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A kind of boundary value problem for inhomogeneous partial differential system

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Abstract

In this article, we first define a kind of generalized singular integral operator and discuss its properties. Then we propose a kind of boundary value problem for an inhomogeneous partial differential system in R^4. Finally, the integral representation of the solution to a boundary value problem for the inhomogeneous partial differential system is obtained using the above singular integral operator.

Keywords: generalized holomorphic function; singular integral operators; inhomogeneous partial differential system; boundary value problem; integral representation

1 Introduction

Partial differential equations are encountered in many problems of physics, mechanics, mathematical finance, mathematical biology, and other branches of mathematics [1, 2]. It has been a popular topic since the 1960s. So boundary value problems for partial differential system have always been an important and meaningful topics. There are many scholars who studied on it, such as Keldysh [3], Wen [4, 5], Čanić and Kim [6], Taira [7], and so on. In addition, singular integral operators are the core components of solutions of the boundary value problems for a partial differential system and a degenerate partial differential system. So, for many years, many scholars and experts have studied some properties of all kinds of singular integral operators, and they obtained the integral representations of solutions of some partial differential equations. For example, Vekua [8] first discussed in detail some properties of the Teodorescu operator, and Hile [9] studied some properties of the Teodorescu operator in R^n. Then Gilbert *et al.* [10] and Meng [11] studied its many properties in high dimensional complex space. Gürlebeck and Sprössig [12], and Yang [13] discussed its properties and corresponding boundary value problems in the real quaternion analysis.

In this article, we will study the Riemann boundary value problem for a kind of inhomogeneous partial differential system of first order equations in R^4 using the Clifford analysis approach. In Section 2, we recall some basic knowledge of Clifford analysis. In Section 3, we construct a singular integral operator and study some of its properties. In Section 4, we first propose the Riemann boundary value problem for a kind of inhomogeneous partial differential system, then we obtain an integral representation of the solution to the Riemann boundary value problem using the relation between the theory of Clifford-valued generalized holomorphic functions and that of the inhomogeneous partial differential system's solutions.

2 Preliminaries

Let $\{e_0, e_1, e_2, e_3\}$ be an orthogonal basis of the Euclidean space R^4 and $Cl_{0,3}$ be the Clifford algebra with basis

$$\{e_0, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, e_1e_2e_3\},$$

where e_0 is the real scalar identity element, e_1, e_2, e_3 satisfy the following multiplication rule:

$$e_i^2 = -e_0, \qquad e_ie_j = -e_je_i, \quad i,j = 1,2,3, i \neq j.$$

If we denote $e_1e_2 = e_4$, $e_1e_3 = e_5$, $e_2e_3 = e_6$, $e_1e_2e_3 = e_7$, then an arbitrary element of the Clifford algebra space $Cl_{0,3}$ can be written as $a = \sum_{j=0}^{7} a_je_j$, $a_j \in R$. The Clifford conjugation is defined by $\bar{a} = a_0 - \sum_{j=1}^{6} a_je_j + a_7e_7$. The norm for an element $a \in Cl_{0,3}$ is taken to be $|a| = \sqrt{\sum_{j=0}^{7} |a_j|^2}$. Moreover, if $a\bar{a} = \bar{a}a = |a|^2$ and $|a| \neq 0$, then we have

$$a \cdot \frac{\bar{a}}{|a|^2} = \frac{\bar{a}}{|a|^2} \cdot a = 1.$$

Thus, we say that a is reversible if $a\bar{a} = \bar{a}a = |a|^2$ and $|a| \neq 0$. Obviously, its inverse element can be written as $a^{-1} = \frac{\bar{a}}{|a|^2}$.

Suppose $\Omega \subset R^4$ is a bounded connected domain and the boundary $\partial\Omega$ is a differentiable, oriented, and compact Liapunov surface. An arbitrary element $x \in \Omega$ is denoted by $x = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3$. The function w which is defined in Ω with values in the Clifford algebra space $Cl_{0,3}$ can be expressed as $w = \sum_{j=0}^{7} w_j(x)e_j$, herein $w_j(x)$ $(j = 0, \dots, 7)$ are real-functions defined on Ω.

Let $C^{(m)}(\Omega, Cl_{0,3}) = \{w | w : \Omega \to Cl_{0,3}, w(x) = \sum_{j=0}^{7} w_j(x)e_j, w_j(x) \in C^{(m)}(\Omega, R)\}$. We introduce the generalized Cauchy-Riemann operator on $C^{(1)}(\Omega, Cl_{0,3})$ as follows:

$$\partial_x w = \sum_{i=0}^{3} e_i \frac{\partial w}{\partial x_i} = \sum_{i=0}^{3} \sum_{j=0}^{7} e_ie_j \frac{\partial w_j}{\partial x_i},$$

$$w\partial_x = \sum_{i=0}^{3} \frac{\partial w}{\partial x_i} e_i = \sum_{i=0}^{3} \sum_{j=0}^{7} e_je_i \frac{\partial w_j}{\partial x_i}.$$

w is called a left (right) Clifford holomorphic function, if $\partial_x w(x) = 0$ $(w(x)\partial_x = 0)$ in Ω. w is called a left (right) generalized Clifford holomorphic function, if $\partial_x w(x) = c(x)$ $(w(x)\partial_x = c(x))$ in Ω, herein $c(x) = \sum_{j=0}^{7} c_j(x)e_j$. Usually a left Clifford holomorphic function and a left generalized Clifford holomorphic function are called a Clifford holomorphic function and a generalized Clifford holomorphic function for short, respectively. And $w(x) \in L^{p,\sigma}(R^4, Cl_{0,3})$ means that $w(x) \in L^p(B, Cl_{0,3})$, $w^{(\sigma)}(x) = |x|^{-\sigma} w(\frac{\bar{x}}{|x|^2}) \in L^p(B, Cl_{0,3})$, in which $B = \{x | |x| < 1\}$, σ is a real number, $\|w\|_{p,\sigma} = \|w\|_{L^p(B)} + \|w^{(\sigma)}\|_{L^p(B)}$, $p \geq 1$.

