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# Multiplicity results for a fractional Kirchhoff equation involving sign-changing weight function

Chuanzhi Bai*

*Correspondence:
czbai8@sohu.com
Department of Mathematics, Huaiyin Normal University, Huaian, Jiangsu 223300, P.R. China

**Abstract**

In this paper, we prove the existence and multiplicity of solutions for a fractional Kirchhoff equation involving a sign-changing weight function which generalizes the corresponding result of Tsung-fang Wu (Rocky Mt. J. Math. 39:995-1011, 2009). Our main results are based on the method of a Nehari manifold.

**MSC:** 35J50; 35J60; 47G20

**Keywords:** fractional $p$-Laplacian; Kirchhoff type problem; sign-changing weight; Nehari manifold

## 1 Introduction

In this paper, we consider the following fractional elliptic equation with sign-changing weight functions:

$$\begin{cases} M\left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\right)(-\Delta)_p^s u = \lambda f(x)u^q + g(x)u^r, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $N > 2s$, $0 < s < 1$, $0 \leq q < 1 < r < p_s^* - 1$ ($p_s^* = \frac{pN}{N-ps}$); $\lambda > 0$, $M(t) = a + bt^{p-1}$, $(-\Delta)_p^s$ is the fractional $p$-Laplacian operator defined as

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{B_\varepsilon(x)^c} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}}\,dy, \quad x \in \mathbb{R}^N.$$

We may assume that the weight functions $f(x)$ and $g(x)$ are as follows:

(H1) $f^+ = \max\{f, 0\} \not\equiv 0$, and $f \in L^{\mu_q}(\Omega)$ where $\mu_q = \frac{\mu}{\mu-(q+1)}$ for some $\mu \in (q+1, p_s^*)$, with in addition $f(x) \geq 0$ a.e. in $\Omega$ in the case $q = 0$;

(H2) $g^+ = \max\{g, 0\} \not\equiv 0$, and $g \in L^{\nu_r}(\Omega)$ where $\nu_r = \frac{\nu}{\nu-(r+1)}$ for some $\nu \in (r+1, p_s^*)$.

The fractional Kirchhoff type problems have been studied by many authors in recent years; see [2–6] and references therein. In the subcritical case, Pucci and Saldi in [5] studied the following Kirchhoff type problem in $\mathbb{R}^N$:

$$\begin{cases} M(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy)(-\Delta)_p^s u + V(x)|u|^{p-2}u \\ \quad = \lambda w(x)|u|^{q-2}u - h(x)|u|^{r-2}u, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

with $n > ps$, $s \in (0,1)$, and they established the existence and multiplicity of entire solutions using variational methods and topological degree theory for the above problem with a real parameter $\lambda$ under the suitable integrability assumptions of the weights $V$, $w$, and $h$. In [7], Mishra and Sreenadh have studied the following Kirchhoff problem with sign-changing weights:

$$\begin{cases} M(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy)(-\Delta)_p^s u = \lambda f(x)|u|^{q-2}u + |u|^{\alpha-2}u, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

and they obtained the multiplicity of non-negative solutions in the subcritical case $\alpha < p_s^*$ by minimizing the energy functional over non-empty decompositions of Nehari manifold.

When $p = 2$, $s = 1$, $a = 1$ and $b = 0$, problem (1.1) is reduced to the following semilinear elliptic equation:

$$\begin{cases} -\Delta u = \lambda f(x)u^q + g(x)u^r, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

In [1], Wu proved that equation (1.2) involving a sign-changing weight function has at least two solutions by using the Nehari manifold.

Motivated by the above work, in this paper, we investigate the existence and multiplicity of solutions for a fractional Kirchhoff equation (1.1) and extend the main results of Wu [1].

This article is organized as follows. In Section 2, we give some notations and preliminaries. Section 3 is devoted to the proof that problem (1.1) has at least two solutions for $\lambda$ sufficiently small.

## 2 Preliminaries

For any $s \in (0,1)$, $1 < p < \infty$, we define

$$X = \left\{ u | u : \mathbb{R}^N \to \mathbb{R} \text{ is measurable}, u|_\Omega \in L^p(\Omega), \text{ and } \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy < \infty \right\},$$

where $Q = \mathbb{R}^{2N} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ with $\mathcal{C}\Omega = \mathbb{R}^N \setminus \Omega$. The space $X$ is endowed with the norm defined by

$$\|u\|_X = \|u\|_{L^p(\Omega)} + \left( \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy \right)^{1/p}.$$

The functional space $X_0$ denotes the closure of $C_0^\infty(\Omega)$ in $X$. By [8], the space $X_0$ is a Hilbert space with scalar product

$$\langle u, v \rangle_{X_0} = \int_Q \frac{|u(x)-u(y)|^{p-1}(v(x)-v(y))}{|x-y|^{n+ps}}\,dx\,dy, \quad \forall u, v \in X_0,$$

and the norm

$$\|u\|_{X_0} = \left(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy\right)^{1/p}.$$

For further details on $X$ and $X_0$ and also for their properties, we refer to [8] and the references therein.

Throughout this section, we denote the best Sobolev constant by $S_l$ for the embedding of $X_0$ into $L^l(\Omega)$, which is defined as

$$S_l = \inf_{X_0\setminus\{0\}} \frac{\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy}{(\int_{\mathbb{R}^N} |u|^l\,dx)^{\frac{p}{l}}} > 0,$$

where $l \in [p, p_s^*]$.

A function $u \in X_0$ is a weak solution of problem (1.1) if

$$\begin{aligned} &M\left(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\right)\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,dx\,dy \\ &\quad = \lambda\int_\Omega f(x)|u|^{q-1}uv\,dx + \int_\Omega g(x)|u|^{r-1}uv\,dx, \quad \forall v \in X_0. \end{aligned}$$

Associated with equation (1.1), we consider the energy functional $\mathcal{J}_{\lambda,M}$ in $X_0$

$$\mathcal{J}_{\lambda,M}(u) = \frac{1}{p}\hat{M}\left(\|u\|_{X_0}^p\right) - \frac{\lambda}{q+1}\int_\Omega f|u|^{q+1}\,dx - \frac{1}{r+1}\int_\Omega g|u|^{r+1}\,dx,$$

where $\hat{M}(t) = \int_0^t M(\mu)\,d\mu$.

It is easy to see that the solutions of equation (1.1) are the critical points of the energy functional $\mathcal{J}_{\lambda,M}$.

The Nehari manifold for $\mathcal{J}_{\lambda,M}$ is defined as

$$\begin{aligned} \mathcal{N}_{\lambda,M}(\Omega) &= \left\{u \in X_0 \setminus \{0\} : \left\langle \mathcal{J}'_{\lambda,M}(u), u\right\rangle = 0\right\} \\ &= \left\{u \in X_0 \setminus \{0\} | M\left(\|u\|_{X_0}^p\right)\|u\|_{X_0}^p - \lambda\int_\Omega f|u|^{q+1}\,dx - \int_\Omega g|u|^{r+1}\,dx = 0\right\}. \end{aligned}$$

