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Discrete boundary value problem based on the fractional Gâteaux derivative

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Abstract
This article aims to organize positive solutions to discrete fractional boundary value problems for continuous Gâteaux differentiable functions. A generalized Gâteaux derivative is introduced using a fractional discrete operator for a Jumarie fractional operator. The method of finding solutions is based on critical point theorems of finite dimensional Banach spaces.

1 Introduction

In 1974, Diaz and Osler [1] imposed a discrete fractional difference operator based on an infinite series. In 1988, Gray and Zhang [2] presented a class of fractional difference operators and introduced the Leibniz formula. In 2009, Atici and Eloe [3] introduced methods for composing fractional sums and differences. Mathematicians, physicists, and engineers have recently employed fractional calculus to solve varieties of models of applied problems in different fields. In a previous work, we developed various classes of discrete fractional operators with one or two parameters for image and signal processing [4–6].

Researchers in the fields of computer science, neural networks, control systems, food processing, and economics rely on mathematical modeling because it naturally entails nonlinear difference equations. Therefore, many authors have widely developed various procedures and patterns, such as upper and lower solutions, fixed point theorems, and the Brouwer degree, to study discrete models [7–11]. Critical point theory has recently attracted the attention of many researchers. The theory guarantees the outcomes of both ordinary and partial differential problems. Hence, critical point theory is the main tool for finding solutions to fractional discrete nonlinear equations.

This study aims to establish positive solutions to discrete fractional boundary value problems for continuous Gâteaux differentiable functions. A generalized Gâteaux derivative is introduced using a fractional discrete operator. Some results are refined in this direction. The method for finding solutions is based on critical point theorems of finite dimensional Banach spaces. The applications are illustrated to study the abstract outcomes.

2 Main methods

This section deals with preliminaries and some concepts.

Definition 2.1 [12] For a continuous function $g : \mathbb{R} \to \mathbb{R}$ and a constant $\ell > 0$, the forward operator $FW(\ell)$ is defined by the equality

$$FW(\ell)g(x) := g(x + \ell).$$

The fractional difference on the right and of the order $\wp$, $0 < \wp < 1$ of $g(x)$ is defined by the formula

$$\triangle^{\wp} g(x) := \bigl(FW(\ell) - 1\bigr)^{\wp} g(x)$$

with its fractional derivative on the right

$$g_+^{(\wp)}(x) = \lim_{\ell \to 0} \frac{\triangle^{\wp}[g(x) - g(0)]}{\ell^{\wp}}.$$

Definition 2.2 [13] Let $(\Xi, \|\cdot\|)$ be a real Banach space with the dual space Ξ^*. A function $\varphi : \Xi \to \mathbb{R}$ is called Gâteaux differentiable at $x \in \Xi$ if $\chi := \varphi'(x) \in \Xi^*$ satisfies

$$\lim_{\ell \to 0^+} \frac{\varphi(x + \ell y) - \varphi(x)}{\ell} = \varphi'(x)(y), \quad \forall y \in \Xi.$$

Definition 2.3 [13] A function $\varphi : \Xi \to \mathbb{R}$ is called Gâteaux differentiable and verifies the Palais-Smale condition ((PS)-condition) if any bounded sequence $\{x_n\}$ with $\lim_{n\to\infty} \|\varphi'(x_n)\|_{\Xi^*} = 0$ has a convergent subsequence.

Combining Definitions 2.1 and 2.2, we obtain a fractional Gâteaux derivative.

Definition 2.4 Let $(\Xi, \|\cdot\|)$ be a real Banach space with the dual space Ξ^*. A function $\varphi : \Xi \to \mathbb{R}$ has a fractional Gâteaux derivative, of the order $0 < \wp < 1$ at $x \in \Xi$ if $\varphi^{(\wp)}(x) \in \Xi^*$ exists such that for a constant $\ell > 0$, the forward operator $FW_y(\ell)$, $y \in \Xi$ is defined by the equality

$$FW_y(\ell)\varphi(x) := \varphi(x + \ell y),$$

with the fractional difference on the right

$$\triangle_y^{\wp} \varphi(x) := \bigl(FW_y(\ell) - 1\bigr)^{\wp} \varphi(x),$$

and its fractional derivative on the right

$$\lim_{\ell \to 0^+} \frac{\triangle_y^{\wp}[\varphi(x) - \varphi(0)]}{\ell^{\wp}} = \varphi^{(\wp)}(x)(y), \quad \forall y \in \Xi.$$

The next results show some properties of the fractional Gâteaux derivative, which basically are generalizations of some results given in [14]. Therefore, we skip the proofs.