Definition 2.1 Suppose that the functions u, v, φ are defined in Ω with values in $Cl_{0,3}$, and $u, v \in L^1(\Omega, Cl_{0,3})$. If for arbitrary $\varphi \in C_0^\infty(\Omega, Cl_{0,3})$, u, v satisfy

$$\int_\Omega (\varphi\partial_x)v(x)\,dx + \int_\Omega \varphi(x)u(x)\,dx = 0,$$

then u is called a generalized derivative of the function v, where we denote $u = \partial_x v$.

Lemma 2.1 ([14]) *Let Ω, $\partial\Omega$ be as stated above. If $f \in C^{(m)}(\overline{\Omega}, Cl_{0,3})$, then for each $x \in \Omega$, we have*

$$\frac{1}{2\pi^2}\int_{\partial\Omega} f(y)\,d\sigma_y E(x,y) - \frac{1}{2\pi^2}\int_\Omega (f\partial_y)E(x,y)\,dy = f(x),$$

where $E(x,y) = \frac{\bar{y}-\bar{x}}{|y-x|^4}$.

Lemma 2.2 ([15]) *If $\sigma_1, \sigma_2 > 0$, $0 \le \gamma \le 1$, then we have*

$$\left|\sigma_1^\gamma - \sigma_2^\gamma\right| \le |\sigma_1 - \sigma_2|^\gamma.$$

Lemma 2.3 ([16]) *Suppose Ω is a bounded domain in R^4, and let α', β' satisfy $0 < \alpha', \beta' < 4$, $\alpha' + \beta' > 4$. Then for all $x_1, x_2 \in R^4$ and $x_1 \ne x_2$, we have*

$$\int_\Omega |t - x_1|^{-\alpha'}|t - x_2|^{-\beta'}\,dt \le M_0(\alpha', \beta')|x_1 - x_2|^{4-\alpha'-\beta'}.$$

3 Some properties of the singular integral operator

In this section, we will discuss some properties of the singular integral operator as follows:

$$\begin{aligned}\bigl(T[g]\bigr)(x) &= -\frac{1}{2\pi^2}\int_B \frac{(\bar{y}-\bar{x})g(y)}{|y-x|^4}\,dy - \frac{1}{2\pi^2}\int_B \frac{\frac{\bar{y}}{|y|^2} - \bar{x}}{|\frac{\bar{y}}{|y|^2} - x|^4} g\left(\frac{\bar{y}}{|y|^2}\right)\frac{1}{|y|^8}\,dy \\ &= \bigl(T_1[g]\bigr)(x) + \bigl(T_2[g]\bigr)(x), \end{aligned} \tag{3.1}$$

where $B = \{x \mid |x| < 1\}$.

Theorem 3.1 *Assume B to be as stated above. If $g \in L^p(B, Cl_{0,3})$, $4 < p < +\infty$, then*

(1) $|(T_1[g])(x)| \le M_1(p)\|g\|_{L^p(B)}$,

(2) $|(T_1[g])(x^{(1)}) - (T_1[g])(x^{(2)})| \le M_2(p)\|g\|_{L^p(B)}|x^{(1)} - x^{(2)}|^\beta$, $x^{(1)}, x^{(2)} \in R^4$,

(3) $\partial_x(T_1[g])(x) = g(x)$, $x \in B$, $\partial_x(T_1[g])(x) = 0$, $x \in R^4\backslash\overline{B}$,

where $0 < \beta = \frac{p-4}{p} < 1$.

Proof (1) By the Hölder inequality, we have

$$\left|\bigl(T_1[g]\bigr)(x)\right| \le J_1\|g\|_{L^p(B)}\left[\int_B \frac{1}{|y-x|^{3q}}|dy|\right]^{\frac{1}{q}}.$$

When $x \in \overline{B}$, because of $p > 4$, $\frac{1}{p} + \frac{1}{q} = 1$, then $1 < q < \frac{4}{3}$. Thus $\int_B \frac{1}{|y-x|^{3q}}|dy|$ is bounded. Hence we suppose

$$\int_B \frac{1}{|y-x|^{3q}}|dy| \le J_2.$$

When $x \in R^4 - \overline{B}$, by Lemma 2.2 and the generalized spherical coordinate, we have

$$\left[\int_B \frac{1}{|y-x|^{3q}}|dy|\right]^{\frac{1}{q}} \le J_3\left[\int_{d_0}^{d_0+2} \rho^{3-3q}\,d\rho\right]^{\frac{1}{q}} \le J_4,$$

where $\rho = |y-x|$, $d_0 = d(x,B)$.

Therefore, for arbitrary $x \in R^4$, we can obtain

$$\left|\left(T_1[g]\right)(x)\right| \le M_1(p)\|g\|_{L^p(B)}, \quad x \in R^4,$$

where $M_1(p) = \max\{J_1J_2, J_1J_4\}$.

(2) For arbitrary $x^{(1)}, x^{(2)} \in R^4$, $x^{(1)} \neq x^{(2)}$, by the Hile lemma [9] and the Hölder inequality, we can obtain

$$\begin{aligned}
&\left|\left(T_1[g]\right)\left(x^{(1)}\right) - \left(T_1[g]\right)\left(x^{(2)}\right)\right| \\
&\quad \le J_5\int_B \left|g(y)\right|\left|\frac{\bar{y}-\bar{x}^{(1)}}{|y-x^{(1)}|^4} - \frac{\bar{y}-\bar{x}^{(2)}}{|y-x^{(2)}|^4}\right||dy| \\
&\quad \le J_5\int_B \left|g(y)\right|\frac{\sum_{k=1}^3 |y-x^{(1)}|^{3-k}|y-x^{(2)}|^{k-1}}{|y-x^{(1)}|^3|y-x^{(2)}|^3}|dy|\left|x^{(1)}-x^{(2)}\right| \\
&\quad \le J_6\|g\|_{L^p(B)}\sum_{k=1}^3\left[\int_B \frac{1}{|y-x^{(1)}|^{kq}|y-x^{(2)}|^{(4-k)q}}|dy|\right]^{\frac{1}{q}}\left|x^{(1)}-x^{(2)}\right| \\
&\quad = J_6\|g\|_{L^p(B)}\left|x^{(1)}-x^{(2)}\right|\sum_{k=1}^3 I_k^{\frac{1}{q}}.
\end{aligned}$$

