Huang et al. *Journal of Inequalities and Applications* (2016) 2016:60
DOI 10.1186/s13660-016-0993-4

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Rates of convergence of lognormal extremes under power normalization

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Abstract

Let $\{X_n, n \geq 1\}$ be an independent and identically distributed random sequence with common distribution F obeying the lognormal distribution. In this paper, we obtain the exact uniform convergence rate of the distribution of maxima to its extreme value limit under power normalization.

MSC: Primary 62E20; 60E05; secondary 60F15; 60G15

Keywords: *P*-max stable laws; logarithmic normal distribution; maximum; uniform convergence rate

1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with common distribution function (df) $F(x)$. Suppose that there exist constants $a_n > 0$, $b_n \in \mathbb{R}$ and a non-degenerate distribution $G(x)$ such that

$$\lim_{n\to\infty} P(M_n \leq a_n x + b_n) = \lim_{n\to\infty} F^n(a_n x + b_n) = G(x) \tag{1.1}$$

for all $x \in C(G)$, the set of all continuity points of G, where $M_n = \max_{1\leq i\leq n} X_i$ denotes the largest of the first n. Then $G(x)$ must belong to one of the following three classes:

$$\Phi_\alpha(x) = \begin{cases} 0, & \text{if } x < 0, \\ \exp\{-x^{-\alpha}\}, & \text{if } x \geq 0, \end{cases}$$

$$\Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\}, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0, \end{cases}$$

$$\Lambda(x) = \exp\{-e^{-x}\}, \quad x \in \mathbb{R},$$

where α is one positive parameter. We say that F is in the max domain of attraction of G if (1.1) holds, denoted by $F \in D_l(G)$. Criteria for $F \in D_l(G)$ and the choice of normalizing constants a_n and b_n can be found in Galambos [1], Leadbetter *et al.* [2], Resnick [3], and De Haan and Ferreira [4].

The limit distributions of maxima under power normalization was first derived by Pancheva [5]. A df F is said to belong to the max domain of attraction of a non-degenerate *df* H under power normalization, written as $F \in D_p(H)$, if there exist constants $\alpha_n > 0$ and $\beta_n > 0$ such that

$$\lim_{n\to\infty} P\left(\left|\frac{M_n}{\alpha_n}\right|^{\frac{1}{\beta_n}} \operatorname{sign}(M_n) \leq x\right) = \lim_{n\to\infty} F^n\left(\alpha_n |x|^{\beta_n} \operatorname{sign}(x)\right) = H(x), \tag{1.2}$$

where $\operatorname{sign}(x) = -1, 0$ or 1 according to $x < 0$, $x = 0$ or $x > 0$. Pancheva [5] showed that H can be only of power type of the *df*'s, that is,

$$H_{1,\alpha}(x) = \begin{cases} 0, & \text{if } x \leq 1, \\ \exp\{-(\log x)^{-\alpha}\}, & \text{if } x > 1, \end{cases}$$

$$H_{2,\alpha}(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ \exp\{-(-\log x)^{\alpha}\}, & \text{if } 0 < x < 1, \\ 1, & \text{if } x \geq 1, \end{cases}$$

$$H_{3,\alpha}(x) = \begin{cases} 0, & \text{if } x \leq -1, \\ \exp\{-(-\log(-x))^{-\alpha}\}, & \text{if } -1 < x < 0, \\ 1, & \text{if } x \geq 0, \end{cases}$$

$$H_{4,\alpha}(x) = \begin{cases} \exp\{-(\log(-x))^{\alpha}\}, & \text{if } x < -1, \\ 1, & \text{if } x \geq -1, \end{cases}$$

$$H_{5,\alpha}(x) = \Phi_1(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ \exp\{-x^{-1}\}, & \text{if } x > 0, \end{cases}$$

$$H_{6,\alpha}(x) = \Psi_1(x) = \begin{cases} \exp\{x\}, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0, \end{cases}$$

where α is a positive parameter. Necessary and sufficient conditions for F to satisfy (1.2) have been given by Christoph and Falk [6], Mohan and Ravi [7], Mohan and Subramanya [8] and Subramanya [9].