Theorem 2.1 *Suppose that Ξ is a real Banach space and $F, G : \Xi \to \mathbb{R}$ are two continuously fractional Gâteaux differentiable functions. Assume that*

$$T := F - G$$

and that $x_0 \in \Xi$ and $\rho_1, \rho_2 \in \mathbb{R}$ exist, with $\rho_1 < F(x_0) < \rho_2$, thereby obtaining

$$\sup_{y\in F^{-1}]\rho_1,\rho_2[} G(y) < G(x_0) + \rho_2 - F(x_0), \tag{1}$$

$$\sup_{y\in F^{-1}]-\infty,\rho_1]} G(y) < G(x_0) + \rho_1 - F(x_0). \tag{2}$$

Presume that T satisfies the (PS)-condition. Then $y_0 \in F^{-1}]\rho_1, \rho_2[$ such that $T(y_0) < T(y)$, $y \in F^{-1}]\rho_1, \rho_2[$, and $T'(y_0) = 0$.

Theorem 2.2 *Let Ξ be a real Banach space, and let $F, G : \Xi \to \mathbb{R}$ be two continuous fractional Gâteaux differentiable functions. Assume that*

$$T = F - G$$

and that $x_0 \in \Xi$ and $\rho \in \mathbb{R}$ such that $\rho > F(x_0)$

$$\sup_{y\in F^{-1}]-\infty,\rho[} G(y) < G(x_0) + \rho - F(x_0). \tag{3}$$

We infer that T satisfies the (PS)-condition. Then $y_0 \in F^{-1}]-\infty, \rho[$ such that $T(y_0) < T(y)$, $y \in F^{-1}]-\infty, \rho[$, and $T'(y_0) = 0$.

Theorem 2.3 *Let Ξ be a real Banach space and let $F, G : \Xi \to \mathbb{R}$ be two continuous fractional Gâteaux differentiable functions. Assume that*

$$T = F - G,$$

where T is bounded from below, and $x_1 \in \Xi$ and $\rho \in \mathbb{R}$ with $\rho < F(x_1)$ such that

$$\sup_{y\in F^{-1}]-\infty,\rho]} G(y) < G(x_1) + \rho - F(x_1). \tag{4}$$

We deduce that T satisfies the (PS)-condition. Then $y_1 \in F^{-1}]\rho, +\infty[$ such that $T(y_1) \leq T(y)$, $y \in F^{-1}]\rho, +\infty[$, and $T'(y_1) = 0$.

We apply the above critical point theorems to find solutions to the fractional difference equation, taking the type

$$\begin{aligned} &- \triangle_y^{\wp} \left(\psi_p\left(\triangle_y^{\wp} \mu_{k-1}\right)\right) + \sigma_k \psi_p(\mu_k) = h(k, \mu_k), \quad k \in [1, N], \\ &\mu_0 = \mu_{N+1} = 0, \end{aligned} \tag{5}$$

where $h : [1, N] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\sigma_k \geq 0$, and $\psi_p(\varsigma) := |\varsigma|^{p-1}\varsigma$, $1 < p < +\infty$.

Define the Banach space as

$$\mathcal{B} := \left\{\mu : [0, N + 1] \to \mathbb{R} : \mu_0 = \mu_{n+1} = 0\right\}$$

endowed with a discrete norm

$$\|\mu\| := \left(\sum_{k=1}^{N+1} \left| \triangle_{y,k-1}^{\wp} \mu \right|^p + \sigma_k |\mu|^p \right)^{1/p}$$

such that (see [7, Lemma 2.2])

$$\max_{k\in[1,N]} |\mu_k| \leq \frac{(N+1)^{(p-1)/p}}{2} \|\mu\|, \quad \forall \mu \in \mathcal{B}.$$

Let

$$F(\mu) := \frac{\|\mu\|^p}{p}, \qquad G(\mu) := \sum_{k=1}^{N} H(k,\mu_k), \qquad T(\mu) := F(\mu) - G(\mu), \quad \forall \mu \in \mathcal{B},$$

where

$$H(k,t) := \int_0^t h(k,u)\,du, \quad \forall (k,t) \in [1,N] \times \mathbb{R}.$$

Note that $T \in C^1(\mathcal{B},\mathbb{R})$, $T(0) = 0$, and that all the critical points of T are the solutions of (5).