We suppose $\alpha' = kq$, $\beta' = (4-k)q$ $(k = 1,2,3)$. From $1 \le q < \frac{4}{3}$, we have

$$\begin{aligned}
&\alpha' = kq \le 3q < 4, \qquad \beta' = (4-k)q \le 3q < 4, \\
&\alpha' + \beta' = 4q > 4 \quad (k = 1,2,3).
\end{aligned}$$

Hence, by Lemma 2.3, we have

$$\begin{aligned}
I_k &= \int_B \frac{1}{|y-x^{(1)}|^{kq}|y-x^{(2)}|^{(4-k)q}}|dy| \\
&\le M_0\left(\alpha',\beta'\right)\left|x^{(1)}-x^{(2)}\right|^{4-4q} \quad (k = 1,2,3).
\end{aligned}$$

So we have

$$\left|\left(T_1[g]\right)\left(x^{(1)}\right) - \left(T_1[g]\right)\left(x^{(2)}\right)\right| \le J_7\|g\|_{L^p(B)}\left|x^{(1)}-x^{(2)}\right|^{1+\frac{4-4q}{q}} = M_2(p)\|g\|_{L^p(B)}|x_1-x_2|^{\beta},$$

where $M_2(p) = J_7$, $0 < \beta = \frac{p-4}{p} < 1$.

(3) For arbitrary $\varphi \in C_0^\infty(B, Cl_{0,3})$, by Definition 2.1, Lemma 2.1, and the Fubini theorem, we have

$$\begin{aligned}
&\int_B (\varphi\partial_x)\left(T_1[g]\right)(x)\,dx \\
&\quad = \int_B (\varphi\partial_x)\left[-\frac{1}{2\pi^2}\int_B \frac{\bar{y}-\bar{x}}{|y-x|^4}g(y)\,dy\right]dx
\end{aligned}$$

$$= \int_B \left[\frac{1}{2\pi^2} \int_B (\varphi \partial_x) \frac{\bar{x} - \bar{y}}{|x-y|^4} \, dx \right] g(y) \, dy$$
$$= \int_B \left[\frac{1}{2\pi^2} \int_{\partial B} \varphi(x) \, d\sigma_x \frac{\bar{x} - \bar{y}}{|x-y|^4} - \varphi(y) \right] g(y) \, dy$$
$$= -\int_B \varphi(y) g(y) \, dy = -\int_B \varphi(x) g(x) \, dx.$$

Hence, in the sense of generalized derivatives, $\partial_x(T_1[g])(x) = g(x)$, $x \in B$. It is easy to see $\partial_x(T_1[g])(x) = 0$, $x \in R^4 \backslash B$. □

Theorem 3.2 *Let B be as stated above. If* $g \in L^{p,4}(R^4, Cl_{0,3})$, $4 < p < +\infty$, *then we have the following results*:

(1) $|(T_2[g])(x)| \leq M_3(p) \|g^{(4)}\|_{L^p(B)}$, $x \in R^4$,

(2) $|(T_2[g])(x^{(1)}) - (T_2[g])(x^{(2)})| \leq M_4(p) \|g^{(4)}\|_{L^p(B)} |x^{(1)} - x^{(2)}|^\beta$, $x^{(1)}, x^{(2)} \in R^4$,

(3) $\partial_x(T_2[g])(x) = 0$, $x \in B$, $\partial_x(T_2[g])(x) = g(x)$, $x \in R^4 \backslash \overline{B}$,

where $0 < \beta = \frac{p-4}{p} < 1$.

Proof (1) By the Hölder inequality, we have

$$\begin{aligned} \left|(T_2[g])(x)\right| &= \left| -\frac{1}{2\pi^2} \int_B \frac{\frac{\bar{y}}{|y|^2} - \bar{x}}{|\frac{\bar{y}}{|y|^2} - x|^4} g\left(\frac{\bar{y}}{|y|^2}\right) \frac{1}{|y|^8} \, dy \right| \\ &\leq J_8 \int_B \frac{1}{|\frac{\bar{y}}{|y|^2} - x|^3} \left| g\left(\frac{\bar{y}}{|y|^2}\right) \right| \frac{1}{|y|^8} |dy| \\ &= J_8 \int_B \frac{1}{|\frac{\bar{y}}{|y|^2} - x|^3 |y|^4} |y|^{-4} \left| g\left(\frac{\bar{y}}{|y|^2}\right) \right| |dy| \\ &\leq J_8 \left[\int_B \left(|y|^{-4} \left| g\left(\frac{\bar{y}}{|y|^2}\right) \right| \right)^p |dy| \right]^{\frac{1}{p}} \left[\int_B \frac{1}{|\frac{\bar{y}}{|y|^2} - x|^{3q} |y|^{4q}} |dy| \right]^{\frac{1}{q}} \\ &= J_8 \left\| g^{(4)} \right\|_{L^p(B)} \left(O(x)\right)^{\frac{1}{q}}, \end{aligned} \tag{3.2}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Next we will discuss $O(x)$ in two cases.

(i) When $|x| \geq \frac{1}{2}$, since

$$\begin{aligned} &|y|^{-4q} \left| \frac{\bar{y}}{|y|^2} - x \right|^{-3q} \\ &= |y|^{-4q} |y|^{3q} |y|^{-3q} \left| \frac{\bar{y}}{|y|^2} - x \right|^{-3q} |x|^{3q} |x|^{-3q} \\ &= |y|^{-q} \left[|y|^{-3q} \left| \frac{\bar{y}}{|y|^2} - x \right|^{-3q} \left| \frac{\bar{x}}{|x|^2} \right|^{-3q} \right] |x|^{-3q} \\ &\leq J_9 |y|^{-q} \left| y \left(\frac{\bar{y}}{|y|^2} - x \right) \frac{\bar{x}}{|x|^2} \right|^{-3q} |x|^{-3q} \\ &= J_9 |y|^{-q} \left| \frac{\bar{x}}{|x|^2} - y \right|^{-3q} |x|^{-3q}, \end{aligned}$$

we have

$$O(x) \le \int_B J_9 |y|^{-q} \left| \frac{\bar{x}}{|x|^2} - y \right|^{-3q} |x|^{-3q} |dy|$$
$$= J_9 |x|^{-3q} \int_B |y|^{-q} \left| \frac{\bar{x}}{|x|^2} - y \right|^{-3q} |dy|.$$