The logarithmic normal distribution (*lognormal distribution* for short) is one of the most widely applied distributions in statistics, biology, and some other disciplines. In this paper, we are interested in considering the uniform rate of convergence of (1.2) with X_n following the lognormal distribution. The probability density function of the lognormal distribution is given by

$$F'(x) = \frac{x^{-1}}{\sqrt{2\pi}} \exp\left\{-\frac{(\log x)^2}{2}\right\}, \quad x > 0.$$

One interesting problem in extreme value analysis is to estimate the rate of uniform convergence of $F^n(\cdot)$ to its extreme value distribution. For a power normalization, Chen *et al.* [10] derived the convergence rates of the distribution of maxima for random variables obeying the general error distribution. For convergence rates of distributions of extremes under linear normalization, see De Haan and Resnick [11] under second-order regular variation and for special cases see Hall [12] and Nair [13] for the normal distribution, which also is extended to those such as general error distribution, logarithmic general error distribution, see recent work of Peng *et al.* [14] and Liao and Peng [15]. For other related work on the convergence rates of some given distributions, see Castro [16] for the gamma distribution, Lin *et al.* [17] for the short-tailed symmetric distribution due to Tiku and Vaughan [18], and Liao *et al.* [19] for the skew normal distribution which extended the results of Nair [13]. The aim of this paper is to study the uniform and point-wise convergence rates of the distribution of power normalized maxima to its limits, respectively.

The contents of this article is organized as follows: some auxiliary results are given in Section 2. In Section 3, we provide our main results with related proofs deferred to Section 4.

2 Preliminaries

To prove our results, we first cite some results from Liao and Peng [15] and Mohan and Ravi [7].

In the sequel, let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables with common *df* F which follows the lognormal distribution. As before, let $M_n = \max_{1\leq i\leq n} X_i$ represent the partial maximum of $\{X_n, n \geq 1\}$. Liao and Peng [15] defined

$$a_n = \frac{\exp((2\log n)^{1/2})}{(2\log n)^{1/2}}, \qquad b_n = \bigl(\exp\bigl((2\log n)^{1/2}\bigr)\bigr)\left(1 - \frac{\log 4\pi + \log\log n}{2(2\log n)^{1/2}}\right), \tag{2.1}$$

and they obtained

$$\lim_{n\to\infty} P\bigl((M_n - b_n)/a_n \leq x\bigr) = \exp\bigl(-e^{-x}\bigr) =: \Lambda(x). \tag{2.2}$$

From (2.2) we immediately derive $F \in D_l(\Lambda)$. The following Mills ratio of the lognormal distribution is due to Liao and Peng [15]:

$$\frac{1 - F(x)}{F'(x)} \sim \frac{x}{\log x}, \tag{2.3}$$

as $x \to \infty$, where $F'(x)$ is the density function of the lognormal distribution $F(x)$. According to Liao and Peng [15], we have

$$1 - F(x) = c(x)\exp\left(-\int_e^x \frac{g(t)}{f(t)}\,\mathrm{d}t\right)$$

for sufficiently large x, where $c(x) \to (2\pi e)^{-1/2}$ as $x \to \infty$, $g(x) = 1 + (\log x)^{-2}$ and

$$f(x) = \frac{x}{\log x}. \tag{2.4}$$

Note that $f'(x) \to 0$ and $g(x) \to 1$ as $x \to \infty$.

Lemma 2.1 [15] *Let F denote the lognormal distribution function. Then*

$$1 - F(x) = \frac{1}{\sqrt{2\pi}}(\log x)^{-1}\exp\left(-\frac{(\log x)^2}{2}\right) - \gamma(x) \tag{2.5}$$

$$= \frac{1}{\sqrt{2\pi}}(\log x)^{-1}\exp\left(-\frac{(\log x)^2}{2}\right)\bigl(1 - (\log x)^{-2}\bigr) + \mathcal{S}(x) \tag{2.6}$$

for $x > 1$, *where*

$$0 < \gamma(x) < \frac{1}{\sqrt{2\pi}}(\log x)^{-3}\exp\left(-\frac{(\log x)^2}{2}\right) \tag{2.7}$$

and

$$0 < \mathcal{S}(x) < \frac{3}{\sqrt{2\pi}}(\log x)^{-5}\exp\left(-\frac{(\log x)^2}{2}\right). \tag{2.8}$$

In order to obtain the main results, we need the following two lemmas.