3 Main findings

In this section, we introduce at least one solution to problem (5) with the following results.

Theorem 3.1 *Consider three positive constants* C, C_1, *and* C_2 *such that*

$$C_1 < \frac{(2+\overline{\sigma})^{1/p}(N+1)^{(p-1)/p}C}{2} < C_2, \tag{6}$$

where $\overline{\sigma} := \sum_{k=1}^{N} \sigma_k$. *If*

$$\beta_1 := \frac{\sum_{k=1}^{N} \max_t H(k,t) - \sum_{k=1}^{N} H(k,C)}{(2C_1)^p - (2+\overline{\sigma})(N+1)^{p-1}C^p} \leq \frac{1}{p(N+1)^{p-1}}$$

and

$$\beta_2 := \frac{\sum_{k=1}^{N} \max_t H(k,t) - \sum_{k=1}^{N} H(k,C)}{(2C_2)^p - (2+\overline{\sigma})(N+1)^{p-1}C^p} \leq \frac{1}{p(N+1)^{p-1}},$$

where $(2C_i)^p \neq (2+\overline{\sigma})(N+1)^{p-1}C^p$, $i = 1,2$, *then* (5) *has at least one non-trivial solution* μ^* *such that*

$$\frac{2C_1}{(N+1)^{(p-1)/p}} < \left\|\mu^*\right\| < \frac{2C_2}{(N+1)^{(p-1)/p}}.$$

Proof We aim to apply Theorem 2.1 on T, because the critical points of T are solutions to (5). Choose $\omega \in \mathcal{B}$ and define

$$\omega_k := \begin{cases} C, & k \in [1,N], \\ 0, & k = 0, k = N+1. \end{cases}$$

Thus, we obtain

$$F(\omega) = \frac{2+\overline{\sigma}}{p} C^p.$$

According to condition (6), a computation implies that

$$\rho_1 := \frac{(2C_1)^p}{p(N+1)^{p-1}} < F(\omega) < \frac{(2C_2)^p}{p(N+1)^{p-1}} := \rho_2.$$

We proceed to consider conditions (1) and (2):

$$\begin{aligned}
\sup_{y \in F^{-1}]-\infty,\rho_1]} \left(\frac{G(y) - G(\omega)}{\rho_1 - F(\omega)} \right) &\leq \frac{\sum_{k=1}^N \max_t H(k,t) - \sum_{k=1}^N H(k,C)}{\frac{(2C_1)^p}{p(N+1)^{p-1}} - \frac{2+\overline{\sigma}C^p}{p}} \\
&= p(N+1)^{p-1} \frac{\sum_{k=1}^N \max_t H(k,t) - \sum_{k=1}^N H(k,C)}{(2C_1)^p - (2+\overline{\sigma})(N+1)^{p-1}C^p} \\
&:= p(N+1)^{p-1}\beta_1 \leq 1.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\sup_{y \in F^{-1}]\rho_1,\rho_2[} \left(\frac{G(y) - G(\omega)}{\rho_2 - F(\omega)} \right) &\leq \frac{\sum_{k=1}^N \max_t H(k,t) - \sum_{k=1}^N H(k,C)}{\frac{(2C_2)^p}{p(N+1)^{p-1}} - \frac{2+\overline{\sigma}C^p}{p}} \\
&= p(N+1)^{p-1} \frac{\sum_{k=1}^N \max_t H(k,t) - \sum_{k=1}^N H(k,C)}{(2C_2)^p - (2+\overline{\sigma})(N+1)^{p-1}C^p} \\
&:= p(N+1)^{p-1}\beta_2 \leq 1.
\end{aligned}$$

Thus, in virtue of Theorem 2.1, we conclude that T has at least one critical point μ^* such that

$$\rho_1 < F\big(\mu^*\big) < \rho_2,$$

and we obtain

$$\frac{2C_1}{(N+1)^{(p-1)/p}} < \|\mu^*\| < \frac{2C_2}{(N+1)^{(p-1)/p}}. \qquad \square$$

By letting $C_1 = 0$, $C_2 = C$, we obtain the following result.