Let $\alpha' = q$, $\beta' = 3q$, by $1 < q < \frac{4}{3}$. we have

$$0 < \alpha' < 4, \qquad 0 < \beta' < 4, \qquad \alpha' + \beta' = 4q > 4.$$

Thus, by Lemma 2.3, we have

$$O(x) \le J_9 M_0(\alpha', \beta') |x|^{-3q} \left| \frac{\bar{x}}{|x|^2} \right|^{4-4q} \le J_9 M_0(\alpha', \beta') 2^{4-q} = J_{10}. \tag{3.3}$$

(ii) When $|x| < \frac{1}{2}$, by $|y| \le 1$, we have $|1 - yx| \ge \frac{1}{2}$. Thus

$$O(x) = \int_B \left| \frac{\bar{y}}{|y|^2} - x \right|^{-3q} |y|^{-4q} |dy|$$
$$= \int_B |y|^{3q} |y|^{-3q} \left| \frac{\bar{y}}{|y|^2} - x \right|^{-3q} |y|^{-4q} |dy|$$
$$\le J_{11} \int_B |y|^{-q} \left| y\left(\frac{\bar{y}}{|y|^2} - x \right) \right|^{-3q} |dy|$$
$$= J_{11} \int_B |y|^{-q} |1 - yx|^{-3q} |dy|$$
$$\le J_{11} \int_B |y|^{-q} 2^{3q} |dy| \le J_{12} \int_B |y|^{-q} |dy| \le J_{13}. \tag{3.4}$$

Therefore, by (3.2)-(3.4), we have

$$\left| \left(T_2[g] \right)(x) \right| \le M_3(p) \left\| g^{(4)} \right\|_{L^p(B)},$$

where $M_3(p) = \max\{J_8 J_{10}^{\frac{1}{q}}, J_8 J_{13}^{\frac{1}{q}}\}$.

(2) By the Hile lemma [9], we have

$$\left| \left(T_2[g] \right)\left(x^{(1)} \right) - \left(T_2[g] \right)\left(x^{(2)} \right) \right|$$
$$\le J_{14} \int_B \left| \frac{\frac{y}{|y|^2} - \bar{x}^{(1)}}{|\frac{\bar{y}}{|y|^2} - x^{(1)}|^4} - \frac{\frac{y}{|y|^2} - \bar{x}^{(2)}}{|\frac{\bar{y}}{|y|^2} - x^{(2)}|^4} \right| \left| g\left(\frac{\bar{y}}{|y|^2} \right) \right| \frac{1}{|y|^8} |dy|$$
$$\le J_{14} \int_B \sum_{k=1}^{3} \frac{|x^{(1)} - x^{(2)}|}{|\frac{\bar{y}}{|y|^2} - x^{(1)}|^k |\frac{\bar{y}}{|y|^2} - x^{(2)}|^{4-k}} \left| g\left(\frac{\bar{y}}{|y|^2} \right) \right| \frac{1}{|y|^8} |dy|.$$

Again, because of

$$\left| \frac{\bar{y}}{|y|^2} - x^{(1)} \right|^{-k} = |y|^k |y|^{-k} \left| \frac{\bar{y}}{|y|^2} - x^{(1)} \right|^{-k} = |y|^k \left| 1 - yx^{(1)} \right|^{-k},$$

$$\left|\frac{\bar{y}}{|y|^2} - x^{(2)}\right|^{-(4-k)} = |y|^{4-k}\left|1 - yx^{(2)}\right|^{-(4-k)},$$

by the Hölder inequality, we have

$$\begin{aligned}
&\left|\left(T_2[g]\right)\left(x^{(1)}\right) - \left(T_2[g]\right)\left(x^{(2)}\right)\right| \\
&\quad\le J_{14}\sum_{k=1}^{3}\int_B\left|1 - yx^{(1)}\right|^{-k}\left|1 - yx^{(2)}\right|^{-(4-k)}|y|^{-4}\left|g\left(\frac{\bar{y}}{|y|^2}\right)\right||dy|\left|x^{(1)} - x^{(2)}\right| \\
&\quad\le J_{14}\left|x^{(1)} - x^{(2)}\right|\left\|g^{(4)}\right\|_{L^p(B)}\sum_{k=1}^{3}\left[\int_B\left|1 - yx^{(1)}\right|^{-kq}\left|1 - yx^{(2)}\right|^{-(4-k)q}|dy|\right]^{\frac{1}{q}} \\
&\quad= J_{14}\left|x^{(1)} - x^{(2)}\right|\left\|g^{(4)}\right\|_{L^p(B)}\sum_{k=1}^{3}\left[\widetilde{O}_k\left(x^{(1)}, x^{(2)}\right)\right]^{\frac{1}{q}},
\end{aligned} \tag{3.5}$$

where

$$\widetilde{O}_k\left(x^{(1)}, x^{(2)}\right) = \int_B\left|1 - yx^{(1)}\right|^{-kq}\left|1 - yx^{(2)}\right|^{-(4-k)q}|dy|.$$

In the following, we discuss $\widetilde{O}_k(x^{(1)}, x^{(2)})$ in four cases.