The Nehari manifold $\mathcal{N}_{\lambda,M}(\Omega)$ is closely linked to the behavior of functions of the form $h_{\lambda,M} : t \to \mathcal{J}_{\lambda,M}(tu)$ for $t > 0$, named fibering maps [9]. If $u \in X_0$, we have

$$\begin{aligned} h_{\lambda,M}(t) &= \frac{1}{p}\hat{M}\left(t^p\|u\|_{X_0}^p\right) - \lambda\frac{t^{q+1}}{q+1}\int_\Omega f|u|^{q+1}\,dx - \frac{t^{r+1}}{r+1}\int_\Omega g|u|^{r+1}\,dx, \\ h'_{\lambda,M}(t) &= t^{p-1}M\left(t^p\|u\|_{X_0}^p\right)\|u\|_{X_0}^p - \lambda t^q\int_\Omega f|u|^{q+1}\,dx - t^r\int_\Omega g|u|^{r+1}\,dx, \end{aligned}$$

and

$$\begin{aligned} h''_{\lambda,M}(t) &= (p-1)t^{p-2}M\left(t^p\|u\|_{X_0}^p\right)\|u\|_{X_0}^p + pt^{2p-2}M'\left(t^p\|u\|_{X_0}^p\right)\|u\|_{X_0}^{2p} \\ &\quad - q\lambda t^{q-1}\int_\Omega f|u|^{q+1}\,dx - rt^{r-1}\int_\Omega g|u|^{r+1}\,dx. \end{aligned}$$

Obviously,

$$\begin{aligned} th'_{\lambda,M}(t) &= M\big(t^p\|u\|_{X_0}^p\big)\|tu\|_{X_0}^p - \lambda\int_\Omega f|tu|^{q+1}\,dx - \int_\Omega g|tu|^{r+1}\,dx \\ &= \big\langle \mathcal{J}_{\lambda,M}(tu), tu\big\rangle, \end{aligned}$$

which implies that for $u \in X_0 \setminus \{0\}$ and $t > 0$, $h_{\lambda,M}(t) = 0$ if and only if $tu \in \mathcal{N}_{\lambda,M}(\Omega)$, *i.e.*, positive critical points of $h_{\lambda,M}$ correspond to points on the Nehari manifold. In particular, $h_{\lambda,M}(1) = 0$ if and only if $u \in \mathcal{N}_{\lambda,M}(\Omega)$. Hence, we define

$$\begin{aligned} \mathcal{N}^+_{\lambda,M}(\Omega) &= \big\{u \in \mathcal{N}_{\lambda,M}(\Omega) : h''_{u,M}(1) > 0\big\}, \\ \mathcal{N}^0_{\lambda,M}(\Omega) &= \big\{u \in \mathcal{N}_{\lambda,M}(\Omega) : h''_{u,M}(1) = 0\big\}, \\ \mathcal{N}^-_{\lambda,M}(\Omega) &= \big\{u \in \mathcal{N}_{\lambda,M}(\Omega) : h''_{u,M}(1) < 0\big\}. \end{aligned}$$

For each $u \in \mathcal{N}_{\lambda,M}(\Omega)$, we have

$$\begin{aligned} h''_{\lambda,M}(1) &= (p-1)M\big(\|u\|_{X_0}^p\big)\|u\|_{X_0}^p + pM'\big(\|u\|_{X_0}^p\big)\|u\|_{X_0}^{2p} \\ &\quad - q\lambda\int_\Omega f|u|^{q+1}\,dx - r\int_\Omega g|u|^{r+1}\,dx \\ &= (p-r-1)M\big(\|u\|_{X_0}^p\big)\|u\|_{X_0}^p + pM'\big(\|u\|_{X_0}^p\big)\|u\|_{X_0}^{2p} - \lambda(q-r)\int_\Omega f|u|^{q+1}\,dx \quad (2.1) \\ &= (p-q-1)M\big(\|u\|_{X_0}^p\big)\|u\|_{X_0}^p + pM'\big(\|u\|_{X_0}^p\big)\|u\|_{X_0}^{2p} - (r-q)\int_\Omega g|u|^{r+1}\,dx. \quad (2.2) \end{aligned}$$

Let $M(t) = a + bt^{p-1}$, where $a > 0$, $b \geq 0$ and $p > 1$. If $u \in \mathcal{N}^0_{\lambda,M}(\Omega)$, then $h''_{\lambda,M}(1) = 0$, and we have by (2.1) and (2.2)

$$a(p-r-1)\|u\|_{X_0}^p + b\big(p^2-r-1\big)\|u\|_{X_0}^{p^2} - \lambda(q-r)\int_\Omega f|u|^{q+1}\,dx = 0, \quad (2.3)$$

$$a(p-q-1)\|u\|_{X_0}^p + b\big(p^2-q-1\big)\|u\|_{X_0}^{p^2} - (r-q)\int_\Omega g|u|^{r+1}\,dx = 0. \quad (2.4)$$

For convenience, we let

(H3) $0 < q < 1$, $p > 1+q$ and $p_s^* - 1 > r \begin{cases} > p^2 - 1, & b \neq 0, \\ > p - 1, & b = 0. \end{cases}$

**Lemma 2.1** *If* (H1) *and* (H3) *hold, then the energy functional* $\mathcal{J}_{\lambda,M}$ *is coercive and bounded below on* $\mathcal{N}_{\lambda,M}(\Omega)$.

*Proof* For $u \in \mathcal{N}_{\lambda,M}(\Omega)$, we have by the Hölder and Sobolev inequalities

$$\begin{aligned} \mathcal{J}_{\lambda,M}(u) &= a\left(\frac{1}{p} - \frac{1}{r+1}\right)\|u\|_{X_0}^p + b\left(\frac{1}{p^2} - \frac{1}{r+1}\right)\|u\|_{X_0}^{p^2} \\ &\quad - \lambda\left(\frac{1}{q+1} - \frac{1}{r+1}\right)\int_\Omega f|u|^{q+1}\,dx \\ &= a\left(\frac{1}{p} - \frac{1}{r+1}\right)\|u\|_{X_0}^p + b\left(\frac{1}{p^2} - \frac{1}{r+1}\right)\|u\|_{X_0}^{p^2} \end{aligned}$$

$$\begin{aligned}
&- \lambda \frac{r-q}{(q+1)(r+1)} \int_\Omega f|u|^{q+1}\,dx \\
&\geq a\left(\frac{1}{p} - \frac{1}{r+1}\right)\|u\|_{X_0}^p + b\left(\frac{1}{p^2} - \frac{1}{r+1}\right)\|u\|_{X_0}^{p^2} \\
&- \lambda \frac{r-q}{(q+1)(r+1)} \|f\|_{L^{\mu_q}} S_\mu^{q+1} \|u\|_{X_0}^{q+1},
\end{aligned}$$

where $\mu_q = \frac{\mu}{\mu-(q+1)}$, $\mu \in (q+1, p_s^*)$. Thus $\mathcal{J}_{\lambda,M}$ is coercive and bounded below on $\mathcal{N}_{\lambda,M}(\Omega)$. □

**Lemma 2.2** *Let* (H1)-(H3) *hold. There exists* $\lambda_1 > 0$ *such that for any* $\lambda \in (0, \lambda_1)$, *we have* $\mathcal{N}_{\lambda,M}^0(\Omega) = \emptyset$.