Lemma 2.2 [7] *Let F denote a df and $r(F) = \sup\{x : F(x) < 1\}$. Suppose that $F \in D_l(\Lambda)$ and $r(F) = \infty$, then $F \in D_p(\Phi_1)$, where normalizing constants $\alpha_n = b_n$, $\beta_n = a_n/b_n$.*

Lemma 2.3 [7] *Let F denote a df, if $F \in D_p(\Phi_1)$ if and only if*

(i) *$r(F) > 0$, and*

(ii) *$\lim_{t\uparrow r(F)} \frac{1-F(t\exp(y\bar{f}(t)))}{1-F(t)} = e^{-y}$, for some positive valued function $\bar{f}$.*

If (ii) *holds for some $\bar{f}$, then $\int_a^{r(F)}((1-F(x))/x)\,\mathrm{d}x < \infty$ for $0 < a < r(F)$ and* (ii) *holds with the choice $\bar{f}(t) = \int_t^{r(F)}((1-F(x))/x)\,\mathrm{d}x/(1-F(t))$. The normalizing constants may be chosen as $\alpha_n = F^{\leftarrow}(1-1/n)$ and $\beta_n = \bar{f}(\alpha_n)$, where $F^{\leftarrow}(x) = \inf\{y : F(y) \geq x\}$.*

Theorem 2.1 *Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed lognormal random variables. Then $F \in D_p(\Phi_1)$ and the normalizing constants can be chosen as $\alpha_n^* = b_n$, $\beta_n^* = a_n/b_n$, where a_n and b_n are given by* (2.1).

Proof Note that F follows the lognormal distribution, which implies $F \in D_p(\Phi_1)$ and $\alpha_n^* = b_n$, $\beta_n^* = a_n/b_n$ by Lemma 2.2, where a_n and b_n are defined by (2.1). □

By Lemma 2.3 and (2.3) and combining with Proposition 1.1(a) in [3], a natural way to choose constants α_n and β_n is to solve the following equations:

$$2\pi(\log\alpha_n)^2\exp\bigl((\log\alpha_n)^2\bigr) = n^2 \tag{2.9}$$

and

$$\beta_n = \frac{f(\alpha_n)}{\alpha_n} = \frac{1}{\log\alpha_n}, \tag{2.10}$$

where f is given by (2.4). The solution of (2.9) may be expressed as

$$\alpha_n = \bigl(\exp\bigl((2\log n)^{1/2}\bigr)\bigr)\left(1 - \frac{\log 4\pi + \log\log n}{2(2\log n)^{1/2}} + o\left(\frac{1}{(\log n)^{1/2}}\right)\right) \tag{2.11}$$

and we easily check that $\beta_n \sim (2\log n)^{-1/2}$.

3 Main results

In this section, we give two main results. Theorem 3.1 proves the result that the rate of uniform convergence of $F^n(\alpha_n x^{\beta_n})$ to its extreme value limit is proportional to $1/\log n$. Theorem 3.2 establishes the result that the point-wise rate of convergence of $|M_n/\alpha_n|^{1/\beta_n}\operatorname{sign}(M_n)$ to the extreme value *df* $\exp(-x^{-1})$ is of the order of $O(x^{-1}(\log x)^2 \times e^{-1/x}(\log n)^{-1})$.

Theorem 3.1 *Let* $\{X_n, n \geq 1\}$ *denote an independent identically distributed random variables sequence with common df F following the lognormal distribution. Then there exist absolute constants* $0 < \mathcal{C}_1 < \mathcal{C}_2$ *such that*

$$\frac{\mathcal{C}_1}{\log n} < \sup_{x>0}\left|F^n\left(\alpha_n x^{\beta_n}\right) - \Phi_1(x)\right| < \frac{\mathcal{C}_2}{\log n}$$

for large $n > n_0$, *where* α_n *and* β_n *are determined by* (2.9) *and* (2.10), *respectively.*

Theorem 3.2 *Let* α_n *and* β_n *be given by* (2.9) *and* (2.10). *Then, for fixed* $x > 0$,

$$\left|F^n\left(\alpha_n x^{\beta_n}\right) - \Phi_1(x)\right| \sim x^{-1}e^{-1/x}\left(1 + \left(1 + \frac{1}{2}\log x\right)\log x\right)\frac{1}{2\log n},$$

as $n \to \infty$.

4 Proofs

First of all, we provide the proof of Theorem 3.2, for it is relatively easy.