(i) When $|x^{(1)}| \le \frac{1}{2}$, $|x^{(2)}| \le \frac{1}{2}$, we have $|1 - yx^{(1)}| \ge \frac{1}{2}$, $|1 - yx^{(2)}| \ge \frac{1}{2}$ and $|x^{(1)} - x^{(2)}| \le 1$. Hence

$$\widetilde{O}_k\left(x^{(1)}, x^{(2)}\right) \le \int_B 2^{kq}2^{(4-k)q}|dy| = 2^{4q}\int_B|dy| = J_{15}.$$

From $|x^{(1)}| - |x^{(2)}| \le 1$, $0 \le \beta = 1 - \frac{4}{p} < 1$, we have $|x^{(1)} - x^{(2)}| \le |x^{(1)} - x^{(2)}|^{\beta}$. Therefore, by (3.5), we have

$$\begin{aligned}
&\left|\left(T_2[g]\right)\left(x^{(1)}\right) - \left(T_2[g]\right)\left(x^{(2)}\right)\right| \\
&\quad\le J_{14}\left|x^{(1)} - x^{(2)}\right|^{\beta}\left\|g^{(4)}\right\|_{L^{(p)}(B)}\sum_{k=1}^{3}J_{15}^{\frac{1}{q}} \\
&\quad= J_{16}\left\|g^{(4)}\right\|_{L^p(B)}\left|x^{(1)} - x^{(2)}\right|^{\beta}.
\end{aligned} \tag{3.6}$$

(ii) When $|x^{(1)}| \ge \frac{1}{2}$, $|x^{(2)}| \le \frac{1}{2}$, we have $|1 - yx^{(2)}| \ge \frac{1}{2}$, $\frac{1}{|x^{(1)}|} \le 2$, $\frac{|x^{(2)}|}{|x^{(1)}|} \le 1$. Thus

$$\begin{aligned}
\widetilde{O}_k\left(x^{(1)}, x^{(2)}\right) &\le 2^{(4-k)q}\int_B\left|1 - yx^{(1)}\right|^{-kq}|dy| \\
&= 2^{(4-k)q}\int_B\left|1 - yx^{(1)}\right|^{-kq}\left|x^{(1)}\right|^{-kq}\left|x^{(1)}\right|^{kq}|dy| \\
&\le J_{17}2^{(4-k)q}\left|x^{(1)}\right|^{-kq}\int_B\left|\left(1 - yx^{(1)}\right)\frac{\bar{x}^{(1)}}{|x^{(1)}|^2}\right|^{-kq}|dy| \\
&= J_{17}2^{(4-k)q}\left|x^{(1)}\right|^{-kq}\int_B\left|\frac{\bar{x}^{(1)}}{|x^{(1)}|^2} - y\right|^{-kq}|dy|,
\end{aligned}$$

where

$$\left(\frac{1}{|x^{(1)}|}\right)^{kq} = 2^{kq}\left(\frac{1}{2|x^{(1)}|}\right)^{kq} \leq 2^{kq}\left(\frac{1}{2|x^{(1)}|}\right)^{q} = 2^{(k-1)q}\left(\frac{1}{|x^{(1)}|}\right)^{q}.$$

Again, since

$$\begin{aligned}
\frac{1}{|x^{(1)}|} &= \left(\frac{1}{|x^{(1)}|}\right)^{\beta}\left(\frac{1}{|x^{(1)}|}\right)^{1-\beta} = \frac{1}{|x^{(1)}|^{\beta}}\left(\frac{\bar{x}^{(1)}}{|x^{(1)}|^2}\right)^{1-\beta} \\
&= \frac{1}{|x^{(1)}|^{\beta}}\left|\frac{\bar{x}^{(1)}(x^{(1)}-x^{(2)})(\bar{x}^{(1)}-\bar{x}^{(2)})}{|x^{(1)}|^2|x^{(1)}-x^{(2)}|^2}\right|^{1-\beta} \\
&\leq J_{18}\frac{1}{|x^{(1)}|^{\beta}}\left|\frac{\bar{x}^{(1)}(x^{(1)}-x^{(2)})}{|x^{(1)}|^2}\right|^{1-\beta}\frac{1}{|x^{(1)}-x^{(2)}|^{1-\beta}} \\
&= J_{18}\left|x^{(1)}\right|^{-\beta}\left|1-\frac{\bar{x}^{(1)}x^{(2)}}{|x^{(1)}|^2}\right|^{1-\beta}\left|x^{(1)}-x^{(2)}\right|^{\beta-1} \\
&\leq J_{19}\left|x^{(1)}\right|^{-\beta}\left(1+\frac{|x^{(2)}|}{|x^{(1)}|}\right)^{1-\beta}\left|x^{(1)}-x^{(2)}\right|^{\beta-1} \\
&\leq J_{20}\left|x^{(1)}-x^{(2)}\right|^{\beta-1},
\end{aligned}$$

we have

$$\left(\frac{1}{|x^{(1)}|}\right)^{kq} \leq 2^{(k-1)q}\left(J_{20}\left|x^{(1)}-x^{(2)}\right|^{\beta-1}\right)^{q} \leq J_{21}\left|x^{(1)}-x^{(2)}\right|^{(\beta-1)q}.$$

Again from $1 < q < \frac{4}{3}$, we have $kq < 4$ ($k = 1,2,3$). Thus $\int_B |\frac{\bar{x}^{(1)}}{|x^{(1)}|^2} - y|^{-kq}|dy|$ is bounded. Hence, we obtain

$$\widetilde{O}_k\left(x^{(1)}, x^{(2)}\right) \leq J_{22}\left|x^{(1)}-x^{(2)}\right|^{(\beta-1)q}.$$

Therefore, by (3.5), we have

$$\begin{aligned}
&\left|\left(T_2[g]\right)\left(x^{(1)}\right)-\left(T_2[g]\right)\left(x^{(2)}\right)\right| \\
&\quad\leq J_{14}\left|x^{(1)}-x^{(2)}\right|\left\|g^{(4)}\right\|_{L^{(p)}(B)}\sum_{k=1}^{3}\left(J_{22}\left|x^{(1)}-x^{(2)}\right|^{(\beta-1)q}\right)^{\frac{1}{q}} \\
&\quad= J_{23}\left\|g^{(4)}\right\|_{L^{p}(B)}\left|x^{(1)}-x^{(2)}\right|^{\beta}.
\end{aligned} \tag{3.7}$$

(iii) When $|x^{(1)}| \leq \frac{1}{2}$, $|x^{(2)}| \geq \frac{1}{2}$, we have $|1 - yx^{(1)}| \geq \frac{1}{2}$, $\frac{1}{|x^{(2)}|} \leq 2$, $\frac{|x^{(1)}|}{|x^{(2)}|} \leq 1$. Similar to (ii), we have

$$\left|\left(T_2[g]\right)\left(x^{(1)}\right)-\left(T_2[g]\right)\left(x^{(2)}\right)\right| = J_{24}\left\|g^{(4)}\right\|_{L^{p}(B)}\left|x^{(1)}-x^{(2)}\right|^{\beta}. \tag{3.8}$$

(V) When $|x^{(1)}| \geq \frac{1}{2}$, $|x^{(2)}| \geq \frac{1}{2}$, we have $\frac{1}{|x^{(1)}|} \leq 2$, $\frac{1}{|x^{(2)}|} \leq 2$.