Corollary 3.1 *Consider two positive constants C_1 and C with*

$$(2+\overline{\sigma})^{1/p}(N+1)^{(p-1)/p} < 2. \tag{7}$$

If

$$\beta_1 := \frac{\sum_{k=1}^N H(k,C) - \sum_{k=1}^N \max_t H(k,t)}{(2+\overline{\sigma})(N+1)^{p-1}} \leq \frac{C^p}{p(N+1)^{p-1}}$$

and

$$\beta_2 := \frac{\sum_{k=1}^{N} \max_t H(k,t) - \sum_{k=1}^{N} H(k,C)}{2^p - (2+\overline{\sigma})(N+1)^{p-1}} \le \frac{C^p}{p(N+1)^{p-1}},$$

where $2^p \neq (2+\overline{\sigma})(N+1)^{p-1} \neq 0$, *then* (5) *has at least one non-trivial solution* μ^* *such that*

$$\|\mu^*\| < \frac{2C}{(N+1)^{(p-1)/p}}.$$

Theorem 3.2 *Consider two positive constants C and* ϵ *such that*

$$\frac{(2+\overline{\sigma})^{1/p}(N+1)^{(p-1)/p}C}{2} < \epsilon. \tag{8}$$

If

$$\beta := \frac{\sum_{k=1}^{N} \max_t H(k,t) - \sum_{k=1}^{N} H(k,C)}{(2\epsilon)^p - (2+\overline{\sigma})(N+1)^{p-1}C^p} \le \frac{1}{p(N+1)^{p-1}},$$

where $(2\epsilon)^p \neq (2+\overline{\sigma})(N+1)^{p-1}C^p$, *then* (5) *has at least one non-trivial solution* μ^* *such that*

$$\|\mu^*\| \le \frac{2\epsilon}{(N+1)^{(p-1)/p}}.$$

Proof We aim to employ Theorem 2.2 on T, where all critical points of T are solutions to (5). Choose $\omega \in \mathcal{B}$ as in Theorem 2.1. We have

$$F(\omega) = \frac{2+\overline{\sigma}}{p} C^p.$$

According to condition (8), a calculation yields

$$F(\omega) < \frac{(2\epsilon)^p}{p(N+1)^{p-1}} := \rho.$$

We proceed to consider condition (3):

$$\begin{aligned}
\sup_{y \in F^{-1}]-\infty,\rho[} \left(\frac{G(y) - G(\omega)}{\rho - F(\omega)} \right) &\le \frac{\sum_{k=1}^{N} \max_t H(k,t) - \sum_{k=1}^{N} H(k,C)}{\frac{(2\epsilon)^p}{p(N+1)^{p-1}} - \frac{2+\overline{\sigma}C^p}{p}} \\
&= p(N+1)^{p-1} \frac{\sum_{k=1}^{N} \max_t H(k,t) - \sum_{k=1}^{N} H(k,C)}{(2\epsilon)^p - (2+\overline{\sigma})(N+1)^{p-1}C^p} \\
&:= p(N+1)^{p-1}\beta \le 1.
\end{aligned}$$

Thus, in view of Theorem 2.2, we observe that T has at least one critical point μ^* such that

$$F(\mu^*) < \rho,$$

and we obtain

$$\|\mu^*\| \leq \frac{2\epsilon}{(N+1)^{(p-1)/p}}.$$

This completes the proof. □

Theorem 3.3 *Consider two positive constants C and γ such that*

$$\gamma < \frac{(2+\overline{\sigma})^{1/p}(N+1)^{(p-1)/p}C}{2}. \tag{9}$$

If

$$\eta := \frac{\sum_{k=1}^{N}\max_t H(k,t) - \sum_{k=1}^{N} H(k,C)}{(2\gamma)^p - (2+\overline{\sigma})(N+1)^{p-1}C^p} \leq \frac{1}{p(N+1)^{p-1}},$$

where $(2\gamma)^p \neq (2+\overline{\sigma})(N+1)^{p-1}C^p$, then (5) *has at least one non-trivial solution.*