Proof of Theorem 3.2 By Lemma 2.1, we have

$$\begin{aligned}
1 - F\left(\alpha_n x^{\beta_n}\right) &= \frac{1}{\sqrt{2\pi}}\left(\log\left(\alpha_n x^{\beta_n}\right)\right)^{-1}\exp\left(-\frac{(\log(\alpha_n x^{\beta_n}))^2}{2}\right) \\
&\quad \times \left(1 - \left(\log\left(\alpha_n x^{\beta_n}\right)\right)^{-2}\right) + \mathcal{S}\left(\alpha_n x^{\beta_n}\right) \\
&=: T_1(x)T_2(x) + T_3(x)
\end{aligned}$$

for $x > 0$, where $T_1(x) = \frac{1}{\sqrt{2\pi}}(\log(\alpha_n x^{\beta_n}))^{-1}\exp(-\frac{(\log(\alpha_n x^{\beta_n}))^2}{2})$, $T_2(x) = 1 - (\log(\alpha_n x^{\beta_n}))^{-2}$ and $T_3(x) = \mathcal{S}(\alpha_n x^{\beta_n})$.

First, we calculate $T_1(x)$. By (2.9) and (2.10), we have

$$\begin{aligned}
T_1(x) &= \frac{1}{\sqrt{2\pi}}(\log\alpha_n)^{-1}\exp\left(-\frac{(\log\alpha_n)^2}{2}\right)\left(1 + (\log\alpha_n)^{-1}\beta_n\log x\right)^{-1} \\
&\quad \times \exp\left(-(\log\alpha_n)\beta_n\log x - \frac{\beta_n^2\log^2 x}{2}\right) \\
&= \frac{1}{nx}\left(1 + \beta_n^2\log x\right)^{-1}\exp\left(-\frac{\beta_n^2\log^2 x}{2}\right) \\
&= \frac{1}{nx}\left(1 - \beta_n^2\log x + O\left(\beta_n^4\right)\right)\left(1 - \frac{\beta_n^2\log^2 x}{2} + O\left(\beta_n^4\right)\right) \\
&= \frac{1}{nx}\left(1 - \beta_n^2\left(1 + \frac{1}{2}\log x\right)\log x + O\left(\beta_n^4\right)\right).
\end{aligned} \tag{4.1}$$

Second, we estimate $T_2(x)$ and $T_3(x)$ for $x > 0$. By (2.10), we derive

$$\begin{aligned}
T_2(x) &= 1 - \beta_n^2\left(1 + \beta_n^2\log x\right)^{-2} \\
&= 1 - \beta_n^2\left(1 - 2\beta_n^2\log x + O\left(\beta_n^4\right)\right) \\
&= 1 - \beta_n^2 + O\left(\beta_n^4\right),
\end{aligned} \tag{4.2}$$

and by Lemma 2.1 we have

$$\begin{aligned} T_3(x) &\le \frac{3}{\sqrt{2\pi}}\big(\log(\alpha_n x^{\beta_n})\big)^{-5}\exp\left(-\frac{(\log(\alpha_n x^{\beta_n}))^2}{2}\right) \\ &= 3\beta_n^4\big(1+\beta_n^2\log x\big)^{-4}T_1(x) \\ &= O\big(n^{-1}\beta_n^4\big). \end{aligned} \tag{4.3}$$

By (4.1)-(4.3), we have

$$1-F^n\big(\alpha_n x^{\beta_n}\big) = \frac{1}{nx}\left(1-\beta_n^2\left(1+\left(1+\frac{1}{2}\log x\right)\log x\right)+O\big(\beta_n^4\big)\right).$$

Thus, we obtain

$$\begin{aligned} &F^n\big(\alpha_n x^{\beta_n}\big)-\Phi_1(x) \\ &\quad= \left(1-\frac{1}{nx}\left(1-\beta_n^2\left(1+\left(1+\frac{1}{2}\log x\right)\log x\right)+O\big(\beta_n^4\big)\right)\right)^n-\exp\left(-\frac{1}{x}\right) \\ &\quad= \exp\left(-\frac{1}{x}\right)\left(\exp\left(\frac{1}{x}\left(\beta_n^2\left(1+\left(1+\frac{1}{2}\log x\right)\log x\right)+O\big(\beta_n^4\big)\right)\right)-1\right) \\ &\quad= \exp\left(-\frac{1}{x}\right)\left(\beta_n^2\frac{1}{x}\left(1+\left(1+\frac{1}{2}\log x\right)\log x\right)+O\big(\beta_n^4\big)\right) \end{aligned} \tag{4.4}$$

for large n and $x>0$. We immediately get the result of Theorem 3.2 by (4.4). □

Proof of Theorem 3.1 By Theorem 3.2 we can prove that there exists an absolute constant $\mathcal{C}_1$ such that

$$\sup_{x>0}\big|F^n\big(\alpha_n x^{\beta_n}\big)-\Phi_1(x)\big| > \frac{\mathcal{C}_1}{\log n}.$$