Since

$$\begin{aligned}
\left|1-yx^{(1)}\right|^{-kq} &= \left|1-yx^{(1)}\right|^{-kq}\left|x^{(1)}\right|^{kq}\left|x^{(1)}\right|^{-kq} \\
&= \left|1-yx^{(1)}\right|^{-kq}\left(\frac{\bar{x}^{(1)}}{|x^{(1)}|^2}\right)^{-kq}\left|x^{(1)}\right|^{-kq} \\
&\leq J_{25}\left|y-\frac{\bar{x}^{(1)}}{|x^{(1)}|^2}\right|^{-kq}\left|x^{(1)}\right|^{-kq}
\end{aligned}$$

and

$$\begin{aligned}
\left|1-yx^{(2)}\right|^{-(4-k)q} &= \left|1-yx^{(2)}\right|^{-(4-k)q}\left|x^{(2)}\right|^{(4-k)q}\left|x^{(2)}\right|^{-(4-k)q} \\
&\leq J_{26}\left|y-\frac{\bar{x}^{(2)}}{|x^{(2)}|^2}\right|^{-(4-k)q}\left|x^{(2)}\right|^{-(4-k)q},
\end{aligned}$$

we have

$$\widetilde{O}_k\left(x^{(1)},x^{(2)}\right) \leq J_{27}\int_B\left|y-\frac{\bar{x}^{(1)}}{|x^{(1)}|^2}\right|^{-kq}\left|y-\frac{\bar{x}^{(2)}}{|x^{(2)}|^2}\right|^{-(4-k)q}|dy|.$$

Suppose $\alpha' = kq$, $\beta' = (4-k)q$, then $0 < \alpha' < 3q < 4$, $0 < \beta' < 3q < 4$, $\alpha' + \beta' = 4q \geq 4$. Thus, by Lemma 2.3, we have

$$\begin{aligned}
\widetilde{O}_k\left(x^{(1)},x^{(2)}\right) &\leq J_{27}\left|\frac{\bar{x}^{(1)}}{|x^{(1)}|^2}-\frac{\bar{x}^{(2)}}{|x^{(2)}|^2}\right|^{4-4q} \\
&= J_{28}\left(\frac{|\bar{x}^{(1)}|x^{(2)}|^2-\bar{x}^{(2)}|x^{(1)}|^2|}{|x^{(1)}|^2|x^{(2)}|^2}\right)^{4-4q} \\
&= J_{28}\left(\frac{\bar{x}^{(1)}|x^{(2)}|^2-\bar{x}^{(2)}|x^{(2)}|^2+\bar{x}^{(2)}|x^{(2)}|^2-\bar{x}^{(2)}|x^{(1)}|^2}{|x^{(1)}|^2|x^{(2)}|^2}\right)^{4-4q} \\
&\leq J_{28}\left(\frac{1}{|x^{(1)}|^2}+\frac{|x^{(2)}|+|x^{(1)}|}{|x^{(1)}|^2|x^{(2)}|}\right)^{4-4q}\left|x^{(1)}-x^{(2)}\right|^{4-4q} \\
&= J_{28}\left(\frac{1}{|x^{(1)}|^2}+\frac{1}{|x^{(1)}|^2}+\frac{1}{|x^{(1)}||x^{(2)}|}\right)^{4-4q}\left|x^{(1)}-x^{(2)}\right|^{4-4q} \\
&\leq J_{29}\left|x^{(1)}-x^{(2)}\right|^{4-4q}.
\end{aligned}$$

Therefore, by (3.5), we have

$$\begin{aligned}
&\left|\left(T_2[g]\right)\left(x^{(1)}\right)-\left(T_2[g]\right)\left(x^{(2)}\right)\right| \\
&\quad\leq J_{14}\left|x^{(1)}-x^{(2)}\right|\left\|g^{(4)}\right\|_{L^{(p)}(B)}\sum_{k=1}^{3}\left(J_{29}\left|x^{(1)}-x^{(2)}\right|^{4-4q}\right)^{\frac{1}{q}} \\
&\quad= J_{30}\left\|g^{(4)}\right\|_{L^p(B)}\left|x^{(1)}-x^{(2)}\right|^{1+\frac{4(1-q)}{q}} \\
&\quad= J_{30}\left\|g^{(4)}\right\|_{L^p(B)}\left|x^{(1)}-x^{(2)}\right|^{\beta}, \qquad (3.9)
\end{aligned}$$

where $0 < \beta = 1 + \frac{4(1-q)}{q} = \frac{p-4}{p} < 1$.

Therefore, by (3.6)-(3.9), we obtain

$$\left|\left(T_2[g]\right)\left(x^{(1)}\right)-\left(T_2[g]\right)\left(x^{(2)}\right)\right| \le M_4(p)\left\|g^{(4)}\right\|_{L^p(B)}\left|x^{(1)}-x^{(2)}\right|^{\beta},$$

where $M_4(p) = \max\{J_{16}, J_{23}, J_{24}, J_{30}\}$.

(3) This case is similar to Theorem 3.1, and it is easy to prove. □

Remark 3.1 Let B be as stated above. If $g \in L^{p,4}(R^4, Cl_{0,3})$, $4 < p < +\infty$, then we have the following results:

(1) $|(T[g])(x)| \le M_5(p)\|g\|_{p,4}$, $x \in R^4$,

(2) $|(T[g])(x^{(1)}) - (T[g])(x^{(2)})| \le M_6(p)\|g\|_{p,4}|x^{(1)} - x^{(2)}|^{\beta}$, $x^{(1)}, x^{(2)} \in R^4$,

(3) $\partial_x(T[g])(x) = g(x)$, $x \in R^4\backslash\partial B$,

where $0 < \beta < 1$.