*Proof* If not, that is, $\mathcal{N}_{\lambda,M}^0(\Omega) \neq \emptyset$ for each $\lambda > 0$, then by (2.3) and the Hölder and Sobolev inequalities, we have for $u_0 \in \mathcal{N}_{\lambda,M}^0(\Omega)$

$$\begin{aligned}
a(r-p+1)\|u_0\|_{X_0}^p &\leq a(r-p+1)\|u_0\|_{X_0}^p + b\left(r-p^2+1\right)\|u_0\|_{X_0}^{p^2} \\
&= \lambda(r-q)\int_\Omega f|u_0|^{q+1}\,dx,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|u_0\|_{X_0}^p &\leq \frac{\lambda(r-q)}{a(r-p+1)} \int_\Omega f|u_0|^{q+1}\,dx \\
&\leq \frac{\lambda(r-q)}{a(r-p+1)} \|f\|_{L^{\mu_q}} S_\mu^{q+1} \|u_0\|_{X_0}^{q+1}
\end{aligned}$$

and so

$$\|u_0\|_{X_0} \leq \left(\frac{\lambda(r-q)}{a(r-p+1)} \|f\|_{L^{\mu_q}} S_\mu^{q+1}\right)^{\frac{1}{p-q-1}}. \tag{2.5}$$

Similarly, we obtain by (2.4) and the Hölder and Sobolev inequalities

$$\|u_0\|_{X_0}^p \leq \frac{r-q}{a(p-q+1)} \|g\|_{L^{\nu_r}} S_\nu^{r+1} \|u_0\|_{X_0}^{r+1},$$

which implies that

$$\|u_0\|_{X_0} \geq \left(\frac{a(p-q+1)}{r-q} \|g\|_{L^{\nu_r}}^{-1} S_\nu^{-(r+1)}\right)^{\frac{1}{r-p+1}}. \tag{2.6}$$

But (2.5) contradicts (2.6) if $\lambda$ is sufficiently small. Hence, we conclude that there exists $\lambda_1 > 0$ such that $\mathcal{N}_{\lambda,M}^0(\Omega) = \emptyset$ for $\lambda \in (0, \lambda_1)$. □

Let

$$c_\lambda = \inf_{u \in \mathcal{N}_{\lambda,M}(\Omega)} \mathcal{J}_{\lambda,M}(u).$$

From Lemma 2.2, for $\lambda \in (0,\lambda_1)$, we write $\mathcal{N}_{\lambda,M}(\Omega) = \mathcal{N}^+_{\lambda,M}(\Omega) \cup \mathcal{N}^-_{\lambda,M}(\Omega)$ and define

$$c^+_\lambda = \inf_{u\in\mathcal{N}^+_{\lambda,M}(\Omega)} \mathcal{J}_{\lambda,M}(u) \quad \text{and} \quad c^-_\lambda = \inf_{u\in\mathcal{N}^-_{\lambda,M}(\Omega)} \mathcal{J}_{\lambda,M}(u).$$

**Lemma 2.3** (i) *If* $u \in \mathcal{N}^+_{\lambda,M}(\Omega)$, *then* $\int_\Omega f|u|^{q+1}\,dx > 0$.
(ii) *If* $u \in \mathcal{N}^-_{\lambda,M}(\Omega)$, *then* $\int_\Omega g|u|^{r+1}\,dx > 0$.

The proof is immediate from (2.3) and (2.4).
Define the function $k_u : \mathbb{R}^+ \to \mathbb{R}$ as follows:

$$k_u(t) = t^{p-q-1}M\big(t^p\|u\|^p_{X_0}\big)\|u\|^p_{X_0} - t^{r-q}\int_\Omega g|u|^{r+1}\,dx \quad t > 0. \tag{2.7}$$

Obviously, $tu \in \mathcal{N}_{\lambda,M}(\Omega)$ if and only if $k_u(t) = \lambda\int_\Omega f|u|^{q+1}\,dx$. Moreover,

$$\begin{aligned} k'_u(t) &= (p-q-1)t^{p-q-2}M\big(t^p\|u\|^p_{X_0}\big)\|u\|^p_{X_0} + pt^{2p-q-2}M'\big(t^p\|u\|^p_{X_0}\big)\|u\|^{2p}_{X_0} \\ &\quad - (r-q)t^{r-q-1}\int_\Omega g|u|^{r+1}\,dx, \end{aligned} \tag{2.8}$$

which implies that $t^q k'_u(t) = h''_{\lambda,M}(t)$ for $tu \in \mathcal{N}_{\lambda,M}(\Omega)$. That is, $u \in \mathcal{N}^+_{\lambda,M}(\Omega)$ (or $\mathcal{N}^-_{\lambda,M}(\Omega)$) if and only if $k'_u(t) > 0$ (or $< 0$).
Set

$$\begin{aligned} A &= \frac{a(r-p+1)}{r-q}\left(\frac{a(p-q-1)}{(r-q)\|g\|_{L^{\nu_r}}S_\nu^{r+1}}\right)^{\frac{p-q-1}{r-p+1}} \\ &\quad + \frac{b(r-p^2+1)}{r-q}\left(\frac{a(p-q-1)}{(r-q)\|g\|_{L^{\nu_r}}S_\nu^{r+1}}\right)^{\frac{p^2-q-1}{r-p+1}}. \end{aligned} \tag{2.9}$$

**Lemma 2.4** *Assume that* (H1)-(H3) *hold. Let* $\lambda_2 = \frac{A}{\|f\|_{L^{\mu_q}}S_\mu^{q+1}}$. *Then, for each* $u \in X_0 \setminus \{0\}$ *and* $\lambda \in (0,\lambda_2)$, *we have*:

(1) *If* $\int_\Omega f|u|^{q+1}\,dx \le 0$, *then there exists a unique* $t^- = t^-(u) > t_{\max}(u)$ *such that* $t^-u \in \mathcal{N}^-_{\lambda,M}(\Omega)$ *and*

$$\mathcal{J}_{\lambda,M}\big(t^-u\big) = \sup_{t\ge0}\mathcal{J}_{\lambda,M}(tu) > 0. \tag{2.10}$$

(2) *If* $\int_\Omega f|u|^{q+1}\,dx > 0$, *then there exists a unique* $0 < t^+ = t^+(u) < t_{\max}(u) < t^-$ *such that* $t^+u \in \mathcal{N}^+_{\lambda,M}(\Omega)$, $t^-u \in \mathcal{N}^-_{\lambda,M}(\Omega)$ *and*

$$\mathcal{J}_{\lambda,M}\big(t^+u\big) = \inf_{0\le t\le t_{\max}(u)}\mathcal{J}_{\lambda,M}(tu), \qquad \mathcal{J}_{\lambda,M}\big(t^-u\big) = \sup_{t\ge0}\mathcal{J}_{\lambda,M}(tu). \tag{2.11}$$

*Proof* From (2.7) and (2.8), we have

$$k_u(t) = at^{p-q-1}\|u\|^p_{X_0} + bt^{p^2-q-1}\|u\|^{p^2}_{X_0} - t^{r-q}\int_\Omega g|u|^{r+1}\,dx \quad t \ge 0,$$

and

$$k'_u(t) = t^{-q-1}\left[a(p-q-1)t^{p-1}\|u\|^p_{X_0} + b\big(p^2-q-1\big)t^{p^2-1}\|u\|^{p^2}_{X_0} - (r-q)t^r\int_\Omega g|u|^{r+1}\,dx\right],$$

which implies that $k_u(0) = 0$, $k_u(t) \to -\infty$ as $t \to \infty$, $\lim_{t\to 0^+} k'_u(t) > 0$ and $\lim_{t\to\infty} k'_u(t) < 0$. Thus there exists a unique $t_{\max}(u) := t_{\max} > 0$ such that $k_u(t)$ is increasing on $(0, t_{\max})$, decreasing on $(t_{\max}, \infty)$ and $k'_u(t_{\max}) = 0$. Moreover, $t_{\max}$ is the root of

$$a(p-q-1)t_{\max}^{p-1}\|u\|_{X_0}^p + b\big(p^2-q-1\big)t_{\max}^{p^2-1}\|u\|_{X_0}^{p^2} - (r-q)t_{\max}^r \int_\Omega g|u|^{r+1}\,dx = 0. \qquad (2.12)$$