In order to obtain the upper bound for $x>0$, we need to prove

$$\text{(a)}\quad \sup_{1\le x<\infty}\big|F^n\big(\alpha_n x^{\beta_n}\big)-\Phi_1(x)\big| < d_1\beta_n^2, \tag{4.5}$$

$$\text{(b)}\quad \sup_{c_n\le x<1}\big|F^n\big(\alpha_n x^{\beta_n}\big)-\Phi_1(x)\big| < d_2\beta_n^2, \tag{4.6}$$

$$\text{(c)}\quad \sup_{0<x<c_n}\big|F^n\big(\alpha_n x^{\beta_n}\big)-\Phi_1(x)\big| < d_3\beta_n^2 \tag{4.7}$$

for $n>n_0$, where $d_i>0$, $i=1,2,3$ are absolute constants and

$$c_n = \frac{1}{2\log\log\alpha_n}$$

is positive for $n>n_0$. By (2.9), we have

$$0.4(2\log n)^{1/2} < \log\alpha_n < (2\log n)^{1/2}$$

for $n>n_0$.

First, consider the case of $x \geq c_n$. Set

$$R_n(x) = -\left[n \log F\left(\alpha_n x^{\beta_n}\right) + n\Psi_n(x)\right],$$
$$B_n(x) = \exp(-R_n), \qquad A_n(x) = \exp\left(-n\Psi_n(x) + \frac{1}{x}\right),$$

where $\Psi_n(x) = 1 - F(\alpha_n x^{\beta_n})$ and $A_n(x) \to 1$, as $x \to \infty$. We have

$$\begin{aligned}
\Psi_n(x) \leq \Psi_n(c_n) &< \frac{1}{\sqrt{2\pi}}\left(\log\left(\alpha_n c_n^{\beta_n}\right)\right)^{-1} \exp\left(-\frac{(\log(\alpha_n c_n^{\beta_n}))^2}{2}\right) \\
&= \frac{1}{n}\left(1 + \beta_n^2 \log c_n\right)^{-1} \exp\left(-\log c_n - \frac{\beta_n^2 \log^2 c_n}{2}\right) \\
&< \frac{1}{n}\left(1 + \beta_n^2 \log c_n\right)^{-1} c_n^{-1} \\
&= \left(1 - \frac{\log(2 \log\log \alpha_n)}{(\log \alpha_n)^2}\right)^{-1} \frac{2 \log\log \alpha_n}{n} \\
&< \tilde{c}_4 < 1
\end{aligned}$$

for $n > n_0$. So,

$$\inf_{x > c_n}\left(1 - \Psi_n(x)\right) > 1 - \tilde{c}_4 > 0.$$

Since

$$-x - \frac{x^2}{2(1-x)} < \log(1-x) < -x$$

for $0 < x < 1$, we obtain

$$\begin{aligned}
0 < R_n(x) &\leq \frac{n\Psi_n^2(x)}{2(1 - \Psi_n(x))} < \frac{n\Psi_n^2(c_n)}{2(1 - \Psi_n(x))} \\
&< \frac{n^{-1}(1 + \beta_n^2 \log c_n)^{-2} c_n^{-2}}{2(1 - \Psi_n(x))} \\
&< \frac{n^{-1}(1 + \beta_n^2 \log c_n)^{-2} c_n^{-2} (\log \alpha_n)^2}{2(1 - \tilde{c}_4)\beta_n^{-2}} \\
&= \frac{2}{\sqrt{2\pi}(1 - \tilde{c}_4)}\left(1 - \frac{\log(2 \log\log \alpha_n)}{(\log \alpha_n)^2}\right)^{-2} \frac{(\log\log \alpha_n)^2 \log \alpha_n}{\exp(\frac{(\log \alpha_n)^2}{2})} \beta_n^2 \\
&< \tilde{c}_5 \beta_n^2
\end{aligned}$$

for $n > n_0$.