4 Integral representation of solution to inhomogeneous partial differential system

In this section, we will discuss the inhomogeneous partial differential system of first order equations as follows:

$$\begin{cases} w_{0x_0} - w_{1x_1} - w_{2x_2} - w_{3x_3} = c_0(x), \\ w_{1x_0} + w_{0x_1} + w_{4x_2} + w_{5x_3} = c_1(x), \\ w_{2x_0} - w_{4x_1} + w_{0x_2} + w_{6x_3} = c_2(x), \\ w_{3x_0} - w_{5x_1} - w_{6x_2} + w_{0x_3} = c_3(x), \\ w_{4x_0} + w_{2x_1} - w_{1x_2} - w_{7x_3} = c_4(x), \\ w_{5x_0} + w_{3x_1} + w_{7x_2} - w_{1x_3} = c_5(x), \\ w_{6x_0} - w_{7x_1} + w_{3x_2} - w_{2x_3} = c_6(x), \\ w_{7x_0} + w_{6x_1} - w_{5x_2} + w_{4x_3} = c_7(x), \end{cases} \tag{4.1}$$

where $w_j(x)$, $c_j(x)$ $(j = 0, 1, 2, \ldots, 7)$ are real-value functions.

Problem P Let $B \subset R^4$ be as stated above. The Riemann boundary value problem for system (4.1) is to find a solution $w(x)$ of (4.1) that satisfies the boundary condition

$$w^+(\tau) = w^-(\tau)G + f(\tau), \quad \tau \in \partial B, \tag{4.2}$$

where $w^{\pm}(\tau) = \lim_{x\in B^{\pm}, x\to\tau} w(x)$, $B^+ = B$, $B^- = R^4\backslash\overline{B}$, G is a Clifford constant, G^- exists, and $f \in H^{\nu}_{\partial B}$ $(0 < \nu < 1)$.

In fact,

$$\begin{aligned} \partial_x w &= \sum_{i=0}^{3} e_i \sum_{j=0}^{7} e_j \frac{\partial w_j}{\partial x_i} = \sum_{j=0}^{7}\left(e_0 e_j \frac{\partial w_j}{\partial x_0} + e_1 e_j \frac{\partial w_j}{\partial x_1} + e_2 e_j \frac{\partial w_j}{\partial x_2} + e_3 e_j \frac{\partial w_j}{\partial x_3}\right) \\ &= (w_{0x_0}e_0 + w_{0x_1}e_1 + w_{0x_2}e_2 + w_{0x_3}e_3) + (w_{1x_0}e_1 - w_{1x_1}e_0 - w_{1x_2}e_4 - w_{1x_3}e_5) \\ &\quad + (w_{2x_0}e_2 + w_{2x_1}e_4 - w_{2x_2}e_0 - w_{2x_3}e_6) + (w_{3x_0}e_3 + w_{3x_1}e_5 + w_{3x_2}e_6 - w_{3x_3}e_0) \\ &\quad + (w_{4x_0}e_4 - w_{4x_1}e_2 + w_{4x_2}e_1 + w_{4x_3}e_7) + (w_{5x_0}e_5 - w_{5x_1}e_3 - w_{5x_2}e_7 + w_{5x_3}e_1) \end{aligned}$$

$$\begin{aligned}
&+ (w_{6x_0}e_6 + w_{6x_1}e_7 - w_{6x_2}e_3 + w_{6x_3}e_2) + (w_{7x_0}e_7 - w_{7x_1}e_6 + w_{7x_2}e_5 - w_{7x_3}e_4) \\
&= (w_{0x_0} - w_{1x_1} - w_{2x_2} - w_{3x_3})e_0 + (w_{1x_0} + w_{0x_1} + w_{4x_2} + w_{5x_3})e_1 \\
&+ (w_{2x_0} - w_{4x_1} + w_{0x_2} + w_{6x_3})e_2 + (w_{3x_0} - w_{5x_1} - w_{6x_2} + w_{0x_3})e_3 \\
&+ (w_{4x_0} + w_{2x_1} - w_{1x_2} - w_{7x_3})e_4 + (w_{5x_0} + w_{3x_1} + w_{7x_2} - w_{1x_3})e_5 \\
&+ (w_{6x_0} - w_{7x_1} + w_{3x_2} - w_{2x_3})e_6 + (w_{7x_0} + w_{6x_1} - w_{5x_2} + w_{4x_3})e_7. \qquad (4.3)
\end{aligned}$$

Let

$$\begin{aligned}
g(x) &= c_0(x)e_0 + c_1(x)e_1 + c_2(x)e_2 + c_3(x)e_3 \\
&\quad + c_4(x)e_4 + c_5(x)e_5 + c_6(x)e_6 + c_7(x)e_7 \\
&= \sum_{j=0}^{7} c_i(x)e_i. \qquad (4.4)
\end{aligned}$$

By (4.3) and (4.4), the inhomogeneous partial differential system (4.1) can be transformed to the following equation:

$$\partial_x w = \sum_{i=0}^{7} c_i(x)e_i = g(x). \qquad (4.5)$$

Therefore Problem P as stated above can be transformed to Problem Q.

Problem Q Let $B \subset R^4$ be as stated above. The Riemann boundary value problem for system (4.1) is to find a solution $w(x)$ of (4.5) that satisfies the boundary condition

$$w^+(\tau) = w^-(\tau)G + f(\tau), \quad \tau \in \partial B,$$

where $w^{\pm}(\tau) = \lim_{x\in B^{\pm}, x\to\tau} w(x)$, $B^+ = B$, $B^- = R^4\backslash\overline{B}$, G is a Clifford constant, G^- exists, and $f \in H^{\nu}_{\partial B}$ $(0 < \nu < 1)$.