From (2.12), we obtain

$$t_{\max} \geq \left(\frac{a(p-q-1)\|u\|_{X_0}^p}{(r-q)\int_\Omega g|u|^{r+1}\,dx}\right)^{\frac{1}{r-p+1}} \geq \frac{1}{\|u\|_{X_0}}\left(\frac{a(p-q-1)}{(r-q)\|g\|_{L^{\nu_r}} S_\nu^{r+1}}\right)^{\frac{1}{r-p+1}} := t_*. \qquad (2.13)$$

Hence, we have by (2.12), (2.13), and the Hölder and Sobolev inequalities

$$\begin{aligned} k_u(t_{\max}) &= t_{\max}^{p-q-1}\left[a\|u\|_{X_0}^p + bt_{\max}^{p(p-1)}\|u\|_{X_0}^{p^2} - t_{\max}^{r-p+1}\int_\Omega g|u|^{r+1}\,dx\right] \\ &= \frac{a(r-p+1)}{r-q}t_{\max}^{p-q-1}\|u\|_{X_0}^p + \frac{b(r-p^2+1)}{r-q}t_{\max}^{p^2-q-1}\|u\|_{X_0}^{p^2} \\ &\geq \frac{a(r-p+1)}{r-q}t_*^{p-q-1}\|u\|_{X_0}^p + \frac{b(r-p^2+1)}{r-q}t_*^{p^2-q-1}\|u\|_{X_0}^{p^2} \\ &\geq \frac{a(r-p+1)}{r-q}\left(\frac{a(p-q-1)}{(r-q)\|g\|_{L^{\nu_r}} S_\nu^{r+1}}\right)^{\frac{p-q-1}{r-p+1}}\|u\|_{X_0}^{q+1} \\ &\quad + \frac{b(r-p^2+1)}{r-q}\left(\frac{a(p-q-1)}{(r-q)\|g\|_{L^{\nu_r}} S_\nu^{r+1}}\right)^{\frac{p^2-q-1}{r-p+1}}\|u\|_{X_0}^{q+1} \\ &= A\|u\|_{X_0}^{q+1}. \end{aligned} \qquad (2.14)$$

Case (1): $\int_\Omega f|u|^{q+1}\,dx \leq 0$. Then $k_u(t) = \lambda\int_\Omega f|u|^{q+1}\,dx$ has unique solution $t^- > t_{\max}$ and $k'_u(t^-) < 0$. On the other hand, we have

$$\begin{aligned} &a(p-q-1)\big\|t^-u\big\|_{X_0}^p + b\big(p^2-q-1\big)\big\|t^-u\big\|_{X_0}^{p^2} - (r-q)\int_\Omega g\big|t^-u\big|^{r+1}\,dx \\ &= \big(t^-\big)^{2+q}\bigg[a(p-q-1)\big(t^-\big)^{p-q-2}\|u\|_{X_0}^p + b\big(p^2-q-1\big)\big(t^-\big)^{p^2-q-2}\|u\|_{X_0}^{p^2} \\ &\quad - (r-q)\big(t^-\big)^{r-q-1}\int_\Omega g|u|^{r+1}\,dx\bigg] \\ &= \big(t^-\big)^{2+q}k'_u\big(t^-\big) < 0 \end{aligned}$$

and

$$\begin{aligned} &\big\langle \mathcal{J}'_{\lambda,M}\big(t^-u\big), t^-u\big\rangle \\ &= a\big(t^-\big)^p\|u\|_{X_0}^p + b\big(t^-\big)^{p^2}\|u\|_{X_0}^{p^2} - \lambda\big(t^-\big)^{q+1}\int_\Omega f|u|^{q+1}\,dx - \big(t^-\big)^{r+1}\int_\Omega g|u|^{r+1}\,dx \\ &= \big(t^-\big)^{q+1}\left[k_u\big(t^-\big) - \lambda\int_\Omega f|u|^{q+1}\,dx\right] = 0. \end{aligned}$$

Hence, $t^- u \in \mathcal{N}^-_{\lambda,M}(\Omega)$ or $t^- = 1$. For $t > t_{\max}$, we obtain

$$\begin{aligned}
&a(p-q-1)\|tu\|^p_{X_0} + b\bigl(p^2-q-1\bigr)\|tu\|^{p^2}_{X_0} - (r-q)\int_\Omega g|tu|^{r+1}\,dx < 0,\\
&\frac{d^2}{dt^2}\mathcal{J}_{\lambda,M}(tu) < 0,\\
&\frac{d}{dt}\mathcal{J}_{\lambda,M}(tu) = at^{p-1}\|u\|^p_{X_0} + bt^{p^2-1}\|u\|^{p^2}_{X_0} - \lambda t^q\int_\Omega f|u|^{q+1}\,dx - t^r\int_\Omega g|u|^{r+1}\,dx = 0,
\end{aligned}$$

for $t = t^-$. Thus, $\mathcal{J}_{\lambda,M}(u) = \sup_{t\geq 0}\mathcal{J}_{\lambda,M}(tu)$. Furthermore, we have

$$\mathcal{J}_{\lambda,M}(u) \geq \mathcal{J}_{\lambda,M}(tu) \geq \frac{a}{p}t^p\|u\|^p_{X_0} + \frac{b}{p^2}t^{p^2}\|u\|^{p^2}_{X_0} - \frac{1}{r+1}t^{r+1}\int_\Omega g|u|^{r+1}\,dx, \quad t \geq 0.$$

Let

$$h_u(t) = \frac{a}{p}t^p\|u\|^p_{X_0} + \frac{b}{p^2}t^{p^2}\|u\|^{p^2}_{X_0} - \frac{1}{r+1}t^{r+1}\int_\Omega g|u|^{r+1}\,dx, \quad t \geq 0.$$

Similar to the argument in the function $k_u(t)$, we see that $h_u(t)$ achieves its maximum at $t_m \geq (\frac{a\|u\|^p_{X_0}}{\int_\Omega g|u|^{r+1}\,dx})^{\frac{1}{r-p+1}}$. Thus, we have

$$\mathcal{J}_{\lambda,M}(u) \geq h_u(t_m) \geq \frac{ap(r+1-p) + b(r+1-p^2)}{p^2(r+1)}\left(\frac{a\|u\|^{r+1}_{X_0}}{\int_\Omega g|u|^{r+1}\,dx}\right)^{\frac{p}{r-p+1}} > 0.$$

Case (2): $\int_\Omega f|u|^{q+1}\,dx > 0$. By (2.14) and

$$\begin{aligned}
k_u(0) = 0 < \lambda\int_\Omega f|u|^{q+1}\,dx &\leq \lambda\|f\|_{L^{\mu_q}}S^{q+1}_\mu\|u\|^{q+1}_{X_0}\\
&< \lambda_2\|f\|_{L^{\mu_q}}S^{q+1}_\mu\|u\|^{q+1}_{X_0} = A\|u\|^{q+1}_{X_0} \leq k_u(t_{\max}), \quad \text{for } \lambda \in (0,\lambda_2).
\end{aligned}$$

Then there exist $t^+$ and $t^-$ such that $0 < t^+ < t_{\max} < t^-$,

$$k_u\bigl(t^+\bigr) = \lambda\int_\Omega f|u|^{q+1}\,dx = k_u\bigl(t^-\bigr).$$

Moreover, we have $k'_u(t^+) > 0$ and $k'_u(t^-) < 0$. Thus, there are two multiples of $u$ lying in $\mathcal{N}_{\lambda,M}(\Omega)$, that is, $t^+u \in \mathcal{N}^+_{\lambda,M}(\Omega)$ and $t^-u \in \mathcal{N}^-_{\lambda,M}(\Omega)$, and $\mathcal{J}_{\lambda,M}(t^-u) \geq \mathcal{J}_{\lambda,M}(tu) \geq \mathcal{J}_{\lambda,M}(t^+u)$ for each $t \in [t^+, t^-]$ and $\mathcal{J}_{\lambda,M}(t^+u) \leq \mathcal{J}_{\lambda,M}(tu)$ for each $t \in [0, t^+]$. Hence, $t^- = 1$ and

$$\mathcal{J}_{\lambda,M}(u) = \sup_{t\geq 0}\mathcal{J}_{\lambda,M}(tu), \qquad \mathcal{J}_{\lambda,M}\bigl(t^+u\bigr) = \inf_{0\leq t\leq t_{\max}}\mathcal{J}_{\lambda,M}(tu).$$