Hence, we have

$$n^{-1}\beta_n^{-2}\left(1 + \beta_n^2 \log c_n\right)^{-2} c_n^{-2} < \tilde{c}_6$$

for $n > n_0$. Thus,

$$\left|B_n(x) - 1\right| < R_n < \tilde{c}_5 \beta_n^2 \tag{4.8}$$

for $n > n_0$. By (4.8), we have

$$\begin{aligned}&\left|F^n\left(\alpha_n x^{\beta_n}\right) - \Phi_1(x)\right|\\&\quad\le \Phi_1(x)B_n(x)\left|A_n(x) - 1\right| + \left|B_n(x) - 1\right|\\&\quad< \Phi_1(x)\left|A_n(x) - 1\right| + \tilde{c}_5\beta_n^2\end{aligned}\tag{4.9}$$

for $x \ge c_n$.

We now prove (4.5). By (2.9), (2.10), and the definition of $A_n(x)$, we have

$$A_n'(x) = A_n(x)\frac{1}{x^2}\left(\exp\left(-\frac{1}{2}\beta_n^2\log^2 x\right) - 1\right) < 0$$

for $x > 1$. Since

$$\begin{aligned}&0 < n\gamma(\alpha_n) < \beta_n^2 \quad\text{and}\quad e^x - 1 \le xe^x \quad\text{for } 0 \le x \le 1 \quad\text{and}\\&\exp\bigl(n\gamma(\alpha_n)\bigr) < \exp\bigl(\beta_n^2\bigr) < \exp\left(\frac{25}{8\log n}\right) < \exp\left(\frac{25}{8\log n_0}\right) \quad\text{for } n > n_0,\end{aligned}$$

and by (2.5), (2.9), we have

$$\begin{aligned}\sup_{x\ge 1}\left|A_n(x) - 1\right| &= \left|A_n(1) - 1\right|\\&= \left|\exp\bigl(n\gamma(\alpha_n)\bigr) - 1\right|\\&\le n\gamma(\alpha_n)\exp\bigl(n\gamma(\alpha_n)\bigr)\\&\le \tilde{c}_7\beta_n^2\end{aligned}\tag{4.10}$$

for $n > n_0$.

Combining (4.9) with (4.10), we have

$$\sup_{x\ge 1}\left|F^n\left(\alpha_n x^{\beta_n}\right) - \Phi_1(x)\right| < (\tilde{c}_5 + \tilde{c}_7)\beta_n^2.$$

Second, consider the situation of $c_n \le x < 1$. By Lemma 2.1, we obtain

$$\begin{aligned}-n\Psi_n(x) + \frac{1}{x} &= -n\left(\frac{1}{\sqrt{2\pi}}\left(\log\left(\alpha_n x^{\beta_n}\right)\right)^{-1}\exp\left(-\frac{(\log(\alpha_n x^{\beta_n}))^2}{2}\right) - \gamma\left(\alpha_n x^{\beta_n}\right)\right) + \frac{1}{x}\\&= -n\left(\frac{1}{\sqrt{2\pi}}\left(\log\left(\alpha_n x^{\beta_n}\right)\right)^{-1}\exp\left(-\frac{(\log(\alpha_n x^{\beta_n}))^2}{2}\right)\right.\\&\quad\left. - \frac{1}{\sqrt{2\pi}}\left(\log\left(\alpha_n x^{\beta_n}\right)\right)^{-3}q_n\left(\alpha_n x^{\beta_n}\right)\exp\left(-\frac{(\log(\alpha_n x^{\beta_n}))^2}{2}\right)\right) + \frac{1}{x}\\&= \frac{1}{x}\left(1 + \beta_n^2\log x\right)^{-1}\left(-\left(1 - (\log\alpha_n)^{-2}q_n\left(\alpha_n x^{\beta_n}\right)\left(1 + \beta_n^2\log x\right)^{-2}\right)\right.\\&\quad\left.\times\exp\left(-\frac{1}{2}\beta_n^2\log^2 x\right) + 1 + \beta_n^2\log x\right)\\&= \frac{1}{x}\left(1 + \beta_n^2\log x\right)^{-1}Q_n(x),\end{aligned}$$

where $0 < q_n(x) < 1$ and

$$Q_n(x) = -\left(1-\beta_n^2 q_n\left(\alpha_n x^{\beta_n}\right)\left(1+\beta_n^2\log x\right)^{-2}\right)\exp\left(-\frac{1}{2}\beta_n^2\log^2 x\right)+1+\beta_n^2\log x.$$