Theorem 4.1 *Let B be as stated above. Find a Clifford-valued function $u(x)$ satisfying the system $\partial_x u = 0$ $(x \in R^4\backslash\partial B)$ and vanishing at infinity with the boundary condition*

$$u^+(\tau) = u^-(\tau)G + f(\tau), \quad \tau \in \partial B, \qquad (4.6)$$

where $u^{\pm}(\tau) = \lim_{x\in B^{\pm}, x\to\tau} u(x)$, G is a Clifford constant, G^- exists, and $f \in H^{\lambda}_{\partial B}$ $(0 < \lambda < 1)$. Then the solution can be expressed as

$$u(x) = \begin{cases} \frac{1}{2\pi^2}\int_{\partial B} \frac{\bar{y}-\bar{x}}{|y-x|^4}\, d\sigma_y f(y), & x \in B^+, \\ \frac{1}{2\pi^2}\int_{\partial B} \frac{\bar{y}-\bar{x}}{|y-x|^4}\, d\sigma_y f(y) G^{-1}, & x \in B^-. \end{cases}$$

Proof Define

$$\varphi(x) = \begin{cases} u(x), & x \in B^+, \\ u(x)G, & x \in B^-. \end{cases}$$

Then it is obvious $\partial_x\varphi(x) = 0$, and the Riemann boundary condition (4.6) is equivalent to

$$\varphi^+(\tau) = \varphi^-(\tau) + f(\tau), \quad \tau \in \partial B.$$

Suppose $\psi(x) = \frac{1}{2\pi^2}\int_{\partial\Omega_x} \frac{\bar{y}-\bar{x}}{|y-x|^4}\, d\sigma_y f(y)$, then $\partial_x\psi(x) = 0$. And by the Plemelj formula [14], we have

$$\psi^+(\tau) - \psi^-(\tau) = f(\tau), \quad \tau \in \partial B.$$

Hence $\varphi^+(\tau) - \psi^+(\tau) = \varphi^-(\tau) - \psi^-(\tau)$ ($\tau \in \partial B$). Thus by the Liouville theorem and the extension theorem [17], we obtain $\varphi(x) = \psi(x)$. So, the solution can be expressed as

$$u(x) = \begin{cases} \frac{1}{2\pi^2}\int_{\partial B} \frac{\bar{y}-\bar{x}}{|y-x|^4}\, d\sigma_y f(y), & x \in B^+, \\ \frac{1}{2\pi^2}\int_{\partial B} \frac{\bar{y}-\bar{x}}{|y-x|^4}\, d\sigma_y f(y)G^{-1}, & x \in B^-. \end{cases} \qquad \square$$

Theorem 4.2 *Let B be as stated above, $g \in L^{p,4}(R^4, Cl_{0,3})$, $4 < p < +\infty$. Find a Clifford-valued function $w(x)$ satisfying the system $\partial_x w = g(x)$ ($x \in R^4\backslash\partial B$) and vanishing at infinity with the boundary condition*

$$w^+(\tau) = w^-(\tau)G + f(\tau), \quad \tau \in \partial B, \tag{4.7}$$

where $w^\pm(\tau) = \lim_{x\in B^\pm, x\to\tau} w(x)$, G is a Clifford constant, G^- exists, and $f \in H^\lambda_{\partial B}$ ($0 < \lambda < 1$). Then the solution has the form

$$w(x) = \Psi(x) + \bigl(T[g]\bigr)(x),$$

in which $\partial_x\Psi(x) = 0$ and

$$\Psi(x) = \begin{cases} \frac{1}{2\pi^2}\int_{\partial B} \frac{\bar{y}-\bar{x}}{|y-x|^4}\, d\sigma_y \tilde{f}(y), & x \in B^+, \\ \frac{1}{2\pi^2}\int_{\partial B} \frac{\bar{y}-\bar{x}}{|y-x|^4}\, d\sigma_y \tilde{f}(y)G^{-1}, & x \in B^-, \end{cases}$$

where $\tilde{f} = f + (T[g])(G-1)$, $(T[g])(x)$ is the same as (3.1).

Proof By Remark 3.1, we know

$$\partial_x w = \partial\bigl[\Psi(x) + \bigl(T[g]\bigr)(x)\bigr] = g(x).$$

The boundary condition (4.7) is equivalent to

$$\bigl(\Psi + T[g]\bigr)^+(\tau) = \bigl(\Psi + T[g]\bigr)^-(\tau)G + f(\tau), \quad \tau \in \partial B. \tag{4.8}$$

Again from Remark 3.1, we know that $(T[g])(x)$ has Hölder continuity in R^4. Thus $(T[g])^+ = (T[g])^- = T[g]$. So (4.8) is equivalent to

$$\Psi^+(\tau) = \Psi^-(\tau)G + \bigl(T[g]\bigr)(\tau)(G-1) + f(\tau). \tag{4.9}$$

Suppose $\tilde{f} = f + T[g](G-1)$, then (4.9) has the following form:

$$\Psi^+(\tau) = \Psi^-(\tau)G + \tilde{f}(\tau), \quad \tau \in \partial B. \tag{4.10}$$

Again from Theorem 4.1, the solutions which satisfy the system $\partial_x \Psi(x) = 0$ and boundary condition (4.10) can be represented in the form

$$\Psi(x) = \begin{cases} \frac{1}{2\pi^2} \int_{\partial B} \frac{\bar{y}-\bar{x}}{|y-x|^4}\, d\sigma_y \tilde{f}(y), & x \in B^+, \\ \frac{1}{2\pi^2} \int_{\partial B} \frac{\bar{y}-\bar{x}}{|y-x|^4}\, d\sigma_y \tilde{f}(y) G^{-1}, & x \in B^-. \end{cases}$$

Remark 4.1 From Theorem 4.2, the solution of Problem P can be expressed as

$$w(x) = \Psi(x) + \bigl(T[g]\bigr)(x),$$

in which $\partial_x \Psi(x) = 0$ and

$$\Psi(x) = \begin{cases} \frac{1}{2\pi^2} \int_{\partial \Omega_x} \frac{\bar{y}-\bar{x}}{|y-x|^4}\, d\sigma_y \tilde{f}(y), & x \in B^+, \\ \frac{1}{2\pi^2} \int_{\partial B} \frac{\bar{y}-\bar{x}}{|y-x|^4}\, d\sigma_y \tilde{f}(y) G^{-1}, & x \in B^-, \end{cases}$$

where $\tilde{f} = f + T[g](G-1)$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
LW has presented the main purpose of the article. All authors read and approved the final manuscript.

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