(x)]_n := \min\{n, f(x)\}$ . Since  $0 < {}_0I_\infty^\alpha(x^{\frac{\alpha}{2}(p-2)}f^p(x)) < \infty$ , there exists  $n_0 \in \mathbb{N}$  such that  $0 < \frac{1}{n}I_n^\alpha(x^{\frac{\alpha}{2}(p-2)}[f(x)]_n^p) < \infty$  ( $n \geq n_0$ ). Setting  $g_n(y) := y^{\frac{\alpha(2-q)}{2(q-1)}} \left[ \frac{1}{n}I_n^\alpha \frac{[f(x)]_n}{\max\{x^\alpha, y^\alpha\}} \right]^{\frac{p}{q}}$  ( $\frac{1}{n} < y < n, n \geq n_0$ ), when  $n \geq n_0$ , by (6), we find

$$\begin{aligned}
 & 0 < \frac{1}{n}I_n^\alpha \left( y^{\frac{\alpha}{2}(q-2)}g_n^q(y) \right) \\
 & = \frac{1}{\Gamma(\alpha + 1)} \int_{\frac{1}{n}}^n y^{\frac{\alpha}{2}(q-2)}g_n^{q-1}(y)g_n(y)(dy)^\alpha \\
 & = \frac{1}{\Gamma(\alpha + 1)} \int_{\frac{1}{n}}^n \frac{1}{n}I_n^\alpha \left( \frac{[f(x)]_n}{\max\{x^\alpha, y^\alpha\}} \right) y^{\frac{\alpha(2-q)}{2(q-1)}} \left[ \frac{1}{n}I_n^\alpha \frac{[f(x)]_n}{\max\{x^\alpha, y^\alpha\}} \right]^{\frac{p}{q}} (dy)^\alpha \\
 & = \frac{1}{\Gamma^2(\alpha + 1)} \int_{\frac{1}{n}}^n \int_{\frac{1}{n}}^n \frac{[f(x)]_n g_n(y)}{\max\{x^\alpha, y^\alpha\}} (dx)^\alpha (dy)^\alpha \\
 & < \eta(\alpha) \left\{ \frac{1}{n}I_n^\alpha \left( x^{\frac{\alpha}{2}(p-2)}[f(x)]_n^p \right) \right\}^{\frac{1}{p}} \left\{ \frac{1}{n}I_n^\alpha \left( y^{\frac{\alpha}{2}(q-2)}g_n^q(y) \right) \right\}^{\frac{1}{q}}. \tag{9}
 \end{aligned}$$

Moreover, by (9) we have

$$\begin{aligned}
 0 &< \frac{1}{n} I_n^\alpha \left( y^{\frac{\alpha}{2}(q-2)} g_n^q(y) \right) = \frac{1}{n} I_n^\alpha \left\{ y^{\frac{\alpha(2-q)}{2(q-1)}} \left[ \frac{1}{n} I_n^\alpha \frac{[f(x)]_n}{\max\{x^\alpha, y^\alpha\}} \right]^p \right\} \\
 &< \eta^p(\alpha) \frac{1}{n} I_n^\alpha \left( x^{\frac{\alpha}{2}(p-2)} [f(x)]_n^p \right) < \infty.
 \end{aligned} \tag{10}$$

For  $n \rightarrow \infty$ , it follows that  $0 < {}_0I_\infty^\alpha \left( y^{\frac{\alpha}{2}(q-2)} g_\infty^q(y) \right) < \infty$ , and  $0 < {}_0I_\infty^\alpha \left( x^{\frac{\alpha}{2}(p-2)} f^p(x) \right) < \infty$ , by (6), both (9) and (10) still keep the form of strict inequalities. Hence we have inequality (8).

On the other hand, by Hölder’s inequality on the fractal set and (8), we find

$$\begin{aligned}
 &{}_0I_\infty^\alpha \left[ {}_0I_\infty^\alpha \frac{f(x)g(y)}{\max\{x^\alpha, y^\alpha\}} \right] \\
 &= \frac{1}{\Gamma^2(\alpha + 1)} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\alpha, y^\alpha\}} (dx)^\alpha (dy)^\alpha \\
 &= \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty \left[ y^{\frac{\alpha(2-q)}{2p(q-1)}} \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty \frac{f(x)}{\max\{x^\alpha, y^\alpha\}} (dx)^\alpha \right] \left[ y^{\frac{\alpha(2-q)}{2p(q-1)}} g(y) \right] (dy)^\alpha \\
 &\leq \left\{ \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty y^{\frac{\alpha(2-q)}{2(q-1)}} \left[ \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty \frac{f(x)}{\max\{x^\alpha, y^\alpha\}} (dx)^\alpha \right]^p (dy)^\alpha \right\}^{\frac{1}{p}} \\
 &\quad \times \left\{ \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty y^{\frac{\alpha}{2}(q-2)} g^q(y) (dy)^\alpha \right\}^{\frac{1}{q}} \\
 &< \eta(\alpha) \left\{ {}_0I_\infty^\alpha \left( x^{\frac{\alpha}{2}(p-2)} f^p(x) \right) \right\}^{\frac{1}{p}} \left\{ {}_0I_\infty^\alpha \left( y^{\frac{\alpha}{2}(q-2)} g^q(y) \right) \right\}^{\frac{1}{q}}.
 \end{aligned}$$

The above inequality is (6), therefore inequality (8) is equivalent to inequality (6).

If the constant factor in (8) is not optimal, then by (8) we can get a contradiction that the constant factor in (6) is not the optimal too. Thus the constant factor  $\eta^p(\alpha)$  in (8) is the best possible. □

### 4 Simple applications

Selecting  $\alpha$  values in (6) and (8), and using mathematics software to calculate, some Hilbert-type fractional integral inequalities and their equivalent forms are obtained.