□

**Lemma 2.5** *If* (H3) *holds, then we have* $c_\lambda \leq c^+_\lambda < 0$.

*Proof* For $u \in \mathcal{N}^+_{\lambda,M}$, we get

$$(r-q)\lambda\int_\Omega f|u|^{q+1}\,dx > a(r-p+1)\|u\|^p_{X_0} + b\bigl(r-p^2+1\bigr)\|u\|^{p^2}_{X_0}.$$

Thus, we have

$$\begin{aligned} J_{\lambda,M}(u) &= \frac{a(r-p+1)}{p(r+1)}\|u\|_{X_0}^{p} + \frac{b(r-p^2+1)}{p^2(r+1)}\|u\|_{X_0}^{p^2} - \frac{\lambda(r-q)}{(q+1)(r+1)}\int_\Omega f|u|^{q+1}\,dx \\ &< \frac{a(r-p+1)}{r+1}\left[\frac{1}{p}-\frac{1}{q+1}\right]\|u\|_{X_0}^{p} + \frac{b(r-p^2+1)}{r+1}\left[\frac{1}{p^2}-\frac{1}{q+1}\right]\|u\|_{X_0}^{p^2} < 0, \end{aligned}$$

which implies that $c_\lambda \le c_\lambda^+ < 0$. □

## 3 Main results

Using the idea of Ni-Takagi [10], we have the following.

**Lemma 3.1** *For each $u \in \mathcal{N}_{\lambda,M}(\Omega)$, there exist $\epsilon > 0$ and a differentiable function $\xi : B(0;\epsilon) \subset X_0 \to \mathbb{R}^+$ such that $\xi(0) = 1$, the function $\xi(v)(u-v) \in \mathcal{N}_{\lambda,M}(\Omega)$ and*

$$\langle \xi'(0), v\rangle = \frac{W}{a(p-q-1)\|u\|_{X_0}^{p} + b(p^2-q-1)\|u\|_{X_0}^{p^2} - (r-q)\int_\Omega g|u|^{r+1}\,dx}, \tag{3.1}$$

*for all $v \in X_0$, where*

$$\begin{aligned} W &= ap\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\,dx\,dy \\ &\quad + bp^2\int_Q \frac{|u(x)-u(y)|^{p^2-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp^2}}\,dx\,dy \\ &\quad - (q+1)\lambda\int_\Omega f|u|^{q-1}uv\,dx - (r+1)\int_\Omega g|u|^{r-1}uv\,dx. \end{aligned} \tag{3.2}$$

*Proof* For $u \in \mathcal{N}_{\lambda,M}(\Omega)$, we define a function $\mathcal{F} : \mathbb{R} \times X_0 \to \mathbb{R}$ by

$$\begin{aligned} \mathcal{F}_u(\xi, w) &= \langle \mathcal{J}'_{\lambda,M}(\xi(u-w)), \xi(u-w)\rangle \\ &= \xi^p M(\xi^p\|u-w\|_{X_0}^{p})\|u-w\|_{X_0}^{p} \\ &\quad - \xi^{q+1}\lambda\int_\Omega f|u-w|^{q+1}\,dx - \xi^{r+1}\int_\Omega g|u-w|^{r+1}\,dx \\ &= a\xi^p\|u-w\|_{X_0}^{p} + b\xi^{p^2}\|u-w\|_{X_0}^{p^2} \\ &\quad - \xi^{q+1}\lambda\int_\Omega f|u-w|^{q+1}\,dx - \xi^{r+1}\int_\Omega g|u-w|^{r+1}\,dx. \end{aligned}$$

Then $\mathcal{F}_u(1,0) = \langle \mathcal{J}'_{\lambda,M}(u), u\rangle = 0$ and

$$\begin{aligned} \frac{d}{d\xi}\mathcal{F}_u(1,0) &= ap\|u\|_{X_0}^{p} + bp^2\|u\|_{X_0}^{p^2} - (q+1)\lambda\int_\Omega f|u|^{q+1}\,dx - (r+1)\int_\Omega g|u|^{r+1}\,dx \\ &= a(p-q-1)\|u\|_{X_0}^{p} + b(p^2-q-1)\|u\|_{X_0}^{p^2} - (r-q)\int_\Omega g|u|^{r+1}\,dx \neq 0. \end{aligned}$$

From the implicit function theorem, we know that there exist $\epsilon > 0$ and a differentiable function $\xi : B(0;\epsilon) \subset X_0 \to \mathbb{R}$ such that $\xi(0) = 1$,

$$\langle \xi'(0), v\rangle = \frac{W}{a(p-q-1)\|u\|_{X_0}^{p} + b(p^2-q-1)\|u\|_{X_0}^{p^2} - (r-q)\int_\Omega g|u|^{r+1}\,dx},$$

where $W$ is as in (3.2), and

$$\mathcal{F}_u\big(\xi(v), v\big) = 0 \quad \text{for all } v \in B(0;\epsilon)$$

which is equivalent to

$$\big\langle \mathcal{J}'_{\lambda,M}\big(\xi(v)(u-v)\big), \xi(v)(u-v)\big\rangle = 0 \quad \text{for all } v \in B(0;\epsilon),$$

which implies that $\xi(v)(u-v) \in \mathcal{N}_{\lambda,M}(\Omega)$. □

Similar to the argument in Lemma 3.1, we can obtain the following lemma.

**Lemma 3.2** *For each $u \in \mathcal{N}^-_{\lambda,M}(\Omega)$, there exist $\epsilon > 0$ and a differentiable function $\xi^- : B(0;\epsilon) \subset X_0 \to \mathbb{R}^+$ such that $\xi^-(0) = 1$, the function $\xi^-(v)(u-v) \in \mathcal{N}^-_{\lambda,M}(\Omega)$ and*

$$\big\langle (\xi^-)'(0), v\big\rangle = \frac{W}{a(p-q-1)\|u\|_{X_0}^{p} + b(p^2-q-1)\|u\|_{X_0}^{p^2} - (r-q)\int_\Omega g|u|^{r+1}\,dx},$$

*for all $v \in X_0$, where $W$ is as in* (3.2).

Let

(H4) $p < 2 + \frac{(r-1)q}{r}$.