Proof Our aim is to employ Theorem 2.3 on T, because the critical points of T are solutions to (5). We put $\omega \in \mathcal{B}$ as in Theorem 3.1 such that

$$F(\omega) = \frac{2+\overline{\sigma}}{p}C^p.$$

According to condition (9), a manipulation leads to

$$\rho := \frac{(2\gamma)^p}{p(N+1)^{p-1}} < F(\omega).$$

Thus, we consider condition (4):

$$\begin{aligned}
\sup_{y\in F^{-1}]-\infty,\rho]}\left(\frac{G(y)-G(\omega)}{\rho - F(\omega)}\right) &\leq \frac{\sum_{k=1}^{N}\max_t H(k,t) - \sum_{k=1}^{N} H(k,C)}{\frac{(2\gamma)^p}{p(N+1)^{p-1}} - \frac{2+\overline{\sigma}C^p}{p}} \\
&= p(N+1)^{p-1}\frac{\sum_{k=1}^{N}\max_t H(k,t) - \sum_{k=1}^{N} H(k,C)}{(2\gamma)^p - (2+\overline{\sigma})(N+1)^{p-1}C^p} \\
&:= p(N+1)^{p-1}\eta \leq 1.
\end{aligned}$$

Therefore, in virtue of Theorem 2.3, we conclude that T has at least one critical point. The boundedness of the solution is due to the (SP)-condition. The above completes the proof. □

4 Multiplicity findings

We impose multiplicity solutions in this section.

Theorem 4.1 *Let the hypothesis of Theorem* 3.1 *hold. Assume that $h(k,0) \neq 0$, $k \in [1,N]$. Consider two constants $\kappa_1 > p$ and $\kappa_2 > 0$ such that for $|\chi| \geq \kappa_2$, one has*

$$0 < \kappa_1 H(k,\chi) \leq \chi h(k,\chi). \tag{10}$$

If T' is a Lipschitz continuous functional in $\mathcal{B}$, then (5) *admits at least two solutions.*

Proof In view of Theorem 3.1, (5) has at least one solution, say μ_1. We use the mountain pass theorem to identify the second solution. The functional $T = F - G$ is obviously in class $C^1(\mathcal{B}, \mathbb{R})$ with $T(0) = 0$. Assume that μ_1 is a strict local minimum for T in $\mathcal{B}$. Thus a constant $\varrho > 0$ satisfies

$$\inf_{\|\mu-\mu_1\|=\varrho} T(\mu) > T(\mu_1), \quad \forall \mu \in \mathcal{B}.$$

After integrating (10), we observe that there are two positive constants b_1 and b_2 such that

$$H(k,t) \geq b_1|t|^{\kappa_1} - b_2, \quad \forall k \in [1,N], t \in \mathbb{R}.$$

We then proceed to identify the second solution of (5). Let $\nu \in \mathcal{B}\backslash\{0\}$. A calculation then yields

$$\begin{aligned} T(t\nu) &= (F-G)(t\nu) \\ &= \frac{1}{p}\|t\nu\|^p - \sum_{k=1}^{N} H(k, t\nu_k) \\ &\leq \frac{t^p}{p}\|\nu\|^p - t^{\kappa_1} b_1 \sum_{k=1}^{N} |\nu_k|^{\kappa_1} + b_2 N. \end{aligned}$$

Given that $\kappa_1 > p$, we have

$$\frac{t^p}{p}\|\nu\|^p - t^{\kappa_1} b_1 \sum_{k=1}^{N} |\nu_k|^{\kappa_1} + b_2 N \to -\infty, \quad \text{as } t \to +\infty.$$

This formula implies that the functional T has a critical point $\mu_2 := \inf_{\nu\in\mathcal{B}} \max_{t\in\mathbb{R}} T(\nu(t))$, which is the second solution for (5). This condition completes the proof. □

In the same manner as in Theorem 4.1, we have the following results.

Theorem 4.2 *Let the hypothesis of Theorem* 3.2 *be satisfied. Assume that* $h(k,0) \neq 0$, $k \in [1,N]$. *Consider the assertion of* (10). *If* T' *is a Lipschitz continuous functional in* $\mathcal{B}$, *then* (5) *admits at least two solutions.*

Theorem 4.3 *Let the hypothesis of Theorem* 3.3 *hold. Assume that* $h(k,0) \neq 0$, $k \in [1,N]$. *Consider the inequality* (10). *If* T' *is a Lipschitz continuous functional in* $\mathcal{B}$, *then* (5) *admits at least two solutions.*

5 Positive findings

We focus on the positive results of problem (5). For this purpose, we assume

$$h(k,t) := \begin{cases} \lambda_k \hbar(t), & t \geq 0, \\ 0, & t < 0, \end{cases} \tag{11}$$

where $\lambda : [1,N] \to \mathbb{R}$ is a non-negative, non-zero function and $\hbar : [0,+\infty) \to \mathbb{R}$ is a continuous function with $\hbar(0) = 0$. We have the following result.