**Example 1** Letting  $\alpha = 1, p = q = 2$ , to calculate formula (3), we get  $\eta(1) = 4$ , then we obtain inequalities (1) and (2).

**Example 2** Letting  $\alpha = 0.5, p = q = 2$ , to calculate formula (3), we get  $\eta(0.5) = 4\sqrt{\frac{2}{\pi}}$ . Suppose that  $f, g > 0 \in C_{0.5}(0, \infty), 0 < {}_0I_\infty^{0.5}(f^2(x)) < \infty, 0 < {}_0I_\infty^{0.5}(g^2(y)) < \infty$ , then we have the following equivalence inequalities:

$${}_0I_\infty^{0.5} \left[ {}_0I_\infty^{0.5} \frac{f(x)g(y)}{\max\{\sqrt{x}, \sqrt{y}\}} \right] < 4\sqrt{\frac{2}{\pi}} \left\{ {}_0I_\infty^{0.5}(f^2(x)) \right\}^{\frac{1}{2}} \left\{ {}_0I_\infty^{0.5}(g^2(y)) \right\}^{\frac{1}{2}}, \tag{11}$$

$${}_0I_\infty^{0.5} \left[ {}_0I_\infty^{0.5} \frac{f(x)}{\max\{\sqrt{x}, \sqrt{y}\}} \right]^2 < \frac{32}{\pi} {}_0I_\infty^{0.5}(f^2(x)), \tag{12}$$

where the constant factors  $4\sqrt{\frac{2}{\pi}}, \frac{32}{\pi}$  are the best values.

**Example 3** Letting  $\alpha = 0.1, p = q = 2$ , to calculate formula (3), we find  $\eta(0.1) = \frac{20^{10}\sqrt{2}\Gamma(\frac{\pi}{10})}{\pi \csc(\frac{\pi}{10})} = 2.253161500^+$ . Suppose that  $f, g (> 0) \in C_{0,1}(0, \infty)$ ,  $0 < {}_0I_{\infty}^{0.1}(f^2(x)) < \infty$ ,  $0 < {}_0I_{\infty}^{0.1}(g^2(y)) < \infty$ , then we have the following equivalence inequalities:

$${}_0I_{\infty}^{0.1} \left[ {}_0I_{\infty}^{0.1} \frac{f(x)g(y)}{\max\{x^{0.1}, y^{0.1}\}} \right] < \eta(0.1) \left\{ {}_0I_{\infty}^{0.1}(f^2(x)) \right\}^{\frac{1}{2}} \left\{ {}_0I_{\infty}^{0.1}(g^2(y)) \right\}^{\frac{1}{2}}, \quad (13)$$

$${}_0I_{\infty}^{0.1} \left[ {}_0I_{\infty}^{0.1} \frac{f(x)}{\max\{x^{0.1}, y^n\}} \right]^2 < \eta^2(0.1) {}_0I_{\infty}^{0.1}(f^2(x)), \quad (14)$$

where the constant factors  $\eta(0.1), \eta^2(0.1)$  are the best values.

## 5 Conclusions

In the paper, based on the local fractional calculus theory, a Hilbert-type fractional integral inequality and its equivalent form are tentatively researched. The results show that some methods and skills of the Hilbert-type integral inequality can be transplanted to the research of Hilbert-type fractional integral inequality, which provides a new direction and field to research Hardy-Hilbert's integral inequalities.

### Competing interests

The authors declare that they have no competing interests.

### Authors' contributions

The two authors contributed equally to this work. They all read and approved the final version of the manuscript.

### Acknowledgements

The authors are extremely grateful to the reviewers for a critical reading of the manuscript and making valuable comments and suggestions leading to an overall improvement of the paper. This work was supported by the National Natural Science Foundations of China (No. 11171280) and Scientific Support Project of Hunan Province Education Department of China (No. 10C1186).

### Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 12 December 2016 Accepted: 7 April 2017 Published online: 21 April 2017

### References

- Hardy, GH, Littlewood, JE, Pólya, G: *Inequalities*. Cambridge University Press, Cambridge (1952)
- Mitrinović, DS, Pečarić, JE, Fink, AM: *Inequalities Involving Functions and Their Integrals and Derivatives*. Kluwer Academic, Boston (1991)
- Yang, BC: On the extended Hilbert's integral inequality. *Chin. J. Eng. Math.* **5**(4), 1-8 (2004) (in Chinese)
- Liu, Q, Zhang, XJ: A Hardy-Hilbert's type inequality with two parameters and the best constant factor. *J. Nat. Sci. Hunan Norm. Univ.* **29**(3), 5-8 (2006) (in Chinese)
- Samet, E, Mehmet, ZS: Generalized Pompeiu type inequalities for local fractional integrals and its applications. *Appl. Math. Comput.* **274**, 282-291 (2016)
- Mo, H, Sui, X, Yu, D: Generalized convex functions on fractal sets and two related inequalities. *Abstr. Appl. Anal.* **2014**, Article ID 636751 (2014)
- Yang, XJ: *Advanced Local Fractional Calculus and Its Applications*. World Science Publisher, New York (2012)
- Huang, ZS, Guo, DR: *An Introduction to Special Function*. Beijing Press, Beijing (2000) (in Chinese)
- Yang, J, Baleanu, D, Yang, XJ: Analysis of fractal wave equations by local fractional Fourier series method. *Adv. Math. Phys.* **2013**, Article ID 632309 (2013)
- Chen, GS: Generalizations of Hölder's and some related integral inequalities on fractal space. arXiv:1109.5567v1 [math.GM] (2011)