Since $e^{-x} > 1-x$, as $x > 0$, we have

$$\begin{aligned} Q_n(x) &< -\left(1-\beta_n^2 q_n\left(\alpha_n x^{\beta_n}\right)\left(1+\beta_n^2\log x\right)^{-2}\right)\left(1-\frac{1}{2}\beta_n^2\log^2 x\right)+1+\beta_n^2\log x \\ &< \beta_n^2\left(\left(1+\beta_n^2\log x\right)^{-2}+\frac{1}{2}\log^2 x\right). \end{aligned}$$

But

$$\begin{aligned} Q_n(x) &> \beta_n^2 q_n\left(\alpha_n x^{\beta_n}\right)\left(1+\beta_n^2\log x\right)^{-2}+\beta_n^2\log x \\ &> \beta_n^2\log x. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \left|Q_n(x)\right| &< \beta_n^2\left(\left(1+\beta_n^2\log x\right)^{-2}+\frac{1}{2}\log^2 x+|\log x|\right) \\ &< \beta_n^2\left(\left(1-\frac{\log(2\log\log\alpha_n)}{\log^2\alpha_n}\right)^{-2}+\frac{1}{2}\log^2 x+|\log x|\right) \\ &< \beta_n^2\left(\tilde{c}_8+\frac{1}{2}\log^2 x+|\log x|\right) \end{aligned}$$

for $n > n_0$, where $c_n \le x < 1$. Therefore,

$$\begin{aligned} \left|-n\Psi_n(x)+\frac{1}{x}\right| &< \beta_n^2\left(\tilde{c}_8+\frac{1}{2}\log^2 x+|\log x|\right)x^{-1}\left(1+\beta_n^2\log x\right)^{-1} \\ &< \beta_n^2\left(\tilde{c}_8+\frac{1}{2}\log^2 c_n+|\log c_n|\right)c_n^{-1}\left(1+\beta_n^2\log c_n\right)^{-1} \\ &< \tilde{c}_9 \end{aligned}$$

for $n \ge n_0$. Thus, there exists a positive number θ satisfying $0 < \theta < 1$ such that

$$\begin{aligned} \Phi_1(x)\left|A_n(x)-1\right| &< \Phi_1(x)\exp\left(\theta\left(-n\Psi_n(x)+\frac{1}{x}\right)\right)\left|-n\Psi_n(x)+\frac{1}{x}\right| \\ &< \exp(\tilde{c}_9)\beta_n^2\sup_{c_n\le x<1}\left|\left(\tilde{c}_8+\frac{1}{2}\log^2 x+|\log x|\right)x^{-1}\right|\left(1+\beta_n^2\log c_n\right)^{-1} \\ &< \tilde{c}_{10}\beta_n^2. \end{aligned} \tag{4.11}$$

By (4.9) and (4.11), the proof of (4.6) is complete.

Third, consider the circumstance of $0 < x < c_n$. In this case

$$\Phi_1(x) < \Phi_1(c_n) = \beta_n^2,$$

we have

$$\begin{aligned}\sup_{0<x<c_n}\left|F^n\left(\alpha_n x^{\beta_n}\right)-\Phi_1(x)\right| &< F^n\left(\alpha_n c_n^{\beta_n}\right)+\Phi_1(c_n)\\ &< \sup_{c_n<x<1}\left|F^n\left(\alpha_n x^{\beta_n}\right)-\Phi_1(x)\right|+2\Phi_1(c_n)\\ &< (\tilde{c}_5+\tilde{c}_{10})\beta_n^2+\beta_n^2\\ &< \tilde{c}_{11}\beta_n^2.\end{aligned}$$

The proof of Theorem 3.1 is finished. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JH obtained the theorem and completed the proof. SC and YL corrected and improved the final version. All authors read and approved the final manuscript.

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Acknowledgements
The authors would like to thank the Editor-in-Chief, the Associate Editor, and the referees for carefully reading the paper and for their comments, which greatly improved the paper. This work was supported by the National Natural Science Foundation of China (Grant No. 11171275, No. 71461027), the SWU Grant for Statistics PhD, the Science and Technology fund Project of GZ (Grant No. LKZS[2014]29), the Science and Technology Plan Project of Guizhou Province (Grant No. LH[2015]7001, No. LH[2015]7055), 2013, 2014, and 2015, and the Zunyi 15851 talents elite project funding and Zhunyi innovative talent team (Zunyi KH(2015)38).

Received: 18 March 2015 Accepted: 1 February 2016 Published online: 13 February 2016

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