Moreover, we let

$$p^* = \frac{(p-2)r}{r-1} - q$$

and

$$\lambda_3 = \left(\frac{a(p-q-1)(r-p^2+1)}{(r-q)(p^2-q-1)}\right)\left(\frac{a(p-q-1)}{r-q}\right)^{\frac{(p-q-1)}{(p-q-1-p^*)(r-1)}} \times \left(\frac{1}{\|f\|_{L^{\mu_q}} S_\mu^{q+1}}\right)\left(\frac{1}{\|g\|_{L^{\nu_r}} S_\nu^{r+1}}\right)^{\frac{(p-q-1)}{(r-1)(p-q-1-p^*)}}.$$

**Remark 3.1** By (H4) we know that $p^* < 0$.

**Lemma 3.3** *Assume that* (H1)-(H4) *hold. Let $\Gamma_0 = \min\{\lambda_1, \lambda_2, \lambda_3\}$, then for $\lambda \in (0, \Gamma_0)$:*

(i) *There exists a minimizing sequence $\{u_n\} \subset \mathcal{N}_{\lambda,M}(\Omega)$ such that*

$$\mathcal{J}_{\lambda,M}(u_n) = c_\lambda + o(1), \qquad \mathcal{J}'_{\lambda,M}(u_n) = o(1) \quad \textit{in } (X_0)^*.$$

(ii) *There exists a minimizing sequence* $\{u_n\} \subset \mathcal{N}^-_{\lambda,M}(\Omega)$ *such that*

$$\mathcal{J}_{\lambda,M}(u_n) = c^-_\lambda + o(1), \qquad \mathcal{J}'_{\lambda,M}(u_n) = o(1) \quad \textit{in } (X_0)^*.$$

*Proof* By the Ekeland variational principle [11] and Lemma 2.2, there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}_{\lambda,M}(\Omega)$ such that

$$\mathcal{J}_{\lambda,M}(u_n) < c_\lambda + \frac{1}{n} \tag{3.3}$$

and

$$\mathcal{J}_{\lambda,M}(u_n) < \mathcal{J}_{\lambda,M}(w) + \frac{1}{n}\|w - u_n\|_{X_0} \quad \forall w \in \mathcal{N}_{\lambda,M}(\Omega). \tag{3.4}$$

Let $n$ large enough, by Lemma 2.5, we obtain

$$\begin{aligned} \mathcal{J}_{\lambda,M}(u_n) &= \frac{a(r-p+1)}{p(r+1)}\|u_n\|^p_{X_0} + \frac{b(r-p^2+1)}{p^2(r+1)}\|u_n\|^{p^2}_{X_0} - \frac{\lambda(r-q)}{(q+1)(r+1)}\int_\Omega f|u_n|^{q+1}\,dx \\ &< c_\lambda + \frac{1}{n} < \frac{c_\lambda}{2}, \end{aligned}$$

which implies that

$$\|f\|_{L^{\mu q}} S^{q+1}_\mu \|u_n\|^{q+1}_{X_0} \ge \int_\Omega f|u_n|^{q+1}\,dx > -\frac{(q+1)(r+1)}{\lambda(r-q)}\frac{c_\lambda}{2} > 0. \tag{3.5}$$

This implies $u_n \not\equiv 0$ and by using (3.4), (3.5), and the Hölder inequality, we get

$$\|u_n\|_{X_0} > \left[-\frac{(q+1)(r+1)}{\lambda(r-q)}\frac{c_\lambda}{2}\|f\|^{-1}_{L^{\mu q}} S^{-(q+1)}_\mu\right]^{\frac{1}{q+1}} \tag{3.6}$$

and

$$\|u_n\|_{X_0} < \left[\frac{\lambda p(r-q)(r+1)}{a(q+1)(r+1)(r-p+1)}\|f\|_{L^{\mu q}} S^{q+1}_\mu\right]^{\frac{1}{p-q-1}}. \tag{3.7}$$

In the following, we will prove that

$$\left\|\mathcal{J}'_{\lambda,M}(u_n)\right\|_{(X_0)^*} \to 0 \quad \text{as } n \to \infty.$$

By using Lemma 3.1 with $u_n$ we get the functions $\xi_n : B(0;\epsilon_n) \to \mathbb{R}^+$ for some $\epsilon_n > 0$, such that $\xi_n(w)(u_n - w) \in \mathcal{N}_{\lambda,M}(\Omega)$. For fixed $n \in \mathbb{N}$, we choose $0 < \rho < \epsilon_n$. Let $u \in X_0$ with $u \not\equiv 0$ and let $w_\rho = \frac{\rho u}{\|u\|_{X_0}}$. Set $\eta_\rho = \xi_n(w_\rho)(u_n - w_\rho)$, since $\eta_\rho \in \mathcal{N}_{\lambda,M}(\Omega)$, we deduce from (3.4) that

$$\mathcal{J}_{\lambda,M}(\eta_\rho) - J_{\lambda,M}(u_n) \ge -\frac{1}{n}\|\eta_\rho - u_n\|_{X_0} \quad \forall w \in \mathcal{N}_{\lambda,M}(\Omega),$$

and by the mean value theorem, we obtain

$$\langle \mathcal{J}'_{\lambda,M}(u_n), \eta_\rho - u_n\rangle + o\big(\|\eta_\rho - u_n\|_{X_0}\big) \ge -\frac{1}{n}\|\eta_\rho - u_n\|_{X_0}.$$

Hence,

$$\begin{aligned}&\langle\mathcal{J}'_{\lambda,M}(u_n),-w_\rho\rangle+\bigl(\xi_n(w_\rho)-1\bigr)\langle\mathcal{J}'_{\lambda,M}(u_n),u_n-w_\rho\rangle\\&\quad\ge-\frac{1}{n}\|\eta_\rho-u_n\|_{X_0}+o\bigl(\|\eta_\rho-u_n\|_{X_0}\bigr).\end{aligned}\tag{3.8}$$

By $\xi_n(w_\rho)(u_n-w_\rho)\in\mathcal{N}_{\lambda,M}(\Omega)$ and (3.8) it follows that

$$\begin{aligned}&-\rho\left\langle\mathcal{J}'_{\lambda,M}(u_n),\frac{u}{\|u\|_{X_0}}\right\rangle+\bigl(\xi_n(w_\rho)-1\bigr)\langle\mathcal{J}'_{\lambda,M}(u_n)-\mathcal{J}'_{\lambda,M}(\eta_\rho),u_n-w_\rho\rangle\\&\quad\ge-\frac{1}{n}\|\eta_\rho-u_n\|_{X_0}+o\bigl(\|\eta_\rho-u_n\|_{X_0}\bigr).\end{aligned}$$

Thus,

$$\begin{aligned}\left\langle\mathcal{J}'_{\lambda,M}(u_n),\frac{u}{\|u\|_{X_0}}\right\rangle&\le\frac{1}{n\rho}\|\eta_\rho-u_n\|_{X_0}+\frac{1}{\rho}o\bigl(\|\eta_\rho-u_n\|_{X_0}\bigr)\\&\quad+\frac{(\xi_n(w_\rho)-1)}{\rho}\langle\mathcal{J}'_{\lambda,M}(u_n)-\mathcal{J}'_{\lambda,M}(\eta_\rho),u_n-w_\rho\rangle.\end{aligned}\tag{3.9}$$

Since

$$\|\eta_\rho-u_n\|_{X_0}\le\rho\bigl|\xi_n(w_\rho)\bigr|+\bigl|\xi_n(w_\rho)-1\bigr|\|u_n\|_{X_0}$$

and

$$\lim_{n\to\infty}\frac{|\xi_n(w_\rho)-1|}{\rho}\le\bigl\|\xi'_n(0)\bigr\|,$$

taking the limit $\rho\to0$ in (3.9), we obtain

$$\left\langle\mathcal{J}'_{\lambda,M}(u_n),\frac{u}{\|u\|_{X_0}}\right\rangle\le\frac{C}{n}\bigl(1+\bigl\|\xi'_n(0)\bigr\|\bigr)$$

for some constant $C>0$, independent of $\rho$. In the following, we will show that $\|\xi'_n(0)\|$ is uniformly bounded in $n$. From (3.1), (3.7), and the Hölder inequality, we obtain for some $\kappa>0$

$$\langle\xi'_n(0),v\rangle\le\frac{\kappa\|v\|_{X_0}}{a(p-q-1)\|u_n\|^p_{X_0}+b(p^2-q-1)\|u_n\|^{p^2}_{X_0}-(r-q)\int_\Omega g|u_n|^{r+1}\,dx}.$$