Theorem 5.1 *Let $h(k,t)$ be defined in* (11) *and* (8) *let be satisfied. Then the problem*

$$-\triangle_y^{\wp}\left(\psi_p\left(\triangle_y^{\wp}\mu_{k-1}\right)\right)+\sigma_k\psi_p(\mu_k)=\lambda_k\hbar(\mu_k),\quad k\in[1,N],\tag{12}$$
$$\mu_0=\mu_{N+1}=0$$

has at least one positive solution μ_k.

Proof By Theorem 3.2, we conclude that (12) has at least one solution μ^*. If $\sigma_k = 0$ and $\mu_j := \min_{k\in[1,N]}\mu_k$, then we obtain

$$-\triangle_y^{\wp}\left(\psi_p\left(\triangle_y^{\wp}\mu_{k-1}\right)\right)\geq 0,\quad k\in[1,N].$$

However, ψ_p is strictly monotone, and $\mu_j-\mu_{j-1}=0$ and $\mu_{j+1}-\mu_j=0$. Consequently, we have a positive solution. Let $\sigma_k\neq 0$, we attain

$$\begin{aligned}\psi_p\left(\triangle_y^{\wp}\mu_j\right)&\geq\left|\triangle_y^{\wp}\mu_j\right|^{p-1}(\mu_{j+1}-\mu j)^{\wp}-\left|\triangle_y^{\wp}\mu_{j-1}\right|^{p-1}(\mu_j-\mu_{j-1})^{\wp}\\&\geq 0\end{aligned}$$

and

$$\begin{aligned}\sigma_j\psi_p(\mu_j)&\geq|\mu_j|^{p-1}(\mu_{j+1}-\mu j)^{\wp}-|\mu_{j-1}|^{p-1}(\mu_j-\mu_{j-1})^{\wp},\\&\geq 0\end{aligned}$$

so $\mu_j\geq 0$ and $\mu\geq 0$. This completes the proof. □

In the same manner as Theorem 5.1, we have the following.

Theorem 5.2 *Let $h(k,t)$ be defined in* (11) *and let* (9) *hold. Then problem* (13) *has at least one positive solution μ_k.*

6 Example

Assume the problem

$$-\triangle_y^{\wp}(\mu_{k-1})+\sigma_k\mu_k=\frac{\lambda}{8}+|\mu_k|^2\mu_k,\quad k\in[1,N],\tag{13}$$
$$\mu_0=\mu_{N+1}=0.$$

Let $h(\zeta):=\frac{\lambda}{8}+|\zeta|^2\zeta$, $\zeta\in\mathbb{R}$. It is clear that $h(0)\neq 0$, when $\lambda\neq 0$ and $\lim_{\zeta\to 0^+}\frac{h(\zeta)}{\zeta}=+\infty$. Moreover, we let $\kappa_1=4>p=3$ and $\kappa_2=1\Rightarrow|\zeta|\geq 1$. Thus, we conclude that

$$0<4H(\zeta)\leq\zeta h(\zeta),\quad\forall|\zeta|\geq 1,\lambda>0.$$

Obviously, $T'(\mu_k)=(\frac{\|\mu\|^p}{p}-\sum_{k=1}^N H(k,\mu_k))'$ is a Lipschitz continuous functional. Hence in view of Theorem 4.1, problem (13) has at least two solutions.

Now let $\sigma_k = 0$ and $\lambda = 0$. We then have the following problem:

$$-\triangle_y^{\wp}(\mu_{k-1}) = |\mu_k|^2\mu_k, \quad k \in [1, N], \qquad (14)$$
$$\mu_0 = \mu_{N+1} = 0$$

such that $\hbar(t) := t|t|^2$, $t \in [0, \infty)$. Hence, in view of Theorem 5.1, where inequality (8) is satisfied for $\overline{\sigma} = 0$, problem (14) has at least one positive solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors jointly worked on deriving the results and approved the final manuscript.

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