We only need to prove that

$$\left|a(p-q-1)\|u_n\|^p_{X_0}+b\bigl(p^2-q-1\bigr)\|u_n\|^{p^2}_{X_0}-(r-q)\int_\Omega g|u_n|^{r+1}\,dx\right|>c\tag{3.10}$$

for some $c>0$ and $n$ large enough. If (3.10) is fails, then there exists a subsequence $\{u_n\}$ such that

$$a(p-q-1)\|u_n\|^p_{X_0}+b\bigl(p^2-q-1\bigr)\|u_n\|^{p^2}_{X_0}-(r-q)\int_\Omega g|u_n|^{r+1}\,dx=o(1).\tag{3.11}$$

Combining (3.11) with (3.6), we may find a suitable constant $d > 0$ such that

$$\int_\Omega g|u_n|^{r+1}\,dx \ge d \quad \text{for } n \text{ sufficiently large.} \tag{3.12}$$

By (3.11) and $u_n \in \mathcal{N}_{\lambda,M}(\Omega)$, we have

$$\begin{aligned}
&\lambda \int_\Omega f|u_n|^{q+1}\,dx \\
&= a\|u_n\|_{X_0}^p + b\|u_n\|_{X_0}^{p^2} - \int_\Omega g|u_n|^{r+1}\,dx \\
&= \frac{1}{p^2-q-1}\big(a\big(p^2-q-1\big)\|u_n\|_{X_0}^p + b\big(p^2-q-1\big)\|u_n\|_{X_0}^{p^2}\big) - \int_\Omega g|u_n|^{r+1}\,dx \\
&\ge \frac{1}{p^2-q-1}\big(a(p-q-1)\|u_n\|_{X_0}^p + b\big(p^2-q-1\big)\|u_n\|_{X_0}^{p^2}\big) - \int_\Omega g|u_n|^{r+1}\,dx \\
&= \frac{r-q}{p^2-q-1}\int_\Omega g|u_n|^{r+1}\,dx - \int_\Omega g|u_n|^{r+1}\,dx + o(1) \\
&= \frac{r-p^2+1}{p^2-q-1}\int_\Omega g|u_n|^{r+1}\,dx + o(1).
\end{aligned} \tag{3.13}$$

Moreover, we have by (3.11) and (3.13)

$$\begin{aligned}
a(p-q-1)\|u_n\|_{X_0}^p &\le a(p-q-1)\|u_n\|_{X_0}^p + b\big(p^2-q-1\big)\|u_n\|_{X_0}^{p^2} \\
&= (r-q)\int_\Omega g|u_n|^{r+1}\,dx + o(1) \\
&\le \lambda\frac{(p^2-q-1)(r-q)}{r-p^2+1}\int_\Omega f|u_n|^{q+1}\,dx + o(1) \\
&\le \lambda\frac{(p^2-q-1)(r-q)}{r-p^2+1}\|f\|_{L^{\mu_q}} S_\mu^{q+1}\|u_n\|_{X_0}^{q+1} + o(1),
\end{aligned}$$

which implies that

$$\|u_n\|_{X_0} \le \left(\lambda\frac{(p^2-q-1)(r-q)}{a(p-q-1)(r-p^2+1)}\|f\|_{L^{\mu_q}} S_\mu^{q+1}\right)^{\frac{1}{p-q-1}} + o(1). \tag{3.14}$$

Let

$$\mathcal{I}_{\lambda,M}(u) = K(p,q,r)\left(\frac{\|u\|_{X_0}^{pr}}{\int_\Omega g|u_n|^{r+1}\,dx}\right)^{\frac{1}{r-1}} - \lambda\int_\Omega f|u|^{q+1}\,dx,$$

where

$$K(p,q,r) = \left(\frac{a(p-q-1)}{r-q}\right)^{\frac{r}{r-1}}\frac{r-p^2+1}{p^2-q-1}.$$

From (3.11), it is easy to see that

$$\|u_n\|_{X_0}^p \le \frac{r-q}{a(p-q-1)}\int_\Omega g|u_n|^{r+1}\,dx. \tag{3.15}$$

Thus,

$$\begin{aligned}\mathcal{I}_{\lambda,M}(u_n) &\le \left(\frac{a(p-q-1)}{r-q}\right)^{\frac{r}{r-1}} \frac{r-p^2+1}{p^2-q-1}\left(\frac{(\frac{r-q}{a(p-q-1)})^r(\int_\Omega g|u_n|^{r+1}\,dx)^r}{\int_\Omega g|u_n|^{r+1}\,dx}\right)^{\frac{1}{r-1}} \\ &\quad - \frac{r-p^2+1}{p^2-q-1}\int_\Omega g|u_n|^{r+1}\,dx + o(1) \\ &= o(1). \end{aligned} \tag{3.16}$$

But, by (3.12), (3.14), and $\lambda \in \Gamma_0$,

$$\begin{aligned}\mathcal{I}_{\lambda,M}(u_n) &\ge K(p,q,r)\left(\frac{\|u_n\|_{X_0}^{pr}}{\|g\|_{L^{\nu r}} S_\nu^{r+1}\|u_n\|_{X_0}^{r+1}}\right)^{\frac{1}{r-1}} - \lambda\|f\|_{L^{\mu q}} S_\mu^{q+1}\|u_n\|_{X_0}^{q+1} \\ &= \|u_n\|_{X_0}^{q+1}\left(K(p,q,r)\|g\|_{L^{\nu r}}^{\frac{1}{1-r}} S_\nu^{\frac{r+1}{1-r}}\|u_n\|_{X_0}^{p^*} - \lambda\|f\|_{L^{\mu q}} S_\mu^{q+1}\right) \\ &\ge \|u_n\|_{X_0}^{q+1}\left\{K(p,q,r)\|g\|_{L^{\nu r}}^{\frac{1}{1-r}} S_\nu^{\frac{r+1}{1-r}}\left[\lambda\frac{(p^2-q-1)(r-q)}{a(p-q-1)(r-p^2+1)}\|f\|_{L^{\mu q}} S_\mu^{q+1}\right]^{\frac{p^*}{p-q-1}} \right. \\ &\quad \left. - \lambda\|f\|_{L^{\mu q}} S_\mu^{q+1}\right\}, \end{aligned}$$

which contradicts (3.16), where $p^* = \frac{(p-2)r}{r-1} - q < 0$.

Hence, we obtain

$$\left\langle \mathcal{J}'_{\lambda,M}(u_n), \frac{u}{\|u\|_{X_0}}\right\rangle \le \frac{C}{n}.$$

This completes the proof of (i). Similarly, we can prove (ii) by using Lemma 3.2. □

**Theorem 3.4** *Assume that* (H1)-(H4) *hold*. *For each* $0 < \lambda < \Gamma_0$ ($\Gamma_0$ *is as in Lemma* 3.3), *the functional* $\mathcal{J}_{\lambda,M}$ *has a minimizer* $u_\lambda^+$ *in* $\mathcal{N}_{\lambda,M}^+(\Omega)$ *satisfying*:

(1) $\mathcal{J}_{\lambda,M}(u_\lambda^+) = c_\lambda^+ = c_\lambda$;

(2) $u_\lambda^+$ *is a solution of* (1.1).

*Proof* By Lemma 3.3(i), there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}_{\lambda,M}(\Omega)$ for $\mathcal{J}_{\lambda,M}$ on $\mathcal{N}_{\lambda,M}(\Omega)$ such that

$$\mathcal{J}_{\lambda,M}(u_n) = c_\lambda + o(1), \qquad \mathcal{J}'_{\lambda,M}(u_n) = o(1) \quad \text{in } (X_0)^*.$$

From Lemma 2.5 and the compact embedding theorem, we see that there exist a subsequence $\{u_n\}$ and $u_\lambda^+ \in X_0$ such that

$$u_n \rightharpoonup u_\lambda^+ \quad \text{weakly in } X_0$$

and

$$u_n \to u_\lambda^+ \quad \text{strongly in } L^\eta(\Omega) \text{ for } 1 < \eta < p_s^*. \tag{3.17}$$

In the following we will prove that $\int_\Omega f|u_\lambda^+|^{q+1}\,dx \neq 0$. In fact, if not, by (3.17) and the Hölder inequality we can obtain

$$\int_\Omega f|u_n|^{q+1}\,dx \to \int_\Omega f|u_\lambda^+|^{q+1}\,dx = 0$$

as $n \to \infty$. Hence,

$$a\|u_n\|_{X_0}^p + b\|u_n\|_{X_0}^{p^2} = \int_\Omega g|u_n|^{r+1}\,dx + o(1)$$

and

$$\mathcal{J}_{\lambda,M}(u_n) = a\left(\frac{1}{p} - \frac{1}{r+1}\right)\|u_n\|_{X_0}^p + b\left(\frac{1}{p^2} - \frac{1}{r+1}\right)\|u_n\|_{X_0}^{p^2} + o(1),$$

which contradicts $\mathcal{J}_{\lambda,M}(u_n) \to c_\lambda < 0$ as $n \to \infty$. Furthermore,

$$o(1) = \langle \mathcal{J}'_{\lambda,M}(u_n), \phi\rangle = \langle \mathcal{J}'_{\lambda,M}(u_\lambda^+), \phi\rangle + o(1) \quad \text{for all } \phi \in X_0.$$

Thus, $u_\lambda^+ \in \mathcal{N}_{\lambda,M}(\Omega)$ is a nonzero solution of (1.1) and $\mathcal{J}_{\lambda,M}(u_\lambda^+) \geq c_\lambda$. Next, we will prove that $\mathcal{J}_{\lambda,M}(u_\lambda^+) = c_\lambda$. Since

$$\begin{aligned}
\mathcal{J}_{\lambda,M}(u_\lambda^+) &= \frac{a}{p}\|u_\lambda^+\|_{X_0}^p + \frac{b}{p^2}\|u_\lambda^+\|_{X_0}^{p^2} - \frac{\lambda}{q+1}\int_\Omega f|u_\lambda^+|^{q+1}\,dx - \frac{1}{r+1}\int_\Omega g|u_\lambda^+|^{r+1}\,dx \\
&= \left(\frac{a}{p} - \frac{a}{r+1}\right)\|u_\lambda^+\|_{X_0}^p + \left(\frac{b}{p^2} - \frac{b}{r+1}\right)\|u_\lambda^+\|_{X_0}^{p^2} \\
&\quad + \left(\frac{\lambda}{r+1} - \frac{\lambda}{q+1}\right)\int_\Omega f|u_\lambda^+|^{q+1}\,dx \\
&\leq \liminf_{n\to\infty}\left[\left(\frac{a}{p} - \frac{a}{r+1}\right)\|u_n\|_{X_0}^p + \left(\frac{b}{p^2} - \frac{b}{r+1}\right)\|u_n\|_{X_0}^{p^2}\right. \\
&\quad \left. + \left(\frac{\lambda}{r+1} - \frac{\lambda}{q+1}\right)\int_\Omega f|u_n|^{q+1}\,dx\right] \\
&= \liminf_{n\to\infty}\mathcal{J}_{\lambda,M}(u_n) = c_\lambda.
\end{aligned}$$

Hence, $\mathcal{J}_{\lambda,M}(u_\lambda^+) = c_\lambda$. Moreover, we have $u_\lambda^+ \in \mathcal{N}_{\lambda,M}^+(\Omega)$. In fact, if $u_\lambda^+ \in \mathcal{N}_{\lambda,M}^-(\Omega)$, by Lemma 2.4, there are unique $t^+$ and $t^-$ such that $t^+u_\lambda^+ \in \mathcal{N}_{\lambda,M}^+(\Omega)$ and $t^-u_\lambda^+ \in \mathcal{N}_{\lambda,M}^-(\Omega)$, we have $t_\lambda^+ < t_\lambda^- = 1$. Since

$$\frac{d}{dt}\mathcal{J}_{\lambda,M}(t_\lambda^+u_\lambda^+) = 0 \quad \text{and} \quad \frac{d^2}{dt^2}\mathcal{J}_{\lambda,M}(t_\lambda^+u_\lambda^+) > 0,$$

there exists $t_\lambda^+ < t^* \leq t_\lambda^-$ such that $\mathcal{J}_{\lambda,M}(t_\lambda^+u_\lambda^+) < \mathcal{J}_{\lambda,M}(t^*u_\lambda^+)$. By Lemma 2.4, we get

$$\mathcal{J}_{\lambda,M}(t_\lambda^+u_\lambda^+) < \mathcal{J}_{\lambda,M}(t^*u_\lambda^+) \leq \mathcal{J}_{\lambda,M}(t_\lambda^-u_\lambda^+) = \mathcal{J}_{\lambda,M}(u_\lambda^+),$$

which is a contradiction. Since $\mathcal{J}_{\lambda,M}(u_\lambda^+) = \mathcal{J}_{\lambda,M}(|u_\lambda^+|)$ and $|u_\lambda^+| \in \mathcal{N}_{\lambda,M}^+(\Omega)$, we see that $u_\lambda^+$ is a solution of (1.1) by Lemma 2.3. □

Similarly, we can obtain the theorem of existence of a local minimum for $\mathcal{J}_{\lambda,M}$ on $\mathcal{N}^{-}_{\lambda,M}(\Omega)$ as follows.

**Theorem 3.5** *Assume that* (H1)-(H4) *hold*. *For each* $0 < \lambda < \Gamma_0$ ($\Gamma_0$ *is as in Lemma* 3.3), *the functional* $\mathcal{J}_{\lambda,M}$ *has a minimizer* $u^{-}_{\lambda}$ *in* $\mathcal{N}^{-}_{\lambda,M}(\Omega)$ *satisfying*:

(1) $\mathcal{J}_{\lambda,M}(u^{-}_{\lambda}) = c^{-}_{\lambda}$;

(2) $u^{-}_{\lambda}$ *is a solution of* (1.1).

Finally, we give the main result of this paper as follows.

**Theorem 3.6** *Suppose that the conditions* (H1)-(H4) *hold*. *Then there exists* $\Gamma_0 > 0$ *such that for* $\lambda \in (0, \Gamma_0)$, (1.1) *has at least two solutions*.

*Proof* From Theorems 3.4, 3.5, we see that (1.1) has two solutions $u^{+}_{\lambda}$ and $u^{-}_{\lambda}$ such that $u^{+}_{\lambda} \in \mathcal{N}^{+}_{\lambda,M}(\Omega)$, $u^{-}_{\lambda} \in \mathcal{N}^{-}_{\lambda,M}(\Omega)$. Since $\mathcal{N}^{+}_{\lambda,M}(\Omega) \cap \mathcal{N}^{-}_{\lambda,M}(\Omega) = \emptyset$, we see that $u^{+}_{\lambda}$ and $u^{-}_{\lambda}$ are different. □

**Remark 3.2** Obviously, if $p = 2$, then (H3) and (H4) hold. Moreover, if $p = 2$, $s = 1$, $a = 1$, and $b = 0$, then Theorem 3.6 is in agreement with Theorem 1.2 in [1].

**Competing interests**
The author declares that he has no competing interests.

**Author's contributions**
All results belong to CB.

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